

SOBOLEV METRICS ON THE MANIFOLD OF ALL RIEMANNIAN METRICS

MARTIN BAUER, PHILIPP HARMS & PETER W. MICHOR

Abstract

On the manifold $\mathcal{M}(M)$ of all Riemannian metrics on a compact manifold M , one can consider the natural L^2 -metric as described first by [11]. In this paper we consider variants of this metric, which in general are of higher order. We derive the geodesic equations; we show that they are well-posed under some conditions and induce a locally diffeomorphic geodesic exponential mapping. We give a condition when Ricci flow is a gradient flow for one of these metrics.

1. Introduction

On the manifold $\mathcal{M}(M)$ of all Riemannian metrics on a compact manifold M , one can consider the natural L^2 -metric. It was first described by [11]. Geodesics and curvature on it were described by [14] and [15], who also described the Jacobi fields and the exponential mapping. This was extended to the space of non-degenerate bilinear structures on M in [16] and restricted to the space of almost Hermitian structures in [17]. In his thesis [8] that was published in two subsequent papers [9, 10], Brian Clarke showed that geodesic distance for the L^2 -metric is a positive topological metric on $\mathcal{M}(M)$, and he determined the metric completion of $\mathcal{M}(M)$. In contrast, it was shown in [24, 23] that the natural L^2 -metric on the space of immersions from a compact manifold into a Riemannian manifold has indeed vanishing geodesic distance. This also holds for the right invariant L^2 -metric on diffeomorphism groups [23], and even on the Virasoro-Bott group [5] where the geodesic equation is the KdV-equation.

In this paper, guided by the results of [2, 3, 4], we investigate stronger metrics on $\mathcal{M}(M)$ than the L^2 -metric. These are metrics of the following form:

$$G_g(h, k) = \Phi(\text{Vol}) \int_M g_2^0(h, k) \text{vol}(g) \quad \text{see Section 4.2}$$

$$\text{or} = \int_M \Phi(\text{Scal}) \cdot g_2^0(h, k) \text{vol}(g) \quad \text{see Section 4.3}$$

Received 2/16/2011.

$$\text{or} = \int_M g_2^0((1 + \Delta)^p h, k) \text{vol}(g) \quad \text{see Section 4.4}$$

where Φ is a suitable real-valued function, $\text{Vol} = \int_M \text{vol}(g)$ is the total volume of (M, g) , Scal is the scalar curvature of (M, g) , and g_2^0 is the induced metric on $\binom{0}{2}$ -tensors. We describe all these metrics uniformly as

$$G_g^P(h, k) = \int_M g_2^0(P_g h, k) \text{vol}(g) = \int_M \text{Tr}(g^{-1} \cdot P_g(h) \cdot g^{-1} \cdot k) \text{vol}(g),$$

where $P_g : \Gamma(S^2 T^* M) \rightarrow \Gamma(S^2 T^* M)$ is a positive, symmetric, bijective pseudo-differential operator of order $2p, p \geq 0$, depending smoothly on the metric g . We derive the geodesic equation for the general metric and all particular cases. We show that under certain assumptions on P_g , the geodesic equation is well-posed, and that the geodesic exponential mapping is a diffeomorphism from a neighborhood of the 0 section in the tangent bundle $T\mathcal{M}(M)$ onto a neighborhood of the diagonal in $\mathcal{M}(M) \times \mathcal{M}(M)$. The assumptions are satisfied for the metrics in Sections 4.2 and 4.4, but not for the metric in Section 4.3. In many cases the curve $(1-t)g_0$ can be reparameterized as a geodesic. In each case we can estimate its length, getting conclusions about geodesic incompleteness.

Finally we derive a condition on P_g that is sufficient for the Ricci vector field to be a gradient field in the G^P -metric.

Acknowledgments. We thank the referee for very helpful remarks. All authors were supported by FWF Project 21030. Bauer was supported by FWF Project P24625.

2. Notation

2.1. Metric on tensor spaces. A Riemannian metric $g : TM \times_M TM \rightarrow \mathbb{R}$ will equivalently be interpreted as

$$\flat = g : TM \rightarrow T^*M \quad \text{and} \quad \sharp = g^{-1} : T^*M \rightarrow TM.$$

The metric g can be extended to the cotangent bundle $T^*M = T_1^0 M$ by setting

$$g^{-1}(\alpha, \beta) = g_1^0(\alpha, \beta) = \alpha(\beta^\sharp)$$

for $\alpha, \beta \in T^*M$, and the product metric

$$g_s^r = \bigotimes^r g \otimes \bigotimes^s g^{-1}$$

extends g to all tensor spaces $T_s^r M$. A useful formula is

$$g_2^0(h, k) = \text{Tr}(g^{-1} h g^{-1} k) \quad \text{for } h, k \in T_2^0 M \text{ if } h \text{ or } k \text{ is symmetric.}$$

For a proof using orthonormal frames, see [3]. In this work, traces always contract the first two free appropriate tensor slots:

$$\text{Tr} : T_s^r M \rightarrow T_{s-1}^{r-1}, \quad \text{Tr}^g : T_s^r M \rightarrow T_s^{r-2} M.$$

2.2. Directional derivatives of functions. We will use the following ways to denote directional derivatives of functions, in particular in infinite dimensions. Given a function $F(x, y)$, for instance, we will write:

$$D_{(x,h)}F \text{ or } dF(x)(h) \text{ as shorthand for } \partial_t|_0 F(x + th, y).$$

Here (x, h) in the subscript denotes the tangent vector with foot point x and direction h . If F takes values in some linear space, we will identify this linear space and its tangent space. We use calculus in infinite dimensions as explained in [20].

2.3. Volume density. The *volume density* on M induced by the metric g is given by $\text{vol}(g) = \text{vol}(g) \in \Gamma(\text{vol}(M))$, where $\text{vol}(M)$ denotes the volume bundle. The *volume* of the manifold with respect to the metric g is given by $\text{Vol} = \int_M \text{vol}(g)$. The integral is well-defined since M is compact. If M is oriented, we may identify the volume density with a differential form. Furthermore, we have the following formula for the first variation of the volume density (see for example [2, Section 3.6] for the proof):

Lemma. *The differential of the volume density*

$$\begin{cases} \Gamma(S_+^2 T^* M) & \rightarrow \Gamma(\text{vol}(M)) \\ g & \mapsto \text{vol}(g) \end{cases}$$

is given by

$$D_{(g,m)} \text{vol}(g) = \frac{1}{2} \text{Tr}(g^{-1} \cdot m) \text{vol}(g).$$

2.4. Metric on tensor fields. A metric on a space of tensor fields is defined by integrating the appropriate metric on the tensor space with respect to the volume density:

$$\tilde{g}_s^r(h, k) = \int_M g_s^r(h(x), k(x)) \text{vol}(g)(x)$$

for $h, k \in \Gamma(T_s^r M)$. According to Section 2.1, if h and k are tensor fields of type $\binom{0}{2}$ and h or k is symmetric, then

$$\tilde{g}_2^0(h, k) = \int_M \text{Tr}(g^{-1}h(x)g^{-1}k(x)) \text{vol}(g)(x).$$

2.5. Covariant derivative on M . We will use covariant derivatives on vector bundles as explained in [22, especially Section 19.12]. Let X be a vector field on M . The Levi-Civita covariant derivative ∇_X on (M, g) can be extended uniquely to an operator on the space $\Gamma(T_s^r M)$ of all tensor fields on M . This covariant derivative depends on the metric g .

We define its derivative with respect to g as

$$(1) \quad N_s^r(m) = N_s^r(g, m) = D_{(g,m)} \nabla,$$

where

$$\nabla \in L(\Gamma(T_s^r M), \Gamma(T_{s+1}^r M))$$

and where m is a tangent vector to $\mathcal{M}(M)$ with foot point g . The operator $N_s^r(m) \in \Gamma(L(T_s^r M, T_{s+1}^r M))$ is tensorial since

$$D_{(g,m)} \nabla(fh) = D_{(g,m)}(df \otimes h + f \nabla h) = f D_{(g,m)} \nabla h$$

holds for $f \in C^\infty(M)$ and $h \in \Gamma(T_s^r M)$. In abstract index notation, one has

$$(2) \quad (N_0^1(m))_{jk}^i = \frac{1}{2} g^{il} ((\nabla m)_{jkl} + (\nabla m)_{kjl} - (\nabla m)_{ljk}),$$

as can be seen from the formula [6, theorem 1.174]:

$$g(D_{(g,m)}(\nabla_X Y), Z) = \frac{1}{2} ((\nabla_X m)(Y, Z) + (\nabla_Y m)(X, Z) - (\nabla_Z m)(X, Y)).$$

Furthermore, $(N_1^0(m))_{jk}^i = -(N_0^1(m))_{kj}^i$, since one has for $\alpha \in \Omega^1(M)$ and $X, Y \in \mathfrak{X}(M)$:

$$\begin{aligned} (N_1^0(m)\alpha)(X, Y) &= (D_{(g,m)} \nabla_X \alpha)(Y) = D_{(g,m)}(d(\alpha(Y)) \cdot X - \alpha(\nabla_X Y)) \\ &= -\alpha(D_{(g,m)} \nabla_X Y) = -(N_0^1 Y)(\alpha, X). \end{aligned}$$

Since ∇_X is a derivation on tensor products, one gets a similar property for $N_s^r(m)$:

$$(3) \quad (N_s^r(m))_{jk_1 \dots k_r k_{r+1} \dots k_{r+s}}^{i_1 \dots i_r i_{r+1} \dots i_{r+s}} = \\ = (N_0^1(m))_{jk_1}^{i_1} \delta_{k_2}^{i_2} \dots \delta_{k_{r+s}}^{i_{r+s}} + \dots + \delta_{k_1}^{i_1} \dots \delta_{k_{r+s-1}}^{i_{r+s-1}} (N_1^0(m))_{jk_{r+s}}^{i_{r+s}},$$

where one has N_0^1 in the first r summands and N_1^0 in the last s summands.

