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ZERO-ENERGY FIELDS ON COMPLEX PROJECTIVE SPACE

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Abstract

We consider complex projective space with its Fubini–Study metric and the X-ray transform defined by integration over its geodesics. We identify the kernel of this transform acting on symmetric tensor fields.

1. Introduction

Suppose $\omega_{ab\cdots c}$ is a smooth symmetric covariant tensor field defined on a Riemannian manifold M. Suppose γ is a smooth oriented curve on M joining points p and q. Let X^a denote the unit vector field defined along γ and tangent to γ consistent with its orientation. We obtain a real number $\int_\gamma \omega_{ab\cdots c}$ by integrating the function $X^aX^b\cdots X^c\omega_{ab\cdots c}$ along γ with respect to arc length.

Suppose that $\phi_{b\cdots c}$ is a symmetric covariant tensor field such that $\omega_{ab\cdots c} = \nabla_{(a}\phi_{b\cdots c)}$ where ∇_a is the Levi-Civita connection and round brackets denote symmetrization over the indices they enclose. Suppose γ is a geodesic. This means that $X^a\nabla_aX^b = 0$ and, in this case,

$$X^a\nabla_a(X^b\cdots X^c\phi_{b\cdots c}) = X^aX^b\cdots X^c\nabla_{(a}\phi_{b\cdots c)} = X^aX^b\cdots X^c\omega_{ab\cdots c}.$$

Therefore $\int_\gamma \omega_{ab\cdots c} = \big[X^b\cdots X^c\phi_{b\cdots c}\big]_p^q$, and, in particular, if γ is a closed geodesic, then $\int_\gamma \omega_{ab\cdots c} = 0$. On complex projective space $\mathbb{CP}_n$ with its standard Fubini–Study metric [**3**, **8**], all geodesics are closed and the *X-ray transform* on symmetric tensor fields associates to $\omega_{ab\cdots c}$ the function

$$\gamma \longmapsto \int_\gamma \omega_{ab\cdots c}$$

defined on the space of geodesics on $\mathbb{CP}_n$. We shall refer to fields in the kernel of this transform as having *zero energy*. We have just observed that fields of the form $\nabla_{(a}\phi_{b\cdots c)}$ have zero energy. The main aim of this article is to prove the converse—namely, the following theorem.

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Theorem 1. *On $\mathbb{CP}_n$ for $n \geq 2$, a smooth symmetric covariant tensor field $\omega_{ab\cdots c}$ of valence $p \geq 1$ having zero energy must be of the form $\nabla_{(a}\phi_{b\cdots c)}$ for some smooth symmetric field $\phi_{b\cdots c}$ of valence $p-1$.*

This theorem was first proved for $p = 1$ in [**21**]. In case $p = 2$, it was first proved by Tsukamoto [**37**]; other proofs in this case can be found in [**18**, **19**] and Chapter III of [**22**]. Tsukamoto's proof for $p = 2$ heavily relied on harmonic analysis on $\mathbb{CP}_2$. In [**22**, Theorem 3.40] harmonic analysis on complex projective space was eliminated from the proof of case $p = 1$ (and already in [**19**] harmonic analysis was severely reduced for the case $p = 2$). In a proof of Theorem 1 for $n = 2$ given in [**12**], harmonic analysis on $\mathbb{CP}_2$ arose in the guise of twistor theory; this proof relied on the Fubini–Study metric on $\mathbb{CP}_2$ being (anti-)self-dual and was therefore limited to the case $n = 2$. In this article, our proof is uniform for all n and p; it completely eliminates any harmonic analysis on $\mathbb{CP}_n$.

Inspired by a remark of J.-P. Demailly (see [**19**, Introduction]) in case $p = 1$, our plan is to deduce Theorem 1 from the corresponding statement for real projective space $\mathbb{RP}_n$ with its usual round metric (inherited from the round n-sphere). The truth of this statement for $\mathbb{RP}_n$ has been shown by various means [**1**, **3**, **11**, **15**, **22**, **23**, **29**, **30**]. (The precise method of proof for $\mathbb{RP}_n$ will not enter our discussion for $\mathbb{CP}_n$.) The point is that the standard embedding $\mathbb{RP}_n \hookrightarrow \mathbb{CP}_n$ induced by $\mathbb{R}^{n+1} \hookrightarrow \mathbb{C}^{n+1}$ is totally geodesic. Furthermore, all translates of this standard $\mathbb{RP}_n \hookrightarrow \mathbb{CP}_n$ by $\mathrm{SU}(n+1)$, the isometry group of $\mathbb{CP}_n$, are totally geodesic, and so this provides many submanifolds on which the kernel of the X-ray transform is known. It is immediate, for example, that injectivity of the X-ray transform on smooth functions on $\mathbb{RP}_n$ implies that the same is true on $\mathbb{CP}_n$. More generally, we shall require some algebraic link between tensors on $\mathbb{CP}_n$ and on these embedded real projective spaces. Such a link is the subject of the following two sections.

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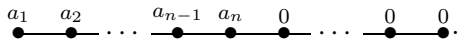
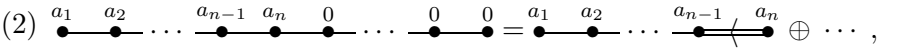
2. Some linear algebra

Let us call a tensor bundle on a smooth $2n$-dimensional manifold M *irreducible* if and only if it is induced from the co-frame bundle by an irreducible representation of $\mathrm{SL}(2n, \mathbb{R})$. Sections of such a bundle will be called *irreducible tensors*. Now the irreducible representations of $\mathrm{SL}(2n, \mathbb{R})$ are classified by their highest weight [**16**], which we may write as an integral combination of fundamental weights, the coefficients

of which may be written over the corresponding nodes of the Dynkin diagram. Let us restrict attention to those tensor bundles arising from representations of the form

(1) 

These representations have the property that their restriction to the subgroup $\mathrm{Sp}(2n,\mathbb{R}) \subset \mathrm{SL}(2n,\mathbb{R})$ has a leading term

(2) 

which is easily described in terms of tensors, namely

$$\psi_{abc\cdots d} = \psi^{\perp}_{abc\cdots d} + \text{terms of the form } J_{ab} \bowtie \theta_{c\cdots d}. \tag{3}$$

Here, J_{ab} is the non-degenerate skew form preserved by $\mathrm{Sp}(2n,\mathbb{R})$, the tensor $\psi^{\perp}_{abc\cdots d}$ satisfies the same symmetries as does $\psi_{abc\cdots d}$ but is, in addition, totally trace-free with respect to the inverse form J^{ab}, and $\bowtie$ is some symmetry operation on the indices $abc\cdots d$. Suppose, for example, that R_{abcd} has Riemann tensor symmetries. The relevant branching for $\mathrm{Sp}(2n,\mathbb{R}) \subset \mathrm{SL}(2n,\mathbb{R})$ is

and may be written explicitly as

$$\begin{aligned} R_{abcd} \;=\; & X_{abcd} \\ & + \Psi_{ac}J_{bd} - \Psi_{bc}J_{ad} - \Psi_{ad}J_{bc} + \Psi_{bd}J_{ac} + 2\Psi_{ab}J_{cd} + 2\Psi_{cd}J_{ab} \\ & + L(J_{ac}J_{bd} - J_{bc}J_{ad} + 2J_{ab}J_{cd}), \end{aligned} \tag{4}$$

where X_{abcd} has Riemann tensor symmetries and is trace-free while Ψ_{ab} is skew and trace-free (where "trace-free" means with respect to J^{ab}). It is the symplectic counterpart to the well-known decomposition

$$R_{abcd} = W_{abcd} + \Phi_{ac}g_{bd} - \Phi_{bc}g_{ad} - \Phi_{ad}g_{bc} + \Phi_{bd}g_{ac} + K(g_{ac}g_{bd} - g_{bc}g_{ad})$$

of the Riemann tensor under $\mathrm{SO}(2n,\mathbb{R}) \subset \mathrm{SL}(2n,\mathbb{R})$.

Proposition 1. *Suppose $\psi_{abc\cdots d}$ is an irreducible covariant tensor under $\mathrm{SL}(2n,\mathbb{R})$ with symmetries of the form* (1)*. Then its totally trace-free part $\psi^{\perp}_{abc\cdots d}$, defined by* (3)*, vanishes if and only if the pullback of $\psi_{abc\cdots d}$ to every Lagrangian subspace of $\mathbb{R}^{2n}$ vanishes.*

Proof. Considering the right-hand side of (3), it is clear that that all terms except $\psi^{\perp}_{abc\cdots d}$ vanish when restricted to a Lagrangian subspace simply because J_{ab} has this property. Conversely, requiring that $\psi_{abc\cdots d}$ vanish on all Lagrangian subspaces is a manifestly $\mathrm{Sp}(2n,\mathbb{R})$-invariant restriction. Bearing in mind that the leading term of (2) is irreducible, it follows that either our proposition is true or all tensors with symmetries of the form (1) vanish on all Lagrangian subspaces. If we first consider fundamental representations of the form (1), then we are done because the corresponding tensors are precisely the k-forms for $k \leq n$. More

specifically, we can choose a basis $\{e_1, e_2, \cdots, e_n, e_{n+1}, e_{n+2}, \cdots, e_{2n}\}$ of $\mathbb{R}^{2n}$ such that

$$J_{ab} = \left[\begin{array}{c|c} 0 & \mathrm{Id} \\ \hline -\mathrm{Id} & 0 \end{array}\right]$$

and consider the Lagrangian subspace

$$\Pi \equiv \mathrm{span}\{e_1, e_2, \cdots, e_n\},$$

noticing that the highest weight vector $\omega^k \in \Lambda^k\mathbb{R}^{2n}$ restricts to a non-zero form on Π. The general case follows because

$$(\omega^1)^{\otimes a_i} \otimes (\omega^2)^{\otimes a_2} \otimes \cdots \otimes (\omega^n)^{\otimes a_n} \in \overset{a_1}{\bullet}\!\!-\!\!\overset{a_2}{\bullet}\!\!-\cdots-\!\!\overset{a_{n-1}}{\bullet}\!\!-\!\!\overset{a_n}{\bullet}\!\!-\!\!\overset{0}{\bullet}\!\!-\cdots-\!\!\overset{0}{\bullet}\!\!-\!\!\overset{0}{\bullet}$$

is non-zero when restricted to Π. q.e.d.

3. Symplectic geometry on complex projective space

In this article, complex projective space may always be regarded as a Riemannian manifold with its Fubini–Study metric, which we shall denote by g_{ab}. From this point of view, we have the equality

$$\mathbb{CP}_n = \mathrm{SU}(n+1)/\mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(n)). \tag{5}$$

However, $\mathbb{CP}_n$ may also be viewed in other well-known guises as follows.

Structure	Quantity	Name	Formula
Riemannian	g_{ab}	Fubini–Study metric	$g_{ab} = J_a{}^c J_{bc}$
complex	$J_a{}^b$	complex structure	$J_a{}^b = g^{bc} J_{ac}$
symplectic	J_{ab}	Kähler form	$J_{ab} = J_a{}^c g_{bc}$

The formulæ show that any two of these structures determine the third. As already remarked in the Introduction, there is a useful family of totally geodesic embeddings $\mathbb{RP}_n \hookrightarrow \mathbb{CP}_n$ obtained from the standard embedding by the action of the isometry group $\mathrm{SU}(n+1)$. For want of a better terminology, let us refer to these as *model embeddings*.

Proposition 2. *Suppose that $\mathbb{RP}_n \hookrightarrow \mathbb{CP}_n$ is a model embedding. Then $T_p\mathbb{RP}_n \hookrightarrow T_p\mathbb{CP}_n$ is a Lagrangian subspace for all $p \in \mathbb{RP}_n$. Conversely, for each $p \in \mathbb{CP}_n$ the model embeddings passing through p are in 1–1 correspondence with the Lagrangian subspaces of $T_p\mathbb{CP}_n$.*

Proof. The Kähler form J_{ab} is of type $(1,1)$ and therefore vanishes on any totally real submanifold of $\mathbb{CP}_n$. In particular, it vanishes on any model embedding. Thus, these embeddings are Lagrangian. Conversely, we may as well take p to be the basepoint of $\mathbb{CP}_n$ as the homogeneous space (5), in which case it is easily calculated that the isotropy group acts on $T_p\mathbb{CP}_n$ as

$$\mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(n)) \ni (\lambda, A) \mapsto \lambda^{-1}A \text{ acting on } \mathbb{C}^n.$$

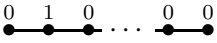
In particular, every transformation in $\mathrm{U}(n)$ is obtained in this way. Also note that the Kähler form is realized as the standard symplectic form on $\mathbb{C}^n$. Therefore, it remains to be seen that $\mathrm{U}(n)$ acts transitively on the Lagrangian subspaces of $\mathbb{C}^n$. This is, indeed, a well-known fact (see, for example, [**26**, Exercise I.A.4(ii)]); otherwise said, the Lagrangian Grassmannian may be realized as the homogeneous space $\mathrm{U}(n)/\mathrm{O}(n)$. q.e.d.

Theorem 2. *Let $\psi_{abc\cdots d}$ be an irreducible tensor on $\mathbb{CP}_n$ corresponding to a representation of $\mathrm{SL}(2n,\mathbb{R})$ of the form* (1)*. Suppose that $\psi_{abc\cdots d}$ vanishes when restricted to all model embeddings $\mathbb{RP}_n \hookrightarrow \mathbb{CP}_n$. Then its totally trace-free part $\psi^{\perp}_{abc\cdots d}$ defined by* (3) *vanishes.*

Proof. Immediate by combining Propositions 1 and 2. q.e.d.

