Heron Triangles and Perfect Cuboids

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Abstract: In this paper, we first derive a new result which improves Normann-Erdös Theorem, then we investigate the structures of Heron triangle with rational angle bisectors. The result goes as follows: There are infinitely many Heron triangles whose angle bisectors are rational numbers; For any Heron triangle (a,b,c), there exists a Heron triangle (m,n,k) whose angle bisectors are rational numbers, where

$$\begin{cases} m = f(a,b,c) \\ n = g(a,b,c) \\ k = h(a,b,c) \end{cases} \qquad (0.1)$$

We also prove that: For any Heron triangle (m',n',k') whose angle bisectors are rational numbers, there exists a Heron triangle (a,b,c) such that the Heron triangle (m',n',k') is similar to the Heron triangle (m,n,k) given by formula (0.1). Moreover, by the representation of (0.1), we obtain its relation with **the Perfect Square Triangular Problem (PSTP i.e., Is there a triangle whose sides are perfect squares and whose angle bisectors are integers**? **).**

Mexico mathematician Luca proved that PSTP is equivalent to the **Perfect Cuboid Problem (** PCP i.e., **Is there a rectangular box with all edges, face diagonals, and the main diagonal integers**?). We obtain four Diophantine systems which are equivalent to PCP. Therefore we wish we could provide some useful results for solving PCP.

Keywords: Heron triangles，PCP，PSTP，angle bisectors，triangular

excircle, square numbers.

Heron 三角形与完全长方体

摘要：本文从研究诺尔曼—埃尔德什定理入手, 得到了新的结论。尔后探求角平分线是有理数的 Heron 三角形的构造。得出的结论是：角平分线为有理数的 Heron 三角形有无数多个；对任意一组 Heron 数组(a,b,c)都对应一组角平分线为有理数的 Heron 数组(m,n,k)，其中

$$\begin{cases} m = f(a,b,c) \\ n = g(a,b,c) \\ k = h(a,b,c) \end{cases} \qquad (0.1)$$

我们还证明了：对任意一组角平分线为有理数的 Heron 数组(m',n',k')，必存在 Heron 数组(a,b,c)，使得三角形(m',n',k')相似于由（0.1）给出的三角形(m,n,k)。而且，我们由（0.1）的表示，发现其与著名的完全平方三角形问题（PSTP）的联系。然而，墨西哥数学家 Luca 证明了 PSTP 问题与完全长方体问题（PCP）的存在性是等价的。由此我们得到了与 PCP 问题等价的几个不定方程组。希望为解决 PCP 问题提供些有用的结论。

关键词：Heron 三角形，PCP，PSTP，角平分线，三角形外接圆，平方数，海伦数。

1. Introduction

A *Heron triangle* is a triangle having the property that the lengths of its sides as well as its area are positive integers. If the great common divisor of three sides of a Heron triangle is 1, then the Heron triangle is called a *primitive* Heron triangle.

There are several open questions concerning the existence of Heron triangles with certain properties. For example (see [1, Problem D21]), it is not known whether there exist Heron triangles having the property that the lengths of all of their medians are positive integers. A different unsolved problem which asks for the existence of a perfect cuboid, i.e., rectangular box having the lengths of all the sides, face diagonals, and main diagonal integers has been related(see [5]) to the existence of a Heron triangle having the lengths of its sides perfect squares and the lengths of its angle bisectors positive integers.

In this paper, we focus our attention on the Heron triangles with rational angle bisectors. First, by using some idea of [3], we prove the following Theorem, which improves the results of [2] and [3].

Theorem 1.1: For any given integer $n(n \geq 3)$, **there exists** n points on a circle such that any triangles whose vertices are three points among the $n(n \geq 3)$ points is a Heron triangle with three integer **angle bisectors.**

We will prove Theorem 1.1 in Section 3. Next, we will investigate the characterization of Heron triangle with rational angle bisectors. By using some known results on triangles and some constructions, we will prove in Section 4 the following three Theorems.

Theorem 1.2 Let (a,b,c) be a primitive Heron block such that the related triangle is a acute triangle. Then (m,n,k) given by the following formula

$$\begin{cases} m = a^2(b^2+c^2-a^2) \\ n = b^2(a^2+c^2-b^2) \\ k = c^2(a^2+b^2-c^2) \end{cases} \qquad \textbf{(1.1)}$$

is a Heron tuple such that the related Heron triangle is a triangle with three rational angle bisectors.

Theorem 1.3 Let (a,b,c) be a primitive Heron block such that the related triangle is a obtuse triangle with $c^2 > a^2+b^2$. Then (m,n,k) given by the following formula

$$\begin{cases} m = a^2(b^2+c^2-a^2) \\ n = b^2(a^2+c^2-b^2) \\ k = c^2(c^2-a^2-b^2) \end{cases} \qquad \textbf{(1.2)}$$

is a Heron tuple such that the related Heron triangle is a triangle with three rational angle bisectors.