2.6. The adjoint of the covariant derivative. The covariant derivative, seen as a mapping $\nabla : \Gamma(T_s^r M) \rightarrow \Gamma(T_{s+1}^r M)$, admits an adjoint $\nabla^* : \Gamma(T_{s+1}^r M) \rightarrow \Gamma(T_s^r M)$ with respect to the metric \tilde{g} , i.e., $\tilde{g}_{s+1}^r(\nabla B, C) = \tilde{g}_s^r(B, \nabla^* C)$. It is given by $\nabla^* B = -\text{Tr}^g(\nabla B)$, where the trace contracts the first two tensor slots. This formula is proven in [3].

2.7. Second covariant derivative. When the covariant derivative is seen as a mapping $\nabla : \Gamma(T_s^r M) \rightarrow \Gamma(T_{s+1}^r M)$, then the *second covariant derivative* is simply $\nabla \nabla = \nabla^2 : \Gamma(T_s^r M) \rightarrow \Gamma(T_{s+2}^r M)$. For $X, Y \in \mathfrak{X}(M)$, it is given by $\nabla_{X,Y}^2 = \iota_Y \iota_X \nabla^2 = \iota_Y \nabla_X \nabla = \nabla_X \nabla_Y - \nabla_{\nabla_X Y}$. Higher covariant derivatives are defined accordingly.

2.8. Laplacian. The *Bochner-Laplacian* is defined as $\Delta h := \nabla^* \nabla h = -\text{Tr}^g(\nabla^2 h)$. It can act on all tensor fields h , and it respects the degree of the tensor field it is acting on. Using Section 2.5 we get:

Lemma. *The differential of the Laplacian acting on $\binom{r}{s}$ -tensors is given by:*

$$\begin{aligned} D_{(g,m)} \Delta h &= -D_{(g,m)} \text{Tr}^g(\nabla^2 h) \\ &= \text{Tr}(g^{-1} m g^{-1} \nabla^2 h) - \text{Tr}^g(N_{s+1}^r(m) \nabla h) - \text{Tr}^g(\nabla N_s^r(m) h). \end{aligned}$$

Here the trace contracts the first two tensor slots, for example

$$\mathrm{Tr}(g^{-1}mg^{-1}\nabla^2h) = g^{ij}m_{jk}g^{kl}\nabla_l^2h.$$

2.9. Curvature. The Riemann curvature tensor is given by

$$R(X, Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X, Y]}Z.$$

The Ricci tensor field $\mathrm{Ricci}(X, Y)$ is the trace of $Z \mapsto R(Z, X)Y$. The scalar curvature is $\mathrm{Scal} = \mathrm{Tr}^g(\mathrm{Ricci})$.

Lemma. [6, theorem 1.174] *The differential of the scalar curvature*

$$\begin{cases} \Gamma(S_+^2T^*M) & \rightarrow C^\infty(M), \\ g & \mapsto \mathrm{Scal} \end{cases}$$

is given by

$$D_{(g,m)}\mathrm{Scal} = \Delta(\mathrm{Tr}(g^{-1}.m)) + \nabla^*(\nabla^*(m)) - g_2^0(\mathrm{Ricci}, m).$$

3. Riemannian metrics on the manifold of Riemannian metrics

Let $P_g : \Gamma(S^2T^*M) \rightarrow \Gamma(S^2T^*M)$ be a positive, symmetric, bijective pseudo-differential operator of order $2p$ depending smoothly on the metric g . Then the operator P induces a metric on the manifold of Riemannian metrics, namely

$$G_g^P(h, k) = \int_M g_2^0(P_g h, k) \mathrm{vol}(g) = \int_M \mathrm{Tr}(g^{-1}.P_g h.g^{-1}.k) \mathrm{vol}(g).$$

3.1. Geodesic equation. Given $(1, 2)$ -tensors H and K on $\mathcal{M}(M)$ such that

$$D_{(g,m)}G_g^P(h, k) = G_g^P(K_g(h, m), k) = G_g^P(m, H_g(h, k)),$$

the geodesic equation is given by the following variant of the Christoffel symbols:

$$g_{tt} = \frac{1}{2}H_g(g_t, g_t) - K_g(g_t, g_t).$$

See [25, 2, 3].

We will now compute the metric gradients H and K . The calculations at the same time show the existence of the gradients. For this aim, let $m, h, k \in T_g\mathcal{M}$ be constant vector fields on $\mathcal{M}(M)$. Using the formula for the variation of the volume density from Section 2.3, we get

$$\begin{aligned} G_g^P(K_g(h, m), k) &= D_{(g,m)}G_g^P(h, k) \\ &= D_{(g,m)}\int_M \mathrm{Tr}(g^{-1}.Ph.g^{-1}.k) \mathrm{vol}(g) \\ &= \int_M \mathrm{Tr}((D_{(g,m)}g^{-1}).Ph.g^{-1}.k) \mathrm{vol}(g) \\ &\quad + \int_M \mathrm{Tr}(g^{-1}.(D_{(g,m)}P)h.g^{-1}.k) \mathrm{vol}(g) \end{aligned}$$

$$\begin{aligned}
& + \int_M \operatorname{Tr} (g^{-1}.Ph.(D_{(g,m)}g^{-1}).k) \operatorname{vol}(g) \\
& + \int_M \operatorname{Tr} (g^{-1}.Ph.g^{-1}.k) D_{(g,m)} \operatorname{vol}(g) \\
= & \int_M \left[- \operatorname{Tr} (g^{-1}.m.g^{-1}.Ph.g^{-1}.k) \right. \\
& \quad + \operatorname{Tr} (g^{-1}.(D_{(g,m)}P)h.g^{-1}.k) - \operatorname{Tr} (g^{-1}.Ph.g^{-1}.m.g^{-1}.k) \\
& \quad \left. + \operatorname{Tr} (g^{-1}.Ph.g^{-1}.k) \frac{1}{2} \operatorname{Tr}(g^{-1}.m) \right] \operatorname{vol}(g) \\
= & \int_M g_2^0 \left(- m.g^{-1}.Ph + (D_{(g,m)}P)h - Ph.g^{-1}.m \right. \\
& \quad \left. + \frac{1}{2} \operatorname{Tr}(g^{-1}.m).Ph, k \right) \operatorname{vol}(g).
\end{aligned}$$

Therefore the K -gradient is given by

$$\begin{aligned}
K_g(h, m) = P^{-1} \left[- m.g^{-1}.Ph \right. \\
\quad \left. + (D_{(g,m)}P)h - Ph.g^{-1}.m + \frac{1}{2} \operatorname{Tr}(g^{-1}.m).Ph \right].
\end{aligned}$$

To calculate the H -gradient we will assume that there exists an *adjoint* in the following sense:

$$(1) \quad \boxed{\int_M g_2^0((D_{(g,m)}P)h, k) \operatorname{vol}(g) = \int_M g_2^0(m, (D_{(g,\cdot)}Ph)^*(k)) \operatorname{vol}(g)}$$

which is smooth in (g, h, k) and bilinear in (h, k) . The existence of the adjoint, needs to be checked for each specific operator P , usually by partial integration. Using the adjoint we can rewrite the equation above as follows:

$$\begin{aligned}
G_g^P(H_g(h, k), m) & = (D_{(g,m)}G_g^P)(h, k) = D_{(g,m)} \int_M g_2^0(Ph, k) \operatorname{vol}(g) \\
& = \int_M g_2^0 \left(- m.g^{-1}.Ph + (D_{(g,m)}P)h - Ph.g^{-1}.m \right. \\
& \quad \left. + \frac{1}{2} \operatorname{Tr}(g^{-1}.m).Ph, k \right) \operatorname{vol}(g) \\
& = \int_M \left(g_2^0(m, -Ph.g^{-1}.k) + g_2^0(m, (D_{(g,\cdot)}Ph)^*(k)) \right. \\
& \quad \left. + g_2^0(m, -k.g^{-1}.Ph) + \frac{1}{2} g_2^0(m, g, \operatorname{Tr}(g^{-1}.Ph.g^{-1}.k)) \right) \operatorname{vol}(g).
\end{aligned}$$

Here we can easily read off the H -gradient:

$$\begin{aligned}
H_g(h, k) = P^{-1} \left[(D_{(g,\cdot)}Ph)^*(k) \right. \\
\quad \left. - Ph.g^{-1}.k - k.g^{-1}.Ph + \frac{1}{2}.g. \operatorname{Tr}(g^{-1}.Ph.g^{-1}.k) \right].
\end{aligned}$$

Therefore the geodesic equation on the manifold of Riemannian metrics reads as:

$$\begin{aligned}
 g_{tt} &= \frac{1}{2}H_g(g_t, g_t) - K_g(g_t, g_t) \\
 &= P^{-1} \left[\frac{1}{2}(D_{(g,\cdot)}Pg_t)^*(g_t) + \frac{1}{4}g \cdot \text{Tr}(g^{-1} \cdot Pg_t \cdot g^{-1} \cdot g_t) + \frac{1}{2}g_t \cdot g^{-1} \cdot Pg_t \right. \\
 &\quad \left. + \frac{1}{2}Pg_t \cdot g^{-1} \cdot g_t - (D_{(g,g_t)}P)g_t - \frac{1}{2}\text{Tr}(g^{-1} \cdot g_t) \cdot Pg_t \right]
 \end{aligned}$$

