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# KÄHLER $C$-SPACES AND QUADRATIC BISECTIONAL CURVATURE

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**Abstract**

In this article we give necessary and sufficient conditions for an irreducible Kähler $C$-space with $b_2 = 1$ to have nonnegative or positive quadratic bisectional curvature, assuming the space is not Hermitian symmetric. In the cases of the five exceptional Lie groups $E_6, E_7, E_8, F_4, G_2$, the computer package MAPLE is used to assist our calculations. The results are related to two conjectures of Li-Wu-Zheng.

## 1. Introduction

Let $(M^n, g)$ be a Kähler manifold of complex dimension $n$ and let $o \in M$. $M$ is said to have *nonnegative quadratic orthogonal bisectional curvature* at $o$ if for any unitary frame $e_i$ at $o$ and real numbers $\xi_i$ we have

$$\sum_{i,j} R_{i\bar{i}j\bar{j}}(\xi^i - \xi^j)^2 \geq 0. \tag{1.1}$$

Here $R_{i\bar{i}j\bar{j}} = R(e_i, \bar{e}_i, e_j, \bar{e}_j)$. Recall that $M$ is said to have nonnegative bisectional curvature at $o$ if for any $X, Y \in T_o^{(1,0)}(M)$, $R(X, \bar{X}, Y, \bar{Y}) \geq 0$, and $M$ is said to have nonnegative *orthogonal* bisectional curvature at $o$ if $R(X, \bar{X}, Y, \bar{Y}) \geq 0$ for all unitary pairs $X, Y \in T_o^{(1,0)}(M)$. Following [**16**] we abbreviate by $QB \geq 0$ for nonnegative quadratic orthogonal bisectional curvature, $B \geq 0$ for nonnegative bisectional curvature, and $B^\perp \geq 0$ for nonnegative orthogonal bisectional curvature. It is obvious that $B \geq 0 \Rightarrow B^\perp \geq 0 \Rightarrow QB \geq 0$. Note that in dimension $n = 2$, the conditions $B^\perp \geq 0$ and $QB \geq 0$ are the same.

It is well-known that compact manifolds with $B \geq 0$ have been completely classified by the works [**18**, **20**, **14**, **1**, **17**]. By these works, we know that any compact simply connected irreducible Kähler manifold with $B \geq 0$ is either biholomorphic to $\mathbb{CP}^n$ or is isometrically biholomorphic to an irreducible compact Hermitian symmetric space of rank at least 2. While the condition $B^\perp \geq 0$ seems weaker, by the works of Chen [**10**] (see also [**22**]) and Gu-Zhang [**13**] we know that a compact

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simply connected irreducible Kähler manifold with $B^\perp \geq 0$ is also either biholomorphic to $\mathbb{CP}^n$ or is isometrically biholomorphic to an irreducible compact Hermitian symmetric space of rank at least 2. In this sense, no new compact complex manifolds are introduced when we weaken the condition $B \geq 0$ to the condition $B^\perp \geq 0$.

The condition $QB \geq 0$ was first considered by Wu-Yau-Zheng [**24**] where they proved that on a compact Kähler manifold with $QB \geq 0$ any class in the boundary of the Kähler cone can be represented by a smooth closed (1,1) form which is everywhere nonnegative. There are other interesting properties satisfied by compact Kähler manifolds with $QB \geq 0$. A fundamental property of such manifolds, implicit from earlier works [**3**] (see [**8**] for additional references) is that all harmonic real (1,1) forms are parallel. Recently it has been proved in [**8**] that the scalar curvature of such a manifold must be nonnegative, and if the manifold is irreducible, then the first Chern class is positive.

The ultimate goal is to classify Kähler manifolds with $QB \geq 0$. For the compact case, a partial classification of the de Rham factors of the universal cover of such a manifold is given in [**8**]. Hence it remains to study the structure of compact simply connected irreducible Kähler manifolds with $QB \geq 0$. By the parallelness of real harmonic (1,1) forms mentioned above, such Kähler manifolds also have $b_2 = 1$ (see [**14**]). In view of the above results for $B^\perp \geq 0$, one may wonder if any new compact complex manifolds are introduced when we weaken the condition $B^\perp \geq 0$ to the condition $QB \geq 0$. To address this, Li, Wu, and Zheng [**16**] constructed the first example of a simply connected irreducible compact Kähler manifold having $QB \geq 0$, which does not support a Kähler metric having $B^\perp \geq 0$. Their example is $(B_3, \alpha_2)$, a classical Kähler $C$-space with second Betti number $b_2 = 1$. It was further conjectured that all Kähler $C$-spaces with second Betti number $b_2 = 1$ must have $QB \geq 0$, and the following conjectures were raised in [**16**]:

