POISSON 2-GROUPS

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Abstract

We prove a 2-categorical analogue of a classical result of Drinfeld: there is a one-to-one correspondence between connected, simply connected Poisson Lie 2-groups and Lie 2-bialgebras. In fact, we also prove that there is a one-to-one correspondence between connected, simply connected quasi-Poisson 2-groups and quasi-Lie 2-bialgebras. Our approach relies on a "universal lifting theorem" for Lie 2-groups: an isomorphism between the graded Lie algebras of multiplicative polyvector fields on the Lie 2-group on one hand and of polydifferentials on the corresponding Lie 2-algebra on the other hand.

Introduction

A Poisson group is a Lie group equipped with a compatible Poisson structure. Poisson groups are the classical limit of quantum groups and have been extensively studied in the past two decades. For instance, Drinfeld proved that there is a bijection between connected, simply connected Poisson groups and Lie bialgebras [**7**, **8**].

Lie 2-groups (also called strict Lie 2-groups in the literature) are Lie group objects in the category of Lie groupoids, or equivalently Lie groupoid objects in the category of Lie groups. More explicitly, a Lie 2-group is a Lie groupoid $\Gamma_1 \rightrightarrows \Gamma_0$, where both Γ_1 and Γ_0 are Lie groups and all the groupoid structure maps are group homomorphisms. Lie 2-groups are special instances of Mackenzie's double groupoids [**14**].

The recent categorification trend motivates the search for an appropriate notion of quantum 2-groups. Poisson 2-groups are a natural first step in that direction. By a Poisson 2-group, we mean a Lie 2-group equipped with a Poisson structure $\mathbf{\Pi}$ on Γ_1, which is multiplicative with respect to both the group and the groupoid structures on Γ_1. In other words, $(\Gamma_1, \mathbf{\Pi})$ is simultaneously both a Poisson group [**7**, **8**] and a Poisson groupoid [**20**].

Lie 2-algebras are Lie algebra objects in the category of Lie algebroids [**1**]. They can be identified with Lie algebra crossed modules: pairs of Lie algebras θ and $\mathfrak{g}$ together with a linear map $\phi : \theta \to \mathfrak{g}$ and an action

of $\mathfrak{g}$ on θ by derivations satisfying a certain compatibility condition. Likewise, a Lie 2-bialgebra can be considered as a Lie bialgebra crossed module, i.e. a pair of Lie algebra crossed modules in duality: $(\theta \xrightarrow{\phi} \mathfrak{g})$ and $(\mathfrak{g}^* \xrightarrow{-\phi^*} \theta^*)$ are both Lie algebra crossed modules, and $(\mathfrak{g} \ltimes \theta, \theta^* \ltimes \mathfrak{g}^*)$ is a Lie bialgebra.

We prove that, at the infinitesimal level, Poisson 2-groups induce Lie 2-bialgebras. More precisely, we prove the following analogue of Drinfeld's theorem:

Theorem A. *There is a one-to-one correspondence between connected, simply connected Poisson 2-groups and Lie 2-bialgebras.*

We will also prove a more general result:

Theorem B. *There is a one-to-one correspondence between connected, simply connected quasi-Poisson 2-groups and quasi-Lie 2-bialgebras.*

Quasi-Poisson 2-groups are, in a certain sense, the 2-categorical analogues of Kosmann-Schwarzbach's quasi-Poisson groups [**10**]. A quasi-Poisson 2-group is a Lie 2-group $\Gamma_1 \rightrightarrows \Gamma_0$ endowed with a multiplicative quasi-Poisson structure on Γ_1, i.e. a multiplicative bivector field $\mathbf{\Pi}$ on Γ_1 such that the Schouten bracket $[\mathbf{\Pi}, \mathbf{\Pi}]$ is some sort of coboundary.

A natural generalization of Lie 2-algebras (or Lie algebra crossed modules), weak Lie 2-algebras are two-term L_∞ algebras. They can be described concisely in terms of the shifted degree "big bracket," which is a Gerstenhaber bracket on $S^\bullet(V_{[2]} \oplus V^*_{[1]})$. Here $V = \theta \oplus \mathfrak{g}$ is a graded vector space, where θ is of degree 1 and $\mathfrak{g}$ is of degree 0. Identifying $S^\bullet(V_{[2]} \oplus V^*_{[1]})$ with the space $\Gamma(\wedge^\bullet T_{[4]}M)$ of polyvector fields on $M = V^*_{[-2]}$ with polynomial coefficients, the big bracket can be simply described as the Schouten bracket of polyvector fields on M.

In [**6**], we developed a notion of weak Lie 2-bialgebras: objects that are simultaneously weak Lie 2-algebras as well as weak Lie 2-coalgebras, both structures being compatible with one another in a certain sense. In terms of the big bracket, a weak Lie 2-bialgebra on a graded vector space V is a degree-(-4) element t of $S^\bullet(V_{[2]} \oplus V^*_{[1]})$ satisfying $\{t, t\} = 0$. Quasi-Lie 2-bialgebras are a special instance of weak Lie 2-bialgebras.

Our proofs of Theorems A and B rely on the following "universal lifting theorem," which should be of independent interest:

Theorem C. *Given a Lie 2-group $\Gamma_1 \rightrightarrows \Gamma_0$, if both Γ_1 and Γ_0 are connected and simply connected, then the graded Lie algebras $\bigoplus_{k \geq 0} \mathfrak{X}^k_{\mathrm{mult}}(\Gamma_1)$ and $\bigoplus_{k \geq 0} \mathcal{A}_k$ are isomorphic.*

Here $\bigoplus_{k \geq 0} \mathfrak{X}^k_{\mathrm{mult}}(\Gamma_1)$ denotes the space of multiplicative polyvector fields on Γ_1 which, being closed with respect to the Schouten bracket, is naturally a graded Lie algebra. On the other hand, $\bigoplus_{k \geq 0} \mathcal{A}_k$ denotes

the graded Lie algebra formed by the polydifferentials on the associated Lie 2-algebra—the infinitesimal counterparts of the multiplicative polyvector fields on the Lie 2-group.

Theorems A and B are proved simply by expressing the algebraic data defining the weak Lie 2-bialgebra structure in terms of the graded Lie algebra $\bigoplus_{k \geq 0} \mathcal{A}_k$.

We refer to the recent papers [**13**, **16**, **18**] on integration of Courant algebroids to symplectic 2-groupoids, which may have a close connection to our work.

1. Quasi-Lie 2-bialgebras

In this section, we recall some basic notions regarding quasi-Lie 2-bialgebras developed in [**6**].

1.1. The big bracket. We will introduce a graded version of the big bracket [**11**, **12**] involving graded vector spaces.

Let $V = \bigoplus_{k \in \mathbb{Z}} V^{(k)}$ be a graded vector space. Consider the $\mathbb{Z}$-graded manifold $M = V^*[-2]$ and the shifted tangent space

$$T_{[4]}M \cong (M \times V^*[-2])[4] \cong M \times V^*[2].$$

Consider the space of polyvector fields on M with polynomial coefficients:

$$\Gamma(\wedge^\bullet T_{[4]}M) \cong S^\bullet(M^*) \otimes S^\bullet((V^*[2])[-1])$$
$$\cong S^\bullet(V[2]) \otimes S^\bullet(V^*[1]) \cong S^\bullet(V[2] \oplus V^*[1]).$$

We write $\mathcal{S}^\bullet$ for $S^\bullet(V[2] \oplus V^*[1])$ and $\odot$ for the symmetric tensor product in $\mathcal{S}^\bullet$.

There is a standard way to endow $\mathcal{S}^\bullet = \Gamma(\wedge^\bullet T_{[4]}M)$ with a graded Lie bracket, i.e. the Schouten bracket, which is denoted by $\{\cdot, \cdot\}$. It is a bilinear map $\{\cdot, \cdot\} : \mathcal{S}^\bullet \otimes \mathcal{S}^\bullet \to \mathcal{S}^\bullet$ satisfying the following properties:

1) $\{v, v'\} = \{\epsilon, \epsilon'\} = 0$, for all $v, v' \in V[2]$ and $\epsilon, \epsilon' \in V^*[1]$;
2) $\{v, \epsilon\} = (-1)^{|v|} \langle v | \epsilon \rangle$, for all $v \in V[2]$ and $\epsilon \in V^*[1]$;

3) $\{e_1, e_2\} = -(-1)^{(|e_1|+3)(|e_2|+3)} \{e_2, e_1\}$, for all $e_1, e_2 \in \mathcal{S}^\bullet$;

4) $\{e_1, e_2 \odot e_3\} = \{e_1, e_2\} \odot e_3 + (-1)^{(|e_1|+3)|e_2|} e_2 \odot \{e_1, e_3\}$, for all $e_1, e_2, e_3 \in \mathcal{S}^\bullet$.

It is clear that $\{\cdot, \cdot\}$ is of degree 3, i.e.

$$|\{e_1, e_2\}| = |e_1| + |e_2| + 3,$$

for all homogeneous $e_i \in \mathcal{S}^\bullet$, and the following graded Jacobi identity holds:

$$\{e_1, \{e_2, e_3\}\} = \{\{e_1, e_2\}, e_3\} + (-1)^{(|e_1|+3)(|e_2|+3)} \{e_2, \{e_1, e_3\}\}.$$

Hence $(\mathcal{S}^\bullet, \odot, \{\cdot, \cdot\})$ is a Schouten algebra, also known as an odd Poisson algebra, or a Gerstenhaber algebra [19].

Due to our degree convention, when V is an ordinary vector space considered as a graded vector space concentrated at degree 0, the bracket above is different from the usual big bracket in the literature [11].

1.2. Quasi-Lie 2-bialgebras. Following Baez-Crans [1], a weak Lie 2-algebra is an L_∞-algebra on the 2-term graded vector space $V = \theta \oplus \mathfrak{g}$, where θ is of degree 1 and $\mathfrak{g}$ is of degree 0. Unfolding the L_∞-structure, one can define a weak Lie 2-algebra as a pair of vector spaces θ and $\mathfrak{g}$ endowed with the following structures:

1) a linear map $\phi \colon \theta \to \mathfrak{g}$;
2) a bilinear skewsymmetric map $[\cdot, \cdot] \colon \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$;
3) a bilinear map $\cdot \triangleright \cdot \colon \mathfrak{g} \otimes \theta \to \theta$;
4) a trilinear skewsymmetric map $h \colon \mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g} \to \theta$, called the homotopy map.

These maps are required to satisfy the following compatibility conditions: for all $w, x, y, z \in \mathfrak{g}$ and $u, v \in \theta$,

$$[[x, y], z] + [[y, z], x] + [[z, x], y] + (\phi \circ h)(x, y, z) = 0,$$

$$y \triangleright (x \triangleright u) - x \triangleright (y \triangleright u) + [x, y] \triangleright u + h(\phi(u), x, y) = 0,$$

$$\phi(u) \triangleright v + \phi(v) \triangleright u = 0,$$

$$\phi(x \triangleright u) = [x, \phi(u)],$$

and

$$-w \triangleright h(x, y, z) - y \triangleright h(x, z, w) + z \triangleright h(x, y, w) + x \triangleright h(y, z, w)$$
$$= h([x, y], z, w) - h([x, z], y, w) + h([x, w], y, z)$$
$$+ h([y, z], x, w) - h([y, w], x, z) + h([z, w], x, y).$$

If h vanishes, we call it a strict Lie 2-algebra, or simply a Lie 2-algebra.

Now consider the degree shifted vector spaces $V[2]$ and $V^*[1]$. Under such a degree convention, the degrees of $\mathfrak{g}$, θ, $\mathfrak{g}^*$, and θ^* are specified as follows:

space	$\mathfrak{g}$	θ	$\mathfrak{g}^*$	θ^*
degree	-2	-1	-1	-2

We will maintain this convention throughout this section. We remind the reader that the abbreviation $\mathcal{S}^\bullet$ stands for $S^\bullet(V^*[1] \oplus V[2])$.

Proposition 1.1 ([**6**]). *Under the above degree convention, a weak Lie 2-algebra structure is equivalent to a solution to the equation*

$$\{s, s\} = 0, \tag{1}$$

where $s = \check{\phi} + \check{b} + \check{a} + \check{h}$ is an element in $\mathcal{S}^{(-4)}$ such that

$$\text{(2)} \qquad \begin{cases} \check{\phi} \in \theta^* \odot \mathfrak{g}, \\ \check{b} \in (\odot^2 \mathfrak{g}^*) \odot \mathfrak{g}, \\ \check{a} \in \mathfrak{g}^* \odot \theta^* \odot \theta, \\ \check{h} \in (\odot^3 \mathfrak{g}^*) \odot \theta. \end{cases}$$

Here the bracket in Eq. (1) *stands for the big bracket as in Section* 1.1.

In the sequel, we denote a weak Lie 2-algebra by $(\theta \to \mathfrak{g}, s)$ in order to emphasize the map from θ to $\mathfrak{g}$. Sometimes, we will omit s and denote a weak Lie 2-algebra simply by $(\theta \to \mathfrak{g})$. If $(\mathfrak{g}^* \to \theta^*)$ is a weak Lie 2-algebra, then $(\theta \to \mathfrak{g})$ is called a weak Lie 2-coalgebra. Equivalently, a weak Lie 2-coalgebra is a 2-term L_∞-structure on $\mathfrak{g}^* \oplus \theta^*$, where $\mathfrak{g}^*$ has degree 1 and θ^* has degree 0.

Similarly, we have the following

Proposition 1.2 ([**6**]). *A weak Lie 2-coalgebra is equivalent to a solution to the equation*

$$\{c, c\} = 0,$$

where $c = \check{\phi} + \check{\epsilon} + \check{\alpha} + \check{\eta} \in \mathcal{S}^{(-4)}$ such that

$$\text{(3)} \qquad \begin{cases} \check{\phi} \in \theta^* \odot \mathfrak{g}, \\ \check{\epsilon} \in \theta^* \odot (\odot^2 \theta), \\ \check{\alpha} \in \mathfrak{g}^* \odot \mathfrak{g} \odot \theta, \\ \check{\eta} \in \mathfrak{g}^* \odot (\odot^3 \theta). \end{cases}$$

We denote such a weak Lie 2-coalgebra by $(\theta \to \mathfrak{g}, c)$.