We can rewrite this equation to get it in a slightly more compact form:

$$\begin{aligned}
 (Pg_t)_t &= (D_{(g,g_t)}P)g_t + Pg_{tt} \\
 (2) \quad &= \frac{1}{2}(D_{(g,\cdot)}Pg_t)^*(g_t) + \frac{1}{4}g \cdot \text{Tr}(g^{-1} \cdot Pg_t \cdot g^{-1} \cdot g_t) \\
 &\quad + \frac{1}{2}g_t \cdot g^{-1} \cdot Pg_t + \frac{1}{2}Pg_t \cdot g^{-1} \cdot g_t - \frac{1}{2}\text{Tr}(g^{-1} \cdot g_t) \cdot Pg_t
 \end{aligned}$$

3.2. Well-posedness of some geodesic equations. For any fixed background Riemann metric \hat{g} on M and its Levi-Civita covariant derivative $\hat{\nabla}$, the Sobolev space $H^k(S^2T^*M)$ is the Hilbert space completion of the space $\Gamma(S^2T^*M)$ of smooth sections, in the Sobolev norm

$$\|h\|_k^2 = \sum_{j=0}^k \int_M \hat{g}_{2+j}^0((\hat{\nabla})^j h, (\hat{\nabla})^j h) \text{vol}(\hat{g}).$$

The topology of the Sobolev space does not depend on the choice of \hat{g} ; the resulting norms are equivalent. See [26] for more information. The following results hold:

- *Sobolev lemma.* If $k > \frac{\dim(M)}{2}$ then the identity on $\Gamma(S^2T^*M)$ extends to an injective bounded linear mapping $H^{k+p}(S^2T^*M) \rightarrow C^p(S^2T^*M)$ where $C^p(S^2T^*M)$ carries the supremum norm of all derivatives up to order p .
- *Module property of Sobolev spaces.* If $k > \frac{\dim(M)}{2}$ then the evaluation $H^k(L(S^2T^*M, S^2T^*M)) \times H^k(S^2T^*M) \rightarrow H^k(S^2T^*M)$ is bounded and bilinear. Likewise, all other point wise contraction operations are multilinear bounded operations. See [13], or [12, 1.3.12].

The Sobolev lemma allows us to define the Sobolev space $\mathcal{M}^k(M) := H^k(S_+^2T^*M)$ for $k > \frac{\dim(M)}{2}$.

Assumptions. In the following we assume the natural condition that $h \mapsto P_g h$ is an elliptic and self-adjoint pseudo-differential operator of order $2p \geq 0$. Then it is Fredholm and it has vanishing index by [26, theorem 26.2]. Thus it is invertible and $g \mapsto P_g^{-1}$ is a smooth mapping

$$H^k(S_+^2T^*M) \rightarrow L(H^k(S^2T^*M), H^{k+2p}(S^2T^*M))$$

by the implicit function theorem on Banach spaces.

We assume that $(D_{(g,\cdot)}Ph)^*(m)$ exists and is a linear pseudo-differential operator of order $2p$ in m, h .

As (non-linear) mappings in the foot point g , we assume that $P_g h$, $(P_g)^{-1}h$, $(D_{(g,\cdot)}Ph)^*(m)$ are compositions of operators of the following type:

(a) non-linear differential operators of order $l \leq 2p$, i.e.

$$A(g)(x) = A(x, g(x), (\hat{\nabla}g)(x), \dots, (\hat{\nabla}^l g)(x)),$$

(b) linear pseudo-differential operators of order $\leq 2p$,

such that the total (top) order of the composition is $\leq 2p$.

Theorem. *Let the assumptions above hold. Then for $k > \frac{\dim(M)}{2} + 1$, the initial value problem for the geodesic equation (3.1.2) has unique local solutions in the Sobolev manifold $\mathcal{M}^{k+2p}(M)$ of H^{k+2p} -metrics. The solutions depend smoothly on t and on the initial conditions $g(0, \cdot) \in \mathcal{M}^{k+2p}(M)$ and $g_t(0, \cdot) \in H^{k+2p}(S^2T^*M)$. The domain of existence (in t) is uniform in k and thus this also holds in $\mathcal{M}(M)$.*

Moreover, in each Sobolev completion $\mathcal{M}^{k+2p}(M)$, the Riemannian exponential mapping \exp^P exists and is smooth on a neighborhood of the zero section in the tangent bundle, and (π, \exp^P) is a diffeomorphism from a (smaller) neighborhood of the zero section to a neighborhood of the diagonal in $\mathcal{M}^{k+2p}(M) \times \mathcal{M}^{k+2p}(M)$. All these neighborhoods are uniform in $k > \frac{\dim(M)}{2}$ and can be chosen H^{k_0+2p} -open, where $k_0 > \frac{\dim(M)}{2}$. Thus all properties of the exponential mapping continue to hold in $\mathcal{M}(M)$.

This proof is an adaptation of [3, Section 4.2].

Proof. We consider the geodesic equation as the flow equation of a smooth (C^∞) vector field X on the open set

$$\mathcal{M}^{k+2p} \times H^k(S^2T^*M) \subset H^{k+2p}(S^2T^*M) \times H^k(S^2T^*M).$$

We now write the geodesic equation as the flow equation of an autonomous smooth vector field $X = (X_1, X_2)$ on $\mathcal{M}^{k+2p} \times H^k$, as follows (using (3.1.2)):

$$\begin{aligned} g_t &= (P_g)^{-1}h =: X_1(g, h) \\ h_t &= \frac{1}{2}((D_{(g,\cdot)}P_g)(P_g)^{-1}h)^*((P_g)^{-1}h) + \frac{1}{4}.g. \operatorname{Tr}(g^{-1}.h.g^{-1}.(P_g)^{-1}h) \\ (1) \quad &+ \frac{1}{2}(P_g)^{-1}h.g^{-1}.h + \frac{1}{2}h.g^{-1}.(P_g)^{-1}h - \frac{1}{2} \operatorname{Tr}(g^{-1}.(P_g)^{-1}h).h \\ &=: X_2(g, h). \end{aligned}$$

For $(g, h) \in \mathcal{M}^{k+2p} \times H^k$ we have $(P_g)^{-1}h \in H^{k+2p}$. Thus a term-by-term investigation of (1), using the assumptions on the orders, shows

that $X_2(g, h)$ is smooth in $(g, h) \in \mathcal{M}^{k+2p} \times H^k$ with values in H^k . Likewise $X_1(g, h)$ is smooth in $(f, h) \in \mathcal{M}^{k+2p} \times H^k$ with values in H^{k+2p} . Thus by the theory of smooth ODE's on Banach spaces, the flow Fl^k exists on $\mathcal{M}^{k+2p} \times H^k$ and is smooth in t and the initial conditions for fixed $k > \frac{\dim(M)}{2} + 1$.

We choose C^∞ initial conditions $g_0 = g(0, \quad)$ and $h_0 = P_{g_0}g_t(0, \quad) = h(0, \quad)$ for the flow equation (1) in $\mathcal{M}(M) \times \Gamma(S^2T^*M)$. Suppose the trajectory $\text{Fl}_t^k(g_0, h_0)$ of X through these initial conditions in $\mathcal{M}^{k+2p} \times H^k$ maximally exists for $t \in (-a_k, b_k)$, and the trajectory $\text{Fl}_t^{k+1}(g_0, h_0)$ in $\mathcal{M}^{k+1+2p} \times H^{k+1}$ maximally exists for $t \in (-a_{k+1}, b_{k+1})$ with $a_{k+1} < a_k$ and $b_{k+1} < b_k$, say. Since solutions are unique, $\text{Fl}_t^{k+1}(g_0, h_0) = \text{Fl}_t^k(g_0, h_0)$ for $t \in (-a_{k+1}, b_{k+1})$. We now apply the background derivative $\hat{\nabla}$ to both equations (1):

$$\begin{aligned} (\hat{\nabla}g)_t &= \hat{\nabla}g_t = \hat{\nabla}X_1(g, h) \\ (\hat{\nabla}h)_t &= \hat{\nabla}h_t = \hat{\nabla}X_2(g, h). \end{aligned}$$

We claim that for $i = 1, 2$ we have

$$\hat{\nabla}X_i(g, h) = X_{i,1}(g, h)(\hat{\nabla}^{2p+1}g) + X_{i,2}(g, h)(\hat{\nabla}^{2p+1}h) + X_{i,3}(g, h)$$

where all $X_{i,j}(g, h)(l)$ and $X_{i,3}(g, h)$ ($i, j = 1, 2$) are smooth in all variables, of highest order $2p$ in g and h , linear and algebraic (i.e., of order 0) in l . This claim follows from the assumptions: (a) For a local operator we can apply the chain rule: The highest order derivative of g appears only linearly. (b) For a linear pseudo differential operator B of order q the commutator $[\nabla_Y, B]$ is a pseudo differential operator of order q again for any vector field Y .

We write $\hat{\nabla}^{2p+1}g = \hat{\nabla}^{2p}\tilde{g}$ and $\hat{\nabla}^{2p+1}h = \hat{\nabla}^{2p}\tilde{h}$ for the highest derivatives only. Then \tilde{g} and \tilde{h} satisfy

$$\begin{aligned} \tilde{g}_t &= X_{1,1}(g, h)(\hat{\nabla}^{2p}\tilde{g}) + X_{1,2}(g, h)(\hat{\nabla}^{2p}\tilde{h}) + X_{1,3}(g, h) \\ \tilde{h}_t &= X_{2,1}(g, h)(\hat{\nabla}^{2p}\tilde{g}) + X_{2,2}(g, h)(\hat{\nabla}^{2p}\tilde{h}) + X_{2,3}(g, h). \end{aligned}$$

This ODE is inhomogeneous bounded and linear in $(\tilde{g}, \tilde{h}) \in \mathcal{M}^{k+2p} \times H^k$ with coefficients that are bounded linear operators on H^{k+2p} and H^k , respectively. These coefficients are C^∞ functions of $(g, h) \in \mathcal{M}^{k+2p} \times H^k \subset C^1$ which we already know on the interval $(-a_k, b_k)$. This equation therefore has a solution $(\tilde{g}(t, \quad), \tilde{h}(t, \quad)) \in \mathcal{M}^{k+2p} \times H^k$ for all t for which the coefficients exists, and thus for all $t \in (-a_k, b_k)$. Obviously, $(\tilde{g}, \tilde{h}) = (\hat{\nabla}g, \hat{\nabla}h)$ for $t \in (-a_{k+1}, b_{k+1})$. By continuity, this holds also for $t \in [-a_{k+1}, b_{k+1}]$ which contradicts that the interval $(-a_{k+1}, b_{k+1})$ is maximal. We can iterate this and conclude that the flow of X exists in $\bigcap_{m \geq k} \mathcal{M}^{m+2p} \times H^m = \mathcal{M} \times \Gamma$.