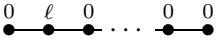
Corollary 1. *A smooth two-form on $\mathbb{CP}_n$ vanishes upon restriction to every model embedding $\mathbb{RP}_n \hookrightarrow \mathbb{CP}_n$ if and only if it is of the form θJ_{ab}, where θ is a some smooth function and J_{ab} is the Kähler form.*

Proof. If $n = 1$, then the hypothesis and conclusion are always trivially satisfied. For $n \geq 2$ the representation of $\mathrm{SL}(2n,\mathbb{R})$ corresponding to two-forms is



and Theorem 2 applies. q.e.d.

For the purposes of this article, we shall need Theorem 2 for tensors having symmetries of the form

(6)


for $\ell \geq 1$, and it is worthwhile stating what this means more explicitly in this case. Writing square brackets to denote skew symmetrization over the indices they enclose, the tensors themselves may realized in the form

$$R_{paqb\cdots rc} = R_{[pa][qb]\cdots[rc]}, \tag{7}$$

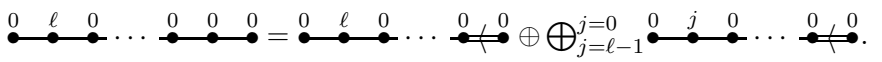
with ℓ pairs of skew indices, symmetric in these pairs, and such that

$$R_{[paq]b\cdots rc} = 0, \tag{8}$$

generalizing the symmetries of a Riemann tensor, which is the case $\ell = 2$. Such tensors enjoy a decomposition

$$R_{paqb\cdots rc} = X_{paqb\cdots rc} + J\text{-trace terms}$$

generalising (4), where $X_{paqb\cdots rc}$ is totally trace-free and the J-trace terms follow the branching



For later use, we record the result that we shall require.

Corollary 2. *For $n \geq 2$, a smooth tensor on $\mathbb{CP}_n$ of the form* (6) *vanishes upon restriction to every model embedding $\mathbb{RP}_n \hookrightarrow \mathbb{CP}_n$ if and only if its trace-free part with respect to the Kähler form vanishes.*

Proof. Immediate from Theorem 2 and our discussion above. q.e.d.

4. Integrability conditions on $\mathbb{CP}_n$

In this section, we explore some necessary local conditions in order that a smooth symmetric tensor $\omega_{ab\cdots c}$ on $\mathbb{CP}_n$ be of the form $\nabla_{(a}\phi_{b\cdots c)}$ for some symmetric $\phi_{b\cdots c}$, where ∇_a is the Fubini–Study connection. To do this, we shall need to know the curvature of the Fubini–Study metric and also of the round metric on $\mathbb{RP}_n$, as induced by a model embedding $\mathbb{RP}_n \hookrightarrow \mathbb{CP}_n$. With suitable normalizations, the following formulæ

$$\begin{array}{lrcl} \mathbb{RP}_n\colon & R_{abcd} & = & g_{ac}g_{bd} - g_{bc}g_{ad} \\ \mathbb{CP}_n\colon & R_{abcd} & = & g_{ac}g_{bd} - g_{bc}g_{ad} + J_{ac}J_{bd} - J_{bc}J_{ad} + 2J_{ab}J_{cd} \end{array} \tag{9}$$

are well-known. While the metric tensors g_{ab} have different meanings on the $2n$-dimensional Riemannian manifold $\mathbb{CP}_n$ and the n-dimensional Riemannian manifold $\mathbb{RP}_n$, they coincide under restriction to a model embedding $\mathbb{RP}_n \hookrightarrow \mathbb{CP}_n$, and, in this sense, our abuse of notation is a legitimate convenience, which should cause no confusion.

The corresponding exploration on $\mathbb{RP}_n$ has already been done. To state its conclusions the following notation is useful. Suppose $T_{pq\cdots rab\cdots c}$ is a tensor that is symmetric in two groups of ℓ indices:

$$T_{pq\cdots rab\cdots c} = T_{(pq\cdots r)(ab\cdots c)}.$$

We may define a new tensor by re-ordering its indices

$$S_{paqb\cdots rc} = T_{pq\cdots rab\cdots c}$$

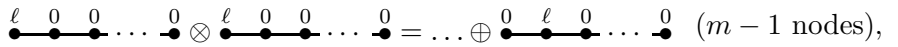
and then manufacturing yet another tensor by

$$R_{paqb\cdots rc} = S_{[pa][qb]\cdots[rc]}.$$

Let us write π for this homomorphism of tensors

$$T_{pq\cdots rab\cdots c} \overset{\pi}{\longmapsto} R_{paqb\cdots rc}.$$

Again, the notation applies equally well to tensors on $\mathbb{RP}_n$ as it does on $\mathbb{CP}_n$. Furthermore, it is evident that π commutes with pull-back to a model embedding $\mathbb{RP}_n \hookrightarrow \mathbb{CP}_n$. On the level of $\mathrm{SL}(m,\mathbb{R})$-modules, where $m = 2n$ for tensors on $\mathbb{CP}_n$ and $m = n$ for tensors on $\mathbb{RP}_n$, the homomorphism π is induced by projection onto the last factor of



($m-1$ nodes),

the last module being trivial when $m = 2$. On any manifold M (but, in particular, on $\mathbb{RP}_n$ and $\mathbb{CP}_n$), let us introduce the notation Y^ℓ for the bundle of 2ℓ-tensors $R_{paqb\cdots rc}$ satisfying the symmetries (7), symmetric

in the pairs pa, qb, …, rc, and such that (8) holds. It is the target for the homomorphism

$$\pi : \bigodot^\ell \Lambda^1 \otimes \bigodot^\ell \Lambda^1 \longrightarrow Y^\ell,$$

which reduces to the exterior product $\wedge : \Lambda^1 \otimes \Lambda^1 \to \Lambda^2$ when $\ell = 1$.

As will be explained in the proof, the following theorem may be derived from a special case of the *Bernstein–Gelfand–Gelfand resolution*. The cases $\ell = 2$ and $\ell = 3$ were established by Calabi [**5**] (cf. [**17**]) and Estezet [**15**], respectively.

Theorem 3. *Fix $n \geq 2$. Suppose $\omega_{abc\cdots d}$ is a smooth symmetric ℓ-tensor on $\mathbb{RP}_n$. Then we can find a smooth symmetric tensor $\phi_{bc\cdots d}$ such that $\nabla_{(a}\phi_{bc\cdots d)} = \omega_{abc\cdots d}$ if and only if*

$$\pi \left(\begin{array}{c} \nabla_{(p}\nabla_q\nabla_r\cdots\nabla_{s)}\omega_{abc\cdots d} \\ \quad + \frac{(\ell-1)\ell(\ell+1)}{6} g_{(pq}\nabla_r\cdots\nabla_{s)}\omega_{abc\cdots d} \\ \quad\quad + \text{lower-order terms} \end{array} \right) = 0. \tag{10}$$

More explicitly, the operator in (10) *with its lower-order terms may be determined as follows. If ℓ is even, then* (10) *is*

$$\pi\big((\nabla^2 + (\ell-1)^2 g)(\nabla^2 + (\ell-3)^2 g)\cdots(\nabla^2 + 9g)(\nabla^2 + g)\big) \tag{11}$$

where

$$\bigodot^{p-1}\Lambda^1 \otimes \bigodot^\ell \Lambda^1 \xrightarrow{\nabla^2 + p^2 g} \bigodot^{p+1}\Lambda^1 \otimes \bigodot^\ell \Lambda^1$$

is given by

$$\omega_{v\cdots wabc\cdots d} \mapsto \nabla_{(t}\nabla_u\omega_{v\cdots w)abc\cdots d} + p^2 g_{(tu}\omega_{v\cdots w)abc\cdots d}.$$

If ℓ is odd, then (10) *is*

$$\pi\big((\nabla^2 + (\ell-1)^2 g)(\nabla^2 + (\ell-3)^2 g)\cdots(\nabla^2 + 16g)(\nabla^2 + 4g)\nabla\big).$$

Proof. Let us write ∇ to stand for the operator $\phi_{bc\cdots d} \mapsto \nabla_{(a}\phi_{bc\cdots d)}$ and $\nabla^{(\ell)}$ for the differential operator in (10). We claim that the sequence

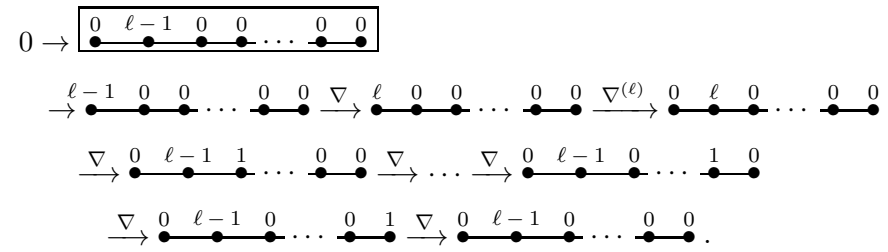
$$\bigodot^{\ell-1}\Lambda^1 \xrightarrow{\nabla} \bigodot^\ell \Lambda^1 \xrightarrow{\nabla^{(\ell)}} Y^\ell \tag{12}$$

on the level of sheaves is part of a fine resolution of a certain locally constant sheaf on $\mathbb{RP}_n$. When $\ell = 1$, we have in mind the de Rham resolution

$$0 \to \mathbb{R} \to \Lambda^0 \xrightarrow{\nabla} \Lambda^1 \xrightarrow{\nabla} \Lambda^2 \xrightarrow{\nabla} \Lambda^3 \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \Lambda^{n-2} \xrightarrow{\nabla} \Lambda^{n-1} \xrightarrow{\nabla} \Lambda^n \to 0,$$

and Theorem 3 follows because $H^1(\mathbb{RP}_n, \mathbb{R}) = 0$ for $n \geq 2$. For $\ell \geq 2$, the BGG (Bernstein–Gelfand–Gelfand) resolution [**6**, **10**] replaces de Rham.

The key point is that the round metric on $\mathbb{RP}_n$ is projectively flat. As detailed in [**13**, **14**], the BGG resolution is



Here, $\boxed{}$ denotes the locally constant sheaf on $\mathbb{RP}_n = \mathrm{SL}(n+1,\mathbb{R})/P$ induced by the given representation of $\mathrm{SL}(n+1,\mathbb{R})$ restricted to P and then induced back up to a homogeneous bundle on $\mathbb{RP}_n$. After $\boxed{}$, we are writing irreducible representations of $\mathrm{SL}(n,\mathbb{R})$ instead of the corresponding induced bundles. The second row coincides with (12). Formulæ for the operators $\nabla^{(\ell)}$ are given inductively in [**7**, **9**], but the products given in the statement of Theorem 3 are most easily deduced from Gover's method [**25**] (and these factorizations hold for any Einstein metric). In the expanded form,

$$\pi\Big(\nabla^\ell\omega + \tfrac{(\ell-1)\ell(\ell+1)}{6}g\nabla^{\ell-2}\omega + \tfrac{(\ell-3)(\ell-2)(\ell-1)\ell(\ell+1)(5\ell+7)}{360}g^2\nabla^{\ell-4}\omega + \cdots\Big),$$

the coefficients are quite complicated (although the array generated by (11) appears as triangle A008956 in the On-Line Encyclopedia of Integer Sequences at `www.research.att.com/~njas/sequences`). Fortunately, we shall not need the details of the operators $\nabla^{(\ell)}$ but only their general form and how they may be manufactured, which is as follows. Let $\mathbb{T}$ denote the bundle $\Lambda^0 \oplus \Lambda^1$ on $\mathbb{RP}_n$ equipped with the connection

$$\mathbb{T} = \begin{matrix}\Lambda^0\\ \oplus \\ \Lambda^1\end{matrix} \ni \begin{bmatrix} \sigma \\ \mu_b \end{bmatrix} \overset{\nabla_a}{\longmapsto} \begin{bmatrix} \nabla_a\sigma - \mu_a \\ \nabla_a\mu_b + g_{ab}\sigma \end{bmatrix} \in \Lambda^1 \otimes \mathbb{T}. \tag{13}$$

We compute that

$$\nabla_a\nabla_b \begin{bmatrix} \sigma \\ \mu_c \end{bmatrix} = \begin{bmatrix} \nabla_a\nabla_b\sigma - \nabla_a\mu_b - \nabla_b\mu_a - g_{ab}\sigma \\ \nabla_a\nabla_b\mu_c + g_{bc}\nabla_a\sigma + g_{ac}\nabla_b\sigma - g_{ac}\mu_b \end{bmatrix}$$

and observe from (9) that

$$(\nabla_a\nabla_b - \nabla_b\nabla_a)\mu_c = R_{abc}{}^d\mu_d = g_{ac}\mu_b - g_{bc}\mu_a\,,$$

and hence that the connection on $\mathbb{T}$ is flat. It follows that the coupled de Rham complex

$$\Lambda^0 \otimes \mathbb{T} \overset{\nabla}{\longrightarrow} \Lambda^1 \otimes \mathbb{T} \overset{\nabla}{\longrightarrow} \Lambda^2 \otimes \mathbb{T} \overset{\nabla}{\longrightarrow} \Lambda^3 \otimes \mathbb{T} \overset{\nabla}{\longrightarrow} \cdots \overset{\nabla}{\longrightarrow} \Lambda^n \otimes \mathbb{T} \to 0$$

is a fine resolution of the locally covariant constant sections of $\mathbb{T}$ with a similar conclusion for any associated vector bundle such as $\Lambda^2\mathbb{T}$. Let us

examine the induced connection on $\Lambda^2\mathbb{T}$ in more detail:

$$\Lambda^2\mathbb{T} = \begin{matrix}\Lambda^1\\ \oplus\\ \Lambda^2\end{matrix} \ni \begin{bmatrix} \sigma_b \\ \mu_{bc} \end{bmatrix} \overset{\nabla_a}{\longmapsto} \begin{bmatrix} \nabla_a\sigma_b - \mu_{ab} \\ \nabla_a\mu_{bc} + g_{ab}\sigma_c - g_{ac}\sigma_b \end{bmatrix} \in \Lambda^1\otimes\Lambda^2\mathbb{T}. \tag{14}$$