**Conjecture 1.1.** (1) *Any Kähler $C$-space with $b_2 = 1$ satisfies $QB \geq 0$ everywhere.*

(2) *A compact simply connected irreducible Kähler manifold $(M^n, g)$ with $QB \geq 0$ is biholomorphic to a Kähler $C$-space with $b_2 = 1$.*

(3) *In (2), if the manifold is not $\mathbb{CP}^n$, then $g$ is a constant multiple of the standard metric.*

A Kähler $C$-space is a compact simply connected Kähler manifold such that the group of holomorphic isometries acts transitively on the manifold; see [**21**, **15**]. There is a complete classification of Kähler $C$-spaces with $b_2 = 1$, and this is associated with the classification of simple complex Lie algebras which are just $A_n = \mathfrak{sl}_{n+1}, B_n = \mathfrak{so}_{2n+1}, C_n = \mathfrak{sp}_{2n}, D_n = \mathfrak{so}_{2n}$ and the exceptional cases $E_6, E_7, E_8, F_4, G_2$. Motivated by the work [**16**], we establish the following theorems related to conjectures (1) and (3). For the classical types we have the following:

**Theorem 1.1.**

(i) *The Kähler $C$-space $(B_n, \alpha_p)$, $n \geq 3$, $1 < p < n$ satisfies $QB \geq 0$ if and only if $5p + 1 \leq 4n$. Moreover, $QB > 0$ if and only if $5p + 1 < 4n$.*

(ii) *The Kähler $C$-space $(C_n, \alpha_p)$, $n \geq 3$, $1 < p < n$ satisfies $QB \geq 0$ if and only if $5p \leq 4n + 3$. Moreover, $QB > 0$ if and only if $5p < 4n + 3$.*

(iii) *The Kähler $C$-space $(D_n, \alpha_p)$, $n \geq 4$, $1 < p < n - 1$ satisfies $QB \geq 0$ if and only if $5p + 3 \leq 4n$. Moreover, $QB > 0$ if and only if $5p + 3 < 4n$.*

For the exceptional cases, we have the following:

**Theorem 1.2.**

(i) *The Kähler $C$-space $(G_2, \alpha_2)$ satisfies $QB > 0$.*

(ii) *The Kähler $C$-space $(F_4, \alpha_p)$, $1 \leq p \leq 4$ satisfies $QB \geq 0$ iff $p = 1, 2, 4$, in which cases $QB > 0$.*

(iii) *The Kähler $C$-space $(E_6, \alpha_p)$, $2 \leq p \leq 5$ satisfies $QB \geq 0$ iff $p = 2, 3, 5$, in which cases $QB > 0$.*

(iv) *The Kähler $C$-space $(E_7, \alpha_p)$, $1 \leq p \leq 6$ satisfies $QB \geq 0$ iff $p = 1, 2, 5$, in which cases $QB > 0$.*

(v) *The Kähler $C$-space $(E_8, \alpha_p)$, $1 \leq p \leq 8$ satisfies $QB \geq 0$ iff $p = 1, 2, 8$, in which cases $QB > 0$.*

We only consider Kähler $C$-spaces that are not Hermitian symmetric. According to Itoh [**15**], Theorem 1.1 and 1.2 include all such Kähler $C$-spaces with $b_2 = 1$. Here $QB > 0$ means that (1.1) is a strict inequality unless all $\xi_i$ are the same. Note that if $QB > 0$, then a small perturbation of the Kähler metric will still satisfy $QB > 0$; see Lemma 2.6 (and Remark 2.1). Hence conjecture (1) for the classical types is true only under some restrictions mentioned in Theorem 1.1, while conjecture (3) is too strong. Conjecture (2), however, may still be true in general.

Theorems 1.1 and 1.2 give more information on the curvature properties of Kähler $C$-spaces with $b_2 = 1$. It is well-known that $\mathbb{C}P^n$ has $B > 0$, and Hermitian symmetric spaces with rank at least 2 have $B \geq 0$ but not $B > 0$. All other Kähler $C$-spaces which are non-Hermitian symmetric spaces do not have $B \geq 0$ or even $B^{\perp} \geq 0$. On the other hand, Itoh [**15**] proved that a Kähler $C$-space with $b_2 = 1$ is a Hermitian symmetric space if and only if its curvature operator has at most two distinct eigenvalues. Our results show that as far as the sign of curvature is concerned, Kähler $C$-spaces with $b_2 = 1$ which are not Hermitian symmetric are further divided into two groups: some of them satisfy $QB \geq 0$ and others do not have such a property.