It remains to check the properties of the Riemannian exponential mapping \exp^P . It is given by $\exp_g^P(h) = c(1)$ where $c(t)$ is the geodesic

emanating from value g with initial velocity h . From the form $g_{tt} = \frac{1}{2}H_g(g_t, g_t) - K_g(g_t, g_t) =: \Gamma_g(g_t, g_t)$ (see Section 3.1), namely linearity in g_{tt} and bilinearity in g_t , and from local existence and uniqueness on each space $\mathcal{M}^{k+2p}(M)$ the properties claimed follow: see for example [22, 22.6 and 22.7] for a detailed proof in terms of the spray vector field $S(g, h) = (g, h; h, \Gamma_g(h, h))$ on a finite dimensional manifold. This proof carries over to infinite dimensional convenient manifolds without any change in notation. So we check this on the largest of these spaces $\mathcal{M}^{k_0}(M)$ (with the smallest k). Since the spray on $\mathcal{M}^{k_0}(M)$ restricts to the spray on each $\mathcal{M}^{k+2p}(M)$, the exponential mapping \exp^P and the inverse $(\pi, \exp^P)^{-1}$ on $\mathcal{M}^{k_0}(M)$ restrict to the corresponding mappings on each $\mathcal{M}^{k+2p}(M)$. Thus the neighborhoods of existence are uniform in k . q.e.d.

3.3. Conserved Quantities. Consider the right action of the diffeomorphism group $\text{Diff}(M)$ on $\mathcal{M}(M)$ given by $(g, \phi) \mapsto \phi^*g$ with fundamental vector field

$$\zeta_X(g) = \mathcal{L}_X g = 2 \text{Sym} \nabla(g(X)).$$

For a proof of the last equality, see [6, Section 1]. If the metric G^P is invariant under this action, we have the following conserved quantities (see for example [2]):

$$\begin{aligned} \text{const} &= G^P(g_t, \zeta_X(g)) = \int_M g_2^0(Pg_t, 2 \text{Sym} \nabla(g(X))) \text{vol}(g) \\ &= 2 \int_M g_1^0(\nabla^* \text{Sym} Pg_t, g(X)) \text{vol}(g) = 2 \int_M (\nabla^* Pg_t)(X) \text{vol}(g) \\ &= 2 \int_M g(g^{-1} \nabla^* Pg_t, X) \text{vol}(g). \end{aligned}$$

Since this equation holds for all vector fields X , this yields

$$\boxed{(\nabla^* Pg_t) \text{vol}(g) \in \Gamma(T^*M \otimes_M \text{vol}(M)) \text{ is const. in time.}}$$

The geometric interpretation of this conserved quantity is carried by the expression $G^P(g_t, \zeta_X)$. After normalization, this gives a formula for the cosine of the angle between the geodesic and any vector field ζ_X . If the constant vanishes, then this geodesic is G^P -perpendicular to each $\text{Diff}(M)$ -orbit it meets.

3.4. Geodesics of pure scalings. In this section we want to investigate when $r(t)g_0$ is a geodesic for some real function r and some fixed metric g_0 . This will help us to determine the geodesic completeness of the space $\mathcal{M}(M)$ under various metrics.

Lemma. *Let $g_0 \in \mathcal{M}(M)$ and $\mathcal{N} = \mathbb{R}_{>0}g_0 = \{rg_0 : r > 0\} \subset \mathcal{M}(M)$. If P viewed as a $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ -tensor field on $\mathcal{M}(M)$ ‘restricts’ to the submanifold \mathcal{N} in the sense that $P_g h$ is tangential to \mathcal{N} for all $g \in \mathcal{N}$ and $h \in T_g \mathcal{N}$, then the following statements are equivalent:*

- (a) \mathcal{N} is totally geodesic.
 (b) $(D_{(g,\cdot)}Ph)^*(k)$ is tangential to \mathcal{N} for all $g \in \mathcal{N}$ and $h, k \in T_g\mathcal{N}$.
 (c) $(D_{(g,m)}P)(h)$ is \tilde{g}_2^0 -normal to \mathcal{N} for all $g \in \mathcal{N}$, $h \in T_g\mathcal{N}$, $m \in T_g\mathcal{M}(M)$ such that m is \tilde{g}_2^0 -normal to \mathcal{N} .

If P restricts to \mathcal{N} and (a)–(c) hold, then there are $\Psi, f : \mathbb{R}_{>0} \rightarrow \mathbb{R}$ such that

$$P_{rg_0}(g_0) = \Psi(r)g_0, \quad ((D_{(rg_0,\cdot)}P)g_0)^*(g_0) = f(r)g_0$$

holds for all $r > 0$. Then the path $g(t, \cdot) = r(t) \cdot g_0$ is a geodesic in $\mathcal{M}(M)$ if and only if the function r satisfies

$$r''\Psi(r) = r'^2\left(\frac{1}{2}f(r) - \Psi'(r) + (1 - \dim(M)/4)\Psi(r)r^{-1}\right).$$

Along these geodesics, the conserved quantity vanishes, i.e.,

$$(\nabla^*Pg_t)\text{vol}(g) = 0.$$

Remark. Note that $(D_{(g,m)}P)(h)$ and $(D_{(g,\cdot)}Ph)^*(k)$ are tensorial in h, k and that for $g \in \mathcal{N}$, all tangent vectors in $T_g\mathcal{N}$ can be written as real multiples of g . Therefore conditions (b) and (c) are equivalent to

- (b') $(D_{(g,\cdot)}Pg)^*(g)$ is tangential to \mathcal{N} for all $g \in \mathcal{N}$.
 (c') $(D_{(g,m)}P)(g)$ is \tilde{g}_2^0 -normal to \mathcal{N} for all $g \in \mathcal{N}$ and $m \in T_g\mathcal{M}(M)$ such that m is \tilde{g}_2^0 -normal to \mathcal{N} .

Proof. The submanifold \mathcal{N} is totally geodesic iff $\frac{1}{2}H_g(h, k) - K_g(h, k)$ is tangential to \mathcal{N} for all $g \in \mathcal{N}$ and $h, k \in T_g\mathcal{N}$. We now look at the formulas for H and K from Section 3.1. Since P_g is bijective and preserves $T_g\mathcal{N}$, the above condition is equivalent to $P_g(\frac{1}{2}H_g(h, k) - K_g(h, k))$ being tangential. A term-by-term investigation shows that this is the case if and only if $((D_{(g,\cdot)}P)h)^*(k)$ is tangential, in which case it can be expressed using a function f . A test for the latter condition is

$$\tilde{g}_2^0(((D_{(g,\cdot)}P)h)^*(k), m) = \tilde{g}_2^0((D_{(g,m)}P)h, k) = 0$$

for all $m \in T_g\mathcal{M}(M)$ that are \tilde{g}_2^0 -normal to \mathcal{N} . Equivalently, $(D_{(g,m)}P)h$ has to be \tilde{g}_2^0 -normal to \mathcal{N} whenever m is \tilde{g}_2^0 -normal to \mathcal{N} and h is tangential to \mathcal{N} .

It remains to check the form of the geodesic equation. We use the geodesic equation (2) from Section 3.1 and substitute

$$g = r(t)g_0, \quad g_t = r'(t)g_0, \quad P_g(g_t) = \Psi(r(t))r'(t)g_0.$$

Dropping the dependence on t in our notation, we get for the left-hand side of the geodesic equation:

$$\partial_t(P_gg_t) = r''\Psi(r)g_0 + \Psi'(r)r'^2g_0$$

The previous substitutions and

$$((D_{(g,\cdot)}P)g_t)^*(g_t) = f(r(t))r'(t)^2g_0.$$

yield the right-hand side of the geodesic equation:

$$\begin{aligned}
& \frac{1}{2}(D_{(g,\cdot)}Pg_t)^*(g_t) + \frac{1}{4}g \cdot \text{Tr}(g^{-1} \cdot Pg_t \cdot g^{-1} \cdot g_t) + \frac{1}{2}g_t \cdot g^{-1} \cdot Pg_t \\
& \quad + \frac{1}{2}Pg_t \cdot g^{-1} \cdot g_t - \frac{1}{2}\text{Tr}(g^{-1} \cdot g_t) \cdot Pg_t \\
& = \frac{1}{2}f(r)r'^2g_0 + \frac{1}{4}\Psi(r)\dim(M)r^{-1}r'^2g_0 + \frac{1}{2}\Psi(r)r^{-1}r'^2g_0 \\
& \quad + \frac{1}{2}\Psi(r)r^{-1}r'^2g_0 - \frac{1}{2}\Psi(r)\dim(M)r^{-1}r'^2g_0 \\
& = \frac{1}{2}f(r)r'^2g_0 + (1 - \dim(M)/4)\Psi(r)r^{-1}r'^2g_0.
\end{aligned}$$

For the conserved quantity, we calculate:

$$\begin{aligned}
(\nabla^*Pg_t)\text{vol}(g) &= \text{Tr}(g^{-1}\nabla^g(r'(t)g_0))\text{vol}(g) \\
&= \frac{r'(t)}{r(t)}\text{Tr}(g^{-1}\nabla^g(g))\text{vol}(g) = 0.
\end{aligned}$$

q.e.d.