We shall show that exactness of the coupled de Rham complex

$$\Gamma(\mathbb{RP}_n, \Lambda^2\mathbb{T}) \overset{\nabla}{\longrightarrow} \Gamma(\mathbb{RP}_n, \Lambda^1\otimes\Lambda^2\mathbb{T}) \overset{\nabla}{\longrightarrow} \Gamma(\mathbb{RP}_n, \Lambda^2\otimes\Lambda^2\mathbb{T})$$

is sufficient to deduce the case $\ell = 2$ of Theorem 3. First, we need a formula for $\nabla_a : \Lambda^1\otimes\Lambda^2\mathbb{T} \to \Lambda^2\otimes\Lambda^2\mathbb{T}$. It is immediate from (14) that

$$\begin{bmatrix} \xi_{bc} \\ \nu_{bcd} \end{bmatrix} \overset{\nabla_a}{\longmapsto} \begin{bmatrix} \nabla_{[a}\xi_{b]c} + \nu_{[ab]c} \\ \nabla_{[a}\nu_{b]cd} + g_{c[a}\xi_{b]d} - g_{d[a}\xi_{b]c} \end{bmatrix}. \tag{15}$$

Now suppose ω_{ab} is a symmetric tensor on $\mathbb{RP}_n$. We claim that the following are equivalent:

i) $\begin{bmatrix} \omega_{bc} \\ \nabla_c\omega_{db} - \nabla_d\omega_{cb} \end{bmatrix} \in \Gamma(\mathbb{RP}_n, \Lambda^1\otimes\Lambda^2\mathbb{T})$ is in the range of the coupled connection $\nabla_b : \Gamma(\mathbb{RP}_n, \Lambda^2\mathbb{T}) \to \Gamma(\mathbb{RP}_n, \Lambda^1\otimes\Lambda^2\mathbb{T})$.

ii) $\omega_{ab} = \nabla_{(a}\phi_{b)}$ for some $\phi_a \in \Gamma(\mathbb{RP}_n, \Lambda^1)$.

iii) $\begin{bmatrix} \omega_{bc} \\ \nu_{bcd} \end{bmatrix} \in \Gamma(\mathbb{RP}_n, \Lambda^1\otimes\Lambda^2\mathbb{T})$ for some $\nu_{bcd} \in \Gamma(\mathbb{RP}_n, \Lambda^1\otimes\Lambda^2)$ is in the range of $\nabla_b : \Gamma(\mathbb{RP}_n, \Lambda^2\mathbb{T}) \to \Gamma(\mathbb{RP}_n, \Lambda^1\otimes\Lambda^2\mathbb{T})$.

It is clear from (14) that (i)⇒(ii)⇒(iii). It remains to show (iii)⇒(i). To see this, recall that the curvature of the connection on $\Lambda^2\mathbb{T}$ is flat and so if (iii) holds, then we must have

$$\begin{bmatrix} \omega_{bc} \\ \nu_{bcd} \end{bmatrix} \overset{\nabla_a}{\longmapsto} 0 \in \Gamma(\mathbb{RP}_n, \Lambda^2\otimes\Lambda^2\mathbb{T}).$$

In particular, we read off from the first row of (15) that

$$\nabla_{[a}\omega_{b]c} + \nu_{[ab]c} = 0.$$

From this, bearing in mind that $\nu_{abc} = \nu_{a[bc]}$, it follows that

$$\nu_{abc} = 3\nu_{[abc]} - 2\nu_{[bc]a} = 2\nabla_{[b}\omega_{c]a} = \nabla_b\omega_{ca} - \nabla_c\omega_{ba},$$

as required. Finally, to deduce Theorem 3 we suppose that (i) holds and consider the second row of (15):

$$\begin{bmatrix} \omega_{bc} \\ \nabla_c\omega_{db} - \nabla_d\omega_{cb} \end{bmatrix} \overset{\nabla_a}{\longmapsto} \begin{bmatrix} 0 \\ \nabla_{[a}\nabla_{|c|}\omega_{b]d} - \nabla_{[a}\nabla_{|d|}\omega_{b]c} + g_{c[a}\omega_{b]d} - g_{d[a}\omega_{b]c} \end{bmatrix}$$

where, following [**32**, pp. 132–136], the vertical bars in $\nabla_{|c|}$ and $\nabla_{|d|}$ exclude the indices they enclose from skew symmetrization. Again, since the connection is flat, we conclude that the vanishing of

$$\nabla_{[a}\nabla_{|c|}\omega_{b]d} - \nabla_{[a}\nabla_{|d|}\omega_{b]c} + g_{c[a}\omega_{b]d} - g_{d[a}\omega_{b]c} \tag{16}$$

is a necessary and sufficient condition in order that $\omega_{ab} = \nabla_{(a}\phi_{b)}$ for some smooth 1-form ϕ_b on $\mathbb{RP}_n$. However, it is easy to check that this coincides with

$$2 \times \pi(\nabla_{(a}\nabla_{c)}\omega_{bd} + g_{ac}\omega_{bd}); \tag{17}$$

thus Theorem 3 is proved for the case $\ell = 2$. The general case follows similarly by considering the induced flat connection on the associated bundle (in terms of Young diagrams)

$$\underbrace{\begin{array}{|c|c|c|c|c|c|}\hline \; & \; & \; & \cdots & \; & \; \\\hline \; & \; & \; & \cdots & \; & \; \\\hline\end{array}}_{\ell-1\text{ columns}}\ \mathbb{T} \tag{18}$$

or, equivalently, (since Λ^n and hence $\Lambda^{n+1}\mathbb{T} = \Lambda^0 \otimes \Lambda^n$ are trivialized by the round volume form), the bundle induced from the special frame-bundle of $\mathbb{T}$ by the representation

$$\overset{0}{\bullet}\!\!-\!\!\overset{\ell-1}{\bullet}\!\!-\!\!\overset{0}{\bullet}\!\!-\!\!\overset{0}{\bullet}\cdots\overset{0}{\bullet}\!\!-\!\!\overset{0}{\bullet} \tag{19}$$

of $\mathrm{SL}(n+1,\mathbb{R})$. A salient feature of this bundle is its structure when written as ordinary tensor bundles on $\mathbb{RP}_n$:

$$\begin{aligned}\begin{array}{|c|c|c|c|c|c|}\hline \; & \; & \; & \cdots & \; & \; \\\hline \; & \; & \; & \cdots & \; & \; \\\hline\end{array}\ \mathbb{T} = {} & \overset{\ell-1}{\bullet}\!\!-\!\!\overset{0}{\bullet}\!\!-\!\!\overset{0}{\bullet}\cdots\overset{0}{\bullet}\!\!-\!\!\overset{0}{\bullet} \oplus \overset{\ell-2}{\bullet}\!\!-\!\!\overset{1}{\bullet}\!\!-\!\!\overset{0}{\bullet}\cdots\overset{0}{\bullet}\!\!-\!\!\overset{0}{\bullet}\\ & \oplus \overset{\ell-3}{\bullet}\!\!-\!\!\overset{2}{\bullet}\!\!-\!\!\overset{0}{\bullet}\cdots\overset{0}{\bullet}\!\!-\!\!\overset{0}{\bullet} \oplus \cdots \oplus \overset{0}{\bullet}\!\!-\!\!\overset{\ell-1}{\bullet}\!\!-\!\!\overset{0}{\bullet}\cdots\overset{0}{\bullet}\!\!-\!\!\overset{0}{\bullet}\end{aligned}$$

in terms of which the induced connection takes the form

$$\begin{bmatrix}\sigma\\ \mu\\ \rho\\ \vdots\end{bmatrix} \overset{\nabla}{\longmapsto} \begin{bmatrix}\nabla\sigma - \partial\mu\\ \nabla\mu - \partial\rho + g \bowtie \sigma\\ \vdots\\ \; \end{bmatrix} \tag{20}$$

for some appropriate tensor combination $g \bowtie \sigma$ where, following [**4**], the homomorphism

$$\partial : \begin{array}{|c|c|c|c|c|c|}\hline \; & \; & \; & \cdots & \; & \; \\\hline \; & \; & \; & \cdots & \; & \; \\\hline\end{array}\ \mathbb{T} \longrightarrow \Lambda^1 \otimes \begin{array}{|c|c|c|c|c|c|}\hline \; & \; & \; & \cdots & \; & \; \\\hline \; & \; & \; & \cdots & \; & \; \\\hline\end{array}\ \mathbb{T}$$

is best regarded as induced by the Lie algebra differential

$$\partial : \overset{0}{\bullet}\!\!-\!\!\overset{\ell-1}{\bullet}\!\!-\!\!\overset{0}{\bullet}\!\!-\!\!\overset{0}{\bullet}\cdots\overset{0}{\bullet}\!\!-\!\!\overset{0}{\bullet} \longrightarrow \mathfrak{g}_1 \otimes \overset{0}{\bullet}\!\!-\!\!\overset{\ell-1}{\bullet}\!\!-\!\!\overset{0}{\bullet}\!\!-\!\!\overset{0}{\bullet}\cdots\overset{0}{\bullet}\!\!-\!\!\overset{0}{\bullet}. \tag{21}$$

Here, $\mathfrak{sl}(n+1,\mathbb{R})$ is regarded as a $|1|$-graded Lie algebra

$$\begin{aligned}&\mathfrak{sl}(n+1,\mathbb{R}) = \mathfrak{g}_{-1} \oplus \mathfrak{g} \oplus \mathfrak{g}_1\\ &= \left\{\left(\begin{array}{c|ccc} 0 & 0 & \cdots & 0\\\hline * & & & \\ \vdots & & 0 & \\ * & & & \end{array}\right)\right\} \oplus \left\{\left(\begin{array}{c|ccc} * & 0 & \cdots & 0\\\hline 0 & & & \\ \vdots & & * & \\ 0 & & & \end{array}\right)\right\} \oplus \left\{\left(\begin{array}{c|ccc} 0 & * & \cdots & *\\\hline 0 & & & \\ \vdots & & 0 & \\ 0 & & & \end{array}\right)\right\}\end{aligned}$$

and the $\mathfrak{sl}(n+1,\mathbb{R})$-module (19) is restricted to $\mathfrak{g}_{-1}$ for the purposes of (21). The operator $\phi_{bc\cdots d} \mapsto \omega_{abc\cdots d} = \nabla_{(a}\phi_{bc\cdots d)}$ appears within (20) as the equation $\nabla\sigma - \partial\mu = \omega$ for some μ, and the analogs of (i), (ii),

(iii) from the case $\ell = 2$ readily arise. As detailed in [**4**], the Lie algebra cohomologies

$$H^0(\mathfrak{g}_{-1}, \overset{0}{\bullet}\!\!-\!\!\overset{\ell-1}{\bullet}\!\!-\!\!\overset{0}{\bullet}\!\!-\!\!\overset{0}{\bullet}\!\!-\cdots-\!\!\overset{0}{\bullet}\!\!-\!\!\overset{0}{\bullet}) \;=\; \overset{\ell-1}{\bullet}\!\!-\!\!\overset{0}{\bullet}\!\!-\!\!\overset{0}{\bullet}\!\!-\cdots-\!\!\overset{0}{\bullet}\!\!-\!\!\overset{0}{\bullet}$$

$$H^1(\mathfrak{g}_{-1}, \overset{0}{\bullet}\!\!-\!\!\overset{\ell-1}{\bullet}\!\!-\!\!\overset{0}{\bullet}\!\!-\!\!\overset{0}{\bullet}\!\!-\cdots-\!\!\overset{0}{\bullet}\!\!-\!\!\overset{0}{\bullet}) \;=\; \overset{\ell}{\bullet}\!\!-\!\!\overset{0}{\bullet}\!\!-\!\!\overset{0}{\bullet}\!\!-\cdots-\!\!\overset{0}{\bullet}\!\!-\!\!\overset{0}{\bullet}$$

provide the representations of $\mathrm{SL}(n, \mathbb{R})$ inducing the vector bundles on $\mathbb{RP}_n$ between which the differential operator $\phi_{bc\cdots d} \mapsto \nabla_{(a}\phi_{bc\cdots d)}$ acts. Reasoning as for the case $\ell = 2$ above, it is the second cohomology

$$H^2(\mathfrak{g}_{-1}, \overset{0}{\bullet}\!\!-\!\!\overset{\ell-1}{\bullet}\!\!-\!\!\overset{0}{\bullet}\!\!-\!\!\overset{0}{\bullet}\!\!-\cdots-\!\!\overset{0}{\bullet}\!\!-\!\!\overset{0}{\bullet}) \;=\; \overset{0}{\bullet}\!\!-\!\!\overset{\ell}{\bullet}\!\!-\!\!\overset{0}{\bullet}\!\!-\cdots-\!\!\overset{0}{\bullet}\!\!-\!\!\overset{0}{\bullet}$$

(computed according to Kostant's Theorem [**28**]) that induces the vector bundle providing the obstruction to being in the range of this operator.

For the purposes of this article, it is not necessary to know the exact formula for this obstruction but only how it arises from (18) with its flat connection and that it has the form

$$\pi\left(\nabla_{(p}\nabla_q\nabla_r\cdots\nabla_{s)}\omega_{abc\cdots d} + \text{lower-order } g\text{-trace terms}\right) = 0,$$

and this much is clear by construction. q.e.d.