We give here an idea of the proof and refer to §2 for details. Consider a Lie algebra as above, and a corresponding system $\Delta \subset \mathbb{R}^n$ of root vectors in $\mathbb{R}^n$ where the induced Killing form is induced by the standard Euclidean inner product. Then each associated Kähler $C$-space corresponds to a certain subset of $\mathfrak{m}^+ \subset \Delta$ representing a unitary frame in which curvature approximation reduces to taking sums and inner products of the vectors in $\mathfrak{m}^+ \subset \Delta$ (we can calculate exact values in the case of bisectional curvatures). We combine this with symmetry, counting, and eigenvalue estimate arguments to obtain Theorem 1.1. For the five exceptional cases in Theorem 1.2, the computer package MAPLE was used to assist our calculations and details are provided in the appendix. Theorem 1.1 was proved in an earlier version of this article [**9**], and while similar in spirit, our proof here is somewhat simpler, eliminating the need for many calculations from the appendix of [**9**].

The organization of the paper is as follows. In §2 we will state basic properties and formulae for Kähler $C$-spaces that will be used throughout the paper. We will discuss the conditions $QB \geq 0$ and $QB > 0$ in general, then in relation to the Kähler $C$-spaces. In §3 we prove Theorem 1.1 for the classical Kähler $C$-spaces; details for some of the calculations in these sections can be found in the appendices of [**9**], which is an earlier version of this article. In §4 we present the details of our results on Theorem 1.2 for the exceptional Kähler $C$-spaces, with details of our use of MAPLE provided in the appendix.

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## 2. Basic facts

**2.1. The Kähler C-spaces and curvature formulae.** Consider a compact Kähler $C$-space $(M, \omega)$ with transitive holomorphic isometry group $G$, and suppose $b_2(M) = 1$. Then any real $(1,1)$ form $\rho$ on $M$ is given by $\rho = c\omega + \sqrt{-1}\partial\overline{\partial} f$ for some constant $c$ and function $f$ where $\omega$ is the Kähler form. Now if $\rho$ is $G$ invariant, then $\Delta_g f$ is also $G$ invariant and hence constant on $M$. Thus $f$ is constant on $M$ and $\rho = c\omega$. In particular, $g$ is the unique $G$ invariant Kähler metric on $M$ and it is Kähler Einstein. For more discussions on Kähler $C$-space, see [**2**, **15**, **21**, **16**].

Kähler $C$-spaces with second Betti number $b_2 = 1$ are obtained as follows (see [**4**, **5**, **6**, **15**, **16**, **21**]). Let $G$ be a simply connected, complex Lie group, and let $\mathfrak{g}$ be its Lie algebra with Cartan subalgebra $\mathfrak{h}$ and corresponding root system $\Delta \subset \mathfrak{h}^*$. Then $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta} \mathbb{C}E_\alpha$, where $E_\alpha$ is a root vector of $\alpha$. Let $l = \dim_{\mathbb{C}} \mathfrak{h}$ and fix a fundamental root system

$\alpha_1, \dots, \alpha_l \subset \Delta$. This gives an ordering of roots in $\Delta$. Let $\Delta^+$ and $\Delta^-$ be the set of positive and negative roots, respectively. Let $K$ be the Killing form for $\mathfrak{g}$. Then we may choose root vectors $\{E_\alpha\}, \alpha \in \Delta$ such that

$$K(E_\alpha, E_{-\alpha}) = -1, \alpha \in \Delta^+; \quad [E_\alpha, E_\beta] = n_{\alpha,\beta} E_{\alpha+\beta}$$

such that $n_{\alpha,\beta} = n_{-\alpha,-\beta} \in \mathbb{R}$ with $n_{\alpha,\beta} = 0$ if $\alpha + \beta$ is not a root. Together with a suitable basis in $\mathfrak{h}$, they form a *Weyl canonical basis* for $\mathfrak{g}$. Now for any $1 \le r \le l$, let

$$\Delta_r^+(k) = \{\sum_i n_i \alpha_i \in \Delta^+ | \ n_r = k\}, \quad \Delta_r^+ = \bigcup_{k>0} \Delta_r^+(k).$$