3.5. Length of pure scalings.

Lemma. *Given g_0 such that $P_{rg_0}(g_0) = \Psi(r) \cdot g_0$, the length of the curve $g(r) = rg_0$ for $r \in [0, 1]$ is given by*

$$\text{Len}_0^1(g) = \sqrt{\dim(M)\text{Vol}(g_0)} \int_0^1 \sqrt{\Psi(r)r^{\dim(M)/2-2}} dr.$$

If $\Psi(r) = O(r^\alpha)$ for some $\alpha > -\dim(M)/2$, then $\mathbb{R}_{>0}g_0 \subset \mathcal{M}(M)$ is an incomplete metric space under G^P . If in addition P and g_0 satisfy the conditions of lemma 3.4, then $(\mathcal{M}(M), G^P)$ is geodesically incomplete.

Note that $(\mathcal{M}(M), G^P)$ is always an incomplete metric space since it does not contain Sobolev class H^p metrics.

Proof. For the length of the curve we calculate:

$$\begin{aligned}
\text{Len}_0^1(g) &= \int_0^1 G_{r \cdot g_0}^P(g_0, g_0)^{1/2} dr \\
&= \int_0^1 \left(\int_M \text{Tr}((rg_0)^{-1} \cdot P_{rg_0}(g_0) \cdot (rg_0)^{-1} \cdot g_0) \text{vol}(rg_0) \right)^{1/2} dr \\
&= \int_0^1 r^{\dim(M)/4-1} \left(\int_M \text{Tr}((g_0)^{-1} \cdot P_{rg_0}(g_0)) \text{vol}(g_0) \right)^{1/2} dr.
\end{aligned}$$

Using the assumption $P_{rg_0}(g_0) = \Psi(r) \cdot g_0$, we can compute this as

$$\text{Len}_0^1(g) = \int_0^1 r^{\dim(M)/4-1} \sqrt{\dim(M)} \left(\int_M \Psi(r) \text{vol}(g_0) \right)^{1/2} dr.$$

Note that the metric space $(\mathcal{M}(M), G^P)$ is geodesically incomplete if $\mathbb{R}_{>0}g_0$ contains a geodesic in $\mathcal{M}(M)$ which connects g_0 to 0 in finite time. q.e.d.

4. Special cases of P

In this section we present various interesting examples of metrics. These special choices are motivated by related metrics on spaces of immersions and shape spaces, see [3, 2, 1]. We will use the notation $n = \dim(M)$ for all of this section.

4.1. The H^0 -metric. The simplest and most natural example is the operator P of order zero given by $P_g(h) = h$ for $g \in \mathcal{M}(M)$ and $h \in T_g\mathcal{M}(M)$. With this choice of P , the metric G^P equals \tilde{g}_2^0 . It is the so called L^2 -metric or H^0 -metric, which is well studied as mentioned in the introduction. We can easily read off the geodesic equation from the previous section:

$$g_{tt} = \frac{1}{4} \cdot g \cdot g_2^0(g_t, g_t) + g_t \cdot g^{-1} \cdot g_t - \frac{1}{2} \text{Tr}(g^{-1} \cdot g_t) \cdot g_t.$$

This coincides with the equation derived in [14] and [15]. All conditions from Section 3.2 are obviously satisfied. Thus the geodesic equation is well-posed. Here the geodesic equation evolves in each set $S_+^2 T_x^* M$ separately. The conserved quantities have the form

$$(\nabla^* g_t) \text{vol}(g) \in \Gamma(T^* M \otimes_M \text{vol}(M)) \text{ is const. in time.}$$

The conditions of lemma 3.4 are obviously satisfied for all $g_0 \in \mathcal{M}(M)$ and we get again the result from [15] that $\mathbb{R}_{>0} g_0$ is the image of a geodesic. The geodesic is $r(t) \cdot g_0$ where $r(t)$ satisfies

$$r''(t) = \frac{r'(t)^2}{r(t)} \left(1 - \frac{n}{4}\right), \text{ i.e., } r(t) = \left(t(r(1)^{n/4} - r(0)^{n/4}) + r(0)^{n/4}\right)^{4/n}.$$

This geodesic connects g_0 with 0 in finite time. Thus it follows that the space $(\mathcal{M}(M), \tilde{g}_2^0)$ is geodesically incomplete.

4.2. Conformal metrics. Here we consider metrics of the form

$$G_g^P(h, k) = \Phi(\text{Vol}(g)) \int_M g_2^0(h, k) \text{vol}(g),$$

where $\Phi \in C^\infty(\mathbb{R}_{>0}, \mathbb{R}_{>0})$ and $\text{Vol}(g) = \int_M \text{vol}(g)$. To calculate the adjoint, we will use the variational formula for the volume form from Section 2.3:

$$\begin{aligned} \int_M g_2^0(m, (D_{(g,\cdot)} P h)^*(k)) \text{vol}(g) &= \int_M g_2^0((D_{(g,m)} P)h, k) \text{vol}(g) \\ &= \Phi' \cdot (D_{(g,m)} \text{Vol}) \cdot \int_M g_2^0(h, k) \text{vol}(g) \\ &= \frac{1}{2} \Phi' \cdot \int_M \text{Tr}(g^{-1} \cdot m) \text{vol}(g) \cdot \int_M g_2^0(h, k) \text{vol}(g) \\ &= \frac{1}{2} \int_M \text{Tr} \left(g^{-1} \cdot m \cdot \Phi' \cdot \int_M g_2^0(h, k) \text{vol}(g) \right) \text{vol}(g) \end{aligned}$$

$$= \frac{1}{2} \int_M g_2^0 \left(m, \Phi' \cdot g \cdot \int_M g_2^0(h, k) \text{vol}(g) \right) \text{vol}(g).$$

Using this formula for the adjoint, the geodesic equation reads as:

$$\begin{aligned} g_{tt} = & \frac{\Phi'}{4\Phi} \cdot g \cdot \int_M g_2^0(g_t, g_t) \text{vol}(g) + \frac{1}{4} \cdot g \cdot g_2^0(g_t, g_t) + g_t \cdot g^{-1} \cdot g_t \\ & - \frac{\Phi'}{2\Phi} \cdot g_t \cdot \int_M g_2^0(g_t, g) \text{vol}(g) - \frac{1}{2} g_2^0(g_t, g) \cdot g_t \end{aligned}$$

or

$$\begin{aligned} (\Phi \cdot g_t)_t = & \frac{\Phi'}{4} \cdot g \cdot \int_M g_2^0(g_t, g_t) \text{vol}(g) + \frac{\Phi}{4} \cdot g \cdot g_2^0(g_t, g_t) \\ & + \Phi \cdot g_t \cdot g^{-1} \cdot g_t - \frac{\Phi}{2} g_2^0(g_t, g) \cdot g_t \end{aligned}$$

All conditions of theorem 3.2 are satisfied. Thus the geodesic equation is well-posed and the geodesic exponential mapping exists and is a local diffeomorphism. Since the total volume $\text{Vol}(M)$ does not depend on the point $x \in M$, the conserved quantities are:

$$\boxed{\Phi(\text{Vol}) \text{Tr}(g^{-1} \nabla g_t) \text{vol}(g) \in \Gamma(T^*M \otimes_M \text{vol}(M)) \text{ is const. in time.}}$$

Now we want to study again whether there exist metrics g_0 and positive real function r such that $r(t)g_0$ is a geodesic. Therefore we check whether the conditions of lemma 3.4 are satisfied. P obviously restricts to the submanifold $\mathbb{R}_{>0} g_0$ for every $g_0 \in \mathcal{M}(M)$. Using again the variational formula for Vol , we get

$$\begin{aligned} \tilde{g}_2^0((D_{(g,m)}P)g, g) &= \frac{1}{2} \Phi' \cdot \int_M \text{Tr}(g^{-1} \cdot m) \text{vol}(g) \int_M g_2^0(g, g) \text{vol}(g) \\ &= \frac{1}{2} \Phi' \int_M \text{Tr}(g^{-1} \cdot m \cdot g^{-1} \cdot g) \text{vol}(g) \int_M n \text{vol}(g) \\ &= \frac{n}{2} \Phi' \text{Vol} \tilde{g}_2^0(m, g) = 0, \end{aligned}$$

if m is \tilde{g}_2^0 -normal to $\mathbb{R}_{>0} g_0$. Thus $\mathbb{R}_{>0} g_0$ is a totally geodesic submanifold for any $g_0 \in \mathcal{M}(M)$. For the corresponding functions Ψ and f we obtain:

$$\begin{aligned} P_{r g_0} g_0 &= \Psi(r) g_0 \text{ with } \Psi(r) := \Phi(r^{\frac{n}{2}} \text{Vol}(g_0)) \\ ((D_{(r g_0, \cdot)}P)g_0)^*(g_0) &= f(r) g_0 \text{ with } f(r) := \frac{n}{2} \Phi'(r^{\frac{n}{2}} \text{Vol}(g_0)) r^{\frac{n}{2}-1} \text{Vol}(g_0). \end{aligned}$$

The geodesic equation on $\mathbb{R}_{>0} g_0$ is then given by

$$r'' \Phi(\text{Vol}(r g_0)) = \frac{r'^2}{r} \left(-\frac{n}{4} \Phi'(\text{Vol}(r g_0)) \text{Vol}(r g_0) + \left(1 - \frac{n}{4}\right) \Phi(\text{Vol}(r g_0)) \right).$$

Let us now consider the special case $\Phi(\text{Vol}) = \text{Vol}^k$ for real k . Then the ODE for $r(t)$ simplifies to

$$r'' = \frac{r'^2}{r} \left(1 - \frac{n}{4}(k+1)\right)$$

with solution

$$r(t) = \left(t(r(1)^a - r(0)^a) + r(0)^a\right)^{\frac{1}{a}} \quad \text{where} \quad a = \frac{n}{4}(1+k).$$

This geodesic connects g_0 with 0 in finite time if and only if $k > -1$. Thus $(\mathcal{M}(M), G^{\Phi(\text{Vol})})$ is geodesically incomplete if $\Phi(r) = O(r^k)$ for $r \searrow 0$, for some $k > -1$. Note that this would also follow from lemma 3.5, since $\Psi(r) = \Phi(r^{\frac{n}{2}} \text{Vol}(g_0))$.