We are now in a position to consider the corresponding problem on $\mathbb{CP}_n$ as posed at the beginning of this section. Recall that Y^ℓ denotes a certain tensor bundle on any manifold, and, in particular,on $\mathbb{CP}_n$. Therefore, parallel to (12) on $\mathbb{RP}_n$, we may consider the sequence of bundles and linear differential operators

$$\bigodot^{\ell-1}\Lambda^1 \xrightarrow{\nabla} \bigodot^{\ell}\Lambda^1 \xrightarrow{\nabla^{(\ell)}} Y^\ell$$

on $\mathbb{CP}_n$, where $\nabla^{(\ell)}$ is given by exactly the same formula (10) as on $\mathbb{RP}_n$ except that ∇_a now refers to the Fubini–Study connection and g_{ab} to the Fubini–Study metric. This sequence is no longer exact. Instead, if we expand the composition $\nabla^{(\ell)}\circ\nabla$ using (9) on $\mathbb{CP}_n$, bearing in mind that both g_{ab} and J_{ab} are covariant constant, and compare the result with the total cancellation that we know occurs on $\mathbb{RP}_n$, then we conclude that the result is forced to be of the form

$$\phi \mapsto J \circledcirc D\phi,$$

where $D : \bigodot^{\ell-1}\Lambda^1 \to Y^{\ell-1}$ is some linear differential operator and

$$\circledcirc : \Lambda^2 \otimes Y^{\ell-1} \to Y^\ell$$

is induced by projection onto the first factor in the decomposition

$$\overset{0}{\bullet}\!\!-\!\!\overset{1}{\bullet}\!\!-\!\!\overset{0}{\bullet}\!\!-\cdots-\!\!\overset{0}{\bullet}\!\!-\!\!\overset{0}{\bullet} \otimes \overset{0}{\bullet}\!\!-\!\!\overset{\ell-1}{\bullet}\!\!-\!\!\overset{0}{\bullet}\!\!-\cdots-\!\!\overset{0}{\bullet}\!\!-\!\!\overset{0}{\bullet} = \overset{0}{\bullet}\!\!-\!\!\overset{\ell}{\bullet}\!\!-\!\!\overset{0}{\bullet}\!\!-\cdots-\!\!\overset{0}{\bullet}\!\!-\!\!\overset{0}{\bullet} \oplus \cdots$$

of $\mathrm{SL}(2n, \mathbb{R})$-modules. In particular, we conclude that the composition

$$\bigodot^{\ell-1}\Lambda^1 \xrightarrow{\nabla} \bigodot^{\ell}\Lambda^1 \xrightarrow{\nabla^{(\ell)}} Y^\ell \xrightarrow{\perp} Y^\ell_\perp$$

vanishes on $\mathbb{CP}_n$, where $\perp$ is the homomorphism of vector bundles on $\mathbb{CP}_n$ induced by (3) and $Y^\ell_\perp$ is induced by the irreducible $\mathrm{Sp}(2n,\mathbb{R})$-module $\overset{0}{\bullet}\!\!-\!\!\overset{\ell}{\bullet}\!\!-\!\!\overset{0}{\bullet}\!\!-\cdots-\!\!\overset{0}{\bullet}\!\!\Longleftarrow\!\!\overset{0}{\bullet}$. Writing $\nabla^{(\ell)}_\perp$ for the composition

$$\bigodot^\ell \Lambda^1 \xrightarrow{\nabla^{(\ell)}} Y^\ell \xrightarrow{\perp} Y^\ell_\perp, \tag{22}$$

we have proved the following.

Theorem 4. *On $\mathbb{CP}_n$ for $n \geq 2$, there is a complex of linear differential operators*

$$\bigodot^{\ell-1}\Lambda^1 \xrightarrow{\nabla} \bigodot^\ell \Lambda^1 \xrightarrow{\nabla^{(\ell)}_\perp} Y^\ell_\perp.$$

The operator $\nabla^{(\ell)}_\perp$ has the form

$$\pi_\perp\left(\nabla_{(p}\nabla_q\nabla_r\cdots\nabla_{s)}\omega_{abc\cdots d} + \text{lower-order } g\text{-trace terms}\right),$$

where $\pi_\perp$ is the composition $\bigodot^\ell\Lambda^1 \otimes \bigodot^\ell\Lambda^1 \xrightarrow{\pi} Y^\ell \xrightarrow{\perp} Y^\ell_\perp$.

In particular, Theorem 4 provides necessary conditions for a globally defined smooth symmetric tensor $\omega_{ab\cdots c}$ on $\mathbb{CP}_n$ to be expressible in the form $\nabla_{(a}\phi_{b\cdots c)}$ for some globally defined smooth symmetric tensor $\phi_{b\cdots c}$. The following section will show that these conditions are also sufficient.

5. Sufficiency on $\mathbb{CP}_n$

5.1. The case $\ell = 1$. In this case Theorem 4 merely states that

$$\Lambda^0 \xrightarrow{d} \Lambda^1 \xrightarrow{d_\perp} \Lambda^2_\perp \tag{23}$$

is a complex on $\mathbb{CP}_n$, where $\Lambda^2_\perp$ denotes the bundle of 2-forms trace-free with respect to the Kähler form J_{ab}. This much is clear, and it is easy to identify the local cohomology of (23) as follows.

Proposition 3. *As a complex of sheaves, the cohomology of* (23) *may be identified with the locally constant sheaf $\mathbb{R}$.*

Proof. Suppose ω is a locally defined 1-form with $d_\perp\omega = 0$. This means that $d\omega = \theta J$ for some smooth function θ. Applying d gives

$$0 = d\theta \wedge J + \theta\, dJ = d\theta \wedge J$$

because J is closed. But since J is non-degenerate and $n \geq 2$, it follows by linear algebra that $d\theta = 0$. Hence θ is locally constant. As J is closed, locally we may choose a 1-form α so that $J = d\alpha$. Then $d(\omega - \theta\alpha) = 0$, and we conclude that locally we may always write

$$\omega = d\phi + \theta\alpha \qquad \text{for some some function } \phi, \tag{24}$$

where $d\alpha = J$ and θ is locally constant. Although α is not determined by J, the only freedom in its choice is $\alpha \mapsto \alpha + d\psi$ for some smooth function ψ, which may be absorbed into the decomposition (24) as

$$\omega = d\phi + \theta\alpha = d(\phi - \theta\psi) + \theta(\alpha + d\psi).$$

In particular, the locally constant function θ is well defined by the local cohomology of the complex (23). q.e.d.

Globally on $\mathbb{CP}_n$, however, there is no 1-form α with $d\alpha = J$. Hence, as was already observed by J.-P. Demailly (see [**19**, Introduction] or [**22**, Theorem 3.40]), the same line of argument shows that globally there is no cohomology. In other words, we have proved our desired global result as follows.

Theorem 5. *The following complex*

$$\Gamma(\mathbb{CP}_n, \Lambda^0) \xrightarrow{d} \Gamma(\mathbb{CP}_n, \Lambda^1) \xrightarrow{d_\perp} \Gamma(\mathbb{CP}_n, \Lambda^2_\perp)$$

is exact.

For the cases $\ell \geq 2$, it will be useful to extend (23) as follows. Let us first suppose that $n \geq 3$. Then there is a naturally defined complex

$$\Lambda^0 \xrightarrow{d} \Lambda^1 \xrightarrow{d_\perp} \Lambda^2_\perp \xrightarrow{d_\perp} \Lambda^3_\perp \tag{25}$$

where $\Lambda^3_\perp$ denotes the bundle of 3-forms trace-free with respect to J_{ab} and $d_\perp : \Lambda^2_\perp \to \Lambda^3_\perp$ is defined as the composition

$$\Lambda^2_\perp \hookrightarrow \Lambda^2 \xrightarrow{d} \Lambda^3 = \Lambda^3_\perp \oplus \Lambda^1 \to \Lambda^3_\perp.$$

In the case $n = 2$, notice that $\Lambda^3_\perp = 0$. Otherwise, we have found the integrability conditions for the range of $d_\perp : \Lambda^1 \to \Lambda^2_\perp$.

Proposition 4. *On $\mathbb{CP}_n$ for $n \geq 3$, the complex*

$$\Lambda^1 \xrightarrow{d_\perp} \Lambda^2_\perp \xrightarrow{d_\perp} \Lambda^3_\perp$$

is exact on the level of sheaves.

Proof. Suppose ξ is a locally defined J-trace-free 2-form with $d_\perp \xi = 0$. This means that $d\xi = \mu \wedge J$ for some 1-form μ. Applying d gives

$$0 = d\mu \wedge J - \mu \wedge dJ = d\mu \wedge J$$

because J is closed. But since J is non-degenerate and $n \geq 3$, it follows by linear algebra that $d\mu = 0$. Locally, therefore, we may find a smooth function ϕ with $d\phi = \mu$. Hence $d(\xi - \phi J) = 0$, and locally we may find a smooth 1-form ω such that $\xi - \phi J = d\omega$. Since ξ is J-trace-free, we conclude that $\xi = d_\perp \omega$. q.e.d.

When $n = 2$ there is a replacement for (25) due to M. Rumin and N. Seshadri [**34**] and defined (on any 4-dimensional symplectic manifold) as follows. [Note added in proof: it was pointed out to us by L.-S. Tseng that this case is, in fact, due to R.T. Smith [**35**]. Its generalization to $d^{(2)}_\perp : \Lambda^n_\perp \to \Lambda^n_\perp$ in dimension $2n$ is due to Rumin and Seshadri and,

independently, Tseng and S.-T. Yau [**36**].] Suppose ξ is a smooth 2-form, and consider $d\xi$. Since $n = 2$, there is an isomorphism

$$\Lambda^1 \xrightarrow{-\wedge J} \Lambda^3,$$

so we may write $d\xi = \mu \wedge J$ for a uniquely defined 1-form μ. Applying d implies $d\mu \wedge J = 0$; thus $d\mu$ is J-trace-free. Let us write $d^{(2)}_\perp : \Lambda^2 \to \Lambda^2_\perp$ for the resulting differential operator. Specifically, we have

$$d^{(2)}_\perp \xi = d\mu, \quad \text{where } \mu \wedge J = d\xi. \tag{26}$$

Notice that if $\xi = \theta J$, then $\mu = d\theta$ and so $d^{(2)}_\perp \xi = 0$. Thus, we obtain a complex of differential operators on $\mathbb{CP}_2$

$$\Lambda^0 \xrightarrow{d} \Lambda^1 \xrightarrow{d_\perp} \Lambda^2_\perp \xrightarrow{d^{(2)}_\perp} \Lambda^2_\perp,$$

which acts as a replacement for (25), especially in view of the following replacement for Proposition 4.

Proposition 5. *On $\mathbb{CP}_2$, the complex*

$$\Lambda^1 \xrightarrow{d_\perp} \Lambda^2_\perp \xrightarrow{d^{(2)}_\perp} \Lambda^2_\perp$$

is exact on the level of sheaves.

Proof. If $d^{(2)}_\perp \xi = 0$ in (26), then locally we may write $d\xi = d\phi \wedge J$ for some smooth function ϕ. In this case, $d(\xi - \phi J) = 0$, and locally we may find a smooth 1-form ω such that $\xi = d\omega + \phi J$. If ξ is also J-trace-free, then it is immediate that $\xi = d_\perp \omega$. q.e.d.

5.2. The case $\ell = 2$. Let $\mathbb{U}$ denote the bundle $\Lambda^0 \oplus \Lambda^1 \oplus \Lambda^0$ on $\mathbb{CP}_n$ equipped with the connection

$$\mathbb{U} = \begin{matrix}\Lambda^0 \\ \oplus \\ \Lambda^1 \\ \oplus \\ \Lambda^0\end{matrix} \ni \begin{bmatrix} \sigma \\ \mu_b \\ \rho \end{bmatrix} \overset{\nabla_a}{\longmapsto} \begin{bmatrix} \nabla_a\sigma - \mu_a \\ \nabla_a\mu_b + g_{ab}\sigma + J_{ab}\rho \\ \nabla_a\rho - J_a{}^c\mu_c \end{bmatrix} \in \Lambda^1 \otimes \mathbb{U}. \tag{27}$$

We compute that

$$\nabla_a\nabla_b \begin{bmatrix} \sigma \\ \mu_c \\ \rho \end{bmatrix} = \begin{bmatrix} J_{ab}\rho + \dots \\ \nabla_a\nabla_b\mu_c - g_{ac}\mu_b - J_{ac}J_b{}^d\mu_d + \dots \\ -J_{ab}\sigma + \dots \end{bmatrix},$$

where the ellipses $\dots$ denote tensors symmetric in the ab indices. From (9) we see that

$$(\nabla_a\nabla_b - \nabla_b\nabla_a)\mu_c = g_{ac}\mu_b - g_{bc}\mu_a + J_{ac}J_b{}^d\mu_d - J_{bc}J_a{}^d\mu_d + 2J_{ab}J_c{}^d\mu_d$$

and hence that the curvature of the connection on $\mathbb{U}$ is given by

$$(\nabla_a\nabla_b - \nabla_b\nabla_a)\begin{bmatrix}\sigma\\ \mu_c\\ \rho\end{bmatrix} = 2J_{ab}\begin{bmatrix}\rho\\ J_c{}^d\mu_d\\ -\sigma\end{bmatrix}. \tag{28}$$

In other words, the curvature of this connection on $\mathbb{U}$ has the form

$$(\nabla_a\nabla_b - \nabla_b\nabla_a)\Sigma = 2J_{ab}\Phi\Sigma, \tag{29}$$

where Φ is some endomorphism of $\mathbb{U}$.