Theorem 1.4 **Let (a,b,c) be a primitive Heron tuple such that the related triangle is a right triangle with $c^2 = a^2 + b^2$ ($a > b$). Then (m,n,k) given by the following formula**

$$\begin{cases} m = 2(a^2 - b^2) \\ n = c^2 \\ k = c^2 \end{cases} \qquad \textbf{(1.3)}$$

is a Heron tuple such that the related Heron triangle is a triangle with three rational angle bisectors.

By Theorems 1.2, 1.3 and 1.4, we have constructed three kinds of Heron triangles with rational angle bisectors. It is natural to ask: Can any Heron triangle with rational angle bisectors be constructed in this way? Theorem 4.1 answers the above question.

Theorem 4.1: If the tuple (m,n,k) is a Heron tuple such that the corresponding Heron triangle ΔMNK has three rational angle bisectors, then the tuple (m,n,k) can be represented as

$$\begin{cases} m = \lambda a^2(b^2 + c^2 - a^2) \\ n = \lambda b^2(a^2 + c^2 - b^2) \\ k = \lambda c^2(a^2 + b^2 - c^2) \end{cases}, \qquad \textbf{(1.4)}$$

where $\lambda \in Q$, and (a,b,c) is a primitive Heron tuple of a acute triangle.

Or

$$\begin{cases} m = \lambda a^2(b^2 + c^2 - a^2) \\ n = \lambda b^2(a^2 + c^2 - b^2) \\ k = \lambda c^2(c^2 - a^2 - b^2) \end{cases}, \qquad \textbf{(1.5)}$$

where $\lambda \in Q$ **, and** (a,b,c) **is a primitive Heron tuple of a obtuse triangle and** $c^2 > a^2 + b^2$**. Or**

$$\begin{cases} m = \lambda 2(a^2 - b^2) \\ n = \lambda c^2 \\ k = \lambda c^2 \end{cases} \qquad \textbf{(1.6)}$$

where $\lambda \in Q$ **, and** (a,b,c) **is a primitive Heron tuple of a right triangle with hypotenuse c, and** $a > b$**. The third case occurs only when** $n = k$**.**

In Section 5, we will connect the above results with the famous Perfect Cuboid Problem. We will prove

Theorem 5.3:

（1） **PCP has a solution if and only if there are positive integers** x, y, z

$$\begin{cases} x^2 + y^2 = c^2 \\ x^2 - z^2 = b^2 \\ y^2 - z^2 = a^2 \end{cases}$$

and $x^2y^2 - y^2z^2 - z^2x^2$ **is a square.**

（2）PCP has a solution if and only if there are positive integers x, y, z

$$\begin{cases} x^2 + y^2 = c^2 \\ x^2 + z^2 = b^2 \\ y^2 + z^2 = a^2 \end{cases}$$

and $x^2y^2 + y^2z^2 + z^2x^2$ **is a square.**

2. Some Lemmas

In this section, we will prove some lemmas that will be used in the subsequent sections.

To begin with, the following results is well-known for a Heron triangle, so we cite them directly without proof.

Known Facts: the values of sine, cosine, tangent and tangent of half-angle of any angle of a Heron triangle are rational numbers; the values of radii of incircle and circumcircle are rational numbers; the lengths of three sides of a Heron triangle must be two odd integers and one even integer.

Lemma 2.1 If $\theta \neq k\pi + \dfrac{\pi}{2}(k$ is an integer), then $\sin 2\theta, \cos 2\theta \in Q$ if and only if $\tan\theta \in Q$。

Proof: If $\theta \neq k\pi + \dfrac{\pi}{2}(k$ is an integer), then $\tan\theta$ makes sense.

(1)Sufficiency: If $\tan\theta \in Q$, we have $\sin 2\theta = \dfrac{2\tan\theta}{1+\tan^2\theta}$, $\cos 2\theta = \dfrac{1-\tan^2\theta}{1+\tan^2\theta} \in Q$.

(2)Necessity: If $\sin 2\theta, \cos 2\theta \in Q$, since $\tan\theta = \dfrac{1-\cos 2\theta}{\sin 2\theta}$, we have $\tan\theta \in Q$。This proves the lemma.

Lemma 2.2 If $x_i, y_i \in Q(i = 1,2,3)$, then the area of the triangle with three non-collinear vertices $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ is rational。

Proof: By the knowledge of analytic geometry, we obtain that the area with three vertices $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ is given by the absolute value of

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

Therefore $\Delta \in Q$。The lemma is proved.

The following lemma is very important in the argument of this paper.

Lemma2.3 Let ABC **be a Heron triangle, the angle bisector** l_a **is**

rational if and only if $\sin\frac{A}{2},\cos\frac{A}{2}$ **are rational numbers。**

Proof: By the sine Theorem, we have

$$\frac{l_a}{\sin B}=\frac{c}{\sin(B+A/2)}。\qquad (2.1)$$

Suppose $\sin\frac{A}{2},\cos\frac{A}{2}$ are rational numbers, since the triangle ABC is a Heron triangle, then $\sin B,\cos B\in Q$, and so $\sin(B+A/2)\in Q, l_a\in Q$。

Conversely, if the angle bisector l_a is rational, since the triangle ABC is a Heron triangle, so $\sin B,\cos B,\tan A/2\in Q$. By (2.1) we obtain $\sin(B+A/2)\in Q$。Therefore

$$\cos\frac{A}{2}(\sin B+\cos B\tan\frac{A}{2})\in Q,$$

consequently $\sin\frac{A}{2},\cos\frac{A}{2}$ are rational numbers。This completes the proof.