Definition 1.3. A weak Lie 2-bialgebra consists of a pair of vector spaces θ and $\mathfrak{g}$ together with a solution $t = \check{b} + \check{a} + \check{h} + \check{\phi} + \check{\epsilon} + \check{\alpha} + \check{\eta} \in \mathcal{S}^{(-4)}$ to the equation $\{t, t\} = 0$. Here $\check{b}, \check{a}, \check{h}, \check{\phi}, \check{\epsilon}, \check{\alpha}, \check{\eta}$ are as in Eqs. (2) and (3). If, moreover, $\check{h} = 0$, it is called a quasi-Lie 2-bialgebra. If both $\check{h}$ and $\check{\eta}$ vanish, we say that the Lie 2-bialgebra is strict, or simply a Lie 2-bialgebra.

Proposition 1.4. *Let $(\theta, \mathfrak{g}, t)$ be a weak Lie 2-bialgebra as in Definition* 1.3. *Then $(\theta \to \mathfrak{g}, l)$, where $l = \check{\phi} + \check{b} + \check{a} + \check{h}$, is a weak Lie 2-algebra, while $(\theta \to \mathfrak{g}, c)$, where $c = \check{\phi} + \check{\epsilon} + \check{\alpha} + \check{\eta}$, is a weak Lie 2-coalgebra.*

Example 1.5. Assume that $\mathfrak{g}$ is a semisimple Lie algebra. Let $(\cdot, \cdot)$ be its Killing form. Then $h(x, y, z) = \hbar(x, [y, z])$, for all $x, y, z \in \mathfrak{g}$, is a Lie algebra 3-cocycle, where $\hbar$ is a constant. Let $\theta = \mathbb{R}$. Then the trivial map $\mathbb{R} \to \mathfrak{g}$ together with h becomes a weak Lie 2-algebra, called the string Lie 2-algebra [**1**]. More precisely, the string Lie 2-algebra is as follows:

1) θ is the abelian Lie algebra $\mathbb{R}$;
2) $\mathfrak{g}$ is a semisimple Lie algebra;
3) $\phi : \theta \to \mathfrak{g}$ is the trivial map;
4) the action map $\triangleright : \mathfrak{g} \otimes \theta \to \theta$ is the trivial map;
5) $h : \wedge^3 \mathfrak{g} \to \theta$ is given by the map $\hbar(\cdot, [\cdot, \cdot])$, where $\hbar$ is a fixed constant.

Now fix an element $x \in \mathfrak{g}$. We endow $\mathbb{R} \to \mathfrak{g}$ with a weak Lie 2-coalgebra structure as follows:

1) $\mathfrak{g}^*$ is an abelian Lie algebra;
2) $\theta^* \cong \mathbb{R}$ is an abelian Lie algebra;
3) $\phi^* : \mathfrak{g}^* \to \theta^*$ is the trivial map;
4) the θ^*-action on $\mathfrak{g}^*$ is given by $\mathbf{1} \triangleright \xi = \mathrm{ad}_x^* \xi$, for all $\xi \in \mathfrak{g}^*$;
5) $\tilde{\eta} : \wedge^3 \theta^* \to \mathfrak{g}^*$ is the trivial map.

One can verify directly that these relations indeed define a weak Lie 2-bialgebra.

1.3. Lie bialgebra crossed modules.

Definition 1.6. A Lie algebra crossed module consists of a pair of Lie algebras θ and $\mathfrak{g}$, a linear map $\phi : \theta \to \mathfrak{g}$, and an action of $\mathfrak{g}$ on θ by derivations satisfying, for all $x, y \in \mathfrak{g}$, $u, v \in \theta$,

1) $\phi(u) \triangleright v = [u, v]$;
2) $\phi(x \triangleright u) = [x, \phi(u)]$,

where $\triangleright$ denotes the $\mathfrak{g}$-action on θ.

Note that 1) and 2) imply that ϕ must be a Lie algebra homomorphism. We write $(\theta \xrightarrow{\phi} \mathfrak{g})$ to denote a Lie algebra crossed module. The associated semidirect product Lie algebra is denoted by $\mathfrak{g} \ltimes \theta$.

The following proposition indicates that crossed modules of Lie algebras are in one-to-one correspondence with Lie 2-algebras. We refer the reader to [**1**] for details.

Proposition 1.7. *Lie algebra crossed modules are equivalent to (strict) Lie 2-algebras.*

Definition 1.8. A Lie bialgebra crossed module is a pair of Lie algebra crossed modules in duality: $(\theta \xrightarrow{\phi} \mathfrak{g})$ and $(\mathfrak{g}^* \xrightarrow{\phi^T} \theta^*)$, where $\phi^T = -\phi^*$, are both Lie algebra crossed modules such that $(\mathfrak{g} \ltimes \theta, \theta^* \ltimes \mathfrak{g}^*)$ is a Lie bialgebra.

Lie bialgebra crossed modules are symmetric, as we see in the next

Proposition 1.9. *If $((\theta \xrightarrow{\phi} \mathfrak{g}), (\mathfrak{g}^* \xrightarrow{\phi^T} \theta^*))$ is a Lie bialgebra crossed module, so is $((\mathfrak{g}^* \xrightarrow{\phi^T} \theta^*), (\theta \xrightarrow{\phi} \mathfrak{g}))$.*

The following result justifies our terminology.

Proposition 1.10. *If $((\theta \xrightarrow{\phi} \mathfrak{g}), (\mathfrak{g}^* \xrightarrow{\phi^T} \theta^*))$ is a Lie bialgebra crossed module, then both pairs (θ, θ^*) and $(\mathfrak{g}, \mathfrak{g}^*)$ are Lie bialgebras.*

Example 1.11. One can construct a Lie bialgebra crossed module from an ordinary Lie bialgebra as follows. Given a Lie bialgebra (θ, θ^*), consider the trivial Lie algebra crossed module $(\theta \xrightarrow{\mathbf{1}} \theta)$, where the second θ acts on the first θ by the adjoint action. In the mean time, consider the dual Lie algebra crossed module $(\theta^* \xrightarrow{-\mathbf{1}} \theta^*)$, where the second θ^* is equipped with the opposite Lie bracket: $-[\cdot, \cdot]_*$, and the action of the second θ^* on the first θ^* is given by $\kappa_2 \triangleright \kappa_1 = -[\kappa_2, \kappa_1]_*$, for all $\kappa_1, \kappa_2 \in \theta^*$. It is simple to see that $((\theta \xrightarrow{\mathbf{1}} \theta), (\theta^* \xrightarrow{-\mathbf{1}} \theta^*))$ is indeed a Lie bialgebra crossed module.

The following theorem was proved in [**6**].

Theorem 1.12. *There is a bijection between Lie bialgebra crossed modules and (strict) Lie 2-bialgebras.*

Example 1.13. Consider the Lie subalgebra $\mathrm{u}(n) \subset \mathfrak{gl}_n(\mathbb{C})$ of $n \times n$ skew-Hermitian matrices. Let $\theta \subset \mathfrak{gl}_n(\mathbb{C})$ be the Lie subalgebra consisting of upper triangular matrices whose diagonal elements are real numbers. It is standard that $(\theta, \mathrm{u}(n))$ is a Lie bialgebra. Indeed $\theta \oplus \mathrm{u}(n) \cong \mathfrak{gl}_n(\mathbb{C})$, and both θ and $\mathrm{u}(n)$ are Lagrangian subalgebras of $\mathfrak{gl}_n(\mathbb{C})$ under the nondegenerate pairing $\langle X | Y \rangle = \mathrm{Im}(\mathrm{Tr}(XY))$, for $X, Y \in \mathfrak{gl}_n(\mathbb{C})$. Hence $(\theta, \mathrm{u}(n), \mathfrak{gl}_n(\mathbb{C}))$ is a Manin triple, and thus $(\theta, \mathrm{u}(n))$ forms a Lie bialgebra.

Let $\mathfrak{g}$ denote the Lie algebra of traceless upper triangular matrices with real diagonal elements. It turns out that $(\theta \xrightarrow{\phi} \mathfrak{g})$, where ϕ is the map $A \mapsto A - \mathrm{tr}A$, is a Lie bialgebra crossed module.

2. Universal lifting theorem

2.1. Lie 2-groups. A Lie 2-group (also called strict Lie 2-groups in the literature) is a Lie groupoid $\Gamma_1 \rightrightarrows \Gamma_0$, where both Γ_1 and Γ_0 are Lie groups and all the groupoid structure maps are group homomorphisms. A Lie 2-group is a special case of double groupoid in the sense of Mackenzie [**14**].

Definition 2.1 ([**21**, **22**]). A Lie group crossed module consists of a Lie group homomorphism $\Phi : \Theta \to G$ and an action of G on Θ by automorphisms satisfying the following compatibility conditions:

1) $\Phi(\alpha) \triangleright \beta = \alpha \beta \alpha^{-1}$,
2) $\Phi(g \triangleright \beta) = g \Phi(\beta) g^{-1}$,

for all $g \in G$ and $\alpha, \beta \in \Theta$. Here $g \triangleright \beta$ denotes the action of $g \in G$ on $\beta \in \Theta$.

We write $(\Theta \xrightarrow{\Phi} G)$ to denote a Lie group crossed module.

Proposition 2.2. *There is a bijection between Lie 2-groups and crossed modules of Lie groups.*

Proof. This is standard. For instance, see [**2**, **4**, **5**, **15**, **17**]. Here we will sketch the construction of the Lie 2-group out of a crossed module, which will be needed later on.

The Lie 2-group corresponding to a Lie group crossed module $(\Theta \xrightarrow{\Phi} G)$ will be denoted by $G \ltimes \Theta \rightrightarrows G$, or simply $G \ltimes \Theta$, by abuse of notations. Here the group structure on $G \ltimes \Theta$ is as follows:

- group multiplication: $(g, \alpha) \diamond (h, \beta) = (gh, (h^{-1} \triangleright \alpha)\beta)$;
- group unit: $\mathbf{1}_\diamond = (\mathbf{1}_G, \mathbf{1}_\Theta)$, with $\mathbf{1}_G$ and $\mathbf{1}_\Theta$ denoting, respectively, the group unit elements of G and Θ;
- group inversion: $(g, \alpha)_\diamond^{-1} = (g^{-1}, (g \triangleright \alpha)^{-1})$.

The groupoid structure on $G \ltimes \Theta \rightrightarrows G$ is as follows:

- source and target maps: $\mathbf{s}(g, \alpha) = g$, $\mathbf{t}(g, \alpha) = g\Phi(\alpha)$;
- groupoid multiplication: $(g, \alpha) \star (h, \beta) = (g, \alpha\beta)$, if $h = g\Phi(\alpha)$;
- groupoid units: $(g, \mathbf{1}_\Theta)$;
- groupoid inversion: $(g, \alpha)_\star^{-1} = (g\Phi(\alpha), \alpha^{-1})$.

<div align="right">q.e.d.</div>

In the sequel, we will use Lie 2-groups and crossed modules of Lie groups interchangeably.

2.2. Multiplicative polyvector fields on Lie groupoids. We recall some standard results regarding multiplicative polyvector fields on a Lie groupoid. Let $\Gamma \rightrightarrows M$ be a Lie groupoid with source and target maps s and t, respectively. Consider the graph of the groupoid multiplication $\Lambda = \{(p, q, pq) | t(p) = s(q)\}$, which is a submanifold in $\Gamma \times \Gamma \times \Gamma$.

Recall that a k-vector field $\Sigma \in \mathfrak{X}^k(\Gamma)$ is said to be multiplicative if Λ is coisotropic with respect to $\Sigma \times \Sigma \times (-1)^{k+1}\Sigma$ [**9**]. In other words,

$$(\Sigma \times \Sigma \times (-1)^{k+1}\Sigma)(\xi_1, \cdots, \xi_k) = 0, \quad \forall \xi_1, \cdots, \xi_k \in \Lambda^\perp$$

for all $\xi_1, \cdots, \xi_k \in \Lambda^\perp$, where

$$\Lambda^\perp = \left\{\xi \in T_\lambda^*(\Gamma \times \Gamma \times \Gamma) \text{ s.t. } \lambda \in \Lambda, \langle \xi | v \rangle = 0, \forall v \in T_\lambda \Lambda\right\}.$$

A k-vector field Σ on Γ is said to be affine if $[\Sigma, \overleftarrow{X}]$ is left invariant for all $X \in \mathbf{\Gamma}(A)$. Here A denotes the Lie algebroid of Γ, and $\overleftarrow{X}$ denotes the left invariant vector field on Γ corresponding to X.

The following lemma gives a useful characterization of multiplicative polyvector fields.

Lemma 2.3 (Theorem 2.19 in [**9**]). *A k-vector field Σ is multiplicative if and only if the following three conditions hold:*

1) *Σ is affine;*
2) *M is a coisotropic submanifold of Γ;*
3) *for any $\xi \in \Omega^1(M)$, $\iota_{\mathbf{t}^*(\xi)}\Sigma$ is left invariant.*

Remark 2.4. The statement of Theorem 2.19 in [**9**] contains more conditions but some of them are redundant.

Along the base manifold M, the tangent bundle $T\Gamma$ admits a natural decomposition

$$T\Gamma|_M = TM \oplus A,$$

where A is identified with $T^s\Gamma|_M$, the tangent bundle to the s-fibers along M. Denote by $\rho : A \to TM$ the anchor map. Then ρ is equal to $t_* : T^s\Gamma|_M \to TM$.

Let Z_k be the set of all elements w of $TM \wedge (\wedge^{k-1}A)$ satisfying

$$\iota_{\zeta_1}\iota_{\rho^*\zeta_2}w = -\iota_{\zeta_2}\iota_{\rho^*\zeta_1}w, \quad \forall \zeta_1, \zeta_2 \in T^*M.$$

Let D_ρ be a degree-0 derivation of $\Gamma(\wedge^\bullet(TM \oplus A))$ such that $D_\rho(a + b) = \rho(a)$, for all $a \in A$ and $b \in TM$.