It is easily verified that the skew form on $\mathbb{U}$ defined by

$$\langle(\sigma,\mu_a,\rho),(\tilde\sigma,\tilde\mu_b,\tilde\rho)\rangle = \sigma\tilde\rho + J^{ab}\mu_a\tilde\mu_b - \rho\tilde\sigma \tag{30}$$

is compatible with ∇_a in the sense that $\nabla_a\langle\Sigma,\tilde\Sigma\rangle = \langle\nabla_a\Sigma,\tilde\Sigma\rangle + \langle\Sigma,\nabla_a\tilde\Sigma\rangle$. Hence, the structure group for $\mathbb{U}$ can be reduced to $\mathrm{Sp}(2(n+1),\mathbb{R})$ and, just as we did for the connection on $\mathbb{T}$ in §4, we may now consider the connections on vector bundles induced from $\mathbb{U}$ by the irreducible representations of this structure group. Parallel to $\Lambda^2\mathbb{T}$ in §4, we should consider $\Lambda^2_\perp\mathbb{U}$ where $\perp$ denotes the trace-free part with respect to (30). Its connection is easily computed; we hvae

$$\begin{array}{rcl}\Lambda^2_\perp\mathbb{U} = & \begin{matrix}\Lambda^1\\ \oplus\\ \Lambda^2\\ \oplus\\ \Lambda^1\end{matrix} & \ni \begin{bmatrix}\sigma_b\\ \mu_{bc}\\ \rho_b\end{bmatrix} \xmapsto{\nabla_a} \\ & & \begin{bmatrix}\nabla_a\sigma_b - \mu_{ab}\\ \nabla_a\mu_{bc} + g_{ab}\sigma_c - g_{ac}\sigma_b + J_{ab}\rho_c - J_{ac}\rho_b - J_{bc}\rho_a + J_{bc}J_a{}^d\sigma_d\\ \nabla_a\rho_b + J_a{}^d\mu_{bd}\end{bmatrix}\end{array} \tag{31}$$

and we immediately notice the similarity with (14). The curvature of this connection automatically has the form (29) with Φ replaced by the induced endomorphism of $\Lambda^2_\perp\mathbb{U}$. Alternatively, it may be verified by composition with the induced operator $\nabla : \Lambda^1\otimes\Lambda^2_\perp\mathbb{U} \to \Lambda^2\otimes\Lambda^2_\perp\mathbb{U}$ given by

$$\begin{bmatrix}\sigma_{bc}\\ \mu_{bcd}\\ \rho_{bc}\end{bmatrix} \xmapsto{} \begin{bmatrix}\nabla_{[a}\sigma_{b]c} + \mu_{[ab]c}\\ \nabla_{[a}\mu_{b]cd} + g_{c[a}\sigma_{b]d} - g_{d[a}\sigma_{b]c} - J_{c[a}\rho_{b]d} + J_{d[a}\rho_{b]c} + J_{cd}\rho_{[ab]} + J_{cd}J_{[a}{}^e\sigma_{b]e}\\ \nabla_{[a}\rho_{b]c} + J_{[a}{}^e\mu_{b]ce}\end{bmatrix} \tag{32}$$

that

$$(\nabla_a\nabla_b - \nabla_b\nabla_a)\begin{bmatrix}\sigma_c\\ \mu_{cd}\\ \rho_c\end{bmatrix} = 2J_{ab}\begin{bmatrix}\rho_c + J_c{}^e\sigma_e\\ J_c{}^e\mu_{ed} + J_d{}^e\mu_{ce}\\ -\sigma_c + J_c{}^e\rho_e\end{bmatrix}. \tag{33}$$

Theorem 6. *Suppose ω_{ab} is a symmetric tensor on $\mathbb{CP}_n$. The following are (locally or globally) equivalent.*

i) $\begin{bmatrix} \omega_{bc} \\ \nabla_c\omega_{db} - \nabla_d\omega_{cb} \\ \mathcal{L}_{bc}(\omega) \end{bmatrix} \in \Gamma(\mathbb{CP}_n, \Lambda^1 \otimes \Lambda^2_\perp\mathbb{U})$ *is in the range of the induced connection* $\nabla_b : \Gamma(\mathbb{CP}_n, \Lambda^2_\perp\mathbb{U}) \longrightarrow \Gamma(\mathbb{CP}_n, \Lambda^1\otimes\Lambda^2_\perp\mathbb{U})$, *where* $\bigodot^2\Lambda^1 \in \omega_{bc} \mapsto \mathcal{L}_{bc}(\omega) \in \Lambda^1 \otimes \Lambda^1$ *is some explicit linear differential operator (to be determined in the proof).*

ii) $\omega_{ab} = \nabla_{(a}\phi_{b)}$ *for some* $\phi_a \in \Gamma(\mathbb{CP}_n, \Lambda^1)$.

iii) $\begin{bmatrix} \omega_{bc} \\ \mu_{bcd} \\ \rho_{bc} \end{bmatrix} \in \Gamma(\mathbb{CP}_n, \Lambda^1 \otimes \Lambda^2_\perp\mathbb{U})$, *for some* $\mu_{bcd} \in \Gamma(\mathbb{CP}_n, \Lambda^1 \otimes \Lambda^2)$ *and* $\rho_{bc} \in \Gamma(\mathbb{CP}_n, \Lambda^1 \otimes \Lambda^1)$, *is in the range of the connection* $\nabla_b : \Gamma(\mathbb{CP}_n, \Lambda^2_\perp\mathbb{U}) \to \Gamma(\mathbb{CP}_n, \Lambda^1 \otimes \Lambda^2_\perp\mathbb{U})$.

Proof. It is clear from (31) that (i)⇒(ii)⇒(iii). It remains to show (iii)⇒(i). To see this, recall that the curvature of the connection on $\Lambda^2_\perp\mathbb{U}$ has the form (29) and so if (iii) holds, then we read off from the first row of (32) that

$$\nabla_{[a}\omega_{b]c} + \mu_{[ab]c} = J_{ab}\rho_c$$

for some ρ_c. From this, bearing in mind that $\mu_{bcd} = \mu_{[bcd]}$, it follows that

$$\mu_{bcd} = 3\mu_{[bcd]} - 2\mu_{[cd]b} = \nabla_c\omega_{db} - \nabla_d\omega_{cb} + J_{bc}\rho_d - J_{cd}\rho_b - J_{bd}\rho_c$$

and from (31) we see that

$$\begin{bmatrix} \omega_{bc} \\ \mu_{bcd} \\ \rho_{bc} \end{bmatrix} = \begin{bmatrix} \omega_{bc} \\ \nabla_c\omega_{db} - \nabla_d\omega_{cb} \\ * \end{bmatrix} + \nabla_b \begin{bmatrix} 0 \\ 0 \\ \rho_c \end{bmatrix}$$

for some ρ_c. As the second term on the right-hand side is already of the required form, it follows that we may take $\mu_{bcd} = \nabla_c\omega_{db} - \nabla_d\omega_{cb}$ without loss of generality, and it remains to consider ρ_{bc}. In fact, we claim that ρ_{bc} now is uniquely determined by equating the second row of (32) to $J_{ab}\tau_{cd}$ for some $\tau_{cd} = \tau_{[cd]}$, as must be the case by (29). To see this, we need the following purely algebraic result.

Lemma 1. *Suppose T_{abcd} is a tensor with the following symmetries:*

- $T_{abcd} = T_{[ab][cd]}$,
- $T_{[abc]d} = J_{[ab}\psi_{c]d}$ *for some tensor* ψ_{cd}.

Then there are unique tensors

- ρ_{ab},
- $\tau_{ab} = \tau_{[ab]}$,
- $X_{abcd} = X_{[ab][cd]}$ *with* $X_{[abc]d} = 0$ *and* $J^{ab}X_{abcd} = 0$,

such that

$$T_{abcd} = X_{abcd} + J_{c[a}\rho_{b]d} - J_{d[a}\rho_{b]c} - J_{cd}\rho_{[ab]} + J_{ab}\tau_{cd}. \tag{34}$$

Proof. Let us consider a complex consisting of (quotients of) spaces of tensors for $\mathrm{Sp}(2n,\mathbb{R})$ and homomorphisms between them

$$0 \to A \to B \to C \to 0 \tag{35}$$

defined by setting

- $A = \{\rho_{bc}$ with no particular symmetries$\}$
- $B = \{T_{abcd}$ s.t. $T_{abcd} = T_{[ab][cd]}\}/\{T_{abcd} = J_{ab}\tau_{cd}$ for $\tau_{cd} = \tau_{[cd]}\}$
- $C = \{S_{abcd}$ s.t. $S_{abcd} = S_{[abc]d}\}/\{S_{abcd} = J_{[ab}\psi_{c]d}$ for some $\psi_{cd}\}$

and taking

$$A \ni \rho_{bc} \mapsto J_{c[a}\rho_{b]d} - J_{d[a}\rho_{b]c} - J_{cd}\rho_{[ab]} \in B \ni T_{abcd} \mapsto T_{[abc]d} \in C.$$

It is easily verified that

- this is, indeed, a complex,
- $A \to B$ is injective,
- $B \to C$ is surjective,
- if we set $H = \{[X_{abcd}] \in B$ s.t. $X_{[abc]d} = 0$ and $J^{ab}X_{abcd} = 0\}$, then H is transverse to the image of $A \hookrightarrow B$ and is mapped to zero under $B \to C$.

Lemma 1 is precisely the statement that H represents the cohomology of the complex (35). This is most easily verified by computing dimensions

- $\dim A = 4n^2$,
- $\dim B = (n-1)n(2n-1)(2n+1)$,
- $\dim C = 4(n-2)n^2(2n+1)/3$,
- $\dim H = (n-1)n(2n-1)(2n+3)/3$,

(for $n \geq 2$) and the result follows. q.e.d.

To continue the proof of Theorem 6, we claim that Lemma 1 applies to

$$T_{abcd} = \nabla_{[a}\mu_{b]cd} + g_{c[a}\omega_{b]d} - g_{d[a}\omega_{b]c} + J_{cd}J_{[a}{}^{e}\omega_{b]e}, \tag{36}$$

where $\mu_{bcd} = \nabla_c\omega_{bd} - \nabla_d\omega_{bc}$. This is because

$$T_{[abc]d} = 2J_{[ab}\omega_{c]e}J_{d}{}^{e},$$

as can be verified by direct computation from (9) or, more simply, by noticing that

$$\Lambda^1\otimes\Lambda^2_\perp\mathbb{U} \ni \begin{bmatrix} \omega_{bc} \\ \nabla_c\omega_{db} - \nabla_d\omega_{cb} \\ 0 \end{bmatrix} \overset{\nabla}{\mapsto} \begin{bmatrix} 0 \\ T_{abcd} \\ * \end{bmatrix} \overset{\nabla}{\mapsto} \begin{bmatrix} -T_{[abc]d} \\ * \\ * \end{bmatrix} \in \Lambda^3\otimes\Lambda^2_\perp\mathbb{U}$$

and that the curvature of ∇ on $\Lambda^2_\perp\mathbb{U}$ is given by (33) (but, in fact, we only need to know that the curvature has the form (29) in order to see that $T_{[abc]d} = J_{[ab}\psi_{c]d}$ for some ψ_{cd} and be in a position to apply Lemma 1). We conclude from Lemma 1 that T_{abcd} in (36) uniquely

determines tensor fields ρ_{ab}, τ_{ab}, and X_{abcd} satisfying the symmetries specified in Lemma 1 and such that (34) holds. Of course, we could determine ρ_{ab} explicitly from its characterizing properties and especially (34) by tracing over various pairs of indices using the inverse symplectic form J^{ab}. The result is extraordinarily complicated but has the form

$$\rho_{ab} = -\frac{1}{2(n+1)}\left(S_{ab} - \frac{1}{2n+1}J^{cd}S_{cd}J_{ab}\right) + \text{lower-order terms},$$

where $S_{ab} = J^{cd}(\nabla_a\nabla_c\omega_{bd} - \nabla_b\nabla_c\omega_{ad})$. The mapping $\omega_{bc} \mapsto \rho_{bc}$ defines the differential operator $\mathcal{L}_{bc}$ in the statement of Theorem 6, and from (32) we have arranged that

$$\Lambda^1 \otimes \Lambda^2_\perp \mathbb{U} \ni \begin{bmatrix} \omega_{bc} \\ \nabla_c\omega_{db} - \nabla_d\omega_{cb} \\ \mathcal{L}_{bc}(\omega) \end{bmatrix} \overset{\nabla}{\longmapsto} \begin{bmatrix} 0 \\ X_{abcd} + J_{ab}\tau_{cd} \\ * \end{bmatrix} \in \Lambda^2 \otimes \Lambda^2_\perp \mathbb{U}, \tag{37}$$

where X_{abcd} and τ_{cd} are determined by (34). Recall that this conclusion was derived under assumption (iii) in Theorem 6 with the additional constraint, without losing generality, that $\mu_{bcd} = \nabla_c\omega_{db} - \nabla_d\omega_{cb}$. The conclusion that $\rho_{bc} = \mathcal{L}_{bc}(\omega)$, for the differential operator $\mathcal{L}_{bc}$ derived above, was forced by these assumptions. This is enough to complete the proof of Theorem 6. q.e.d.

There are, however, some further conclusions that can be derived from this proof and are worth recording here. First, if (iii) holds, then it is immediate from (29) and (37) that $X_{abcd} = 0$. We can compute X_{abcd} from (34) and (36), but, in fact, we have essentially done this computation already in §4 when we derived (16) leading to (17). The point is that if one simply ignores all terms involving J in the formula (31) for the connection on $\Lambda^2_\perp \mathbb{U}$, then one obtains

$$\begin{bmatrix} \sigma_b \\ \mu_{bc} \\ \rho_b \end{bmatrix} \longmapsto \begin{bmatrix} \nabla_a\sigma_b - \mu_{ab} \\ \nabla_a\mu_{bc} + g_{ab}\sigma_c - g_{ac}\sigma_b \\ \nabla_a\rho_b \end{bmatrix},$$

which, as far as the first two rows are concerned, coincides with (14). But we are planning to remove all J-traces in defining X_{abcd} via (34). Bearing in mind that the Riemann curvature tensors on complex and real projective space differ only by terms involving J, it follows from the derivation of (17) that

$$X_{abcd} = 2 \times \pi_\perp(\nabla_{(a}\nabla_{c)}\omega_{bd} + g_{ac}\omega_{bd}), \tag{38}$$

where recall that the subscript $\perp$ means to remove the J-traces. Of course, this confirms Theorem 4 in case $\ell = 2$. Another key observation from (37) is as follows.