Let $P$ be the subgroup whose Lie algebra is $\mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta \setminus \Delta_r^+} \mathbb{C} E_\alpha$. Then $G/P$ is a complex homogeneous space having $b_2 = 1$. Now let

$$\mathfrak{m}_k^+ = \bigoplus_{\alpha \in \Delta_r^+(k)} \mathbb{C} E_\alpha, \ \ \mathfrak{m}_k^- = \bigoplus_{\alpha \in \Delta_r^-(k)} \mathbb{C} E_\alpha, \ \ \mathfrak{t} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta^+(0)} (\mathbb{C} E_\alpha \oplus \mathbb{C} E_{-\alpha}).$$

Then $\mathfrak{m}^+ = \bigoplus_{k>0} \mathfrak{m}_k^+$ can be identified with the tangent space of $G/P$. As given in [**4**, **15**, **16**], the $G$-invariant Kähler form on $G/P$ is given by:

**Lemma 2.1.**

(i) *In a Weyl canonical basis, let $\omega^\alpha$, $\omega^{\bar\alpha}$ be the dual of $E_\alpha$ and $\bar E_\alpha := -E_{-\alpha}$, $\alpha \in \Delta_r^+$. The $G$ invariant Kähler form on $G/P$ is*

$$g = 2 \sum_{k>0} k \sum_{\alpha \in \Delta_r^+(k)} \omega^\alpha \cdot \omega^{\bar\alpha} = \sum_{k>0} (-kK)|_{\mathfrak{m}_k^+ \times \mathfrak{m}_k^-}.$$

(ii) $[\mathfrak{t}, \mathfrak{m}_k^\pm] \subset \mathfrak{m}_k^\pm$, $[\mathfrak{m}_k^\pm, \mathfrak{m}_l^\pm] \subset \mathfrak{m}_{k+l}^\pm$, $[\mathfrak{m}_k^+, \mathfrak{m}_k^-] \subset \mathfrak{t}$. *If $k > l > 0$, then* $[\mathfrak{m}_k^+, \mathfrak{m}_l^-] \subset \mathfrak{m}_{k-l}^+$, $[\mathfrak{m}_k^-, \mathfrak{m}_l^+] \subset \mathfrak{m}_{k-l}^-$.

The Kähler $C$ space thus obtained is denoted as $(\mathfrak{g}, \alpha_r)$. Conversely, every Kähler $C$ space with $b_2 = 1$ can be obtained by the construction. Thus the set $\{e_\alpha := 1/\sqrt{k} E_\alpha\}$; $\alpha \in \Delta_k^+$, $k \ge 1$ forms a unitary basis for the tangent space of $(\mathfrak{g}, \alpha_r)$ in the metric $g$. We call this basis as a *Weyl frame.* To compute the curvature tensor in this frame, we have the following from Li-Wu-Zheng [**16**, Proposition 2.1], using the method in [**15**]. For the sake of completeness, we give a proof.

**Proposition 2.1.** *[Li-Wu-Zheng] Let $X^i \in \mathfrak{m}_i^+, Y^j \in \mathfrak{m}_j^+, Z^k \in \mathfrak{m}_k^+, W^l \in \mathfrak{m}_l^+$. Suppose $i + k = j + l$. Then*

(2.1)

$$R(X^i, \bar Y^j, Z^k, \bar W^l) = \left((k-j)\xi_{k-j} - \frac{kl}{i+k}\right) K([X^i, Z^k], [\bar Y^j, \bar W^l]) + \left(-(k-j)\xi_{k-j} + k\xi_{i-j} + l\xi_{j-i} + l\delta_{ij}\delta_{kl}\right) K([X^i, \bar Y^j], [Z^k, \bar W^l]).$$

$R(X^i, \bar Y^j, Z^k, \bar W_l) = 0$ *if $i + k \ne j + l$.*

Here $\xi_q = 1$ if $q > 0$ and $\xi_q = 0$ if $q \le 0$.

*Proof.* Note that $g(U,\bar V) = -kK(U,\bar V)$, on $\mathfrak{m}_k^+ \times \mathfrak{m}_k^-$ etc.
Then [**15**, p. 43]

(2.2)
$$\begin{aligned} R(X^i,\bar Y^j,Z^k,\bar W_l) =& g(R(X^i,\bar Y^j)Z^k,\bar W^l)\\ =& g([\Lambda(X^i),\Lambda(\bar Y^j)]Z^k,\bar W^l) - g(\Lambda([X^i,\bar Y^j]_{\mathfrak{m}})Z^k,\bar W^l)\\ &- g([[X^i,\bar Y^j]_{\mathfrak{t}},Z^k],\bar W^l), \end{aligned}$$

where $\Lambda(U)V = n'/(n+n')[U,V]_{\mathfrak{m}^+}$ if $U \in \mathfrak{m}_n^+, V \in \mathfrak{m}_{n'}^+$, and $\Lambda(\bar U)V = [\bar U, V]_{\mathfrak{m}^+}$, for all $U,V \in \mathfrak{m}^+$; see [**15**, p. 45]. Here $[U,V]_{\mathfrak{m}^+}$ is the component of $[U,V]$ in $\mathfrak{m}^+$. Now if $i+k = j+l$,