Lemma 2.5. *For any $w \in Z_k$ and $j \geq 1$, we have*

(4) $\qquad \iota_{\rho^*\zeta}(D_\rho^{j-1}w) = D_\rho^j(\iota_\zeta w) = \dfrac{1}{j+1}\iota_\zeta(D_\rho^j w), \quad \forall \zeta \in T^*M.$

Proof. First, note that we have the following identities:

(5) $\qquad\qquad\qquad\qquad \iota_\zeta \circ D_\rho - D_\rho \circ \iota_\zeta = \iota_{\rho^*\zeta},$

(6) $\qquad\qquad\qquad\qquad \iota_{\rho^*\zeta} \circ D_\rho = D_\rho \circ \iota_{\rho^*\zeta},$

where $\zeta \in T^*M$, and both sides of Eqs. (5) and (6) are considered as linear maps $\wedge^\bullet(TM \oplus A) \to \wedge^{\bullet-1}(TM \oplus A)$.

Now we prove Eq. (4) by induction. If $j = 1$, the equation

$$\iota_{\rho^*\zeta}w = D_\rho(\iota_\zeta w)$$

follows from the definition of Z_k. By Eq. (5), we have

$$(\iota_\zeta \circ D_\rho - D_\rho \circ \iota_\zeta)w = \iota_{\rho^*\zeta}w = D_\rho(\iota_\zeta w).$$

It thus follows that

$$D_\rho(\iota_\zeta w) = \frac{1}{2}\iota_\zeta(D_\rho w).$$

Assume that Eq. (4) is valid for $j \geq 1$. Then, using Eq. (6), we have

$$\iota_{\rho^*\zeta}(D_\rho^j w) = (D_\rho \circ \iota_{\rho^*\zeta})(D_\rho^{j-1}w) = (D_\rho \circ D_\rho^j)(\iota_\zeta w) = D_\rho^{j+1}(\iota_\zeta w).$$

Moreover, using Eq. (5), we have

$$D_\rho^{j+1}(\iota_\zeta w) = D_\rho(D_\rho^j(\iota_\zeta w)) = \frac{1}{j+1}(D_\rho \circ \iota_\zeta \circ D_\rho^j)w$$

$$= \frac{1}{j+1}(\iota_\zeta \circ D_\rho - \iota_{\rho^*\zeta}) \circ D_\rho^j w = \frac{1}{j+1}\big(\iota_\zeta(D_\rho^{j+1}w) - D_\rho^{j+1}(\iota_\zeta w)\big),$$

which implies that

$$D_\rho^{j+1}(\iota_\zeta w) = \frac{1}{j+2}\iota_\zeta(D_\rho^{j+1}w).$$

q.e.d.

Proposition 2.6. *Given a multiplicative k-vector field Σ on Γ, there exists a section $\sigma \in \mathbf{\Gamma}(TM \wedge (\wedge^{k-1}A))$ such that*

$$(7) \quad \Sigma|_M = \frac{1 - e^{-D_\rho}}{D_\rho}(\sigma) = \sigma - \tfrac{1}{2!}D_\rho\sigma + \tfrac{1}{3!}D_\rho^2\sigma + \cdots - \frac{(-1)^k}{k!}D_\rho^{k-1}\sigma.$$

Moreover, σ satisfies the following properties:

$$(8) \quad \partial_\Sigma(f) = (-1)^{k-1}\iota_{\mathrm{d}f}\sigma, \quad \forall f \in C^\infty(M),$$

$$\iota_\zeta\iota_{\rho^*\xi}\sigma = -\iota_\xi\iota_{\rho^*\zeta}\sigma, \quad \forall \xi, \zeta \in \Omega^1(M).$$

Proof. Since M is coisotropic in Γ with respect to Σ, we may write

$$(9) \quad \Sigma|_M = \sigma^{1,k-1} + \sigma^{2,k-2} + \cdots + \sigma^{k,0},$$

where $\sigma^{i,k-i} \in \mathbf{\Gamma}((\wedge^i TM) \wedge (\wedge^{k-i}A))$.

Also observe that, for any 1-form $\xi \in \Omega^1(M)$, we have

$$(10) \quad \mathbf{t}^*(\xi)|_M = \xi + \rho^*\xi \in \mathbf{\Gamma}(T^*M \oplus A^*),$$

where $T^*\Gamma|_M$ is naturally identified with $T^*M \oplus A^*$. On the other hand, Condition 3 of Lemma 2.3 implies that $(\iota_{\mathbf{t}^*(\xi)}\Sigma)|_M$ is tangent to the **s**-fibers, and therefore contains only $\Gamma(\wedge^k A)$-components. Using Eqs. (9) and (10), we obtain

$$(11) \quad \begin{cases} \iota_{\rho^*\xi}\sigma^{1,k-1} = -\iota_\xi\sigma^{2,k-2} \\ \iota_{\rho^*\xi}\sigma^{2,k-2} = -\iota_\xi\sigma^{3,k-3} \\ \quad\vdots \\ \iota_{\rho^*\xi}\sigma^{k-1,1} = -\iota_\xi\sigma^{k,0}. \end{cases}$$

Note that, for all $f \in C^\infty(M)$,

$$(12) \quad \partial_\Sigma(f) = [\Sigma, \mathbf{t}^*f]|_M = (-1)^{k+1}\iota_{\mathbf{t}^*(\mathrm{d}f)}\Sigma|_M = (-1)^{k+1}\iota_{\mathrm{d}f}\sigma^{1,k-1}.$$

Since

$$0 = \partial_\Sigma[f_1, f_2] = [\partial_\Sigma(f_1), f_2] + (-1)^{k-1}[f_1, \partial_\Sigma(f_2)],$$

it follows that

$$(13) \quad \iota_{\rho^*\mathrm{d}f_2}\iota_{\mathrm{d}f_1}\sigma^{1,k-1} + \iota_{\rho^*\mathrm{d}f_1}\iota_{\mathrm{d}f_2}\sigma^{1,k-1} = 0.$$

Let $\sigma = \sigma^{1,k-1}$. We will prove the following identity by induction on i:

$$(14) \qquad\qquad \sigma^{i,k-i} = \frac{(-1)^{i-1}}{i!} D_\rho^{i-1} \sigma.$$

The case $i = 1$ is obvious. Assume that Eq. (14) is valid for i. Then, by Eq. (11),

$$\begin{aligned}
\iota_\xi \sigma^{i+1,k-i-1} &= -\iota_{\rho^*\xi} \sigma^{i,k-i} && \text{(by the induction assumption)}\\
&= \frac{(-1)^i}{i!} \iota_{\rho^*\xi} D_\rho^{i-1} \sigma && \text{(by Eq. (4))}\\
&= \frac{(-1)^i}{i!} \frac{1}{i+1} \iota_\xi D_\rho^i \sigma\\
&= \iota_\xi \Big(\frac{(-1)^i}{(i+1)!} D_\rho^i \sigma \Big).
\end{aligned}$$

Thus Eq. (14) is proved. Therefore, Eq. (9) implies Eq. (7), Eq. (12) implies Eq. (8), and Eq. (13) implies Eq. (2.6).   q.e.d.

2.3. k-differentials of a Lie algebroid. It is known that the Schouten bracket of two multiplicative polyvector fields on a Lie groupoid is still multiplicative. Therefore, the space of multiplicative polyvector fields is a graded Lie algebra [**9**]. On the level of Lie algebroids, multiplicative k-vector fields correspond to k-differentials of the Lie algebroid, whose definition we recall below.

Given a Lie algebroid A, a k-differential is an $\mathbb{R}$-linear map

$$\partial : \mathbf{\Gamma}(\wedge^\bullet A) \to \mathbf{\Gamma}(\wedge^{\bullet + k - 1} A)$$

satisfying

$$\begin{aligned}
\partial(P \wedge Q) &= (\partial P) \wedge Q + (-1)^{|P|(k-1)} P \wedge (\partial Q),\\
\partial[P,Q] &= [\partial P, Q] + (-1)^{(|P|-1)(k-1)} [P, \partial Q],
\end{aligned}$$

for all $P, Q \in \mathbf{\Gamma}(\wedge^\bullet A)$.

The commutator of a k_1 differential ∂_1 and a k_2-differential ∂_2 is the $(k_1 + k_2 - 1)$-differential

$$\lfloor \partial_1, \partial_2 \rfloor = \partial_1 \circ \partial_2 - (-1)^{(k_1 - 1)(k_2 - 1)} \partial_2 \circ \partial_1.$$

Theorem 2.7 ([**9**]). *Let Γ be a Lie groupoid, and A its Lie algebroid. Then every multiplicative k-vector field Σ induces a k-differential ∂_Σ by $\overleftarrow{\partial_\Sigma(P)} = [\Sigma, \overleftarrow{P}]$, for all $P \in \mathbf{\Gamma}(\wedge^\bullet A)$. Here $\overleftarrow{V}$ denotes the left invariant polyvector field on Γ determined by V.*

Moreover, the map $\Sigma \mapsto \partial_\Sigma$ is a homomorphism of graded Lie algebras, which is an isomorphism provided Γ is s-connected and s-simply connected.

In case of Lie groups, k-differentials of multiplicative k-vector fields can be described more explicitly.

Lemma 2.8. *Let Σ be a multiplicative k-vector field on a Lie group G.*

1) *The map $\widehat{\sigma} : G \to \wedge^k \mathfrak{g}$ defined by $\widehat{\sigma}(g) = L_{g^{-1}*}(\Sigma|_g)$ is a Lie group 1-cocycle.*

2) *The Lie algebra 1-cocycle induced by $\widehat{\sigma}$ is the k-differential ∂_Σ:*

$$\frac{d}{dt}\bigg|_{t=0} (\widehat{\sigma}|_{\exp tx}) = \frac{d}{dt}\bigg|_{t=0} L_{\exp^{-1} tx*}(\Sigma|_{\exp tx}) = -\partial_\Sigma(x), \quad \forall x \in \mathfrak{g}.$$

2.4. Infinitesimal data of multiplicative vector fields on Lie 2-groups. This section is devoted to the description of infinitesimal data of multiplicative vector fields on Lie 2-groups.

2.4.1. Multiplicative polyvector fields on Lie 2-groups. Let $(\Theta \xrightarrow{\Phi} G)$ be a Lie group crossed module and $G \ltimes \Theta$ the corresponding Lie 2-group. Consider a k-vector field $\mathbf{V} \in \mathfrak{X}^k(G \ltimes \Theta)$.

Definition 2.9. A multiplicative 0-vector field on $G \ltimes \Theta$ is a smooth function $f \in C^\infty(G \ltimes \Theta)$ subject to the following conditions:

$$f(p \diamond q) = f(p) + f(q), \quad \forall p, q \in G \ltimes \Theta;$$
$$f(p \star q) = f(p) + f(q), \quad \forall p, q \in G \ltimes \Theta \text{ s.t. } \mathbf{t}(p) = \mathbf{s}(q).$$

For $k \geq 1$, a k-vector field $\mathbf{V}$ is called multiplicative if it is multiplicative with respect to both the group and the groupoid structure on $G \ltimes \Theta$. In other words, the graph of the group multiplication

$$\Lambda^{gp} = \{(r_1, r_2, r_1 \diamond r_2)|r_1, r_2 \in G \ltimes \Theta\}$$

and the graph of the groupoid multiplication

$$\Lambda^{gpd} = \{(r_1, r_2, r_1 \star r_2)|r_1, r_2 \in G \ltimes \Theta, \mathbf{t}(r_1) = \mathbf{s}(r_2)\}$$

are both coisotropic with respect to the k-vector field $(\mathbf{V}, \mathbf{V}, (-1)^{k+1}\mathbf{V})$ on $(G \ltimes \Theta) \times (G \ltimes \Theta) \times (G \ltimes \Theta)$.

Denote the space of multiplicative k-vector fields by $\mathfrak{X}^k_{\mathrm{mult}}(G \ltimes \Theta)$. The following lemma follows immediately.

Lemma 2.10. *When endowed with the Schouten bracket, the space of multiplicative polyvector fields*

$$\mathfrak{X}^\bullet_{\mathrm{mult}}(G \ltimes \Theta) := \oplus_{k \geq 0}\mathfrak{X}^k_{\mathrm{mult}}(G \ltimes \Theta)$$

is a graded Lie algebra.

Remark 2.11. It is easy to see that $f \in \mathfrak{X}^0_{\mathrm{mult}}(G \ltimes \Theta)$ if and only if $f(g, \alpha) = \nu(\alpha)$, $\forall \alpha \in \Theta$ and $g \in G$, where $\nu \in C^\infty(\Theta)$ satisfies $\nu|_{\alpha\beta} = \nu|_\alpha + \nu|_\beta$ and $\nu|_{g \triangleright \alpha} = \nu|_\alpha$, $\forall \alpha, \beta \in \Theta$.

2.4.2. The infinitesimal data. It is natural to ask what is the infinitesimal data of a multiplicative k-vector field (with $k \geq 1$) on a Lie 2-group. To answer this question, we need, as a first step, to describe the Lie algebroid A of the groupoid $G \ltimes \Theta \rightrightarrows G$.