3、Proof of Theorem 1.1

Proof of Theorem 1.1 To begin with, we choose n distinct angles $\theta_1,\dots,\theta_n$ in $[0,\pi/4)$ such that $\tan\theta_i\in Q(i=1,\dots,n)$。Hence $8\theta_i\in[0,2\pi)$, and thus we can find the corresponding points $M_i(\cos 8\theta_i,\sin 8\theta_i)(i=1,\dots,n)$ which are on the unit circle。

Since $\tan\theta_i\in Q(i=1,\dots,n)$, by Lemma 2.1 we have

$\sin 2\theta_i, \cos 2\theta_i, \sin 4\theta_i, \cos 4\theta_i, \sin 8\theta_i, \cos 8\theta_i \in Q (i = 1, \ldots, n)$。 It follows that the coordinates of these n points are rational numbers. Now, by Lemma 2.2, the area of any triangle whose vertices are three distinct points among the n points is a rational number.

Now we will prove that the distance between any two points is a rational number. By the distance formula of two points, we have

$$M_k M_l = \sqrt{(\cos 8\theta_k - \cos 8\theta_l)^2 + (\sin 8\theta_k - \sin 8\theta_l)^2} = \sqrt{4\sin^2 4(\theta_k - \theta_l)}$$
$$= 2|\sin 4(\theta_k - \theta_l)| = 2|\sin 4\theta_k \cos 4\theta_l - \cos 4\theta_k \sin 4\theta_l| \in Q.$$

Next, joint $M_k M_l$ and any other point M_m, we obtain a triangle $\Delta M_k M_l M_m$, and $\theta = \angle M_k M_m M_l = 4|\theta_k - \theta_l|$ or $\pi - 4|\theta_k - \theta_l|$。 Therefore

$$\sin\frac{\theta}{2} = \pm\sin 2(\theta_k - \theta_l) = \pm(\sin 2\theta_k \cos 2\theta_l - \cos 2\theta_k \sin 2\theta_l) \in Q,$$

or $\sin\frac{\theta}{2} = \pm\cos 2(\theta_k - \theta_l) = \pm(\sin 2\theta_k \sin 2\theta_l - \cos 2\theta_k \cos 2\theta_l) \in Q,$

and $\cos\frac{\theta}{2} = \pm\cos 2(\theta_k - \theta_l) = \pm(\sin 2\theta_k \sin 2\theta_l - \cos 2\theta_k \cos 2\theta_l) \in Q,$

or $\cos\frac{\theta}{2} = \pm\sin 2(\theta_k - \theta_l) = \pm(\sin 2\theta_k \cos 2\theta_l - \cos 2\theta_k \sin 2\theta_l) \in Q.$

By Lemma 2.3, the angle bisector of $\angle M_k M_m M_l$ is a rational number。

In this way we have proved that there exist n distinct points on the unit circle such that the distance between any two points is a rational number, the area of any triangle whose vertices are three distinct points among the n points is a rational number and any angle bisector of any angle with three distinct point is a rational. Now by suitable similar amplification, all rational numbers can be turned into integers. That is, there exists such a circle and n distinct points on the circle such that the distance between any two points is an integer, the area of any triangle whose vertices are three distinct points among the n points is an integer and any angle bisector of any angle with three distinct point is an integer. Recall that a Heron triangle is a triangle with integral sides and area, which completes the proof.

Therefore, we have proved that there are infinitely many Heron triangle with rational angle bisectors.

4、The construction of Heron triangles with rational angle bisectors.

In this section, we will give the proofs of Theorems 1.2-1.4, which characterize the construction of Heron triangle with rational angle bisectors.

Proof of Theorem 1.2：

Let ΔABC be a primitive Heron triangles whose lengths of three sides are a Heron tuple (a,b,c), by the assumptions, we have that ΔABC is a acute triangle, so the center O of the circumscribed circle of ΔABC lies in the interior of ΔABC。 Do the circumscribed circle O of ΔABC; make three tangential lines of the circle through A,B,C, we obtain the intersection triangle ΔMNK of the three tangential lines （as see the figure (1)）

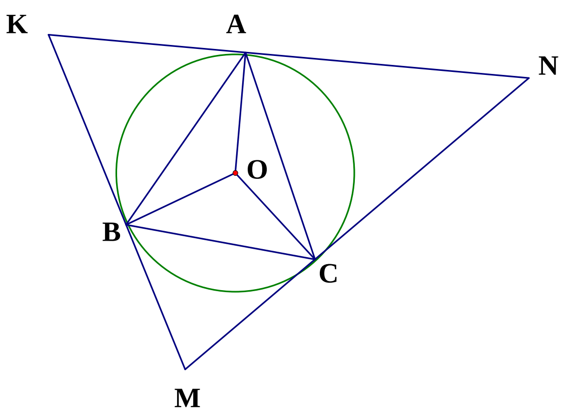


Figure (1)

Notice that $\angle BOC = 2\angle BAC, \tan A = \tan\angle BAC \in Q$,

And let $r = OA = OB = OC$ be the radius of the circle， then $r \in Q$, And so $MC = r\tan A \in Q$。Similarly, $NC \in Q$,

So $MN = MC + CN \in Q$。Similarly $MK \in Q, NK \in Q$,

It follows that all sides of triangle MNK are rational numbers.