(2.3)
$$\begin{aligned} [\Lambda(X^i),\Lambda(\bar Y^j)]Z^k =& \left(\Lambda(X^i)\Lambda(\bar Y^j) - \Lambda(\bar Y^j)\Lambda(X^i)\right) Z^k\\ =& \Lambda(X^i)([\bar Y^j,Z^k]_{\mathfrak{m}^+}) - \frac{k}{i+k}\Lambda(\bar Y^j)([X^i,Z^k])\\ =& \frac{(k-j)}{l}\xi_{k-j}[X^i,[\bar Y^j,Z^k]] - \frac{k}{i+k}[\bar Y^j,[X^i,Z^k]]_{\mathfrak{m}^+}. \end{aligned}$$

Here each term is in $\mathfrak{m}_l^+$. Hence

(2.4)
$$\begin{aligned} &g([\Lambda(X^i),\Lambda(\bar Y^j)]Z^k,\bar W^l)\\ &\quad= -(k-j)\xi_{k-j}K([X^i,[\bar Y^j,Z^k]],\bar W^l) + \frac{kl}{i+k}K([\bar Y^j,[X^i,Z^k]],\bar W^l)\\ &\quad= -(k-j)\xi_{k-j}K(X^i,[[\bar Y^j,Z^k],\bar W^l]) + \frac{kl}{i+k}K([\bar Y^j,[X^i,Z^k]],\bar W^l)\\ &\quad= (k-j)\xi_{k-j}K(X^i,[[\bar W^l,\bar Y^j],Z^k] + [[Z^k,\bar W^l]],\bar Y^j)\\ &\qquad + \frac{kl}{i+k}K([\bar W^l,\bar Y^j],[X^i,Z^k]])\\ &\quad= -(k-j)\xi_{k-j}K([X^i,\bar Y^j],[Z^k,\bar W^l])\\ &\qquad + \left((k-j)\xi_{k-j} - \frac{kl}{i+k}\right)K([X^i,Z^k],[\bar Y^j,\bar W^l]). \end{aligned}$$

Now $[X^i,\bar Y^j]_{\mathfrak{m}}$ is in $\mathfrak{m}_{i-j}^+$ if $i > j$, and $\mathfrak{m}_{j-i}^-$ if $j > i$, and is 0 if $i = j$. So

$$g(\Lambda([X^i,\bar Y^j]_{\mathfrak{m}})Z^k,\bar W^l) = -\left(k\xi_{i-j} + l\xi_{j-i}\right)K\left([X^i,\bar Y^j],[Z^k,\bar W^l]\right).$$

Also, $[X^i,\bar Y^j]_{\mathfrak{t}} = 0$ unless $i = j$. If $i = j$, then $[[X^i,\bar Y^j]_{\mathfrak{t}},Z^k] \in \mathfrak{m}_k^+$. Hence

$$\begin{aligned} g([[X^i,\bar Y^j]_{\mathfrak{t}},Z^k],\bar W^l) &= \delta_{ij}\delta_{kl}g([[X^i,\bar Y^j],Z^k],\bar W^l)\\ &= -l\delta_{ij}\delta_{kl}K\left([X^i,\bar Y^j],[Z^k,\bar W^l]\right). \end{aligned}$$

Also, $R(X^i, \bar{Y}^j, Z^k, \bar{W}_l) = 0$ if $i + k \neq j + l$. q.e.d.

**Lemma 2.2.** *Same notations as in Proposition* 2.1*. Assume* $X, Z, W$ *are canonical Weyl basis vectors and* $Z \neq W$*; then*

$$R(X, \bar{X}, Z, \bar{W}) = 0.$$

*Proof.* Since $i = j$, the lemma is true if $k \neq l$ by Proposition 2.1. Hence we assume $k = l$. We first assume that $k \leq i$. Then

$$R(X, \bar{X}, Z, \bar{W}) = -\frac{k^2}{i+k} K([X, Z], [\bar{X}, \bar{W}]) + kK([X, \bar{X}], [Z, \bar{W}]).$$