It is simple to see that A is the transformation Lie algebroid $G \times \theta \to G$, where the θ-action on G is $u \mapsto \overleftarrow{\phi(u)}, \forall\, u \in \theta$. Here the superscript $\overleftarrow{\cdot}$ stands for the left invariant vector field on G associated to a Lie algebra element in $\mathfrak{g}$. It follows from Theorem 2.7 that a multiplicative k-vector field $\mathbf{V} \in \mathfrak{X}^k_{\text{mult}}(G \ltimes \Theta)$ induces a k-differential

$$(15) \qquad\qquad \partial^{\text{gpd}} : \mathbf{\Gamma}(\wedge^\bullet A) \to \mathbf{\Gamma}(\wedge^{\bullet + k - 1} A)$$

of the Lie algebroid A. In particular, we have a map

$$\partial^{\text{gpd}} : C^\infty(G) \to \mathbf{\Gamma}(\wedge^{k-1} A) \cong C^\infty(G, \wedge^{k-1}\theta).$$

Since ∂^{gpd} is a derivation, i.e. $\partial^{\text{gpd}}(f_1 f_2) = f_2 \partial^{\text{gpd}}(f_1) + f_1 \partial^{\text{gpd}}(f_2)$, for all $f_1, f_2 \in C^\infty(G)$, ∂^{gpd} induces a $\wedge^{k-1}\theta$-valued vector field on G, which in turn can be identified with a $\mathfrak{g} \otimes (\wedge^{k-1}\theta)$-valued function on G. Here we identify the tangent bundle TG with $G \times \mathfrak{g}$ by left translations. By skew symmetrization, we thus obtain a $\mathfrak{g} \wedge (\wedge^{k-1}\theta)$-valued function on G, denoted by $\widehat{\delta}$. More explicitly, we have

$$(16) \qquad \partial^{\text{gpd}}(f)|_g = (-1)^{k-1} \iota_{(L_g^* \mathrm{d}f)} \widehat{\delta}|_g, \qquad \forall\, f \in C^\infty(G),\ g \in G.$$

For $k \geq 1$, let

$$W_k = \left\{ w \in \mathfrak{g} \wedge (\wedge^{k-1}\theta) \text{ s.t. } \iota_{\zeta_1} \iota_{\phi^* \zeta_2} w = -\iota_{\zeta_2} \iota_{\phi^* \zeta_1} w,\ \forall\, \zeta_1, \zeta_2 \in \mathfrak{g}^* \right\}.$$

We have

Lemma 2.12. *The function $\widehat{\delta} : G \to \mathfrak{g} \wedge (\wedge^{k-1}\theta)$ is a group 1-cocycle valued in W_k. Here G acts on $\mathfrak{g}$ by the adjoint action, and on θ by the induced action from the crossed module structure.*

Proof. Proposition 2.6 describes how a multiplicative vector field on a Lie groupoid looks along the base manifold. Now we apply this theorem to the groupoid $G \ltimes \Theta \rightrightarrows G$. Identify TG with $G \times \mathfrak{g}$ via left translations. We write $\mathbf{V}|_g$ for the value of $\mathbf{V}$ at $(g, 1_\Theta)$. For all $g \in G$, we have

$$(17) \quad \mathbf{V}|_g = L_{g*}\left(\frac{\mathbf{1} - e^{-D_\phi}}{D_\phi}\left(\widehat{\delta}|_g \right) \right)$$

$$= L_{g*}\left(\widehat{\delta}|_g - \tfrac{1}{2} D_\phi \widehat{\delta}|_g + \frac{1}{3!} D_\phi^2 \widehat{\delta}|_g + \cdots + \frac{(-1)^{k-1}}{k!} D_\phi^{k-1} \widehat{\delta}|_g \right).$$

Here $D_\phi : \wedge^\bullet(\mathfrak{g} \ltimes \theta) \to \wedge^\bullet(\mathfrak{g} \ltimes \theta)$ is a degree-0 derivation of the exterior algebra $\wedge^\bullet(\mathfrak{g} \ltimes \theta)$ such that $D_\phi(x + u) = \phi(u)$, $\forall\, x \in \mathfrak{g}$, $u \in \theta$, and, by abuse of notation, L_{g*} denotes the tangent map of the left translation by $(g, 1_\Theta)$ on the group $G \ltimes \Theta$.

Since $\mathbf{V}$ is multiplicative with respect to the group structure on $G \ltimes \Theta$, it follows that

$$\mathbf{V}|_{gh} = L_{g*}\mathbf{V}|_h + R_{h*}\mathbf{V}|_g,$$

where R_{h*} denotes the tangent map of the right translation by $(h, 1_\Theta)$ in the group $G \ltimes \Theta$. Substituting Eq. (17) into the equation above, we see that $\widehat{\delta}$ is indeed a Lie group 1-cocycle.

Moreover, Proposition 2.6 implies that

$$(18) \qquad \iota_{\zeta_1}\iota_{\phi^*\zeta_2}\big(\widehat{\delta}|_g\big) = -\iota_{\zeta_2}\iota_{\phi^*\zeta_1}\big(\widehat{\delta}|_g\big), \quad \forall \zeta_1, \zeta_2 \in \mathfrak{g}^*.$$

As a consequence, $\widehat{\delta}$ takes values in W_k. This concludes the proof. q.e.d.

Taking the derivative of $\widehat{\delta}$ at the unit:

$$(19) \qquad\qquad \delta(x) = -\left.\frac{d}{dt}\right|_{t=0} \widehat{\delta}|_{\exp tx}, \quad \forall x \in \mathfrak{g},$$

we obtain the following

Corollary 2.13. *Any multiplicative k-vector field on a Lie 2-group $G \ltimes \Theta$ induces a Lie algebra 1-cocycle $\delta : \mathfrak{g} \to \mathfrak{g} \wedge (\wedge^{k-1}\theta)$.*

Lemma 2.14. *Identify Θ with the Lie subgroup $\{1_G\} \times \Theta$ of $G \ltimes \Theta$, where 1_G is the unit element of G. Then any multiplicative k-vector field $\mathbf{V}$ ($k \geq 1$) is tangent to Θ, and therefore defines a multiplicative k-vector field $\mathbf{V}|_\Theta$ on Θ.*

Proof. Let i denote the inverse map of the groupoid $G \ltimes \Theta \rightrightarrows G$, as described in the proof of Proposition 2.2, i.e. $i(g, \alpha) = (g\Phi(\alpha), \alpha^{-1})$. It is clear that $i_*\mathbf{V} = (-1)^{k+1}\mathbf{V}$, since $\mathbf{V}$ is multiplicative. To prove the lemma, it suffices to prove that, for any function $f \in C^\infty(G)$, $[\mathbf{V}, \mathbf{s}^*f]|_\Theta = 0$. For all $\alpha \in \Theta \subset G \ltimes \Theta$, we have

$$i_*\big([\mathbf{V}, \mathbf{s}^*f]|_\alpha\big) = (-1)^{k+1}[\mathbf{V}, \mathbf{t}^*f]|_{i(\alpha)} = (-1)^{k+1}\big(\overleftarrow{\partial^{\mathrm{gpd}}(f)}\big)|_{i(\alpha)}$$
$$= (-1)^{k+1}L_{i(\alpha)*}\big(\partial^{\mathrm{gpd}}(f)|_{\mathbf{t}\circ i(\alpha)}\big) = (-1)^{k+1}L_{i(\alpha)*}\big(\partial^{\mathrm{gpd}}(f)|_{1_G}\big) = 0.$$

Here 1_G is the unit element of G, and L stands for the left translations with respect to the groupoid structure. The fact that $\partial^{\mathrm{gpd}}(f)|_{1_G} = 0$ is due to Eq. (16) and Lemma 2.12.   q.e.d.

As an immediate consequence, the infinitesimal of $\mathbf{V}|_\Theta$ gives rise to a Lie algebra 1-cocycle

$$(20) \qquad\qquad\qquad \omega : \theta \to \wedge^k\theta.$$

The pair (δ, ω) as defined in Eqs. (19) and (20) constitutes the *infinitesimal data of* $\mathbf{V}$.

2.4.3. Compatibility conditions. This section is devoted to exploring the compatibility condition between the infinitesimal data ω and δ. The main theorem is the following:

Theorem 2.15. *Let $(\Theta \xrightarrow{\Phi} G)$ be a crossed module of Lie groups. A multiplicative k-vector field $\mathbf{V}$ on the Lie 2-group $G \ltimes \Theta$ associated to this crossed module determines a pair of linear maps*

$$\omega : \theta \to \wedge^k \theta,$$

$$\delta : \mathfrak{g} \to \mathfrak{g} \wedge (\wedge^{k-1}\theta)$$

which satisfy the following three properties:

ID1: $D_\phi \circ \omega = \delta \circ \phi$, *i.e. the diagram*

$$\begin{array}{ccc} \theta & \xrightarrow{\phi} & \mathfrak{g} \\ \downarrow{\omega} & & \downarrow{\delta} \\ \wedge^k(\mathfrak{g} \ltimes \theta) & \xrightarrow{D_\phi} & \wedge^k(\mathfrak{g} \ltimes \theta) \end{array}$$

commutes;

ID2: δ *is a Lie algebra 1-cocycle valued in W_k;*

ID3: *for all $x \in \mathfrak{g}$ and $u \in \theta$,*

$$x \triangleright \big(\omega(u)\big) - \omega(x \triangleright u) = \mathrm{pr}_{\wedge^k \theta}\big([u, \delta(x)]\big),$$

where the bracket is taken in $\mathfrak{g} \ltimes \theta$.

First of all, we prove that the k-differentials of $\mathbf{V}$ with respect to both the groupoid and the group structures can be expressed in terms of the infinitesimal data (ω, δ). Since $A \cong G \times \theta$, ∂^{gpd} is completely determined by two $\mathbb{R}$-linear operators: $\partial^{\mathrm{gpd}} : C^\infty(G) \to C^\infty(G, \wedge^{k-1}\theta)$ and $\partial^{\mathrm{gpd}} : C^\infty(G, \theta) \to C^\infty(G, \wedge^k \theta)$. The latter is determined by its value on constant functions due to the Leibniz rule.

Let $\partial^{\mathrm{gp}} : \mathfrak{g} \ltimes \theta \to \wedge^k(\mathfrak{g} \ltimes \theta)$ be the k-differential with respect to the group structure on $G \ltimes \Theta$.

Proposition 2.16. *The map $\omega : \theta \to \wedge^k \theta$ satisfies*

$$(21) \qquad\qquad \omega(u) = \partial^{\mathrm{gp}}(u) = \partial^{\mathrm{gpd}}(u), \quad \forall u \in \theta.$$

Proof. Every $u \in \theta \subset \mathfrak{g} \ltimes \theta$ (considered as a constant section of the Lie algebroid $A \cong G \times \theta$) determines two vector fields on $G \ltimes \Theta$: a vector field $\overleftarrow{u}^{\,\mathrm{gp}}$ invariant under left translations relative to the group structure and a vector field $\overleftarrow{u}$ invariant under left translations relative to the groupoid structure $G \ltimes \Theta \rightrightarrows G$. It is simple to see that

$$(22) \qquad\qquad\qquad \overleftarrow{u}^{\,\mathrm{gp}} = \overleftarrow{u}.$$

Therefore,

$$\overleftarrow{\partial^{\mathrm{gp}}(u)} = \big(\overleftarrow{\partial^{\mathrm{gp}}(u)}\big)^{\mathrm{gp}} = [\mathbf{V}, \overleftarrow{u}^{\,\mathrm{gp}}] = [\mathbf{V}, \overleftarrow{u}] = \overleftarrow{\partial^{\mathrm{gpd}}(u)}.$$

By definition, we have $\omega(u) = \partial^{\mathrm{gp}}(u)$. The conclusion follows.   q.e.d.

Proposition 2.17. *The k-differential $\partial^{\mathrm{gp}} : \mathfrak{g} \ltimes \theta \to \wedge^k(\mathfrak{g} \ltimes \theta)$ satisfies*

$$\partial^{\mathrm{gp}}(u) = \omega(u), \quad \forall u \in \theta,$$

$$\partial^{\mathrm{gp}}(x) = \frac{\mathbf{1} - e^{-D_\phi}}{D_\phi}\big(\delta(x)\big), \quad \forall x \in \mathfrak{g}.$$

Proof. It remains to prove the second equation, which follows from a direct verification by applying Lemma 2.8 (2) and Eq. (17).   q.e.d.

Proof of Theorem 2.15. According to Lemma 2.12, **ID2** holds. It suffices to prove **ID1** and **ID3**.

Consider the k-differential $\partial^{\mathrm{gpd}} : \Gamma(\wedge^\bullet A) \to \Gamma(\wedge^{\bullet+k-1} A)$ induced by **V**. For any $u \in \theta$, $f \in C^\infty(G)$, we have

(23) $$\partial^{\mathrm{gpd}}[u, f] = [\partial^{\mathrm{gpd}}(u), f] + [u, \partial^{\mathrm{gpd}}(f)].$$

Next we evaluate both sides of Eq. (23) at $e = \mathbf{1}_G$. Since $\widehat{\delta}$ is a Lie group 1-cocycle according to Lemma 2.12, we have $\widehat{\delta}|_e = 0$. It thus follows from Eq. (16) that $\partial^{\mathrm{gpd}}[u, f]|_e = 0$. On the other hand, we have

$$[\partial^{\mathrm{gpd}}(u), f]|_e = [\omega(u), f]|_e = (-1)^{k-1}\iota_{\phi^*(\mathrm{d}f)}\omega(u)$$
$$= (-1)^{k-1}\iota_{\mathrm{d}f}\big((D_\phi \circ \omega)(u)\big).$$

Here we have used Proposition 2.16 and the equality $\iota_{\phi^*\xi} = \iota_\xi \circ D_\phi$ in $\mathrm{Hom}(\wedge^k\theta, \wedge^{k-1}\theta)$ valid for all $\xi \in \mathfrak{g}^*$. Moreover, Eq. (16) implies that $\partial^{\mathrm{gpd}}(f)|_e = 0$. Since the Lie algebroid A is the transformation Lie algebroid $G \rtimes \theta$, we have

$$[u, \partial^{\mathrm{gpd}}(f)]|_e = \frac{d}{dt}\bigg|_{t=0}\big(\partial^{\mathrm{gpd}}(f)|_{\exp t\phi(u)}\big)$$
$$= (-1)^{k-1}\frac{d}{dt}\bigg|_{t=0}\Big(\big(L^*_{\exp t\phi(u)}(\mathrm{d}f)\big) \lrcorner \widehat{\delta}|_{\exp t\phi(u)}\Big)$$
$$= -(-1)^{k-1}\iota_{\mathrm{d}f}\delta(\phi(u)).$$

Here $\partial^{\mathrm{gpd}}(f)$ is considered as a $(\wedge^{k-1}\theta)$-valued function on G, and the first equality follows from the Leibniz rule of the Lie algebroid axiom and the identity $\partial^{\mathrm{gpd}}(f)|_e = 0$. Hence **ID1** follows immediately from Eq. (23).