Since $\angle NMK = 2(\frac{\pi}{2} - \angle BAC) = \pi - 2A$,

then $\sin\angle NMK = \sin 2A = 2\sin A\cos A \in Q$。Similarly,

$\angle MKN = \pi - 2C, \angle MNK = \pi - 2B, \sin\angle MKN, \sin\angle MNK \in Q$。

Now in ΔMNK, we have, $\cos\angle NMK = -\cos 2A = \sin^2 A - \cos^2 A \in Q$,

Similarly, $\cos K, \cos N \in Q$。Therefore $S_{\Delta MNK} = \frac{1}{2} MN \times MK \times \sin\angle NMK \in Q$。

For the angle bisector d_M of the angle $\angle NMK$ in ΔMNK, we have

$$\frac{d_M}{\sin N} = \frac{MN}{\sin(K + M/2)} = \frac{MN}{\sin(K + (\pi/2 - A))} = \frac{MN}{\cos(K - A)},$$

Since $\cos(K - A) = \cos K\cos A + \sin K\sin A \in Q$, thus the angle bisector d_M of the angle $\angle NMK$ is a rational number.

Similarly, the two other angle bisectors of ΔMNK are rational numbers.

Summing up, the three sides, area and three angle bisectors of ΔMNK are rational numbers.

Scaling ΔMNK we can obtain a primitive Heron triangle whose lengths of three sides are a primitive Heron tuple with integer area and rational angle bisectors.

Now we prove that the formula (1.1) holds. Since

$$MC = r\tan A, NC = r\tan B, \sin A = \frac{a}{2r}, \sin B = \frac{b}{2r}$$, so

$$\cos A = \frac{b^2+c^2-a^2}{2bc}, \cos B = \frac{a^2+c^2-b^2}{2ac}$$ 。Hence

$$MN = r(\tan A + \tan B) = \frac{abc}{b^2+c^2-a^2} + \frac{abc}{a^2+c^2-b^2} = \frac{2abc^3}{(b^2+c^2-a^2)(a^2+c^2-b^2)}.$$

Similarly,

$$MK = r(\tan A + \tan C) = \frac{abc}{b^2+c^2-a^2} + \frac{abc}{a^2+b^2-c^2} = \frac{2ab\ c}{(b^2+c^2-a^2)(a^2+b^2-c^2)},$$

$$KN = r(\tan C + \tan B) = \frac{abc}{a^2+b^2-c^2} + \frac{abc}{a^2+c^2-b^2} = \frac{2a^3bc}{(a^2+b^2-c^2)(a^2+c^2-b^2)}.$$

By dividing out a common rational divisor of MN, MK, KN,we obtain

$$\begin{cases} m = a^2(b^2+c^2-a^2) \\ n = b^2(a^2+c^2-b^2) \\ k = c^2(a^2+b^2-c^2) \end{cases}$$

This proves Theorem 1.1。

Example 4.1:

For a given Heron tuple $(13,14,15)$， find the corresponding primitive Heron tuple such that the lengths of three angle bisectors of the corresponding Heron triangle are rational numbers.

Solving： It is easy to see that the triangle with primitive Heron $(13,14,15)$ is a acute triangle， by Theorem 1.1, substituting

$a=13, b=14, c=15$ into formula (1.1) to obtain

$$\begin{cases} m=13^2(14^2+15^2-13^2)=2^3\times 3^2\times 7^2\times 11 \\ n=14^2(13^2+15^2-14^2)=2^2\times 3^2\times 7\times 13^2 \\ k=15^2(13^2+14^2-15^2)=2^2\times 3^2\times 5^3\times 7 \end{cases},$$

Therefore we obtain the corresponding primitive Heron tuple $(154,169,125)$. It is easy to check that the semi-perimeter $p=224$, the area

$$S=\sqrt{224\times 70\times 55\times 99}=2^3\times 3\times 5\times 11.$$

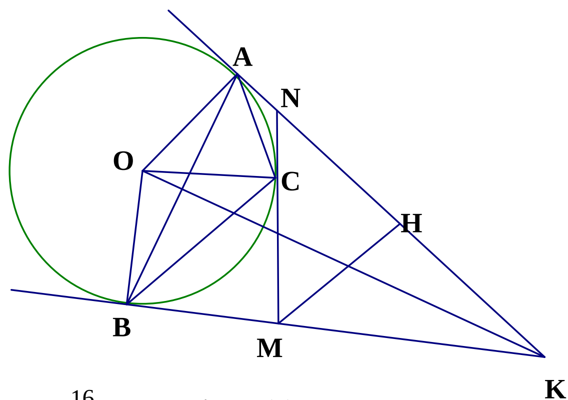
By the formula of angle bisector, the lengths of three angle bisectors are