Theorem 7. *Suppose ω_{ab} is a symmetric tensor on $\mathbb{CP}_n$ and that*

$$\pi_\perp(\nabla_{(a}\nabla_{c)}\omega_{bd} + g_{ac}\omega_{bd}) = 0. \tag{39}$$

Then

$$\left[\begin{array}{c}\omega_{bc}\\ \nabla_c\omega_{db}-\nabla_d\omega_{cb}\\ \mathcal{L}_{bc}(\omega)\end{array}\right]\overset{\nabla}{\longmapsto}\left[\begin{array}{c}0\\ J_{ab}\tau_{cd}\\ J_{ab}\theta_c\end{array}\right]\in\Gamma(\mathbb{CP}_n,\Lambda^2\otimes\Lambda^2_\perp\mathbb{U}),\tag{40}$$

for some $\tau_{cd}\in\Gamma(\mathbb{CP}_n,\Lambda^2)$ *and* $\theta_c\in\Gamma(\mathbb{CP}_n,\Lambda^1)$.

Proof. The first and second rows of the left-hand side of (40) are enough to give the vanishing of the first row of the right-hand side, and $\mathcal{L}_{bc}(\omega)$ is designed so that (37) holds in this case. Now that we know (39), it follows from (38) that the second row of the right-hand side of (40) is of the stated form. It remains to demonstrate that the third row is as stated. Evidently, given the difficulties in writing down an explicit formula for $\mathcal{L}_{bc}(\omega)$, a direct verification is out of the question. Instead, let us observe that

$$\left[\begin{array}{c}0\\ J_{bc}\tau_{de}\\ \rho_{bcd}\end{array}\right]\overset{\nabla}{\longmapsto}\left[\begin{array}{c}-J_{[ab}\tau_{c]d}\\ J_{[ab}\nabla_{c]}\tau_{de}-J_{d[a}\rho_{bc]e}+J_{e[a}\rho_{bc]d}-J_{de}\rho_{[abc]}\\ \nabla_{[a}\rho_{bc]d}+J_{[ab}J_{c]}{}^f\tau_{df}\end{array}\right]$$

under $\Lambda^2\otimes\Lambda^2_\perp\mathbb{U}\overset{\nabla}{\longrightarrow}\Lambda^3\otimes\Lambda^2_\perp\mathbb{U}$ and from (29) conclude that

$$-J_{d[a}\rho_{bc]e}+J_{e[a}\rho_{bc]d}-J_{de}\rho_{[abc]}=J_{[ab}\psi_{c]de}\tag{41}$$

for some $\psi_{cde}=\psi_{c[de]}$. Certainly (41) holds if $\rho_{cde}=J_{cd}\theta_e$. Conversely, it may be verified without too much difficulty that the converse is true on $\mathbb{CP}_n$ for $n\geq 3$. On $\mathbb{CP}_2$, the relation (41) is content-free and a separate argument is needed: it turns out that there is a second-order differential operator on $\Lambda^2\otimes\Lambda^2_\perp\mathbb{U}$ that may be applied to give sufficiently strong algebraic constraints on ρ_{bcd}. This is part of the general theory developed in §5.3 below and will be omitted here (and the general theory will also provide a workaround for the algebraic verification claimed above). q.e.d.

We are at last in a position to prove the sufficiency of the condition given in Theorem 4 in case $\ell=2$.

Theorem 8. *Suppose* ω_{ab} *is a globally defined smooth symmetric tensor on* $\mathbb{CP}_n$ *and that*

$$\pi_\perp(\nabla_{(a}\nabla_{c)}\omega_{bd}+g_{ac}\omega_{bd})=0.$$

Then there is a smooth 1*-form* ϕ_a *on* $\mathbb{CP}_n$ *such that* $\omega_{ab}=\nabla_{(a}\phi_{b)}$.

Proof. According to Theorems 6 and 7 it suffices to show that if

$$\Gamma(\mathbb{CP}_n,\Lambda^1\otimes\Lambda^2_\perp\mathbb{U})\ni\left[\begin{array}{c}\omega_{bc}\\ \mu_{bcd}\\ \rho_{bc}\end{array}\right]\overset{\nabla_a}{\longmapsto}\left[\begin{array}{c}0\\ J_{ab}\tau_{cd}\\ J_{ab}\theta_c\end{array}\right]\in\Gamma(\mathbb{CP}_n,\Lambda^2\otimes\Lambda^2_\perp\mathbb{U})$$

for some $\tau_{cd} \in \Gamma(\mathbb{CP}_n, \Lambda^2)$ and $\theta_c \in \Gamma(\mathbb{CP}_n, \Lambda^1)$, then

$$\begin{bmatrix} \omega_{bc} \\ \mu_{bcd} \\ \rho_{bc} \end{bmatrix} \text{ is in the range of } \nabla_b : \Gamma(\mathbb{CP}_n, \Lambda^2_\perp \mathbb{U}) \to \Gamma(\mathbb{CP}_n, \Lambda^1 \otimes \Lambda^2_\perp \mathbb{U}).$$

In other words, it suffices to show exactness of the complex

$$\Gamma(\mathbb{CP}_n, \Lambda^2_\perp \mathbb{U}) \xrightarrow{\nabla} \Gamma(\mathbb{CP}_n, \Lambda^1 \otimes \Lambda^2_\perp \mathbb{U}) \xrightarrow{\nabla_\perp} \Gamma(\mathbb{CP}_n, \Lambda^2_\perp \otimes \Lambda^2_\perp \mathbb{U}).$$

We already used (33) in observing that this is, indeed, a complex, but now let us analyze (33) more precisely. Certainly, it is of the form (29) for some $\Phi : \Lambda^2_\perp \mathbb{U} \to \Lambda^2_\perp \mathbb{U}$, but this Φ is quite special. From (33) we compute that

$$\begin{bmatrix} \sigma_c \\ \mu_{cd} \\ \rho_c \end{bmatrix} \xmapsto{\Phi^2} 2 \begin{bmatrix} -\sigma_c + J_c{}^e \rho_e \\ -\mu_{cd} + J_c{}^e J_d{}^f \mu_{ef} \\ -\rho_c - J_c{}^e \sigma_e \end{bmatrix},$$

which suggests the decomposition

$$\Lambda^2_\perp \mathbb{U} = \Lambda^2_{\perp,0} \mathbb{U} \oplus \Lambda^2_{\perp,-4} \mathbb{U} \tag{42}$$

according to

$$\begin{bmatrix} \sigma_c \\ \mu_{cd} \\ \rho_c \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \sigma_c + J_c{}^e \rho_e \\ \mu_{cd} + J_c{}^e J_d{}^f \mu_{ef} \\ \rho_c - J_c{}^e \sigma_e \end{bmatrix} + \frac{1}{2} \begin{bmatrix} \sigma_c - J_c{}^e \rho_e \\ \mu_{cd} - J_c{}^e J_d{}^f \mu_{ef} \\ \rho_c + J_c{}^e \sigma_e \end{bmatrix},$$

for then Φ^2 vanishes on $\Lambda^2_{\perp,0} \mathbb{U}$ and coincides with $-4 \times \mathrm{Id}$ on $\Lambda^2_{\perp,-4} \mathbb{U}$. Furthermore, it is readily verified that the connection ∇ respects this decomposition. We are reduced to showing exactness of the following two complexes:

$$\Gamma(\mathbb{CP}_n, \Lambda^2_{\perp,0} \mathbb{U}) \xrightarrow{\nabla} \Gamma(\mathbb{CP}_n, \Lambda^1 \otimes \Lambda^2_{\perp,0} \mathbb{U}) \xrightarrow{\nabla_\perp} \Gamma(\mathbb{CP}_n, \Lambda^2_\perp \otimes \Lambda^2_{\perp,0} \mathbb{U}) \tag{43}$$

and

$$\Gamma(\mathbb{CP}_n, \Lambda^2_{\perp,-4} \mathbb{U}) \xrightarrow{\nabla} \Gamma(\mathbb{CP}_n, \Lambda^1 \otimes \Lambda^2_{\perp,-4} \mathbb{U}) \xrightarrow{\nabla_\perp} \Gamma(\mathbb{CP}_n, \Lambda^2_\perp \otimes \Lambda^2_{\perp,-4} \mathbb{U}). \tag{44}$$

The exactness of (44) is straightforward as follows. Suppose

$$\Omega_a \in \Gamma(\mathbb{CP}_n, \Lambda^1 \otimes \Lambda^2_{\perp,-4} \mathbb{U}) \text{ satisfies } \nabla_\perp \Omega = 0.$$

Then $\nabla_{[a} \Omega_{b]} = J_{ab} \Sigma$ for some $\Sigma \in \Gamma(\mathbb{CP}_n, \Lambda^2_{\perp,-4} \mathbb{U})$. From (29) it follows that $\nabla_{[a} \nabla_{b]} \Omega_c = J_{ab} \Phi \Omega_c$; thus we have

$$J_{[ab} \Phi \Omega_{c]} = \nabla_{[a} \nabla_b \Omega_{c]} = \nabla_{[a} (J_{bc]} \Sigma) = J_{[ab} \nabla_{c]} \Sigma,$$

and, since $\Lambda^1 \xrightarrow{J\wedge -} \Lambda^3$ is injective, we may conclude that $\nabla_c \Sigma = \Phi \Omega_c$. By the Bianchi identity, or by direct calculation, one readily verifies that $\nabla_a \Phi = 0$. It follows that

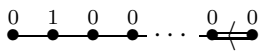
$$\nabla_a (-\Phi \Sigma / 4) = -\Phi^2 \Omega_a / 4 = \Omega_a,$$

as required.

It remains to prove the exactness of (43). Notice from (33) that Φ already vanishes on $\Lambda^2_{\perp,0}\mathbb{U}$, which means that our connection (31) is actually flat on $\Lambda^2_{\perp.0}\mathbb{U}$. As $\mathbb{CP}_n$ is simply connected, the vector bundle $\Lambda^2_{\perp,0}\mathbb{U}$ is trivialized by this connection, and the exactness of (43) follows immediately from the corresponding uncoupled Theorem 5. q.e.d.

The complete proof in case $\ell = 2$ may seem rather complicated, and, indeed, the detailed analysis is necessarily severe. However, as we shall see in the following section, the general argument can be given rather cleanly. In particular, the awkward Lemma 1 can be formulated and finessed by means of suitable Lie algebra cohomology.

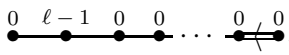
5.3. The case $\ell \geq 2$. The following discussion is a strict generalization of the case $\ell = 2$ given in §5.2 above. First, we generalize the bundle $\Lambda^2_\perp\mathbb{U}$ and its connection constructed from (27). Recall that the natural structure group for $\mathbb{U}$ is $\mathrm{Sp}(2(n+1),\mathbb{R})$ and so we may form an induced bundle with connection for any irreducible representation thereof. In particular, the bundle $\Lambda^2_\perp\mathbb{U}$ arises from the representation



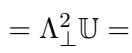
$(n+1$ nodes),

and, more generally, let us consider the bundle $Y^{\ell-1}_\perp\mathbb{U}$ induced by

(45)



As tensor bundles, these are quite complicated, e.g.,

$Y^1_\perp\mathbb{U} = \Lambda^2_\perp\mathbb{U} =$  $\qquad Y^2_\perp\mathbb{U} =$,

but we shall not need to know the details. The curvature of the induced connection on $Y^{\ell-1}_\perp\mathbb{U}$ is given by

$$(\nabla_a\nabla_b - \nabla_b\nabla_a)\Sigma = 2J_{ab}\Psi\Sigma, \tag{46}$$

where $\Psi \in \mathrm{End}(Y^{\ell-1}_\perp\mathbb{U})$ is induced by $\Phi \in \mathrm{End}(\mathbb{U})$ defined by equations (28) and (29). The form of the curvature (46) is all we know in order to proceed with a rather general construction as follows. We shall be mimicking the construction of the Bernstein–Gelfand–Gelfand complex on projective space given in [**14**]. For simplicity, let us suppose that $n \geq 3$, postponing the case $n = 2$ for later discussion. It is clear that the complex (25) can be naturally coupled with $Y^{\ell-1}_\perp\mathbb{U}$ to yield a complex

$$Y^{\ell-1}_\perp\mathbb{U} \xrightarrow{\nabla} \Lambda^1 \otimes Y^{\ell-1}_\perp\mathbb{U} \xrightarrow{\nabla_\perp} \Lambda^2_\perp \otimes Y^{\ell-1}_\perp\mathbb{U} \xrightarrow{\nabla_\perp} \Lambda^3_\perp \otimes Y^{\ell-1}_\perp\mathbb{U}, \tag{47}$$

and we maintain that this complex is naturally filtered. This is because, by (27), the same is evidently true of the $\mathbb{U}$-coupled complex

$$\begin{array}{ccccccc}
\mathbb{U} & \xrightarrow{\nabla} & \Lambda^1\otimes\mathbb{U} & \xrightarrow{\nabla_\perp} & \Lambda^2_\perp\otimes\mathbb{U} & \xrightarrow{\nabla_\perp} & \Lambda^3_\perp\otimes\mathbb{U}\\
\| & & \| & & \| & & \|\\
\Lambda^0 & & \Lambda^1 & & \Lambda^2_\perp & & \Lambda^3_\perp\\
\oplus & \nearrow & \oplus & \nearrow & \oplus & \nearrow & \oplus\\
\Lambda^1 & & \Lambda^1\otimes\Lambda^1 & & \Lambda^2_\perp\otimes\Lambda^1 & & \Lambda^3_\perp\otimes\Lambda^1\\
\oplus & \nearrow & \oplus & \nearrow & \oplus & \nearrow & \oplus\\
\Lambda^0 & & \Lambda^1 & & \Lambda^2_\perp & & \Lambda^3_\perp
\end{array}$$

and the filtration on (47) is inherited therefrom. Writing

$$Y_\perp^{\ell-1}\mathbb{U}=\mathbb{V}=\mathbb{V}_0\oplus\mathbb{V}_1\oplus\mathbb{V}_2\oplus\mathbb{V}_3\oplus\cdots\oplus\mathbb{V}_N$$

for the associated graded vector bundle, the E_0-level of the resulting spectral sequence has the form