On the other hand, the k-differential ∂^{gp} satisfies

$$\partial^{\mathrm{gp}}[x, u] = [\partial^{\mathrm{gp}}(x), u] + [x, \partial^{\mathrm{gp}}(u)], \quad \forall x \in \mathfrak{g}, u \in \theta,$$

where the brackets stand for the Lie algebra bracket on $\mathfrak{g} \ltimes \theta$. Applying Proposition 2.17, and comparing the $\wedge^k\theta$-terms of both sides of the equation above, **ID3** follows immediately.   q.e.d.

Proposition 2.18. *The map $\omega : \theta \to \wedge^k \theta$ is a Lie algebra 1-cocycle, i.e.*

$$\omega[u,v] = [\omega(u), v] + [u, \omega(v)], \quad \forall u, v \in \theta.$$

Proof. Using **ID1** and **ID3** from Theorem 2.15, we have

$$\begin{aligned}
\omega[u,v] &= \omega(\phi(u) \triangleright v) \\
&= \phi(u) \triangleright \omega(v) - \mathrm{pr}_{\wedge^k \theta}([v, \delta(\phi(u))]) \\
&= [u, \omega(v)] - \mathrm{pr}_{\wedge^k \theta}([v, D_\phi(\omega(u))]) \\
&= [u, \omega(v)] - [v, \omega(u)].
\end{aligned}$$

Here, in the last equality, we have used the identity

$$\mathrm{pr}_{\wedge^k \theta}([v, D_\phi(\zeta)]) = [v, \zeta], \quad \forall \zeta \in \wedge^k \theta,$$

which can be verified by a straightforward computation.   q.e.d.

Now we extend the two maps ω and δ to degree-$(k-1)$ derivations (which we denote by the same symbols by abuse of notation) on the exterior algebra $\wedge^\bullet (\mathfrak{g} \ltimes \theta)$ by setting $\omega(\mathfrak{g}) = 0$ and $\delta(\theta) = 0$.

Proposition 2.19. *Assume that $\mathbf{V}_1$ and $\mathbf{V}_2$ are multiplicative k_1- and k_2-vector fields on the Lie 2-group $G \ltimes \Theta$. Let (ω_1, δ_1) and (ω_2, δ_2) be their corresponding infinitesimals. Then the infinitesimal (ω_3, δ_3) of $\mathbf{V}_3 = [\mathbf{V}_1, \mathbf{V}_2]$ is given by the following formulae:*

(24) $\omega_3 = \omega_1 \circ \omega_2 - (-1)^{(k_1-1)(k_2-1)} \omega_2 \circ \omega_1,$

(25) $\delta_3 = (\delta_1 + \omega_1) \circ \delta_2 - (-1)^{(k_1-1)(k_2-1)}(\delta_2 + \omega_2) \circ \delta_1.$

Proof. Note that

$$\begin{aligned}
\partial_{\mathbf{V}_3}^{\mathrm{gp}} = \partial_{[\mathbf{V}_1, \mathbf{V}_2]}^{\mathrm{gp}} &= \lfloor \partial_{\mathbf{V}_1}^{\mathrm{gp}}, \partial_{\mathbf{V}_2}^{\mathrm{gp}} \rfloor \\
&= \partial_{\mathbf{V}_1}^{\mathrm{gp}} \circ \partial_{\mathbf{V}_2}^{\mathrm{gp}} - (-1)^{(k_1-1)(k_2-1)} \partial_{\mathbf{V}_2}^{\mathrm{gp}} \circ \partial_{\mathbf{V}_1}^{\mathrm{gp}}.
\end{aligned}$$

Hence Eqs. (24) and (25) follow immediately from Proposition 2.17. q.e.d.

By $\mathcal{A}_k$ ($k \geq 1$) we denote the space of pairs (ω, δ) of linear maps $\omega : \theta \to \wedge^k \theta$ and $\delta : \mathfrak{g} \to \mathfrak{g} \wedge (\wedge^{k-1} \theta)$ satisfying the three properties **ID1**, **ID2**, and **ID3** listed in Theorem 2.15. By $\mathcal{A}_0$, we denote the space of all pairs (ω, δ), where δ is the trivial map $\mathfrak{g} \to 0$ and $\omega : \theta \to \mathbb{R}$ satisfies $\omega(x \triangleright u) = 0$, for all $x \in \mathfrak{g}$ and $u \in \theta$.

Corollary 2.20. *When endowed with the bracket defined by Eqs. (24) and (25), the direct sum $\bigoplus_{k \geq 0} \mathcal{A}_k$ is a graded Lie algebra.*

2.5. The universal lifting theorem.

2.5.1. Statement of the main theorems. The converse of Theorem 2.15 holds as well.

Theorem 2.21. *Let $G \ltimes \Theta$ be a Lie 2-group, where both G and Θ are connected and simply connected. Given any $(\omega, \delta) \in \mathcal{A}_k$, there exists a unique multiplicative k-vector field $\mathbf{V}$ on $G \ltimes \Theta$, whose infinitesimal corresponds to (ω, δ).*

An immediate consequence is the following main result of the paper:

Theorem 2.22 (universal lifting theorem)**.** *Let $G \ltimes \Theta$ be a Lie 2-group, where both G and Θ are connected and simply connected Lie groups with Lie algebras $\mathfrak{g}$ and θ, respectively. There is a canonical isomorphism of graded Lie algebras*

$$\bigoplus_{k \geq 0} \mathfrak{X}^k_{\mathrm{mult}}(G \ltimes \Theta) \cong \bigoplus_{k \geq 0} \mathcal{A}_k.$$

2.5.2. From infinitesimal data to k-differentials of the Lie algebra $\mathfrak{g} \ltimes \theta$. The $k = 0$ case is obvious, so we will assume $k \geq 1$ below. We will divide the proof of Theorem 2.21 into several steps.

First, since δ is a Lie algebra 1-cocycle, it integrates to a Lie group 1-cocycle

$$\widehat{\delta} : G \to \mathfrak{g} \wedge (\wedge^{k-1}\theta)$$

such that

$$(26) \qquad \delta(x) = - \left. \frac{d}{dt} \right|_{t=0} \widehat{\delta}|_{\exp tx}, \quad \forall x \in \mathfrak{g}.$$

Clearly the map $\widehat{\delta}$ takes values in W_k.

Let $\widehat{\omega} : \Theta \to \wedge^k \theta$ be the group 1-cocycle integrating ω. As a direct consequence of property **ID1** from Theorem 2.15, we have

$$(27) \qquad \widehat{\delta}|_{\Phi(\alpha)} = D_\phi(\widehat{\omega}|_\alpha), \quad \forall g \in G, \alpha \in \Theta.$$

Define a linear map $\partial : \mathfrak{g} \ltimes \theta \to \wedge^k(\mathfrak{g} \ltimes \theta)$ by

$$(28) \qquad \begin{cases} \partial(u) = \omega(u), & \forall u \in \theta; \\[2mm] \partial(x) = \dfrac{\mathbf{1} - e^{-D_\phi}}{D_\phi} \, \delta(x), & \forall x \in \mathfrak{g}. \end{cases}$$

Proposition 2.23. *The operator ∂ defines a Lie algebra k-differential for the Lie algebra $\mathfrak{g} \ltimes \theta$.*

Proof. It suffices to prove that ∂ is a Lie algebra 1-cocycle for the Lie algebra $\mathfrak{g} \ltimes \theta$. In fact, Proposition 2.18 implies that

$$\partial[u, v] = [\partial u, v] + [u, \partial v], \quad \forall u, v \in \theta.$$

On the other hand, it follows from a direct verification that

$$D_\phi[x, w] = [x, D_\phi(w)], \quad \forall x \in \mathfrak{g}, w \in \wedge^\bullet(\mathfrak{g} \ltimes \theta).$$

As a consequence, applying the operator $\frac{1-e^{-D_\phi}}{D_\phi}$ to both sides of the equation:

$$\delta[x,y] = [\delta x, y] + [x, \delta y],$$

we obtain

$$\partial[x,y] = [\partial x, y] + [x, \partial y], \quad \forall x, y \in \mathfrak{g}.$$

It remains to prove the identity

$$\partial[x,u] = [\partial x, u] + [x, \partial u], \quad \forall x \in \mathfrak{g}, u \in \theta.$$

Since $\partial[x,u] - [x, \partial u] = \mathrm{pr}_{\wedge^k \theta}([\delta(x), u])$ according to property **ID3** from Theorem 2.15, it suffices to prove that

$$[\partial x, u] = \mathrm{pr}_{\wedge^k \theta}([\delta(x), u]).$$

Now

$$
\begin{aligned}
[\partial x, u] &= \left[\sum_{i=0}^{k-1} \frac{(-1)^i}{(i+1)!} D_\phi^i(\delta(x)), u \right] \\
&= \sum_{j=0}^{k-1} \mathrm{pr}_{\wedge^j \mathfrak{g} \wedge (\wedge^{k-j} \theta)} \left[\sum_{i=0}^{k-1} \frac{(-1)^i}{(i+1)!} D_\phi^i(\delta(x)), u \right].
\end{aligned}
$$

Using the definitions of δ and D_ϕ, we obtain the following identity:

$$\mathrm{pr}_{\wedge^k \theta} \left[\sum_{i=0}^{k-1} \frac{(-1)^i}{(i+1)!} D_\phi^i(\delta(x)), u \right] = \mathrm{pr}_{\wedge^k \theta}[\delta(x), u].$$

For $1 \le j \le k-1$, the sum $\mathrm{pr}_{\wedge^j \mathfrak{g} \wedge (\wedge^{k-j} \theta)} \left[\sum_{i=0}^{k-1} \frac{(-1)^i}{(i+1)!} D_\phi^i(\delta(x)), u \right]$ contains only the two terms

$$\mathrm{pr}_{\wedge^j \mathfrak{g} \wedge (\wedge^{k-j} \theta)} \left(\left[\frac{(-1)^{j-1}}{j!} D_\phi^{j-1}(\delta(x)), u \right] + \left[\frac{(-1)^j}{(j+1)!} D_\phi^j(\delta(x)), u \right] \right)$$

and thus reduces to

$$(29) \quad \frac{(-1)^{j-1}}{j!} \mathrm{pr}_{\wedge^j \mathfrak{g} \wedge (\wedge^{k-j} \theta)} \left(\left[D_\phi^{j-1}(\delta(x)), u \right] - \frac{1}{j+1} \left[D_\phi^j(\delta(x)), u \right] \right).$$

To prove that it vanishes, we need a couple of lemmas.

Lemma 2.24. *For any $v \in \wedge^{k-1} \theta$ and $1 \le l \le k-1$, we have*

$$\mathrm{pr}_{\wedge^{l-1} \mathfrak{g} \wedge (\wedge^{k-l} \theta)} \left(\left[l D_\phi^{l-1}(v) - D_\phi^l(v), u \right] \right) = 0, \quad \forall u \in \theta.$$

Proof. This follows from a straightforward computation, which is left to the reader. q.e.d.

From Lemma 2.5, it follows that, for any $w \in W_k$ and $j \ge 1$, we have

$$(30) \quad \iota_{\phi^* \zeta}(D_\phi^{j-1} w) = D_\phi^j(\iota_\zeta w) = \frac{1}{j+1} \iota_\zeta(D_\phi^j w), \quad \forall \zeta \in \mathfrak{g}^*.$$

Now we return to the proof of Proposition 2.23. It remains to prove that (29) vanishes. Indeed, for any $\zeta \in \mathfrak{g}^*$, we have

$$\iota_\zeta \operatorname{pr}_{\wedge^j \mathfrak{g} \wedge (\wedge^{k-j}\theta)} \left(\left[D_\phi^{j-1}(\delta(x)), u \right] - \frac{1}{j+1} \left[D_\phi^j(\delta(x)), u \right] \right)$$

$$= \operatorname{pr}_{\wedge^{j-1} \mathfrak{g} \wedge (\wedge^{k-j}\theta)} \left(\left[\iota_\zeta D_\phi^{j-1}(\delta(x)), u \right] - \frac{1}{j+1} \left[\iota_\zeta D_\phi^j(\delta(x)), u \right] \right)$$

$$= \operatorname{pr}_{\wedge^{j-1} \mathfrak{g} \wedge (\wedge^{k-j}\theta)} \left(\left[j D_\phi^{j-1}(\iota_\zeta \delta(x)) - D_\phi^j(\iota_\zeta \delta(x)), u \right] \right) = 0.$$

Here in the last two steps, we have used Eq. (30) and Lemma 2.24. This concludes the proof of the proposition.   q.e.d.

2.5.3. Multiplicative with respect to the groupoid structure. As a consequence of Proposition 2.23, we obtain a k-vector field Σ on $G \ltimes \Theta$, which is multiplicative with respect to the group structure and whose induced k-differential with respect to the group structure on $G \ltimes \Theta$ is ∂. Now we need to prove that Σ is also multiplicative with respect to the groupoid structure on $G \ltimes \Theta \rightrightarrows G$. For this purpose, we need an explicit expression of Σ. Since Σ is multiplicative with respect to the group structure on $G \ltimes \Theta$, it suffices to find an explicit expression of Σ along the subgroups $\{\mathbf{1}_G\} \times \Theta$ and $G \times \{\mathbf{1}_\Theta\}$, respectively. The next two lemmas are devoted to this investigation.

The following lemma is immediate.