4.3. Curvature weighted metrics. We consider metrics weighted by scalar curvature,

$$G_g^P(h, k) = \int_M \Phi(\text{Scal}(g)) \cdot g_2^0(h, k) \text{vol}(g),$$

where $\Phi \in C^\infty(\mathbb{R}, \mathbb{R}_{>0})$. Using the variational formula from Section 2.9, we can calculate the adjoint as follows:

$$\begin{aligned} \int_M g_2^0(m, (D_{(g,\cdot)}Ph)^*(k)) \text{vol}(g) &= \int_M g_2^0((D_{(g,m)}P)h, k) \text{vol}(g) \\ &= \int_M \Phi' \cdot (D_{(g,m)} \text{Scal}) g_2^0(h, k) \text{vol}(g) \\ &= \int_M \Phi' \cdot \left(\Delta(\text{Tr}(g^{-1} \cdot m)) + \nabla^*(\nabla^*(m)) - g_2^0(\text{Ricci}, m) \right) g_2^0(h, k) \text{vol}(g) \\ &= \int_M \Phi' \cdot \left[g_1^0 \left(\nabla \text{Tr}(g^{-1} \cdot m), \nabla g_2^0(h, k) \right) + g_1^0 \left(\nabla^*(m), \nabla g_2^0(h, k) \right) \right. \\ &\quad \left. - g_2^0 \left(g_2^0(h, k) \text{Ricci}, m \right) \right] \text{vol}(g) \\ &= \int_M \Phi' \cdot \left[\text{Tr}(g^{-1} \cdot m) \cdot \nabla^* \nabla g_2^0(h, k) + g_2^0 \left(m, \nabla^2 g_2^0(h, k) \right) \right. \\ &\quad \left. - g_2^0 \left(g_2^0(h, k) \text{Ricci}, m \right) \right] \text{vol}(g) \\ &= \int_M \Phi' \cdot g_2^0 \left(m, g \cdot \Delta g_2^0(h, k) + \nabla^2 g_2^0(h, k) - g_2^0(h, k) \text{Ricci} \right) \text{vol}(g). \end{aligned}$$

Using the formula for the geodesic equation from Section 3.1 yields

$$\boxed{\begin{aligned} (\Phi \cdot g_t)_t &= \frac{\Phi'}{2} \left(g \cdot \Delta^g g_2^0(g_t, g_t) + \nabla^2 g_2^0(g_t, g_t) - g_2^0(g_t, g_t) \text{Ricci} \right) \\ &\quad + \frac{\Phi}{4} \cdot g \cdot g_2^0(g_t, g_t) + \Phi \cdot g_t \cdot g^{-1} \cdot g_t - \frac{\Phi}{2} g_2^0(g_t, g) \cdot g_t. \end{aligned}}$$

The conditions of theorem 3.2 are violated and therefore it is not applicable. We do not know whether the geodesic equation is well-posed. The conserved quantities are given by

$$\begin{aligned} & \nabla^*(\Phi(\text{Scal})g_t) \text{vol}(g) \\ &= \left(\Phi'(\text{Scal}) \text{Tr}(g^{-1} d\text{Scal} \otimes g_t) + \Phi(\text{Scal}) \text{Tr}(g^{-1} \nabla g_t) \right) \text{vol}(g). \end{aligned}$$

The conditions of lemma 3.4 are violated for general g_0 . However, we consider the special case that M admits a metric g_0 such that the Einstein equation $\text{Ricci}(g_0) = Cg_0$ is satisfied. Let $g = r\bar{g}_0 \in \mathbb{R}_{>0}g_0$; then $\text{Scal}(g) = \frac{Cn}{r}$. For $h \in T_g(\mathbb{R}_{>0}g_0)$, we calculate

$$P_g h = \Phi(\text{Scal}(g))h = \Phi\left(\frac{Cn}{r}\right)h \in T_g(\mathbb{R}_{>0}g_0).$$

It remains to show that $(D_{(g,m)}P)(g)$ is \tilde{g}_2^0 -normal to $\mathbb{R}_{>0}g_0$ for all $m \in T_g\mathcal{M}(M)$ such that m is \tilde{g}_2^0 -normal to $\mathbb{R}_{>0}g_0$. This follows from

$$\begin{aligned} & \tilde{g}_2^0((D_{(g,m)}P)g, g) = \\ &= \tilde{g}_2^0\left(\Phi'(\text{Scal}(g))(\Delta(\text{Tr}(g^{-1}.m)) + \nabla^*(\nabla^*(m)) - g_2^0(\text{Ricci}, m))g, g\right) \\ &= \Phi'\left(\frac{Cn}{r}\right) \int_M (\Delta(\text{Tr}(g^{-1}.m)) + \nabla^*(\nabla^*(m)) \\ &\quad - g_2^0(\text{Ricci}, m))g_2^0(g, g) \text{vol}(g) \\ &= \Phi'\left(\frac{Cn}{r}\right)n \int_M (\Delta(\text{Tr}(g^{-1}.m)) + \nabla^*(\nabla^*(m)) - g_2^0(Cg, m)) \text{vol}(g) = 0. \end{aligned}$$

The first two terms vanish because they are divergences, and the last term vanishes by assumption on m . Thus $\mathbb{R}_{>0}g_0$ is a totally geodesic submanifold if g_0 satisfies the Einstein equation. For the corresponding functions Ψ and f we obtain:

$$\Psi(r) := \Phi\left(\frac{1}{r} \text{Scal}(g_0)\right) = \Phi\left(\frac{Cn}{r}\right), \quad f(r) = -\Phi'\left(\frac{Cn}{r}\right)\frac{Cn}{r^2}.$$

Thus $g(t) = r(t)g_0$ is a geodesic iff g_0 is a solution to the Einstein equation and r satisfies

$$r''\Phi\left(\frac{Cn}{r}\right) = \frac{r'^2}{r} \left(\frac{1}{2}\Phi'\left(\frac{Cn}{r}\right)\frac{Cn}{r} + \left(1 - \frac{n}{4}\right)\Phi\left(\frac{Cn}{r}\right) \right).$$

In the case that M does not admit a metric solving the Einstein equation, we cannot use lemma 3.5 to check for geodesic incompleteness, but we can still compute the length of shrinking a metric to zero. Let $g(r) = rg_0$, with $\text{Scal}(g_0)$ not necessarily constant:

$$\text{Len}_0^1(g) = \int_0^1 r^{\frac{n}{4}-1} \sqrt{n} \left(\int_M \Phi\left(\frac{\text{Scal}(g_0)}{r}\right) \text{vol}(g_0) \right)^{1/2} dr.$$

Now let us assume that $\Phi(u) \leq C(1 + |u|^{2k})$ for constants C and k :

$$\begin{aligned} \text{Len}_0^1(g) &\leq \int_0^1 r^{\frac{n}{4}-1} \sqrt{n} \left(C \int_M \left(1 + \frac{|\text{Scal}(g_0)|^{2k}}{r^{2k}} \right) \text{vol}(g_0) \right)^{1/2} dr \\ &= \int_0^1 r^{\frac{n}{4}-1} \sqrt{nC} \left(\text{Vol}(g_0) + \frac{1}{r^{2k}} \int_M |\text{Scal}(g_0)|^{2k} \text{vol}(g_0) \right)^{1/2} dr \\ &= \int_0^1 r^{\frac{n}{4}-1} \sqrt{nC \text{Vol}(g_0)} \left(1 + \frac{1}{2r^{2k} \text{Vol}(g_0)} \int_M |\text{Scal}(g_0)|^{2k} \text{vol}(g_0) \right) dr. \end{aligned}$$

This is finite if and only if $\frac{n}{4} - 1 - 2k > -1$, i.e., $n > 8k$. Thus $(\mathcal{M}(M), G^{\Phi(\text{Scal})})$ is geodesically incomplete if M admits a metric solving the Einstein equation and $\Phi(u) \leq C(1 + |u|^{2k})$ for $k < \dim(M)/8$.

4.4. Sobolev metrics using the Laplacian. We first consider the Sobolev metric of the form

$$G_g^P(h, k) = \int_M g_2^0((1 + \Delta)^P h, k) \text{vol}(g)$$

where Δ^g is the geometric Bochner-Laplacian described in Section 2.8. The adjoint of the derivative of P satisfies

$$\begin{aligned} \int_M g_2^0(m, (D_{(g,\cdot)} P h)^*(k)) \text{vol}(g) &= \int_M g_2^0((D_{(g,m)} P) h, k) \text{vol}(g) \\ &= \sum_{i=1}^p \int_M g_2^0((1 + \Delta)^{i-1} (D_{(g,m)} \Delta) (1 + \Delta)^{p-i} h, k) \text{vol}(g) \\ &= \sum_{i=1}^p \int_M g_2^0((D_{(g,m)} \Delta) (1 + \Delta)^{p-i} h, (1 + \Delta)^{i-1} k) \text{vol}(g) \\ &= \sum_{i=1}^p \int_M g_2^0\left(m, ((D_{(g,\cdot)} \Delta) (1 + \Delta)^{p-i} h)^* (1 + \Delta)^{i-1} k\right) \text{vol}(g) . \end{aligned}$$

Thus it remains to calculate the adjoint of the derivative of Δ .