Now let $X = E_\alpha$, $Z = E_\beta$, $W = E_\gamma$ with $\beta \neq \gamma$. Note that $\bar{E}_\alpha = -E_{-\alpha}$. Then $[X, Z] = n_{\alpha,\beta} E_{\alpha+\beta}$, $[X, W] = n_{\alpha,\gamma} E_{\alpha+\gamma}$. Hence

$$K([X, Z], [\bar{X}, \bar{W}]) = -n_{\alpha,\beta} n_{\alpha,\gamma} K(E_{\alpha+\beta}, E_{-\alpha-\beta}).$$

If $\alpha + \beta$ or $\alpha + \gamma$ is not a root, then $n_{\alpha,\beta} = 0$ or $n_{\alpha,\gamma} = 0$ and $K([X, Z], [\bar{X}, \bar{W}]) = 0$. Otherwise, both $E_{\alpha+\beta}$ and $E_{\alpha+\gamma}$ are canonical Weyl basis vectors and are in $\mathfrak{m}^+_{i+k}$ by Lemma 2.1. Since $\beta \neq \gamma$, and $K$ is proportional to $g$ on $\mathfrak{m}^+_{i+k} \times \mathfrak{m}^-_{i+k}$, we also have $K([X, Z], [\bar{X}, \bar{W}]) = 0$.

On the other hand, by the fact that $K([x, y], z) = K(z, [y, z])$, we have

$$K([X, \bar{X}], [Z, \bar{W}]) = K(X, [\bar{X}, [Z, \bar{W}]]). \tag{2.5}$$

Now $[Z, \bar{W}] = n_{\beta,-\gamma} E_{\beta-\gamma}$, $[\bar{X}, [Z, \bar{W}]] = n_{-\alpha,\beta-\gamma} n_{\beta,-\gamma} E_{-\alpha+\beta-\gamma}$. If $\beta - \gamma$ or $-\alpha + \beta - \gamma$ is not a root, then as before we have $K(X, [\bar{X}, [Z, \bar{W}]]) = 0$. Otherwise, by Lemma 2.1, $[Z, \bar{W}] \in \mathfrak{t}$ and $[\bar{X}, [Z, \bar{W}]] \in \mathfrak{m}^-_i$. Since $-\alpha + \beta - \gamma \neq -\alpha$, so as before

$$K([X, \bar{X}], [Z, \bar{W}]) = K([X, [\bar{X}, [Z, \bar{W}]) = 0.$$

Hence the lemma is true when $k \leq i$.

Suppose $i < k$. Then it is equivalent to prove $R(X, \bar{Y}, Z, \bar{Z}) = 0$, but assuming $i > k$ and $X \neq Y$. In this case,

$$R(X, \bar{Y}, Z, \bar{Z}) = -\frac{k^2}{i+k} K([X, Z], [\bar{Y}, \bar{Z}]) + kK([X, \bar{Y}], [Z, \bar{Z}]).$$

The previous argument implies the lemma is true in this case as well. q.e.d.

To use the formula in Proposition 2.1, we need to compute the Lie bracket and Killing form in the given Weyl basis. Now the Killing form $K$ is negative definite on $\mathfrak{h}$ and thus induces a positive definite bilinear form, denoted also by $K$, on the dual $\mathfrak{h}^*$. We can then identify $\mathfrak{h}^*$ with $\mathbb{R}^l$ (or a subspace of some $\mathbb{R}^n$) so that $K$ becomes the standard Euclidean inner product and the root system is represented by a subset $\Delta \subset \mathbb{R}^l$. It turns out that a corresponding Weyl basis $\{E_\alpha\}_{\alpha \in \Delta^+}$ exists in which the Lie bracket and Killing form are computed in terms of Euclidean inner products and addition of the vectors $\alpha$. We describe this in more detail below.

Let $\mathfrak{g}$ be a semi simple Lie algebra, and let $\Delta = \{\alpha, \beta, \ldots\} \subset \mathbb{R}^n$ be a corresponding root system with standard inner product $(\cdot,\cdot)$ corresponding to the induced Killing form $K$. To the positive roots there corresponds a *Chevalley basis* $\{X_\alpha, X_{-\alpha}, H_\alpha\}_{\alpha\in\Delta^+}$ where for each $\alpha$, $X_\alpha, X_{-\alpha}$ are root vectors for $\alpha, -\alpha$ respectively, $H_\alpha \in \mathfrak{h}$, and the following relations are satisfied (see [**19**, p. 51]):