Lemma 2.25. *Identify Θ with the subgroup $\{\mathbf{1}_G\} \times \Theta$ of $G \ltimes \Theta$. Then Σ is tangent to Θ and therefore induces a multiplicative k-vector field $\Sigma|_\Theta$ on Θ. Moreover, $\Sigma|_\alpha = L_{\alpha*}(\widehat{\omega}|_\alpha)$, for all $\alpha \in \Theta$.*

Next, we have

Lemma 2.26. *Along the Lie subgroup $G \cong G \times \{\mathbf{1}_\Theta\} \subset G \times \Theta$, Σ can be explicitly expressed by the following formula:*

$$(31) \qquad \Sigma|_g = L_{g*}\left(\frac{\mathbf{1} - e^{-D_\phi}}{D_\phi} \left(\widehat{\delta}|_g \right) \right), \quad \forall g \in G.$$

Moreover, for any $\zeta \in \mathfrak{g}^*$, we have

$$(32) \qquad \iota_{(L_{g-1}^* \zeta + \phi^* \zeta)}(\Sigma|_g) = \iota_\zeta(\widehat{\delta}|_g).$$

Here $L_{g^{-1}}^* \zeta \in T_g^* G$ and $\phi^* \zeta \in \theta^* = T_{\mathbf{1}_\Theta}^* \Theta$.

Proof. Eq. (31) follows from integrating $\partial(x)$ in Eq. (28). To prove Eq. (32), according to Eq. (30), we have

$$\iota_{\phi^* \zeta}\left(\frac{(-1)^{j-1}}{j!} D_\phi^{j-1}\left(\widehat{\delta}|_g \right) \right) = \frac{(-1)^{j-1}}{j!} \frac{1}{j+1} \iota_\zeta D_\phi^j\left(\widehat{\delta}|_g \right) = -\iota_\zeta \left(\frac{(-1)^j}{(j+1)!} D_\phi^j\left(\widehat{\delta}|_g \right) \right).$$

The conclusion thus follows immediately by using Eq. (31).   q.e.d.

Proposition 2.27. *The k-vector field Σ is also multiplicative with respect to the groupoid structure on $G \ltimes \Theta \rightrightarrows G$.*

Proof. We divide the proof into three steps.

(1) *The base manifold G is coisotropic with respect to Σ.*

For every $g \in G$, we have $T_{(g,\mathbf{1}_\Theta)}(G \ltimes \Theta) \cong T_g G \oplus \theta$. The conormal space of $T_g G$ can thus be canonically identified with θ^*. It follows that G is coisotropic with respect to Σ since $\Sigma|_g$ does not contain any $(\wedge^k \theta)$-components according to Lemma 2.26.

(2) *For every $\xi \in \Omega^1(G)$, $\iota_{\mathbf{t}^*(\xi)} \Sigma$ is left-invariant with respect to the groupoid structure.*

For every $(g, \alpha) \in G \ltimes \Theta$, we identify $T_{(g,\alpha)}(G \ltimes \Theta)$ with $T_g G \oplus T_\alpha \Theta$. Since $\widehat{\omega}|_\alpha$ takes values in $\wedge^k \theta$, we have

$$\Sigma|_{(g,\alpha)} = \Sigma|_{g \diamond \alpha} = L_{g*}(\Sigma|_\alpha) + R_{\alpha*}(\Sigma|_g) = L_{g*}L_{\alpha*}(\widehat{\omega}|_\alpha) + R_{\alpha*}(\Sigma|_g).$$

Let m denote the point $\mathbf{t}(g, \alpha) = g\Phi(\alpha)$ of G. Choose a $\zeta \in \mathfrak{g}^*$ and set $\xi|_m = L^*_{m^{-1}}\zeta \in T^*_m G$. We have, for all $u \in \theta$,

$$(33) \qquad\qquad \iota_{\mathbf{t}^*(\xi)} L_{g*} L_{\alpha*} u = \iota_{\phi^*(\zeta)} u,$$

which follows from the identity

$$(\mathbf{t} \circ L_g \circ L_\alpha)(\mathbf{1}_G, \beta) = g\Phi(\alpha)\Phi(\beta) = (L_{g\Phi(\alpha)} \circ \Phi)(\beta), \quad \forall \beta \in \Theta.$$

Also note that, for all $V \in T_{(g,\mathbf{1}_\Theta)}(G \ltimes \Theta)$,

$$(34) \qquad \iota_{\mathbf{t}^*(\xi)} R_{\alpha*} V = \iota_{\left(L^*_{g^{-1}} \mathrm{Ad}^*_{\Phi(\alpha)}\zeta\right)} V + \iota_{\left(\phi^* \mathrm{Ad}^*_{\Phi(\alpha)}\zeta\right)} V.$$

To prove this identity, we observe that

$$\left(\mathbf{t} \circ R_\alpha \circ L_g\right)(h, \beta) = gh\Phi(\beta)\Phi(\alpha), \quad \forall h \in G, \beta \in \Theta,$$

which implies that

$$\left(\mathbf{t}_* \circ R_{\alpha*} \circ L_{g*}\right)(x, u) = \left(L_{g\Phi(\alpha)*} \circ \mathrm{Ad}_{\Phi(\alpha^{-1})}\right)(x + \phi(u)), \quad \forall x \in \mathfrak{g}, u \in \theta.$$

Thus Eq. (34) follows from a straightforward verification.

Applying Eq. (33), we obtain

$$\begin{aligned}
\iota_{\mathbf{t}^*(\xi)} L_{g*}(\Sigma|_\alpha) &= \iota_{\mathbf{t}^*(\xi)}\left(L_{g*}L_{\alpha*}(\widehat{\omega}|_\alpha)\right) \\
&= L_{g*}L_{\alpha*}(\iota_{\phi^*\zeta}\widehat{\omega}|_\alpha) \\
&= L_{g*}L_{\alpha*}\left(\iota_\zeta D_\phi \widehat{\omega}|_\alpha\right) \qquad\qquad \text{(by Eq. (27))} \\
&= L_{g*}L_{\alpha*}\left(\iota_\zeta \widehat{\delta}|_{\Phi(\alpha)}\right).
\end{aligned}$$

Using Eq. (34) and Lemma 2.26 we have

$$\begin{aligned}
\iota_{\mathbf{t}^*(\xi)} R_{\alpha*}(\Sigma|_g) &= R_{\alpha*}\left(\iota_{L^*_{g^{-1}}(\mathrm{Ad}^*_{\Phi(\alpha)}\zeta)}(\Sigma|_g) + \iota_{\phi^* \mathrm{Ad}^*_{\Phi(\alpha)}\zeta}(\Sigma|_g)\right) \\
&= L_{g*}R_{\alpha*}\left(\iota_{\mathrm{Ad}^*_{\Phi(\alpha)}\zeta}\,\widehat{\delta}|_g\right).
\end{aligned}$$

Therefore we have

$$
\begin{aligned}
(\iota_{\mathbf{t}^*(\xi)}\Sigma)|_{(g,\alpha)} &= \iota_{\mathbf{t}^*(\xi)}L_{g*}(\Sigma|_\alpha) + \iota_{\mathbf{t}^*(\xi)}R_{\alpha*}(\Sigma|_g) \\
&= L_{g*}L_{\alpha*}\big(\iota_\zeta\widehat{\delta}|_{\Phi(\alpha)}\big) + L_{g*}R_{\alpha*}\big(\iota_{\mathrm{Ad}^*_{\Phi(\alpha)}\zeta}\widehat{\delta}|_g\big) \\
&= L_{g*}L_{\alpha*}\Big(\iota_\zeta\widehat{\delta}|_{\Phi(\alpha)} + \mathrm{Ad}_{\alpha^{-1}}\big(\iota_{\mathrm{Ad}^*_{\Phi(\alpha)}\zeta}\widehat{\delta}|_g\big)\Big) \\
&= L_{g*}L_{\alpha*}\iota_\zeta\Big(\widehat{\delta}|_{\Phi(\alpha)} + (\Phi(\alpha^{-1}))_*\widehat{\delta}|_g\big)\Big) \\
&= L_{g*}L_{\alpha*}\iota_\zeta\widehat{\delta}|_{g\Phi(\alpha)},
\end{aligned}
$$

where, in the last step, we used the fact that $\widehat{\delta}$ is a Lie group 1-cocycle. In particular, we have

$$
(\iota_{\mathbf{t}^*(\xi)}\Sigma)|_{g\Phi(\alpha)} = (\iota_{\mathbf{t}^*(\xi)}\Sigma)|_{(g\Phi(\alpha),\mathbf{1}_\Theta)} = L_{g\Phi(\alpha)*}\iota_\zeta\widehat{\delta}|_{g\Phi(\alpha)}
$$

and therefore

$$
(\iota_{\mathbf{t}^*(\xi)}\Sigma)|_{(g,\alpha)} = L^{\mathrm{gpd}}_{(g,\alpha)*}(\iota_{\mathbf{t}^*(\xi)}\Sigma)|_{g\Phi(\alpha)}.
$$

This proves that $\iota_{\mathbf{t}^*(\xi)}\Sigma$ is indeed left-invariant with respect to the groupoid structure.

(3) *For every $X \in \mathbf{\Gamma}(A)$, $[\Sigma, \overleftarrow{X}]$ is left-invariant with respect to the groupoid structure.*

It suffices to consider $X = fu$, where $f \in C^\infty(G)$ and $u \in \theta$ is considered as a constant section of $A \cong G \times \theta$. Then

$$
\begin{aligned}
[\Sigma, \overleftarrow{X}] &= [\Sigma, (\mathbf{t}^*f)\overleftarrow{u}] \\
&= (\mathbf{t}^*f)[\Sigma, \overleftarrow{u}] + [\Sigma, \mathbf{t}^*f] \wedge \overleftarrow{u} && \text{(by Eq. (22))} \\
&= (\mathbf{t}^*f)[\Sigma, \overleftarrow{u}^{\mathrm{gp}}] + (-1)^{k-1}\iota_{\mathbf{t}^*\mathrm{d}f}\Sigma \wedge \overleftarrow{u} \\
&= (\mathbf{t}^*f)\overleftarrow{(\partial^{\mathrm{gp}}_\Sigma(u))}^{\mathrm{gp}} + (-1)^{k-1}\iota_{\mathbf{t}^*\mathrm{d}f}\Sigma \wedge \overleftarrow{u} && \text{(by Eq. (22))} \\
&= (\mathbf{t}^*f)\overleftarrow{\partial^{\mathrm{gp}}_\Sigma(u)} + (-1)^{k-1}\iota_{\mathbf{t}^*\mathrm{d}f}\Sigma \wedge \overleftarrow{u},
\end{aligned}
$$

which is clearly left-invariant according to Claim **(2)**.

Finally, Claims **(1)**, **(2)**, and **(3)** imply that Σ is indeed multiplicative by Lemma 2.3. q.e.d.

Proof of Theorem 2.21. From the infinitesimal data (ω, δ), we have constructed a multiplicative k-vector field Σ on the 2-group $G \ltimes \Theta$. Assume that the infinitesimal data corresponding to Σ is (ω', δ'). Proposition 2.17 implies that ω' and δ' can be recovered from ∂^{gp}, the k-differential of Σ with respect to the group structure on $G \ltimes \Theta$, by the following relations:

$$
\begin{cases}
\partial^{\mathrm{gp}}(u) = \omega'(u), \\
\partial^{\mathrm{gp}}(x) = \dfrac{\mathbf{1} - e^{-D_\phi}}{D_\phi}\big(\delta'(x)\big).
\end{cases}
$$

Since ∂ is defined by Eq. (28) and Σ integrates ∂, ∂^{gp} must coincide with ∂. Hence it follows that $\omega' = \omega$ and $\delta' = \delta$.

Since both G and Θ are connected and simply connected, so must be $G \ltimes \Theta$. Hence the multiplicative vector field Σ that integrates ∂ must be unique.   q.e.d.

3. Quasi-Poisson Lie 2-groups

Throughout this section, $(\Theta \overset{\Phi}{\to} G)$ denotes a Lie group crossed module, and $G \ltimes \Theta$ its associated Lie 2-group. By $(\theta \overset{\phi}{\to} \mathfrak{g})$, we denote its corresponding Lie algebra crossed module, and by $\mathfrak{g} \ltimes \theta$ the semidirect product Lie algebra.

3.1. Quasi-Poisson Lie 2-groups.

Definition 3.1. A quasi-Poisson structure on a Lie 2-group $G \ltimes \Theta$ is a pair $(\boldsymbol{\Pi}, \widehat{\eta})$, where $\boldsymbol{\Pi} \in \mathfrak{X}^2_{\mathrm{mult}}(G \ltimes \Theta)$ is a multiplicative bivector field, $\widehat{\eta} : G \to \wedge^3 \theta$ is a Lie group 1-cocycle such that

$$\tag{35} \tfrac{1}{2}[\boldsymbol{\Pi}, \boldsymbol{\Pi}] = \overleftarrow{\widehat{\eta}} - \overrightarrow{\widehat{\eta}},$$

and

$$\tag{36} [\boldsymbol{\Pi}, \overleftarrow{\widehat{\eta}}] = 0.$$

Here $\widehat{\eta}$ is considered as a section in $\boldsymbol{\Gamma}(\wedge^3 A)$. When $\widehat{\eta}$ is zero, $\boldsymbol{\Pi}$ defines a Poisson structure on $G \ltimes \Theta$. In this case, we say that $(G \ltimes \Theta, \boldsymbol{\Pi})$ is a Poisson 2-group.

It is clear that $G \ltimes \Theta \rightrightarrows G$ together with $(\boldsymbol{\Pi}, \widehat{\eta})$ is a quasi-Poisson groupoid [**9**].

The main result of this section is the following:

Theorem 3.2. *Any quasi-Poisson Lie 2-group $(\boldsymbol{\Pi}, \widehat{\eta})$ on $G \ltimes \Theta$ naturally induces a quasi-Lie 2-bialgebra.*

Conversely, given a quasi-Lie 2-bialgebra $(\theta, \mathfrak{g}, t)$ as in Definition 1.3, *if both G and Θ are connected and simply connected Lie groups with Lie algebras θ and $\mathfrak{g}$, respectively, then $G \ltimes \Theta$ admits a quasi-Poisson Lie 2-group structure whose infinitesimal is isomorphic to the given quasi-Lie 2-bialgebra.*

The proof is deferred to Section 3.3. In fact, from its proof, it is clear that exactly the same conclusion holds between Poisson Lie 2-groups and Lie 2-bialgebras. Thus, as an immediate consequence, we obtain the following analogue of a classical theorem of Drinfeld in the context of 2-groups.