$$d_m=\frac{2}{n+k}\times\sqrt{p(p-m)nk}=\frac{2}{169+125}\sqrt{224\times 70\times 13^2\times 5^3}=\frac{2600}{21},$$
$$d_n=\frac{2}{m+k}\times\sqrt{p(p-n)mk}=\frac{2}{154+125}\sqrt{224\times 55\times 154\times 5^3}=\frac{30800}{279},$$
$$d_k=\frac{2}{n+m}\times\sqrt{p(p-k)mn}=\frac{2}{169+154}\sqrt{224\times 99\times 13^2\times 154}=\frac{48048}{323}.$$

Therefore the lengths of three angle bisectors of primitive Heron triangle with Heron tuple $(154,169,125)$ are rational numbers.

Proof of Theorem 1.3:

Let ΔABC be the primitive Heron triangle with Heron tuple (a,b,c), by the assumptions, we obtain that ΔABC is a obtuse triangle, hence the center O of the circumcircle of ΔABC lies in the exterior of



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Figure(2)

ΔABC . Do the circumscribed circle O of ΔABC ; make three tangential lines of the circle through A, B, C , we obtain the intersection triangle ΔMNK of the three tangential lines （as see the figure(2)）

Notice that $\angle BOC = 2\angle BAC, \tan A = \tan \angle BAC \in Q$, and let $r = OA = OB = OC$ be the radius of the circle O , then $r \in Q$, thus $MC = r\tan A \in Q$ 。Similarly, $NC \in Q$, and hence $MN = MC + CN \in Q$ 。

Since $\angle AOB = 2\pi - 2C, \tan C \in Q$, so $BK = r\tan(\pi - C) = -r\tan C \in Q$ 。 Hence

$MK = KB - MB = -r\tan C - r\tan A \in Q, NK = KB - NC = -r\tan C - r\tan B \in Q$ 。

So the lengths of three sides of the triangle MNK are rational numbers.

Now $\angle NMK = 2\angle BAC = 2A$, so $\sin \angle NMK = \sin 2A = 2\sin A\cos A \in Q$ 。

Similarly, $\angle MKN = 2C - \pi, \angle MNK = 2B, \sin \angle MKN, \sin \angle MNK \in Q$ 。

In ΔMNK , we have $\cos \angle NMK = \cos 2A = \cos^2 A - \sin^2 A \in Q$,

Similarly, $\cos K, \cos N \in Q$ 。

Therefore the area of ΔMNK is $S_{\Delta MNK} = \frac{1}{2} MN \times MK \times \sin \angle NMK \in Q$ 。

For the angle bisector MH of $\angle NMK$, we have

$$\frac{MH}{\sin N} = \frac{MN}{\sin \angle MHN} = \frac{MN}{\sin(K + (\pi/2 - A))} = \frac{MN}{\cos(K - A)},$$

since $\cos(K - A) = \cos K\cos A + \sin K\sin A \in Q$, thus the angle bisector MH of $\angle NMK$ is rational. Similarly, the two other angle bisectors of

ΔMNK are rational numbers.

Summing up, the three sides, area and three angle bisectors of ΔMNK are rational numbers.

Scaling ΔMNK we can obtain a primitive Heron triangle whose lengths of three sides are a primitive Heron tuple with integer area and rational angle bisectors.

Now we prove that the formula (1.2) holds. Since

$$MC = r\tan A, NC = r\tan B, \sin A = \frac{a}{2r}, \sin B = \frac{b}{2r},$$

$$\cos A = \frac{b^2+c^2-a^2}{2bc}, \cos B = \frac{a^2+c^2-b^2}{2ac}.$$ Hence

$$MN = r(\tan A + \tan B) = \frac{abc}{b^2+c^2-a^2} + \frac{abc}{a^2+c^2-b^2} = \frac{2abc^3}{(b^2+c^2-a^2)(a^2+c^2-b^2)}.$$

Similarly,

$$MK = -r(\tan A + \tan C) = -\frac{abc}{b^2+c^2-a^2} - \frac{abc}{a^2+b^2-c^2} = \frac{2ab^3c}{(b^2+c^2-a^2)(c^2-a^2-b^2)},$$

$$KN = -r(\tan C + \tan B) = -\frac{abc}{a^2+b^2-c^2} - \frac{abc}{a^2+c^2-b^2} = \frac{2a^3bc}{(c^2-a^2-b^2)(a^2+c^2-b^2)}.$$

By dividing out a common rational divisor of MN, MK, KN,we obtain

$$\begin{cases} m = a^2(b^2+c^2-a^2) \\ n = b^2(a^2+c^2-b^2) \\ k = c^2(c^2-a^2-b^2) \end{cases}$$

which completes the proof.

Example 4.2:

For a given primitive Heron tuple $(5,5,8)$, find the corresponding primitive Heron tuple such that the lengths of three angle bisectors of the corresponding Heron triangle are rational numbers.