(48)
$$\begin{array}{ccccccc}
\mathbb{V}_0 & \Lambda^1\otimes\mathbb{V}_0 & \Lambda^2_\perp\otimes\mathbb{V}_0 & \Lambda^3_\perp\otimes\mathbb{V}_0 & & &\\
 & \uparrow\partial & \uparrow\partial_\perp & \uparrow\partial_\perp & & &\\
 & \mathbb{V}_1 & \Lambda^1\otimes\mathbb{V}_1 & \Lambda^2_\perp\otimes\mathbb{V}_1 & \Lambda^3_\perp\otimes\mathbb{V}_1 & &\\
 & & \uparrow\partial & \uparrow\partial_\perp & \uparrow\partial_\perp & &\\
 & & \mathbb{V}_2 & \Lambda^1\otimes\mathbb{V}_2 & \Lambda^2_\perp\otimes\mathbb{V}_2 & \Lambda^3_\perp\otimes\mathbb{V}_2 &\\
 & & & \ddots & \ddots & \ddots & \ddots
\end{array}$$

where all differentials are vector bundle homomorphisms. The key point is that we can identify much of the E_1-level explicitly. Take, for example, the homomorphism $\partial:\mathbb{V}_1\to\Lambda^1\otimes\mathbb{V}_0$. It is induced by the identity mapping $\Lambda^1=\mathbb{U}_1\to\Lambda^1\otimes\mathbb{U}_0=\Lambda^1$ and thus may be identified as the canonical inclusion

$$\mathbb{V}_1=\ydiagram{2,1}\hookrightarrow\square\otimes\underbrace{\ydiagram{3}}_{\ell-1\text{ boxes}}=\Lambda^1\otimes\mathbb{V}_0$$

with quotient $\bigodot^\ell\Lambda^1$. As a more subtle example, when $\ell=2$ we have $\mathbb{V}=\Lambda^1_\perp\mathbb{U}$ with $\mathbb{V}_0\oplus\mathbb{V}_1\oplus\mathbb{V}_2=\Lambda^1\oplus\Lambda^2\oplus\Lambda^1$ and

$$\begin{array}{ccccc}
\Lambda^1\otimes\mathbb{V}_2 & \xrightarrow{\partial_\perp} & \Lambda^2_\perp\otimes\mathbb{V}_1 & \xrightarrow{\partial_\perp} & \Lambda^3_\perp\otimes\mathbb{V}_0\\
\| & & \| & & \|\\
\Lambda^1\otimes\Lambda^1 & \to & \Lambda^2_\perp\otimes\Lambda^2 & \to & \Lambda^3_\perp\otimes\Lambda^1,
\end{array}$$

whose cohomology is actually the subject of Lemma 1, being identified there as

$$H=\{X_{abcd}\text{ s.t. }X_{abcd}=X_{[ab][cd]}\text{ and }X_{[abc]d}=0\text{ and }J^{ab}X_{abcd}=0\}$$

corresponding to the irreducible representation

$$\ydiagram{2,2}_\perp=\overset{0}{\bullet}\!\!-\!\!\overset{2}{\bullet}\!\!-\!\!\overset{0}{\bullet}\!\!-\cdots-\!\!\overset{0}{\bullet}\!\!\Lleftarrow\!\!\overset{0}{\bullet}\qquad(n\text{ nodes})$$

of $\mathrm{Sp}(2n,\mathbb{R})$. Similarly, the proof of Theorem 7 for $n \geq 3$ boils down to

$$\begin{array}{ccc} \Lambda^2_\perp \otimes \mathbb{V}_2 & \xrightarrow{\qquad\qquad\partial_\perp\qquad\qquad} & \Lambda^3_\perp \otimes \mathbb{V}_1 = \Lambda^3_\perp \otimes \Lambda^2 \\ \| & & \uparrow \\ \Lambda^2_\perp \otimes \Lambda^1 & \ni \rho_{bcd} \mapsto J_{d[a}\rho_{bc]e} - J_{e[a}\rho_{bc]d} + \rho_{[abc]}J_{de} \in & \Lambda^3 \otimes \Lambda^2 \end{array}$$

being injective. In general, the E_1-level of the spectral sequence is controlled by Lie algebra cohomology as follows.

Proposition 6. *Suppose $\mathbb{V}$ is a representation of the Heisenberg algebra $\mathfrak{h}_{2n+1} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ of dimension $2n+1$ for $n \geq 3$ where $\mathfrak{g}_{-2}$ denotes the center and for $X, Y \in \mathfrak{g}_{-1}$, we have $[X,Y] = J(X \wedge Y)$ for a non-degenerate symplectic form $J : \Lambda^2\mathfrak{g}_{-1} \to \mathfrak{g}_{-2}$. Then the Lie algebra cohomologies $H^r(\mathfrak{h}_{2n+1}, \mathbb{V})$ for $r = 0, 1, 2$ may be computed by the complex*

$$0 \to \mathbb{V} \xrightarrow{\partial} \mathrm{Hom}(\mathfrak{g}_{-1}, \mathbb{V}) \xrightarrow{\partial_\perp} \mathrm{Hom}(\Lambda^2_\perp\mathfrak{g}_{-1}, \mathbb{V}) \xrightarrow{\partial_\perp} \mathrm{Hom}(\Lambda^3_\perp\mathfrak{g}_{-1}, \mathbb{V}),$$

induced by the action of $\mathfrak{g}_{-1}$ on $\mathbb{V}$.

Proof. Let us introduce abstract indices in the sense of [**32**] to write $\eth_a$ for the action of $\mathfrak{g}_{-1}$ on $\mathbb{V}$, meaning that $Xv = X^a\eth_a v$ for $X \in \mathfrak{g}_{-1}$. Then

$$v \overset{\partial}{\mapsto} \eth_a v \quad v_a \overset{\partial_\perp}{\longmapsto} \eth_{[a}v_{b]} - \tfrac{1}{2n}J^{cd}\eth_c v_d J_{ab} \quad v_{ab} \overset{\partial_\perp}{\longmapsto} \eth_{[a}v_{bc]} - \tfrac{1}{n-1}J^{de}\eth_d v_{e[a}J_{bc]}$$

are the explicit formulæ for the differentials of the complex in question. Let us also write $\eth$ for the action of $\mathfrak{g}_{-2}$ on $\mathbb{V}$. Then to say that $\mathbb{V}$ is an $\mathfrak{h}_{2n+1}$-module is precisely that

$$\eth_a\eth_b - \eth_b\eth_a v = 2J_{ab}\eth v \quad \forall\, v \in \mathbb{V}$$

and the differentials of the usual Koszul complex $\Lambda^\bullet(\mathfrak{h}_{2n+1})^* \otimes \mathbb{V}$ defining the Lie algebra cohomology begin with

$$\mathbb{V} \ni v \mapsto \begin{bmatrix} \eth_a v \\ \eth v \end{bmatrix} \in \begin{array}{c} \mathfrak{g}^*_{-1} \otimes \mathbb{V} \\ \oplus \\ \mathfrak{g}^*_{-2} \otimes \mathbb{V} \end{array} = (\mathfrak{h}_{2n+1})^* \otimes \mathbb{V}$$

and continue with

$$\begin{array}{ccc} \Lambda^p\mathfrak{g}^*_{-1} \otimes \mathbb{V} & & \Lambda^{p+1}\mathfrak{g}^*_{-1} \otimes \mathbb{V} \\ \oplus & \longrightarrow & \oplus \qquad\qquad \text{for } p \geq 1 \\ \mathfrak{g}^*_{-2} \otimes \Lambda^{p-1}\mathfrak{g}^*_{-1} \otimes \mathbb{V} & & \mathfrak{g}^*_{-2} \otimes \Lambda^p\mathfrak{g}^*_{-1} \otimes \mathbb{V} \\ \begin{bmatrix} v_{ab\cdots cd} \\ w_{ab\cdots c} \end{bmatrix} & \longmapsto & \begin{bmatrix} \eth_{[a}v_{bc\cdots de]} + (-1)^p J_{[ab}w_{c\cdots de]} \\ \eth_{[a}w_{bc\cdots d]} + (-1)^p \eth v_{abc\cdots d} \end{bmatrix}. \end{array}$$

Easy diagram chasing gives the desired result. q.e.d.

When $n = 2$, there is a replacement for Proposition 6 as follows.

Proposition 7. *Suppose* $\mathbb{V}$ *is a representation of the Heisenberg algebra* $\mathfrak{h}_5 = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ *of dimension* 5. *Then the Lie algebra cohomologies* $H^r(\mathfrak{h}_5, \mathbb{V})$ *for* $r = 0, 1, 2$ *may be computed by the complex*

$$0 \to \mathbb{V} \xrightarrow{\partial} \mathfrak{g}_{-1}^* \otimes \mathbb{V} \xrightarrow{\partial_\perp} \Lambda^2_\perp \mathfrak{g}_{-1}^* \otimes \mathbb{V} \xrightarrow{\partial^{(2)}_\perp} \mathfrak{g}_{-2}^* \otimes \Lambda^2_\perp \mathfrak{g}_{-1}^* \otimes \mathbb{V}$$

where, adopting the notation from the proof of Proposition 6, *the linear transformation* $\partial^{(2)}_\perp \colon \Lambda^2_\perp \mathfrak{g}_{-1}^* \otimes \mathbb{V} \to \mathfrak{g}_{-2}^* \otimes \Lambda^2_\perp \mathfrak{g}_{-1}^* \otimes \mathbb{V}$ *is given by*

$$v_{ab} \longmapsto J^{cd} \mathfrak{d}_c \mathfrak{d}_{[a} v_{b]d} + 3 \mathfrak{d} v_{ab}.$$

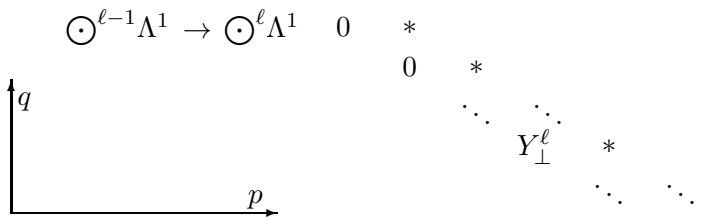
Proof. This is what emerges by following the proof of Proposition 6. The only difference when $n = 2$ is that $\Lambda^3_\perp \mathfrak{g}_{-1}^* = 0$, and so there is one extra step in the resulting diagram chase. q.e.d.

Looking back at the E_0-level (48) of our spectral sequence, we see that for $n \geq 3$, Proposition 6 is exactly what we need to identify much of the E_1-level, provided we are able to identify the Lie algebra cohomology $H^r(\mathfrak{h}_{2n+1}, \mathbb{V})$ for $r = 0, 1, 2$. Kostant's Theorem [**28**] provides such an identification

$$\begin{aligned} H^0(\mathfrak{h}_{2n+1}, Y^{\ell-1}_\perp \mathbb{U}) &= \overset{\ell-1}{\bullet}\!\!-\!\!\overset{0}{\bullet}\!\!-\!\!\overset{0}{\bullet}\cdots\overset{0}{\bullet}\!\!=\!\!\overset{0}{\bullet} \qquad (n \text{ nodes}) \\ H^1(\mathfrak{h}_{2n+1}, Y^{\ell-1}_\perp \mathbb{U}) &= \overset{\ell}{\bullet}\!\!-\!\!\overset{0}{\bullet}\!\!-\!\!\overset{0}{\bullet}\cdots\overset{0}{\bullet}\!\!=\!\!\overset{0}{\bullet} \\ H^2(\mathfrak{h}_{2n+1}, Y^{\ell-1}_\perp \mathbb{U}) &= \overset{0}{\bullet}\!\!-\!\!\overset{\ell}{\bullet}\!\!-\!\!\overset{0}{\bullet}\cdots\overset{0}{\bullet}\!\!=\!\!\overset{0}{\bullet} \end{aligned}$$

as $\mathrm{Sp}(2n, \mathbb{R})$-modules as well as the exact locations of the corresponding induced bundles in the E_1-level

(49)



where $Y^\ell_\perp$ arises as the cohomology of

$$\Lambda^1 \otimes \mathbb{V}_\ell \xrightarrow{\partial_\perp} \Lambda^2_\perp \otimes \mathbb{V}_{\ell-1} \xrightarrow{\partial_\perp} \Lambda^3_\perp \otimes \mathbb{V}_{\ell-2}.$$

This is the right location to be the target of a differential $\bigodot^\ell \Lambda^1 \to Y^\ell_\perp$ at the E_ℓ-level. We conclude immediately that there is a complex

$$\bigodot^{\ell-1} \Lambda^1 \xrightarrow{\nabla} \bigodot^\ell \Lambda^1 \xrightarrow{\nabla^{(\ell)}_\perp} Y^\ell_\perp$$

whose cohomology is the same as that of the original complex

$$Y^{\ell-1}_\perp \mathbb{U} \xrightarrow{\nabla} \Lambda^1 \otimes Y^{\ell-1}_\perp \mathbb{U} \xrightarrow{\nabla_\perp} \Lambda^2_\perp \otimes Y^{\ell-1}_\perp \mathbb{U}.$$

This is true both locally (which confirms abstractly [**31**] that $\nabla^{(\ell)}_\perp$ is a differential operator) and globally, which is what we shall use to prove the sufficiency of the condition given in Theorem 4 as follows.