Corollary 3.3.   1) *There is a one-to-one correspondence between connected and simply connected quasi-Poisson Lie 2-groups and quasi-Lie 2-bialgebras.*

2) *There is a one-to-one correspondence between connected and simply connected Poisson Lie 2-groups and Lie 2-bialgebras.*

3.2. Multiplicative k-vector fields generated by group 1-cocycles.

Lemma 3.4. *For any $u \in \theta$, we have*

$$(37) \qquad \big(\mathrm{Ad}_{(h,\beta)_\diamond^{-1}} \circ (\mathrm{id}_\theta - \phi)\big)(u) = \big((\mathrm{id}_\theta - \phi) \circ h_*^{-1}\big)(u),$$

$$(38) \qquad \mathrm{Ad}_{(h,\beta)_\diamond^{-1}}(u) = \big(\mathrm{Ad}_{\beta^{-1}} \circ h_*^{-1}\big)(u) = \big(h\Phi(\beta)\big)_*^{-1}(u).$$

Here id_θ denotes the identity map on θ.

Proof. A straightforward computation yields that, for all $(g, \alpha) \in G \ltimes \Theta$,

$$\mathrm{Ad}_{(h,\beta)_\diamond^{-1}}(g, \alpha) = (h, \beta)_\diamond^{-1} \diamond (g, \alpha) \diamond (h, \beta)$$

$$= \Big(h^{-1}gh, \big((h^{-1}g^{-1}h) \triangleright \beta^{-1}\big)(h^{-1} \triangleright \alpha)\beta\Big)$$

$$= \Big(\mathrm{Ad}_{h^{-1}}g, \big((\mathrm{Ad}_{h^{-1}}g^{-1}) \triangleright \beta^{-1}\big)(h^{-1} \triangleright \alpha)\beta\Big).$$

In particular, we have

$$\mathrm{Ad}_{(h,\beta)_\diamond^{-1}}(\Phi(\alpha^{-1}), \alpha) = \big(\Phi(h^{-1} \triangleright \alpha^{-1}), h^{-1} \triangleright \alpha\big).$$

Eq. (37) thus follows immediately by taking the tangent map at $\alpha = 1_\Theta$. Similarly, we have

$$\mathrm{Ad}_{(h,\beta)_\diamond^{-1}}(\mathbf{1}_G, \alpha) = \big(\mathbf{1}_G, (h\Phi(\beta))^{-1} \triangleright \alpha\big).$$

Eq. (38) follows by taking the tangent map at $\alpha = 1_\Theta$. q.e.d.

Proposition 3.5. *Let $\widehat{\lambda} : G \to \wedge^l \theta$ be a Lie group 1-cocycle, and $\lambda : \mathfrak{g} \to \wedge^l \theta$ the corresponding Lie algebra 1-cocycle.*

1) *The l-vector field*

$$\mathbf{C}_{\widehat{\lambda}} = \overleftarrow{\widehat{\lambda}} - \overrightarrow{\widehat{\lambda}}$$

on the 2-group $G \ltimes \Theta$ is multiplicative. Here $\widehat{\lambda}$ is considered as a section in $\mathbf{\Gamma}(\wedge^l A)$, and $\overleftarrow{\widehat{\lambda}}$ and $\overrightarrow{\widehat{\lambda}}$, respectively, denote the left- and right-invariant l-vector fields on the groupoid $G \ltimes \Theta \rightrightarrows G$.

2) *The infinitesimal data of $\mathbf{C}_{\widehat{\lambda}}$ is*

$$\omega_\lambda = \lambda \circ \phi : \theta \to \wedge^l \theta,$$

$$\delta_\lambda = D_\phi \circ \lambda : \mathfrak{g} \to \mathfrak{g} \wedge (\wedge^{l-1}\theta).$$

3) *Let $\partial^{\mathrm{gpd}} : \mathbf{\Gamma}(\wedge^\bullet A) \to \mathbf{\Gamma}(\wedge^{\bullet + k - 1}A)$ be the k-differential on the Lie algebroid A induced by a multiplicative k-vector field $\mathbf{V}$ on $G \ltimes \Theta$. Then the section $\widehat{\sigma} = \partial^{\mathrm{gpd}}(\widehat{\lambda}) \in \mathbf{\Gamma}(\wedge^{k+l-1}A)$, considered as a map $G \to \wedge^{k+l-1}\theta$, is a Lie group 1-cocycle. The corresponding Lie algebra 1-cocycle $\sigma : \mathfrak{g} \to \wedge^{k+l-1}\theta$ is*

$$\sigma = \omega \circ \lambda - (-1)^{(k-1)(l-1)}\lambda \circ \delta,$$

where (ω, δ) is the infinitesimal of **V**.

Proof. **1)** It is clear that $\mathbf{C}_{\widehat{\lambda}}$ is multiplicative with respect to the groupoid structure. It suffices to show that $\mathbf{C}_{\widehat{\lambda}}$ is also multiplicative with respect to the group structure on $G \ltimes \Theta$. Define $c : G \ltimes \Theta \to \wedge^l(\mathfrak{g} \ltimes \theta)$ by

$$c|_{(g,\alpha)} = L_{(g,\alpha)_{\diamond}^{-1}*}\big(\mathbf{C}_{\widehat{\lambda}}|_{(g,\alpha)}\big),$$

where L stands for the group left translations. It is well known that $\mathbf{C}_{\widehat{\lambda}}$ is multiplicative with respect to the group structure if and only if c is a group 1-cocycle, i.e.

(39) $$c|_{(g,\alpha)\diamond(h,\beta)} = c|_{(h,\beta)} + \mathrm{Ad}_{(h,\beta)_{\diamond}^{-1}}(c|_{(g,\alpha)}).$$

Now a direct calculation yields

(40) $$c|_{(g,\alpha)} = \widehat{\lambda}|_{g\Phi(\alpha)} - (\mathrm{id}_\theta - \phi)\widehat{\lambda}|_g.$$

Here $(\mathrm{id}_\theta - \phi)$ extends naturally to a map $\wedge^l\theta \to \wedge^l\theta$, i.e.

$$(\mathrm{id}_\theta - \phi)(u_1 \wedge u_2 \wedge \cdots \wedge u_l) = (\mathrm{id}_\theta - \phi)u_1 \wedge (\mathrm{id}_\theta - \phi)u_2 \wedge \cdots \wedge (\mathrm{id}_\theta - \phi)u_l,$$

for all $u_1, \cdots, u_l \in \theta$. Using Eq. (40) and the assumption that $\widehat{\lambda}$ is a 1-cocycle, we have

r.h.s of Eq. (39)
$$= \widehat{\lambda}|_{h\Phi(\beta)} - (\mathrm{id}_\theta - \phi)\widehat{\lambda}|_h + (h\Phi(\beta))_*^{-1}(\widehat{\lambda}|_{g\Phi(\alpha)}) - ((\mathrm{id}_\theta - \phi) \circ h_*^{-1})(\widehat{\lambda}|_g)$$
$$= \widehat{\lambda}|_{g\Phi(\alpha)h\Phi(\beta)} - (\mathrm{id}_\theta - \phi)\widehat{\lambda}|_{gh}$$
$$= c|_{(gh,(h^{-1}\triangleright\alpha)\beta)}$$
$$= \text{l.h.s. of Eq. (39)}.$$

Thus, $\mathbf{C}_{\widehat{\lambda}}$ is indeed multiplicative with respect to the group structure.
2) Let $\partial : \mathfrak{g} \ltimes \theta \to \wedge^l(\mathfrak{g} \ltimes \theta)$ be the l-differential induced by the multiplicative l-vector field $\mathbf{C}_{\widehat{\lambda}}$. According to Lemma 2.8, we have

$$\partial(x + u) = -\left.\frac{d}{dt}\right|_{t=0} c|_{\exp t(x+u)}, \quad \forall x + u \in \mathfrak{g} \ltimes \theta.$$

Assume that $(\omega_\lambda, \delta_\lambda)$ is the infinitesimal data corresponding to $\mathbf{C}_{\widehat{\lambda}}$. According to Proposition 2.17, we have

$$\omega_\lambda(u) = \partial(u) = -\left.\frac{d}{dt}\right|_{t=0}\big(\widehat{\lambda}|_{\Phi(\exp tu)} - (\mathrm{id}_\theta - \phi)\widehat{\lambda}|_{\mathbf{1}_G}\big) = \big(\lambda \circ \phi\big)(u).$$

Moreover,

$$
\begin{aligned}
\delta_\lambda(x) &= \mathrm{pr}_{\mathfrak{g}\wedge(\wedge^{l-1}\theta)}\,\partial(x) \\
&= -\frac{d}{dt}\Big|_{t=0}\mathrm{pr}_{\mathfrak{g}\wedge(\wedge^{l-1}\theta)}\big(\widehat{\lambda}|_{\exp tx} - (\mathrm{id}_\theta-\phi)\widehat{\lambda}|_{\exp tx}\big) \\
&= -\frac{d}{dt}\Big|_{t=0} D_\phi(\widehat{\lambda}|_{\exp tx}) \\
&= \big(D_\phi\circ\lambda\big)(x).
\end{aligned}
$$

Hence it follows that $\delta_\lambda = D_\phi\circ\lambda$.

3) We first prove the following formula:

$$
\tag{41}\qquad \widehat{\sigma}|_g = \omega(\widehat{\lambda}|_g) + (-1)^{(k-1)(l-1)+1}\lambda(\widehat{\delta}_g) - \mathrm{pr}_{\wedge^{k+l-1}\theta}[\widehat{\delta}|_g,\widehat{\lambda}|_g],
$$

where $\widehat{\delta}: G \to \mathfrak{g}\wedge(\wedge^{k-1}\theta)$ is the Lie group 1-cocycle corresponding to δ as in Eq. (26). To prove it, assume that

$$
\widehat{\lambda}|_g = \sum_i f_i(g)u_i \quad\text{and}\quad \widehat{\delta}|_g = \sum_j h_j(g)x_j\wedge w_j, \quad \forall g\in G,
$$

where $f_i, h_j \in C^\infty(G)$, $u_i \in \wedge^l\theta$, $x_j \in \mathfrak{g}$, and $w_j \in \wedge^{k-1}\theta$. According to Eq. (16) and Proposition 2.16, we have

$$
\begin{aligned}
\widehat{\sigma}|_g &= \partial^{\mathrm{gpd}}(\widehat{\lambda})|_g \\
&= \sum_i \Big(f_i(g)\partial^{\mathrm{gpd}}(u_i) + (\partial^{\mathrm{gpd}}f_i)|_g\wedge u_i\Big) \\
&= \sum_i \Big(f_i(g)\omega(u_i) + (-1)^{k-1}\iota_{(L_g^*\mathrm{d}f_i)}\widehat{\delta}|_g\wedge u_i\Big) \\
&= \omega(\widehat{\lambda}|_g) + (-1)^{k-1}\sum_{i,j}\big(\langle L_g^*\mathrm{d}f_i, x_j\rangle h_j(g)w_j\wedge u_i\big) \\
&= \omega(\widehat{\lambda}|_g) + (-1)^{k-1}\sum_j\Big(h_j(g)w_j\wedge\frac{d}{dt}\Big|_{t=0}\widehat{\lambda}|_{g\exp tx_j}\Big) \\
&= \omega(\widehat{\lambda}|_g) + (-1)^{k-1}\sum_j\Big(h_j(g)w_j\wedge\big(-\lambda(x_j)-\mathrm{pr}_{\wedge^{k+l-1}\theta}[x_j,\widehat{\lambda}|_g]\big)\Big) \\
&= \text{r.h.s. of Eq. (41)}.
\end{aligned}
$$

Here in the second from the last equality, we used the identity

$$
\widehat{\lambda}|_{g\exp tx_j} = \widehat{\lambda}|_{\exp tx_j} + (\exp tx_j)_*^{-1}\widehat{\lambda}|_g.
$$

From Eq. (41), it follows that $\widehat{\sigma}$ is indeed a Lie group 1-cocycle. Moreover, the induced Lie algebra 1-cocycle is

$$
\begin{aligned}
\sigma(x) &= -\left.\frac{d}{dt}\right|_{t=0} \widehat{\sigma}|_{\exp tx} \\
&= \left.\frac{d}{dt}\right|_{t=0} \Big((-1)^{(k-1)(l-1)} \lambda(\widehat{\delta}|_{\exp tx}) - \omega(\widehat{\lambda}|_{\exp tx}) \\
&\qquad\qquad + \mathrm{pr}_{\wedge^{k+l-1}\theta}[\widehat{\delta}|_{\exp tx}, \widehat{\lambda}|_{\exp tx}] \Big) \\
&= \omega(\lambda(x)) - (-1)^{(k-1)(l-1)} \lambda(\delta(x)).
\end{aligned}
$$

Here $-\left.\frac{d}{dt}\right|_{t=0} \mathrm{pr}_{\wedge^{k+l-1}\theta}\left[\widehat{\delta}|_{\exp tx}, \widehat{\lambda}|_{\exp tx}\right] = 0$, since both $\widehat{\delta}$ and $\widehat{\lambda}$ are group 1-cocycles. This completes the proof.   q.e.d.

3.3. Proof of the main theorem. The following result describes the infinitesimal data of a quasi-Poisson structure on the 2-group $G \ltimes \Theta$.

Proposition 3.6. *Let $(G \ltimes \Theta, \mathbf{\Pi}, \widehat{\eta})$ be a quasi-Poisson 2-group as in Definition 3.1. Let (ω, δ) be the corresponding infinitesimal of $\mathbf{\Pi}$ and $\eta : \mathfrak{g} \to \wedge^3\theta$ the Lie algebra 1-cocycle induced by $\widehat{\eta}$. Then the following identities hold:*

$$
\tag{42} \omega^2 = \eta \circ \phi,
$$

$$
\tag{43} (\omega + \delta) \circ \delta = D_\phi \circ \eta,
$$

$$
\tag{44} \omega \circ \eta = \eta \circ \delta,
$$

where η is identified with its extension to a degree-2 derivation of the exterior algebra $\wedge^\bullet(\mathfrak{g} \ltimes \theta)$.