(2.6)
$$
\begin{aligned}
[X_\alpha, X_{-\alpha}] =& H_\alpha,\\
[X_\alpha, X_\beta] =& \begin{cases} N_{\alpha,\beta} X_{\alpha+\beta}, & \text{if } \alpha+\beta \text{ is a root, } N_{\alpha,\beta} = -N_{-\alpha,-\beta}\ ;\\ 0, & \text{if } \alpha+\beta \neq 0 \text{ is not a root.} \end{cases}\\
& N_{\alpha,\beta} = \pm (p+1),\ p \text{ is the largest integer so that } \beta - p\alpha \text{ is a root.}\\
[H_\alpha, X_\beta] =& \beta(H_\alpha) X_\beta.
\end{aligned}
$$

We also have:

(a) $\beta(H_\alpha) = 2\frac{(\beta,\alpha)}{(\alpha,\alpha)}$, [**11**, p. 337];

(b) $K(H_\alpha, H_\alpha) = 2K(X_\alpha, X_{-\alpha})$, [**11**, p. 207].

By these and [**11**, p. 207–208], we have the formulas

$$
\begin{aligned}
K(H_\alpha, H_{-\alpha}) &= \frac{4}{(\alpha,\alpha)},\\
K(X_\alpha, X_{-\alpha}) &= \frac{2}{(\alpha,\alpha)},\\
K(H_\alpha, H_\beta) &= \frac{4(\alpha,\beta)}{|\alpha|^2|\beta|^2},
\end{aligned}
\tag{2.7}
$$

where in the last equation we have used the first equation of (2.6), (2.5), (a), and the second formula in (2.7).

**Lemma 2.3.** *For positive roots $\alpha$, let*

$$
E_\alpha = \frac{|\alpha|}{\sqrt{2}} X_\alpha, E_{-\alpha} = -\frac{|\alpha|}{\sqrt{2}} X_{-\alpha}. \tag{2.8}
$$

*Then for positive roots $\alpha, \beta$*

$$
\begin{aligned}
K(E_\alpha, E_{-\alpha}) =& -1,\\
[E_\alpha, E_\beta] =& n_{\alpha,\beta} E_{\alpha+\beta},\\
[E_{-\alpha}, E_{-\beta}] =& n_{-\alpha,-\beta} E_{-\alpha-\beta},\\
[E_\alpha, E_{-\beta}] =& n_{\alpha,-\beta} E_{\alpha-\beta}, \text{if } \alpha-\beta \neq 0,
\end{aligned}
\tag{2.9}
$$

*where*

$$
n_{-\alpha,-\beta} = n_{\alpha,\beta} = \begin{cases} \frac{|\alpha||\beta|}{\sqrt{2}|\alpha+\beta|} N_{\alpha,\beta}, & \textit{if } \alpha+\beta \textit{ is a root,}\\ 0, & \textit{if } \alpha+\beta \textit{ is not a root,} \end{cases}
$$

*and*

$$n_{\alpha,-\beta} = \begin{cases} \frac{|\alpha||\beta|}{\sqrt{2}|\alpha-\beta|} N_{\alpha,-\beta}, & \textit{if } \alpha-\beta \textit{ is a positive root,} \\ -\frac{|\alpha||\beta|}{\sqrt{2}|\alpha-\beta|} N_{\alpha,-\beta}, & \textit{if } \alpha-\beta \textit{ is a negative root,} \\ 0, & \textit{if } \alpha-\beta \textit{ is not a root.} \end{cases}$$

*Hence $\{E_\alpha\}$ form a Weyl canonical basis.*

*Proof.* It is easy to see that $K(E_\alpha, E_{-\alpha}) = -1$. If $\alpha+\beta$ is a root, then

$$\begin{aligned}[E_\alpha, E_\beta] =& \frac{|\alpha||\beta|}{2}[X_\alpha, X_\beta] \\ =& N_{\alpha,\beta}\frac{|\alpha||\beta|}{2}X_{\alpha+\beta} \\ =& \frac{|\alpha||\beta|}{\sqrt{2}|\alpha+\beta|}N_{\alpha,\beta}E_{\alpha+\beta} = n_{\alpha,\beta}E_{\alpha+\beta}\end{aligned}$$

$$\begin{aligned}[E_{-\alpha}, E_{-\beta}] =& \frac{|\alpha||\beta|}{2}[X_{-\alpha}, X_{-\beta}] \\ =& N_{-\alpha,-\beta}\frac{|\alpha||\beta|}{2}X_{-\alpha-\beta} \\ =& N_{\alpha,\beta}\frac{|\alpha||\beta|}{\sqrt{2}|\alpha+\beta|}E_{-\alpha-\beta} = n_{\alpha,\beta}E_{-\alpha-\beta}\end{aligned}$$

where $n_{\alpha,\beta} = \frac{|\alpha||\beta|}{\sqrt{2}|\alpha+\beta|}N_{\alpha,\beta}$. Here we have used the fact that $N_{\alpha,\beta} = -N_{-\alpha,-\beta}$ and $X_{-\alpha-\beta} = -\frac{\sqrt{2}}{|\alpha+\beta|}E_{-\alpha-\beta}$. If $\alpha+\beta$ is not a root, then $[E_,a, E_\beta] = 0$.