Solving： It is easy to see the triangle with primitive Heron tuple $(5,5,8)$ is an obtuse triangle, by Theorem 1.2, substituting $a=5, b=5, c=8$ into the formula (1.2), we obtain

$$\begin{cases} m = 5^2(5^2+8^2-5^2) = 2^6\times 5^2 \\ n = 5^2(5^2+8^2-5^2) = 2^6\times 5^2 \\ k = 8^2(8^2-5^2-5^2) = 2^7\times 7 \end{cases},$$

Therefore we obtain the corresponding primitive Heron tuple $(25,25,14)$。 It is easy to check that the semi-perimeter $p=32$,the eara $S=\sqrt{32\times 8\times 7\times 7}=16\times 7$。

By the formula of angle bisector, the lengths of three angle bisectors are

$$d_m = \frac{2}{n+k}\times\sqrt{p(p-m)nk} = \frac{2}{25+14}\sqrt{32\times 8\times 25^2} = \frac{800}{39},$$
$$d_n = \frac{2}{m+k}\times\sqrt{p(p-n)mk} = \frac{2}{25+14}\sqrt{32\times 8\times 25^2} = \frac{800}{39},$$
$$d_k = \frac{2}{n+m}\times\sqrt{p(p-k)mn} = \frac{2}{25+25}\sqrt{32\times 7\times 14\times 25} = \frac{56}{5}。$$

Therefore the lengths of three angle bisectors of primitive Heron triangle with Heron tuple $(25,25,14)$ are rational numbers.

Proof of Theorem 1.4:

Let ΔABC be the primitive Heron triangle with Heron tuple (a,b,c)， by the assumptions, we obtain that ΔABC is a right triangle, hence the center

O of the circumcircle of ΔABC is on the hypotenuse. Do the circumscribed circle O of ΔABC; make two tangential lines of the circle through B,C ,the two line intersect in K。

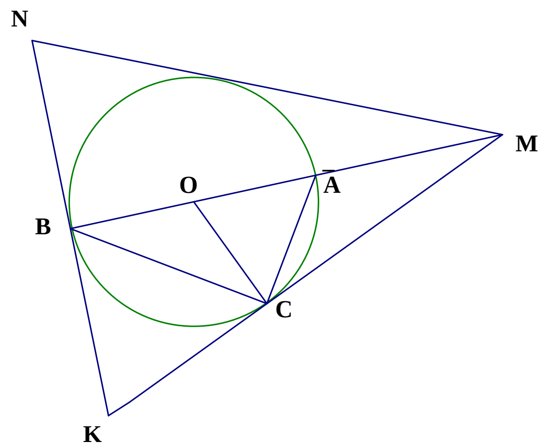

Figure (3)

Prolong BA which intersects M with the tangential line through C, make the tangential line of the circle O through M, which intersects N with the tangential line through B, in this way we obtain a isosceles triangle ΔMNK (as see in figure(3)）.

Notice that $\angle BOC = 2\angle BAC, \tan A = \tan \angle BAC \in Q$, and let $r = OA = OB = OC$ be the radius of the circle O, so $r \in Q$, thus

$KB = KC = r\tan A \in Q$. Similarly, $MC \in Q$, so $MK = MC + CK \in Q$. So the lengths of three sides of the triangle MNK are rational numbers.

Now $\angle NKM = 2\angle BAC = 2A$, so $\sin \angle NKM = \sin 2A = 2\sin A\cos A \in Q$。

Similarly, $\sin \angle MKN, \sin \angle MNK \in Q$。

In ΔMNK, $\cos \angle NKM = \cos 2A = \cos^2 A - \sin^2 A \in Q$,in the same way we have $\cos M, \cos N \in Q$。

Therefore the area of ΔMNK is $S_{\Delta MNK} = \frac{1}{2} MK \times KN \times \sin\angle NKM \in Q$ 。

For the angle bisector MH of $\angle NMK$, we have

$$\frac{MH}{\sin N} = \frac{MN}{\sin\angle MHN} = \frac{MN}{\sin(K + (\pi/2 - A))} = \frac{MN}{\cos(K - A)},$$

since $\cos(K - A) = \cos K \cos A + \sin K \sin A \in Q$, so the angle bisector MH of $\angle NMK$ is a rational number. Similarly, the two other angle bisectors of ΔMNK are rational numbers.

Summing up, the three sides, area and three angle bisectors of ΔMNK are rational numbers.

Scaling ΔMNK we can obtain a primitive Heron triangle whose lengths of three sides are a primitive Heron tuple with integer area and rational angle bisectors.

Similarly, we can prove that the formula (1.3) holds. Since the computations are similar, we omit the details.

Therefore we have constructed a Heron triangle with three rational angle bisectors from any known Heron triangle, and we also give the expressions of the lengths of three sidesof the related Heron triangle..