Lemma. *The differential of the Laplacian acting on $\binom{0}{2}$ -tensors admits an adjoint with respect to the metric \tilde{g}_2^0 , which is given by:*

$$\begin{aligned} \tilde{g}_2^0(D_{(g,m)} \Delta h, k) &=: \tilde{g}_2^0(m, (D_{(g,\cdot)} \Delta h)^*(k)) \\ &= \tilde{g}_2^0\left(m, g^{i_1 j_1} g^{i_2 j_2} (\nabla^2 h)_{..i_1 i_2} k_{j_1 j_2} - (N_3^0(\cdot) \nabla h)^*(g \otimes k) + (N_2^0(\cdot) h)^*(\nabla k)\right). \end{aligned}$$

Here $(N_q^0(\cdot) h)^*$ denotes the adjoint of the differential of the covariant derivative:

$$\tilde{g}_{q+1}^0(N_q^0(m) h, k) =: \tilde{g}_2^0(m, (N_q^0(\cdot) h)^*(k)) = \tilde{g}_2^0(m, \nabla^*(\sigma(N_q^0(\cdot) h)^* k)),$$

where $h \in \Gamma(T_q^0 M)$, $k \in \Gamma(T_{q+1}^0 M)$ and where $\sigma(N_q^0)$ denotes the total symbol of N_q^0 . It is tensorial and of the form

$$\sigma(N_q^0)(\tilde{m})(h)(X_0, \dots, X_q)$$

$$= \frac{-1}{2} \sum_{j=1}^q h \left(X_1, \dots, X_{j-1}, \sum_{i=0}^2 (-1)^i (\tau^i(\tilde{m})(X_0, X_j, \cdot))^{\sharp}, X_{j+1}, \dots, X_q \right),$$

where $\tilde{m} \in \Gamma(T^*M \otimes S^2T^*M)$, $h \in \Gamma(T_q^0M)$, $X_0, \dots, X_q \in \mathfrak{X}(M)$, and where τ^i is the i -th power of the cyclic permutation $\tau(\alpha \otimes \beta \otimes \gamma) = \gamma \otimes \alpha \otimes \beta$.

Proof. To prove the formula for $(N_q^0(\cdot)h)^*$, it suffices to show that

$$N_q^p(m)(h) = \sigma(N_q^p)(\nabla m)(h).$$

This follows from (2) and (3) in Section 2.5. The formula for $D_{(g,\cdot)}\Delta$ follows from Section 2.8. q.e.d.

The above discussion and the formula for the geodesic equation from Section 3.1 yield the geodesic equation for Sobolev-type metrics:

$$\begin{aligned} ((1 + \Delta)^p g_t)_t &= \frac{1}{2} g^{i_1 j_1} g^{i_2 j_2} (\nabla^2 (1 + \Delta)^{p-i} g_t) \dots_{i_1 i_2} (1 + \Delta)^{i-1} (g_t)_{j_1 j_2} \\ &\quad - \frac{1}{2} (N_3^0(\cdot) \nabla (1 + \Delta)^{p-i} g_t)^* (g \otimes (1 + \Delta)^{i-1} g_t) \\ &\quad + \frac{1}{2} (N_2^0(\cdot) (1 + \Delta)^{p-i} g_t)^* (\nabla (1 + \Delta)^{i-1} g_t) \\ &\quad + \frac{1}{4} \cdot g \cdot \text{Tr}(g^{-1} \cdot (1 + \Delta)^p g_t \cdot g^{-1} \cdot g_t) + \frac{1}{2} g_t \cdot g^{-1} \cdot (1 + \Delta)^p g_t \\ &\quad + \frac{1}{2} (1 + \Delta)^p g_t \cdot g^{-1} \cdot g_t - \frac{1}{2} \text{Tr}(g^{-1} \cdot g_t) \cdot (1 + \Delta)^p g_t. \end{aligned}$$

The conditions of theorem 3.2 are valid, so the geodesic equation is well-posed. The conserved quantity is

$$\boxed{\nabla^* ((1 + \Delta)^p g_t) \text{ vol}(g)}.$$

Finally we want to study again the geodesics of pure scaling using lemma 3.4. Let $g_0 \in \mathcal{M}(M)$ and $g = r g_0 \in \mathbb{R}_{>0} g_0$. Since $\nabla g = 0$ and consequently $\Delta g = 0$, one has $P_g g = (1 + \Delta)^p g = g$. It remains to show that $(D_{(g,m)} P)(g)$ is \tilde{g}_2^0 -normal to $\mathbb{R}_{>0} g_0$ for all $m \in T_g \mathcal{M}(M)$ such that m is \tilde{g}_2^0 -normal to $\mathbb{R}_{>0} g_0$. This follows from

$$\begin{aligned} \tilde{g}_2^0 \left((D_{(g,m)} P)g, g \right) &= \sum_{i=1}^p \tilde{g}_2^0 \left((1 + \Delta)^{i-1} (D_{(g,m)} \Delta) (1 + \Delta)^{p-i} g, g \right) \\ &= \sum_{i=1}^p \tilde{g}_2^0 \left((D_{(g,m)} \Delta) (1 + \Delta)^{p-i} g, (1 + \Delta)^{i-1} g \right) = p \tilde{g}_2^0 \left((D_{(g,m)} \Delta) g, g \right) \\ &= p \tilde{g}_2^0 \left(\text{Tr}(g^{-1} m g^{-1} \nabla^2 g) - \text{Tr}^g (N_3^0(m) \nabla g) - \text{Tr}^g (\nabla N_2^0(m) g), g \right) \\ &= 0 - 0 + p \tilde{g}_2^0 \left(\nabla^* (N_2^0(m) g), g \right) = p \tilde{g}_3^0 \left((N_2^0(m) g), \nabla g \right) = 0. \end{aligned}$$

Thus the conditions of lemma 3.4 are satisfied and $\mathbb{R}_{>0}g_0$ is a totally geodesic submanifold for every $g_0 \in \mathcal{M}(M)$. Furthermore, since $(D_{(rg_0, \cdot)}Pg_0)^*(g_0) = g_0$, the equation for geodesics of the form $r(t)g_0$ with respect to Sobolev metrics is equal to that with respect to the L^2 metric, c.f. Section 4.1. In particular, this proves that $\mathcal{M}(M)$ is geodesically incomplete for each Sobolev metric.

4.5. General remarks. The L^2 -metric is the only one of the above discussed examples that is relatively well understood. An explicit analytic formula for geodesics has been derived, e.g. in [15], and as a direct consequence it has been shown that the space of Riemannian metrics is not complete with respect to this metric. Furthermore, the completion of this space has been described and analyzed in [8, 10].

For the other metrics described in this section, the situation is more complicated, since there is no hope to find general analytic solutions to the corresponding geodesic equations. But the equations as presented above are ready for numerical implementation. This has been successfully done for the related spaces of immersions and shapes; see [3, 2, 1]. Another issue is that we still do not know whether there exists a metric such that the space of all Riemannian metrics is geodesically complete.

5. The Ricci vector field

The space of metrics $\mathcal{M}(M)$ is a convex open subset in the Fréchet space $\Gamma(S^2T^*M)$. So it is contractible. A necessary and sufficient condition for Ricci curvature to be a gradient vector field with respect to the G^P -metric is that the following exterior derivative vanishes:

$$\begin{aligned} (dG^P(\text{Ricci}, \cdot))(h, k) &= \\ &= hG^P(\text{Ricci}, k) - kG^P(\text{Ricci}, h) - G^P(\text{Ricci}, [h, k]) = 0. \end{aligned}$$

It suffices to look at constant vector fields h, k , in which case $[h, k] = 0$. We have

$$\begin{aligned} &hG^P(\text{Ricci}, k) - kG^P(\text{Ricci}, h) \\ &= \int \left(-\text{Tr}(g^{-1}hg^{-1}(P \text{Ricci})g^{-1}k) + \text{Tr}(g^{-1}kg^{-1}(P \text{Ricci})g^{-1}h) \right. \\ &\quad + \text{Tr}(g^{-1}D_{g,h}(P \text{Ricci})g^{-1}k) - \text{Tr}(g^{-1}D_{g,k}(P \text{Ricci})g^{-1}h) \\ &\quad - \text{Tr}(g^{-1}(P \text{Ricci})g^{-1}hg^{-1}k) + \text{Tr}(g^{-1}(P \text{Ricci})g^{-1}kg^{-1}h) \\ &\quad + \frac{1}{2} \text{Tr}(g^{-1}(P \text{Ricci})g^{-1}k) \text{Tr}(g^{-1}h) \\ &\quad \left. - \frac{1}{2} \text{Tr}(g^{-1}(P \text{Ricci})g^{-1}h) \text{Tr}(g^{-1}k) \right) \text{vol}(g). \end{aligned}$$

Some terms in this formula cancel out because for symmetric A, B, C one has $\text{Tr}(ABC) = \text{Tr}((ABC)^\top) = \text{Tr}(C^\top B^\top A^\top) = \text{Tr}(A^\top C^\top B^\top) =$