Theorem 9. *Suppose $\omega_{abc\cdots d}$ is a globally defined smooth symmetric ℓ-tensor on $\mathbb{CP}_n$ for $n \geq 2$. Suppose that $\nabla_\perp^{(\ell)}(\omega_{ab\cdots d}) = 0$, where $\nabla_\perp^{(\ell)}$ is the differential operator of Theorem 4 defined as the composition (22). Then there is a smooth symmetric $(\ell-1)$-tensor $\phi_{bc\cdots d}$ on $\mathbb{CP}_n$ such that $\omega_{abc\cdots d} = \nabla_{(a}\phi_{bc\cdots d)}$.*

Proof. For $n \geq 3$, there remain just two facts to verify. The first is that, as our notation already indicates, the differential operator $\nabla_\perp^{(\ell)}$ arising at the E_ℓ-level of the spectral sequence of the filtered complex (47) coincides with the composition (22). The second is that

$$\Gamma(\mathbb{CP}_n, Y_\perp^{\ell-1}\mathbb{U}) \xrightarrow{\nabla} \Gamma(\mathbb{CP}_n, \Lambda^1 \otimes Y_\perp^{\ell-1}\mathbb{U}) \xrightarrow{\nabla_\perp} \Gamma(\mathbb{CP}_n, \Lambda^2_\perp \otimes Y_\perp^{\ell-1}\mathbb{U}) \tag{50}$$

is exact. The first of these two facts follows by comparing the spectral sequence constructions of $\nabla_\perp^{(\ell)}$ on $\mathbb{CP}_n$ and $\nabla^{(\ell)}$ on $\mathbb{RP}_n$ for the model embeddings $\iota : \mathbb{RP}_n \hookrightarrow \mathbb{CP}_n$. Recall, as in Proposition 2, that each ι is totally geodesic and Lagrangian. If $\Sigma = (\sigma, \mu, \rho)$ is a section of $\mathbb{U} \equiv \Lambda^0 \oplus \Lambda^1 \oplus \Lambda^0$ on $\mathbb{CP}_n$, then we can define its "pull-back" $\iota^*\Sigma$ to $\mathbb{RP}_n$ to be the section $(\iota^*\sigma, \iota^*\mu)$ of the bundle $\mathbb{T} \equiv \Lambda^0 \oplus \Lambda^1$ on $\mathbb{RP}_n$. From the formulæ (27) and (13) for the connections on $\mathbb{U}$ and $\mathbb{T}$, we see that pull-back intertwines these connections, i.e., $\iota^*(\nabla\Sigma) = \nabla(\iota^*\Sigma)$. The same is therefore true of pull-back from

$$Y_\perp^{\ell-1}\mathbb{U} = \underbrace{\boxplus\cdots\boxplus}_{\ell-1 \text{ columns}}{}_\perp\mathbb{U} \quad \text{to} \quad \underbrace{\boxplus\cdots\boxplus}_{\ell-1 \text{ columns}}\mathbb{T}.$$

Hence, the resulting operators

$$\bigodot^\ell\Lambda^1 \xrightarrow{\nabla_\perp^{(\ell)}} Y_\perp^\ell \text{ on } \mathbb{CP}_n \quad \text{and} \quad \bigodot^\ell\Lambda^1 \xrightarrow{\nabla^{(\ell)}} Y^\ell \text{ on } \mathbb{RP}_n$$

are also related by pull-back. As this is true for every model embedding $\iota : \mathbb{RP}_n \hookrightarrow \mathbb{CP}_n$, Corollary 2 now shows that $\nabla_\perp^{(\ell)}$ is characterized by this property. By construction, (22) also has this property and so the two operators agree, as required.

Now we must show that (50) is exact. The following reasoning applies to any bundle $\mathbb{V}$ induced from $\mathbb{U}$ by an irreducible representation of $\mathrm{Sp}(2(n+1), \mathbb{R})$ including $Y_\perp^{\ell-1}\mathbb{U}$ (recall that it is induced by (45)). In particular, when applied to $\mathbb{V} = \Lambda^2_\perp\mathbb{U}$ it puts the reasoning in the proof of Theorem 8 in proper context. The endomorphism

$$\begin{bmatrix} \sigma \\ \mu_c \\ \rho \end{bmatrix} \xmapsto{\Phi} \begin{bmatrix} \rho \\ J_c{}^d\mu_d \\ -\sigma \end{bmatrix}$$

of $\mathbb{U}$ defined by (29) is preserved by the connection on $\mathbb{U}$. Evidently, it also satisfies $\Phi^2 = -\mathrm{Id}$. Finally, we compute

$$\langle \Phi(\sigma, \mu_a, \rho), (\tilde\sigma, \tilde\mu_b, \tilde\rho)\rangle = \sigma\tilde\sigma + g^{ab}\mu_a\tilde\mu_b + \rho\tilde\rho$$

where $\langle\ ,\ \rangle$ is the skew form on $\mathbb{U}$ defined by (30) and notice that this is symmetric and positive definite. Hence, the structure group for $\mathbb{U}$ naturally reduces from $\mathrm{Sp}(2(n+1),\mathbb{R})$ to $\mathrm{SU}(2(n+1))$ with Φ providing the complex structure. Accordingly, the decomposition (42) may be seen as follows. The complexified bundles split in familiar fashion as

$$\mathbb{C}\mathbb{U} = \Lambda^{1,0}\mathbb{U} \oplus \Lambda^{0,1}\mathbb{U}$$

and

$$\Lambda^2\mathbb{C}\mathbb{U} = \Lambda^{2,0}\mathbb{U} \oplus \Big(\Lambda^{1,1}_\perp\mathbb{U} \oplus \Lambda^0\mathbb{U}\Big) \oplus \Lambda^{0,2}\mathbb{U}$$

as complex eigenspaces under the action of Φ. Then (42) is simply the real counterpart of the complex decomposition

$$\Lambda^2_\perp\mathbb{C}\mathbb{U} = \Lambda^{1,1}_\perp\mathbb{U} \oplus \big(\Lambda^{2,0}\mathbb{U} \oplus \Lambda^{0,2}\mathbb{U}\big)\,.$$

For our purposes, a sufficient counterpart to (42) in general is to write

$$\mathbb{V} = \mathbb{V}_0 \oplus \mathbb{V}_{\neq 0}$$

where $\mathbb{V}_0 \equiv \ker \Psi : \mathbb{V} \to \mathbb{V}$ (and, following (46), we are writing Ψ for endomorphism of $\mathbb{V}$ induced by $\Phi \in \mathrm{End}(\mathbb{U})$) and $\mathbb{V}_{\neq 0}$ is such that $\mathbb{C}\mathbb{V}_{\neq 0}$ collects all the non-zero eigenspaces of Ψ. In particular, notice that $\Psi|_{\mathbb{V}_{\neq 0}} : \mathbb{V}_{\neq 0} \to \mathbb{V}_{\neq 0}$ is invertible. Now we are in a position to show that

$$\Gamma(\mathbb{C}\mathbb{P}_n, \mathbb{V}) \xrightarrow{\nabla} \Gamma(\mathbb{C}\mathbb{P}_n, \Lambda^1 \otimes \mathbb{V}) \xrightarrow{\nabla_\perp} \Gamma(\mathbb{C}\mathbb{P}_n, \Lambda^2_\perp \otimes \mathbb{V}) \tag{51}$$

in general, and hence (50) in particular, is exact. As in the proof of Theorem 8, this breaks into two cases

$$\begin{array}{ccccc}
\Gamma(\mathbb{C}\mathbb{P}_n, \mathbb{V}_0) & \xrightarrow{\nabla} & \Gamma(\mathbb{C}\mathbb{P}_n, \Lambda^1 \otimes \mathbb{V}_0) & \xrightarrow{\nabla_\perp} & \Gamma(\mathbb{C}\mathbb{P}_n, \Lambda^2_\perp \otimes \mathbb{V}_0) \\
\Gamma(\mathbb{C}\mathbb{P}_n, \mathbb{V}_{\neq 0}) & \xrightarrow{\nabla} & \Gamma(\mathbb{C}\mathbb{P}_n, \Lambda^1 \otimes \mathbb{V}_{\neq 0}) & \xrightarrow{\nabla_\perp} & \Gamma(\mathbb{C}\mathbb{P}_n, \Lambda^2_\perp \otimes \mathbb{V}_{\neq 0}),
\end{array}$$

the counterparts of (43) and (44). The first of these complexes is exact as a consequence of Theorem 5 coupled with the flat connection $\nabla|_{\mathbb{V}_0}$. The curvature of $\nabla|_{\mathbb{V}_{\neq 0}}$ has the form (46) with $\Psi \in \mathrm{End}(\mathbb{V}_{\neq 0})$ crucially being invertible. Exactness of the corresponding complex is established as follows. Suppose $\Omega \in \Gamma(\mathbb{C}\mathbb{P}_n, \Lambda^1 \otimes \mathbb{V}_{\neq 0})$ satisfies $\nabla_\perp\Omega = 0$. Precisely, this means that

$$\nabla_{[a}\Omega_{b]} = J_{ab}\Sigma \quad \text{for some } \Sigma \in \Gamma(\mathbb{C}\mathbb{P}_n, \mathbb{V}_{\neq 0}).$$

Differentiating again, we find that

$$J_{[ab}\nabla_{c]}\Sigma = \nabla_{[a}(J_{bc]}\Sigma) = \nabla_{[a}\nabla_b\Omega_{c]} = J_{[ab}\Psi\Omega_{c]},$$

the last equality being a consequence of (46) as applied to the vector bundle $\mathbb{V}_{\neq 0}$. Since J_{ab} is non-degenerate, we conclude that $\nabla_c\Sigma = \Psi\Omega_c$. But the Bianchi identity $\nabla_{[a}(J_{bc]}\Psi) = 0$ implies that $\nabla_a\Psi = 0$. Finally, recall that Ψ is invertible; therefore we have the equalities

$$\nabla_a(\Psi^{-1}\Sigma) = \Psi^{-1}\nabla_a\Sigma = \Psi^{-1}\Psi\Omega_a = \Omega_a,$$

which is exactly as needed to complete the proof of exactness of (51).

The proof of Theorem 9 is complete save for the case $n = 2$, for which a modification to the argument is needed as follows. We replace the complex (47) by

$$(52)\quad Y^{\ell-1}_\perp\mathbb{U} \xrightarrow{\nabla} \Lambda^1\otimes Y^{\ell-1}_\perp\mathbb{U} \xrightarrow{\nabla_\perp} \Lambda^2_\perp\otimes Y^{\ell-1}_\perp\mathbb{U} \xrightarrow{\nabla^{(2)}_\perp} \Lambda^2_\perp\otimes Y^{\ell-1}_\perp\mathbb{U},$$

where $\nabla^{(2)}_\perp$ is defined by

$$(\nabla^{(2)}_\perp\xi)_{ab} = \nabla_{[a}\mu_{b]} - \Psi\xi_{ab} \quad \text{where } \mu_{[a}J_{bc]} = \nabla_{[a}\xi_{bc]}$$

as a coupled version of the operator $d^{(2)}_\perp : \Lambda^2_\perp \to \Lambda^2_\perp$ defined by (26). It is easy to check that this operator is well defined and that (52) is a complex. It is naturally filtered, and there is spectral sequence whose E_0-level has the form

$$\begin{array}{ccccccc}
\mathbb{V}_0 & \Lambda^1\otimes\mathbb{V}_0 & \Lambda^2_\perp\otimes\mathbb{V}_0 & & & & \\
 & \uparrow\partial & \uparrow\partial_\perp & & & & \\
 & \mathbb{V}_1 & \Lambda^1\otimes\mathbb{V}_1 & \Lambda^2_\perp\otimes\mathbb{V}_1 & \Lambda^2_\perp\otimes\mathbb{V}_0 & & \\
 & & \uparrow\partial & \uparrow\partial_\perp & \uparrow\partial^{(2)}_\perp & & \\
 & & \mathbb{V}_2 & \Lambda^1\otimes\mathbb{V}_2 & \Lambda^2_\perp\otimes\mathbb{V}_2 & \Lambda^2_\perp\otimes\mathbb{V}_1 & \\
 & & & \ddots & \ddots & \ddots & \uparrow\partial^{(2)}_\perp\ddots
\end{array}$$

replacing (48), where $\partial^{(2)}_\perp$ is as in Proposition 7, which is now used together with Kostant's Theorem [**28**] to identify the E_1-level as (49), just as before. The rest of the proof is unchanged. q.e.d.

6. Proof of the main theorem

The proof of Theorem 1 is now a straightforward application of the machinery we have developed. Suppose $\omega_{ab\cdots c}$ is a smooth symmetric ℓ-tensor, globally defined on $\mathbb{CP}_n$ and having zero energy. Then the same is true of $\iota^*\omega_{ab\cdots c}$ for any model embedding $\iota : \mathbb{RP}_n \hookrightarrow \mathbb{CP}_n$. The X-ray transform on $\mathbb{RP}_n$ is well understood, and it is proved in [**1**] that $\iota^*\omega_{ab\cdots c}$ is of the form $\nabla_{(a}\phi_{b\cdots c)}$. By Theorem 3 we conclude that $\nabla^{(\ell)}(\iota^*\omega_{ab\cdots c}) = 0$. As this is true for all model embeddings, we conclude by Corollary 2 that $\nabla^{(\ell)}_\perp(\omega_{ab\cdots c}) = 0$. Theorem 9 finishes our proof.

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