Theorem 4.1: If the tuple (m,n,k) is a Heron tuple such that the corresponding Heron triangle ΔMNK has three rational angle bisectors, then the tuple (m,n,k) can be represented as

$$\begin{cases} m = \lambda a^2 (b^2 + c^2 - a^2) \\ n = \lambda b^2 (a^2 + c^2 - b^2) \text{ ,} \\ k = \lambda c^2 (a^2 + b^2 - c^2) \end{cases} \qquad (4.1)$$

where $\lambda \in Q$ **, and** (a,b,c) **is a primitive Heron tuple of a acute triangle.**

Or

$$\begin{cases} m = \lambda a^2 (b^2 + c^2 - a^2) \\ n = \lambda b^2 (a^2 + c^2 - b^2) \text{ ,} \\ k = \lambda c^2 (c^2 - a^2 - b^2) \end{cases} \qquad (4.2)$$

where $\lambda \in Q$ **, and** (a,b,c) **is a primitive Heron tuple of a obtuse triangle and** $c^2 > a^2 + b^2$ **. Or**

$$\begin{cases} m = \lambda 2(a^2 - b^2) \\ \quad n = \lambda c^2 \\ \quad k = \lambda c^2 \end{cases} \qquad (4.3)$$

where $\lambda \in Q$ **, and** (a,b,c) **is a primitive Heron tuple of a right triangle with hypotenuse c, and** $a > b$ **.**

Proof of Theorem 4.1: Suppose **the tuple** (m,n,k) **is a Heron tuple such that the corresponding Heron triangle** ΔMNK **has three rational angle bisectors, then** ΔMNK has a incircle O。Let the three tangential points of O with ΔMNK be A,B,C, then from the proof of Theorem 1.2, we have

$$\angle NMK = 2(\frac{\pi}{2} - \angle BAC) = \pi - 2A \text{ ,}$$

$$\angle MKN = \pi - 2C, \angle MNK = \pi - 2B \text{ 。}$$

Therefore the triangle ABC is acute. By the assumptions and Lemma 2.3, we have $\sin A, \sin B, \sin C \in Q, r \in Q$, so $BC, AC, AB \in Q$ and $S_{\Delta ABC} \in Q$ 。By scaling the triangle ABC, we can obtain a primitive Heron triangle, denote the lengths of the three sides by a, b, c. By Theorem 1.2, the tuple (m,n,k) can be represented as

$$\begin{cases} m=\lambda a^2(b^2+c^2-a^2) \\ n=\lambda b^2(a^2+c^2-b^2) \\ k=\lambda c^2(a^2+b^2-c^2) \end{cases},$$

where $\lambda\in Q$, and (a,b,c) **is a primitive Heron tuple of a acute triangle.**

Assume that **the tuple** (m,n,k) **is a Heron tuple such that the corresponding Heron triangle** ΔMNK **has three rational angle bisectors, by picture 2 we can construct** a circle O and a triangle ABC。 Now from the proof of Theorem 1.3 we have

$\angle NMK=2\angle BAC=2A$， $\angle MKN=2C-\pi,\angle MNK=2B,$

it follows that C is an obtuse and the triangle ABC is an obtuse. By the argument of Theorem 1.3, we obtain the formula (4.2).

Finally, from picture 3 and the proof of Theorem 1.4, if ΔMNK **is an** isosceles triangle that **three rational angle bisectors, then we obtain a rational Heron right triangle ABC. From the proof of Theorem 1.4, we obtain the formula (4.3). Theorem 4.1 is proved.**

From Theorems 1.2-1.4, we can construct a Heron triangle with rational angle bisector from any Heron triangle (whether it is acute, obtuse or triangle). Moreover, Theorem 4.1 tells us that the lengths of three sides of any Heron triangle with rational angle bisectors can be represented as in the formulas (4.1) and (4.2), and also (4.3) when the

Heron triangle is isosceles.

So far, we have completely solved the problem of constructing Heron triangles with rational angle bisectors.

5. Perfect Cuboids

The following two famous problems are closely related with Heron triangles.

The Perfect Cuboid Problem (PCP):

Is there a rectangular box with all edges, face diagonals, and the main diagonal integers?

The Perfect Square Triangular Problem (PSTP):

Is there a triangle whose sides are perfect squares and whose angle bisectors are integers?

By using Theorem 4.1, we obtain the following two results on PSTP.

Theorem 5.1 Let (a,b,c) be a primitive Heron tuple, if the lengths of three sides of the primitive Heron triangle that is simlar to the triangle with three sides $a^2(b^2+c^2-a^2)$、$b^2(a^2+c^2-b^2)$、$c^2(a^2+b^2-c^2)$ are squares, then there exist positive integers x,y,z such that

$$\begin{cases} x^2+y^2=c^2 \\ x^2+z^2=b^2 \\ y^2+z^2=a^2 \end{cases}$$

And $x^2y^2+y^2z^2+z^2x^2$ is a square。Conversely, if there are positive integers x,y,z satisfying the above conditions, then the primitive triangle which is similar to the triangle with three sides $a^2(b^2+c^2-a^2)$、$b^2(a^2+c^2-b^2)$、$c^2(a^2+b^2-c^2)$ is a primitive Heron triangle whose lengths of three sides are squares and three angle bisectors are rational numbers.