$\text{Tr}(ACB)$. Therefore

$$\begin{aligned} & hG^P(\text{Ricci}, k) - kG^P(\text{Ricci}, h) \\ &= \int \left(\text{Tr}(g^{-1}D_{g,h}(P \text{Ricci})g^{-1}k) - \text{Tr}(g^{-1}D_{g,k}(P \text{Ricci})g^{-1}h) \right. \\ &\quad + \frac{1}{2} \text{Tr}(g^{-1}(P \text{Ricci})g^{-1}k) \text{Tr}(g^{-1}h) \\ &\quad \left. - \frac{1}{2} \text{Tr}(g^{-1}(P \text{Ricci})g^{-1}h) \text{Tr}(g^{-1}k) \right) \text{vol}(g). \end{aligned}$$

We write $D_{g,h}(P \text{Ricci}) = Q(h)$ for some differential operator Q mapping symmetric two-tensors to themselves and Q^* for the adjoint of Q with respect to \tilde{g}_2^0 .

$$\begin{aligned} & hG^P(\text{Ricci}, k) - kG^P(\text{Ricci}, h) = \\ &= \int \left(g_2^0(Q(h), k) - g_2^0(Q(k), h) + \frac{1}{2}g_2^0(P \text{Ricci}, k) \text{Tr}(g^{-1}h) \right. \\ &\quad \left. - \frac{1}{2}g_2^0(P \text{Ricci}, h) \text{Tr}(g^{-1}k) \right) \text{vol}(g) \\ &= \int g_2^0 \left(Q(h) - Q^*(h) + \frac{1}{2}(P \text{Ricci}) \cdot \text{Tr}(g^{-1}h) \right. \\ &\quad \left. - \frac{1}{2}g \cdot g_2^0(P \text{Ricci}, h), k \right) \text{vol}(g). \end{aligned}$$

We have proved:

Lemma. *The Ricci vector field Ricci is a gradient field for the G^P -metric if and only if the equation*

$$(1) \quad 2(Q(h) - Q^*(h)) + (P \text{Ricci}) \cdot \text{Tr}(g^{-1}h) - g \cdot g_2^0(P \text{Ricci}, h) = 0,$$

with $Q(h) = Q_g(h) = D_{g,h}(P_g \text{Ricci}_g)$,

is satisfied for all $g \in \mathcal{M}(M)$ and all symmetric $\binom{0}{2}$ -tensors h .

None of the specific metrics studied in Section 4 of this paper satisfies the lemma in general dimension. Note that the lemma is trivially satisfied in dimension $\dim(M) = 1$. In dimension 2 the equation $\text{Ricci}_g = \frac{1}{2} \text{Scal}_g$ holds and the operator $P_g h = 2 \text{Scal}_g^{-1} h$ satisfies equation (1) on the open subset $\{g : \text{Scal}_g \neq 0\}$. Generally, equation (1) is satisfied if $P_g \text{Ricci}_g = g$, but this cannot hold on the space of all metrics if $\dim(M) > 2$.

References

- [1] M. Bauer & M. Bruveris, *A new Riemannian setting for surface registration*, in *3rd MICCAI Workshop on Mathematical Foundations of Computational Anatomy*, pages 182–194, 2011.
- [2] M. Bauer, P. Harms & P.W. Michor, *Almost local metrics on shape space of hypersurfaces in n -space*, *SIAM J. Imaging Sci.* **5** (2012), pp. 244–310, MR 2902663, Zbl pre06082561.

- [3] M. Bauer, P. Harms & P.W. Michor, *Sobolev metrics on shape space of surfaces*, Journal of Geometric Mechanics **3**, **4** (2011), MR 2888014, Zbl pre06037237.
- [4] M. Bauer, P. Harms & P.W. Michor, *Curvature weighted metrics on shape space of hypersurfaces in n -space*, Differential Geometry and Its Applications, **30** (2012), 33–41, MR 2871702, Zbl 1237.58009.
- [5] M. Bauer, M. Bruveris, P. Harms & P.W. Michor, *The KdV-equation has vanishing geodesic distance*, Ann. Glob. Anal. Geom. **41**, **4** (2012) 461–472, MR 2891297, Zbl 1246.58005.
- [6] A.L. Besse *Einstein manifolds*, Classics in Mathematics. Springer-Verlag, Berlin, 2008, MR 2371700, Zbl 1147.53001.
- [7] J. Cheeger, M. Gromov & M. Taylor, *Finite propagation speed, kernel estimates for functions of the Laplace operator, and the geometry of complete Riemannian manifolds*, J. Differential Geom., **17**, 15–53, 1982, MR 0658471, Zbl 0493.53035.
- [8] B. Clarke, *The Completion of the Manifold of Riemannian Metrics with Respect to its L^2 Metric*, Doctoral Dissertation, Leipzig 2009.
- [9] B. Clarke, *The Metric Geometry of the Manifold of Riemannian Metrics over a Closed Manifold*, Calc. Var. Partial Differential Equations, **39**, 533–545, 2010, MR 2729311, Zbl 1213.58007.
- [10] B. Clarke, *The Completion of the Manifold of Riemannian Metrics*, J. Differential Geom., **93** (2), 203–268, 2013.
- [11] D.G. Ebin, *The manifold of Riemannian metrics*, Global Analysis, Proc. Symp. Pure Math. AMS, **15**, 11–40, 1970, MR 0267604, Zbl 0205.53702.
- [12] J. Eichhorn, *Global analysis on open manifolds*, Nova Science Publishers, Inc., New York, 2007, MR 2343536, Zbl 1188.58001.
- [13] J. Eichhorn & J. Fricke, *The module structure theorem for Sobolev spaces on open manifolds*, Math. Nachr., **194**, 35–47, 1998, MR 1653149, Zbl 0954.46020.
- [14] D.S. Freed & D. Groisser, *The basic geometry of the manifold of Riemannian metrics and of its quotient by the diffeomorphism group*, Michigan Math. J., **36**, 323–344, 1989, MR 1027070, Zbl 0694.58008.
- [15] O. Gil-Medrano & P.W. Michor, *The Riemannian manifold of all Riemannian metrics*, Quarterly J. Math. Oxford, **42**, 183–202, 1991, MR 1107281, Zbl 0739.58010.
- [16] O. Gil-Medrano, P.W. Michor & M. Neuwirther, *Pseudoriemannian metrics on spaces of bilinear structures*, Quarterly J. Math. Oxford, **43**, 201–221, 1992, MR 1164623, Zbl 0799.58013.
- [17] O. Gil-Medrano & P.W. Michor, *Geodesics on spaces of almost hermitian structures*, Israel J. Math., **88**, 319–332, 1994, MR 1303501, Zbl 0821.53031.
- [18] S. Kobayashi & K. Nomizu, *Foundations of differential geometry, Vol. I*, Wiley Classics Library. John Wiley & Sons Inc., New York, 1996, MR 1393940, Zbl 0508.53002.
- [19] I. Kolář, P.W. Michor & J. Slovák, *Natural operations in differential geometry*, Springer-Verlag, Berlin, 1993, MR 1202431, Zbl 0782.53013.
- [20] A. Kriegl & P.W. Michor, *The convenient setting of global analysis*, volume 53 of *Mathematical Surveys and Monographs*, American Mathematical Society, Providence, RI, 1997, MR 1471480, Zbl 0889.58001.

- [21] A.C.G. Menzies, A. Yezzi & G. Sundaramoorthi, *Properties of Sobolev-type metrics in the space of curves*, Interfaces Free Bound., **10** (4), 423–445, 2008, MR 2465268, Zbl 1168.58005.
- [22] P.W. Michor. Topics in differential geometry, *volume 93 of Graduate Studies in Mathematics*, American Mathematical Society, Providence, RI, 2008, MR 2428390, Zbl 1175.53002.
- [23] P.W. Michor & D. Mumford, *Vanishing geodesic distance on spaces of submanifolds and diffeomorphisms*, Doc. Math., **10**, 217–245 (electronic), 2005, MR 2148075, Zbl 1083.58010.
- [24] P.W. Michor & D. Mumford, *Riemannian geometries on spaces of plane curves*, J. Eur. Math. Soc. (JEMS) **8**, (2006), 1–48, 2006, MR 2201275, Zbl 1101.58005.
- [25] P.W. Michor & D. Mumford, *An overview of the Riemannian metrics on spaces of curves using the Hamiltonian approach*, Appl. Comput. Harmon. Anal., **23** (1), 74–113, 2007, MR 2333829, Zbl 1116.58007.
- [26] M.A. Shubin, *Pseudodifferential operators and spectral theory*, Springer Series in Soviet Mathematics. Springer-Verlag, Berlin, 1987, MR 1852334, Zbl 0616.47040.
- [27] A. Trouvé & L. Younes, *Diffeomorphic matching problems in one dimension: Designing and minimizing matching functionals*, in David Vernon, editor, *Computer Vision*, volume 1842. ECCV, 2000.
- [28] S. Verpoort, *The geometry of the second fundamental form: Curvature properties and variational aspects*, PhD thesis, Katholieke Universiteit Leuven, 2008.
- [29] L. Younes, *Computable elastic distances between shapes*, SIAM J. Appl. Math., **58** (2): 565–586 (electronic), 1998, MR 1617630, Zbl 0907.68158.
- [30] L. Younes, P.W. Michor, J. Shah & D. Mumford, *A metric on shape space with explicit geodesics*, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl., **19** (1): 25–57, 2008, MR 2383560, Zbl 1142.58013.

FAKULTÄT FÜR MATHEMATIK
UNIVERSITÄT WIEN
NORDBERGSTRASSE 15
A-1090 WIEN, AUSTRIA

E-mail address: Bauer.Martin@univie.ac.at

EDLABS
HARVARD UNIVERSITY
44 BRATTLE STREET
CAMBRIDGE, MA 02138

E-mail address: pharms@edlabs.harvard.edu

FAKULTÄT FÜR MATHEMATIK
UNIVERSITÄT WIEN
NORDBERGSTRASSE 15
A-1090 WIEN, AUSTRIA

E-mail address: Peter.Michor@univie.ac.at