Proof. Let $\mathbf{C}_{\widehat{\eta}} = \overleftarrow{\widehat{\eta}} - \overrightarrow{\widehat{\eta}}$. According to Proposition 3.5, $\mathbf{C}_{\widehat{\eta}}$ is multiplicative. Moreover, its corresponding infinitesimal is $(\eta \circ \phi, D_\phi \circ \eta)$. By Proposition 2.19, the infinitesimal of $\frac{1}{2}[\mathbf{\Pi}, \mathbf{\Pi}]$ is given by $(\omega^2, (\omega+\delta)\circ\delta)$. Thus Eq. (35) implies Eqs. (42) and (43). On the other hand, Eq. (36) is equivalent to $\partial_{\mathbf{\Pi}}^{\mathrm{gpd}}(\widehat{\eta}) = 0$. By Lemma 3.5 (3), we have $\omega \circ \eta - \eta \circ \delta = 0$. This completes the proof.   q.e.d.

Conversely, we have

Proposition 3.7. *Let $G \ltimes \Theta$ be a Lie 2-group. If both Lie groups G and Θ are connected and simply connected, every triple (ω, δ, η), where (ω, δ) satisfies the conditions of Theorem 2.15, and $\eta : \mathfrak{g} \to \wedge^3\theta$ is a Lie algebra 1-cocycle satisfying the conditions of Proposition 3.6, can be uniquely integrated to a quasi-Poisson structure on $G \ltimes \Theta$.*

Proof. By Theorem 2.21, we obtain a multiplicative bivector field $\mathbf{\Pi}$ on $G \ltimes \Theta$ whose infinitesimal is (ω, δ). Let $\widehat{\eta} : G \to \wedge^3\theta$ be the Lie group 1-cocycle integrating η. By Proposition 3.5 (3), $\partial_{\mathbf{\Pi}}^{\mathrm{gpd}}(\widehat{\eta})$ vanishes, since $\omega \circ \eta - \eta \circ \delta = 0$. Thus $[\mathbf{\Pi}, \overleftarrow{\widehat{\eta}}] = \overleftarrow{\partial_{\mathbf{\Pi}}^{\mathrm{gpd}}(\widehat{\eta})} = 0$. Moreover, Eqs. (42) and (43) imply Eq. (35) according to Proposition 3.5 (1 and 2).   q.e.d.

Finally, we need the following

Lemma 3.8. *A quasi-Lie 2-bialgebra structure on a crossed module of Lie algebras $(\theta \xrightarrow{\phi} \mathfrak{g})$ is equivalent to triples (δ, ω, η) of linear maps $\delta : \mathfrak{g} \to W_2 \subset \mathfrak{g} \wedge \theta$, $\omega : \theta \to \wedge^2 \theta$ and $\eta : \mathfrak{g} \to \wedge^3 \theta$ that satisfy the following properties:*

1) $D_\phi \circ \omega = \delta \circ \phi$;
2) $\omega^2 = \eta \circ \phi$;
3) $(\omega + \delta) \circ \delta = D_\phi \circ \eta$;
4) $\omega \circ \eta = \eta \circ \delta$;
5) *η is a Lie algebra 1-cocycle;*
6) *δ is a Lie algebra 1-cocycle;*
7) $x \triangleright \omega(u) - \omega(x \triangleright u) = \mathrm{pr}_{\wedge^k \theta}([u, \delta(x)])$, *for all $x \in \mathfrak{g}$ and $u \in \theta$.*

Proof. By Proposition 1.2, a weak Lie 2-coalgebra structure underlying $(\theta \xrightarrow{\phi} \mathfrak{g})$ is equivalent to an element $c = \check{\phi} + \check{\epsilon} + \check{\alpha} + \check{\eta} \in \mathcal{S}^{(-4)}$ such that $\{c, c\} = 0$. Here ϕ and $\check{\phi}$ are related by the equation: $\phi(u) = \{\check{\phi}, u\}$, for all $u \in \theta$. And $(\theta \xrightarrow{\phi} \mathfrak{g})$ is a quasi-Lie 2-bialgebra if and only if $\{o + c, o + c\} = 0$, where $o = \check{b} + \check{a}$ is the data defining the crossed module structure of $(\theta \xrightarrow{\phi} \mathfrak{g})$, as a special Lie 2-algebra with $\check{h} = 0$. Introduce the operators δ, ω, and η by the following relations:

$$\langle \delta(x) | \xi \wedge \kappa \rangle = -\{\{\{\check{\alpha}, x\}, \xi\}, \kappa\},$$

$$\langle \omega(u) | \kappa_1 \wedge \kappa_2 \rangle = \{\{\{\check{\epsilon}, u\}, \kappa_1\}, \kappa_2\},$$

$$\langle \eta(x) | \kappa_1 \wedge \kappa_2 \wedge \kappa_3 \rangle = \{\{\{\{\check{\eta}, x\}, \kappa_1\}, \kappa_2\}, \kappa_3\}$$

for all $x \in \mathfrak{g}$, $u \in \theta$, $\xi \in \mathfrak{g}^*$, and $\kappa, \kappa_i \in \theta^*$. Expand $\{o + c, o + c\}$ and consider the result term by term. Immediately, we have the following:

1) the $(\odot^2 \mathfrak{g}) \odot \mathfrak{g}^*$-part is zero if and only if δ is valued in W_2;
2) the $\theta \odot \mathfrak{g} \odot \theta^*$-part is zero if and only if Condition 1) is satisfied;
3) the $(\odot^3 \theta) \odot \theta^*$-part is zero if and only if Condition 2) is satisfied;
4) the $(\odot^2 \theta) \odot \mathfrak{g} \odot \mathfrak{g}^*$-part is zero if and only if Condition 3) is satisfied;
5) the $(\odot^4 \theta) \odot \mathfrak{g}^*$-part is zero if and only if Condition 4) is satisfied;
6) the $(\odot^2 \mathfrak{g}^*) \odot (\odot^3 \theta)$-part is zero if and only if Condition 5) is satisfied;
7) the $(\odot^2 \mathfrak{g}^*) \odot \mathfrak{g} \odot \theta$-part is zero if and only if Condition 6) is satisfied;
8) the $(\odot^2 \theta) \odot \mathfrak{g}^* \odot \theta^*$-part is zero if and only if Condition 7) is satisfied.

This concludes the proof.   q.e.d.

Proof of Theorem 3.2. Lemma 3.8 implies that a quasi-Lie 2-bialgebra underlying the crossed module $(\theta \xrightarrow{\phi} \mathfrak{g})$ is determined by the triple (ω, δ, η) that satisfies the conditions in Theorem 2.15 and Proposition 3.6. Thus Theorem 3.2 follows from Proposition 3.6 and Proposition 3.7. q.e.d.

3.4. Coboundary quasi-Poisson structures. The following proposition describes a class of interesting examples of quasi-Poisson structures on a Lie 2-group.

> **Proposition 3.9.**   1) *Associated to any Lie group 1-cocycle* $\widehat{\lambda}$:
> $G \to \wedge^2\theta$, *there exists a quasi-Poisson structure* $\left(\mathbf{\Pi}, \widehat{\eta}\right)$ *on* $G \ltimes \Theta$
> *given as follows:*

$$\text{(45)} \qquad\qquad\qquad \mathbf{\Pi} = \overleftarrow{\lambda} - \overrightarrow{\lambda},$$

$$\text{(46)} \qquad\qquad\qquad \widehat{\eta} = \tfrac{1}{2}[\widehat{\lambda}, \widehat{\lambda}].$$

> *In Eq.* (45), $\widehat{\lambda}$ *is considered as a section in* $\mathbf{\Gamma}(\wedge^2 A)$ *and the bracket*
> *in Eq.* (46) *stands for the pointwise Schouten bracket on* $\wedge^\bullet\theta$.
> 2) *The infinitesimal* $(\omega_\lambda, \delta_\lambda, \eta_\lambda)$ *of* $\left(\mathbf{\Pi}, \widehat{\eta}\right)$ *as described by Proposi*
> *tion* 3.6 *is as follows:*

$$\omega_\lambda = \lambda \circ \phi,$$
$$\delta_\lambda = D_\phi \circ \lambda,$$
$$\eta_\lambda = \lambda \circ D_\phi \circ \lambda,$$

> *where* $\lambda : \mathfrak{g} \to \wedge^2\theta$ *is the Lie algebra 1-cocycle induced by* $\widehat{\lambda}$.

Proof. The proof is standard, and is left to the reader.   q.e.d.

In particular, any $\mathbf{r} \in \wedge^2\theta$ induces a Lie algebra 1-cocycle $\lambda_\mathbf{r} : \mathfrak{g} \to \wedge^2\theta$:

$$\text{(47)} \qquad\qquad \lambda_\mathbf{r}(x) = -x \triangleright \mathbf{r}, \quad \forall x \in \mathfrak{g}.$$

Therefore by Proposition 3.9, there exists a quasi-Poisson structure $(\mathbf{\Pi_r}, \widehat{\eta}_\mathbf{r})$ on the 2-group $G \ltimes \Theta$. By a straightforward computation, we can describe this quasi-Poisson structure more explicitly:

$$\text{(48)} \quad (\mathbf{\Pi_r})|_{(g,\alpha)} = R_{g*}\Phi_*\mathbf{r} - L_{g*}\Phi_*\mathbf{r} + L_{\alpha*}g_*^{-1}\mathbf{r} - R_{\alpha*}g_*^{-1}\mathbf{r}$$
$$+ \left[(L_{g*} \circ \Phi_*) \otimes (L_\alpha)\right]\mathbf{r} - \left[(R_{g*} \circ \Phi_*) \otimes (L_\alpha \circ g_*^{-1})\right]\mathbf{r},$$

and

$$\text{(49)} \qquad\qquad (\widehat{\eta}_\mathbf{r})|_g = \tfrac{1}{2}\big([\mathbf{r}, \mathbf{r}] - g_*^{-1}[\mathbf{r}, \mathbf{r}]\big).$$

The infinitesimal of $(\mathbf{\Pi_r}, \widehat{\eta}_\mathbf{r})$ is given as follows:

$$\text{(50)} \qquad \omega_\mathbf{r}(u) = [\mathbf{r}, u], \qquad\qquad\qquad \forall u \in \theta;$$

$$\text{(51)} \qquad \delta_\mathbf{r}(x) = -D_\phi(x \triangleright \mathbf{r}) = -x \triangleright (D_\phi\mathbf{r}), \qquad \forall x \in \mathfrak{g};$$

$$\text{(52)} \qquad \eta_\mathbf{r}(x) = -\tfrac{1}{2}x \triangleright [\mathbf{r}, \mathbf{r}], \qquad\qquad \forall x \in \mathfrak{g}.$$

According to Lemma 3.8, the triple $(\omega_\mathbf{r}, \delta_\mathbf{r}, \eta_\mathbf{r})$ also defines a quasi-Lie 2-bialgebra structure underlying $(\theta \to \mathfrak{g})$. In particular, if $\eta_\mathbf{r} = 0$, i.e.

$$\text{(53)} \qquad\qquad x \triangleright [\mathbf{r}, \mathbf{r}] = 0, \quad \forall x \in \mathfrak{g},$$

we obtain a Lie 2-bialgebra.

Definition 3.10. An element $\mathbf{r}$ of $\wedge^2\theta$ is called an r-matrix of a Lie algebra crossed module $(\theta \xrightarrow{\phi} \mathfrak{g})$ if $[\mathbf{r}, \mathbf{r}] \in \wedge^3\theta$ is $\mathfrak{g}$-invariant, i.e. if Eq. (53) holds.

Similar to the Poisson group case, we have the following

Theorem 3.11. *Corresponding to any r-matrix $\mathbf{r}$ as above, there is*

1) *a Poisson Lie 2-group structure $\mathbf{\Pi_r}$ on $G \ltimes \Theta$ such that*

$$\mathbf{\Pi_r} = \overleftarrow{\lambda_\mathbf{r}} - \overrightarrow{\lambda_\mathbf{r}},$$

where $\widehat{\lambda}_\mathbf{r} : G \to \wedge^2\theta$ is given by Eq. (47) and

2) *a Lie bialgebra crossed module underlying $(\theta \xrightarrow{\phi} \mathfrak{g})$.*

In this case, the Lie bracket on θ^* is induced by the r-matrix $\mathbf{r}$:

$$\langle [\kappa_1, \kappa_2]_\mathbf{r} | u \rangle = \langle \kappa_1 \wedge \kappa_2 | [\mathbf{r}, u] \rangle, \quad \forall \kappa_1, \kappa_2 \in \theta^*, u \in \theta,$$

while the action of θ^* on $\mathfrak{g}^*$ is given by

$$\langle \kappa \rhd \xi | x \rangle = \langle \kappa \wedge \phi^*\xi | x \rhd \mathbf{r} \rangle,$$

for all $\kappa \in \theta^*$, $\xi \in \mathfrak{g}^*$, and $x \in \mathfrak{g}$.

Example 3.12. Let $\theta = \mathfrak{gl}(2) \cong \mathbb{R}\,\mathrm{id} \oplus \mathfrak{sl}(2)$ and $\mathfrak{g} = \mathfrak{sl}(2)$. Then the projection $\phi : \mathfrak{gl}(2) \to \mathfrak{sl}(2)$ is a Lie algebra crossed module. It is easy to check that any $\mathbf{r} \in \wedge^2\theta$ is indeed an r-matrix.

Example 3.13. Let $\mathfrak{g}$ be a Lie algebra and $\theta \subseteq \mathfrak{g}$ an ideal. Consider the Lie algebra crossed module $\iota : \theta \to \mathfrak{g}$, where ι is the inclusion. Assume that $\mathbf{r} \in \wedge^2\theta$ such that $[\mathbf{r}, \mathbf{r}] \in \wedge^3\theta$ is $\mathfrak{g}$-invariant. Then $\mathbf{r}$ is clearly an r-matrix. For example, we take $\mathfrak{g} = \mathfrak{gl}(2)$ and $\theta = \mathfrak{sl}(2)$. Then any bivector in $\wedge^2\theta$ is indeed an r-matrix.

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