Proof： Since (a,b,c) is a Heron tuple, then two of a,b,c are odd integers, the other is even and $(a,b,c)=1$。By symmetry, we may assume that b,c are odd integers and a is even.

Suppose **the lengths of three sides of the primitive Heron triangle which is similar to the triangle with three sides $a^2(b^2+c^2-a^2)$、$b^2(a^2+c^2-b^2)$、$c^2(a^2+b^2-c^2)$ are squares, then there are positive integers** λ,u,v,w such that

$$\begin{cases} a^2(b^2+c^2-a^2)=\lambda u^2 \\ b^2(a^2+c^2-b^2)=\lambda v^2 \\ c^2(a^2+b^2-c^2)=\lambda w^2 \end{cases}, \qquad (5.1)$$

where λ is square-free. It follows that there are positive integers x,y,z such that

$$\begin{cases} b^2+c^2-a^2=\lambda x^2 \\ a^2+c^2-b^2=\lambda y^2 \\ a^2+b^2-c^2=\lambda z^2 \end{cases}。$$

That is

$$\begin{cases} 2c^2=\lambda(x^2+y^2) \\ 2b^2=\lambda(x^2+z^2) \\ 2a^2=\lambda(y^2+z^2) \end{cases}。$$

Since $(a,b,c)=1$，then $\lambda \mid 2$。Comparing the parity of formula (5.1), we obtain $\lambda = 2$ and

$$\begin{cases} b^2+c^2-a^2=2x^2 \\ a^2+c^2-b^2=2y^2 \\ a^2+b^2-c^2=2z^2 \end{cases}。$$

Or

$$\begin{cases} x^2+y^2=c^2 \\ x^2+z^2=b^2 \\ y^2+z^2=a^2 \end{cases}。$$

Notice that the triangle whose lengths of three sides are $a^2(b^2+c^2-a^2)$、$b^2(a^2+c^2-b^2)$、$c^2(a^2+b^2-c^2)$ is a Heron triangle，for this triangle, we have the semi-perimeter $p=2(x^2y^2+y^2z^2+z^2x^2)$ and the area $S=x^2y^2z^2\sqrt{(x^2y^2+y^2z^2+z^2x^2)}$ 。 Therefore $x^2y^2+y^2z^2+z^2x^2$ is a square。

Conversely, if there are positive integers x,y,z such that $\begin{cases} x^2+y^2=c^2 \\ x^2+z^2=b^2 \\ y^2+z^2=a^2 \end{cases}$ and $x^2y^2+y^2z^2+z^2x^2$ is a square. It is not difficult to check that the triangles with three sides a^2x^2、b^2y^2、c^2z^2 is a Heron triangle whose lengths of three sides are squares **and three angle bisectors are rational numbers.**

Theorem 5.1 is proved.

Similarly, we have

Theorem 5.2 Let (a,b,c) be a primitive Heron tuple with 满足$c^2>a^2+b^2$，if the lengths of three sides of the primitive Heron

triangle that is simlar to the triangle with three sides $a^2(b^2+c^2-a^2)$ 、$b^2(a^2+c^2-b^2)$ 、$c^2(c^2-a^2-b^2)$ **are squares, then there exist positive integers** x,y,z **such that**

$$\begin{cases} x^2+y^2=c^2 \\ x^2-z^2=b^2 \\ y^2-z^2=a^2 \end{cases}$$

and $x^2y^2-y^2z^2-z^2x^2$ **is a square。Conversely, if there are positive integers** x,y,z **satisfying the above conditions, then the primitive triangle which is similar to the triangle with three sides** $a^2(b^2+c^2-a^2)$ 、 $b^2(a^2+c^2-b^2)$ 、 $c^2(c^2-a^2-b^2)$ **is a primitive Heron triangle whose lengths of three sides are squares and three angle bisectors are rational numbers.**

In 2000， Mexico mathematician Luca proved the following interesting theorem which implies that PCP is equivalent to TSPT.

Theorem L:（**[5] Luca 2000**） **PCP has a solution if and onlyif PSTP has a solution.**

By using Theorems 5.1, 5.2 and Theorem L, we have

Theorem 5.3:（1） **PCP has a solution if and only if there are positive integers** x,y,z

$$\begin{cases} x^2+y^2=c^2 \\ x^2-z^2=b^2 \\ y^2-z^2=a^2 \end{cases}$$

and $x^2y^2 - y^2z^2 - z^2x^2$ **is a square.**

（2）PCP has a solution if and only if there are positive integers x, y, z

$$\begin{cases} x^2 + y^2 = c^2 \\ x^2 + z^2 = b^2 \\ y^2 + z^2 = a^2 \end{cases}$$

and $x^2y^2 + y^2z^2 + z^2x^2$ **is a square.**

Proof: It follows obviously from Theorems 5.1， 5.2 and Theorem L.

Since we can obtain three formulae from three obtuse triangles, by Theorem 5.3, we know that the solution of the PCP Problem is equivalent to the solution of any equation of the four equations in Theorem 5.3. Therefore, if we can derive some results from the four equations, then we can derive the similar results on the famous PCP problem。This is a by-product of this research. We wish we can do something more on this topic in the near future.

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