If $\alpha-\beta \neq 0$ and is a positive root, then

$$\begin{aligned}[E_\alpha, E_{-\beta}] &= \frac{|\alpha||\beta|}{2}[X_\alpha, X_{-\beta}] = N_{\alpha,-\beta}\frac{|\alpha||\beta|}{2}X_{\alpha-\beta} \\ &= \frac{|\alpha||\beta|}{\sqrt{2}|\alpha-\beta|}N_{\alpha,-\beta}E_{\alpha-\beta}.\end{aligned}$$

If $\alpha-\beta \neq 0$ and is a negative root, then

$$\begin{aligned}[E_\alpha, E_{-\beta}] &= \frac{|\alpha||\beta|}{2}[X_\alpha, X_{-\beta}] = N_{\alpha,-\beta}\frac{|\alpha||\beta|}{2}X_{\alpha-\beta} \\ &= -\frac{|\alpha||\beta|}{\sqrt{2}|\alpha-\beta|}N_{\alpha,-\beta}E_{\alpha-\beta}.\end{aligned}$$

q.e.d.

Now let $\eta \in \Delta^+$, and consider the Kähler $C$-space $(\mathfrak{g}, \eta)$ with corresponding Weyl frame (unitary frame for $(\mathfrak{g}, \eta)$) $e_\alpha = \frac{1}{\sqrt{k}}E_\alpha$ for $\alpha \in$

$\Delta^+(k)$. For any positive roots $\alpha, \beta$, define

$$\begin{cases} \widetilde{N}_{\alpha,\beta} = \frac{|\alpha||\beta|}{|\alpha+\beta|} N_{\alpha,\beta}; \\ \widetilde{N}_{\alpha,-\beta} = \frac{|\alpha||\beta|}{|\alpha-\beta|} N_{\alpha,-\beta}, \quad \text{if } \alpha - \beta \neq 0. \end{cases} \tag{2.10}$$

We can now combine Lemma 2.2 and Proposition 2.1 with Lemma 2.3, (2.6), and (2.7) to obtain the following from [**15**, Proposition 2.4]. In the above setting, we denote the curvature tensor $R(e_\alpha, \bar{e}_\beta, e_\gamma, \bar{e}_\delta)$ also by $R(\alpha, \bar{\beta}, \gamma, \bar{\delta})$ or $R_{\alpha,\bar{\beta},\gamma,\bar{\delta}}$.

**Lemma 2.4.** *Let $\alpha \in \Delta^+(i)$, $\beta \in \Delta^+(j)$, with $i \leq j$, and let $R_{\alpha\bar{\alpha}\beta\bar{\beta}} = R(e_\alpha, \bar{e}_\alpha, e_\beta, \bar{e}_\beta)$. Then*

$$R_{\alpha\bar{\alpha}\beta\bar{\beta}} = \frac{1}{j}\left((\alpha,\beta) + \frac{1}{2}\frac{i}{i+j}\widetilde{N}^2_{\alpha,\beta}\right). \tag{2.11}$$

Next let us consider $R(\alpha, \bar{\beta}, \gamma, \bar{\delta}) = R(e_\alpha, \bar{e}_\beta, e_\gamma, \bar{e}_\delta)$ with $\alpha - \beta$, $\gamma - \delta \neq 0$.

**Lemma 2.5.** *Let $e_\alpha \in \Delta^+(i)$, $e_\beta \in \Delta^+(j)$, $e_\gamma \in \Delta^+(k)$, and $e_\delta \in \Delta^+(l)$.*

1) *If $\alpha - \beta \neq \delta - \gamma$, then $R(\alpha, \bar{\beta}, \gamma, \bar{\delta}) = 0$.*

2) *If $\alpha - \beta = \delta - \gamma \neq 0$, then*

(2.12)

$$\begin{aligned} R(\alpha, \bar{\beta}, \gamma, \bar{\delta}) = & -\frac{1}{2\sqrt{ijkl}}\left[\left((k-j)\xi_{k-j} - \frac{kl}{i+k}\right)\widetilde{N}_{\alpha,\gamma}\widetilde{N}_{\beta,\delta}\right] \\ & + \frac{1}{2\sqrt{ijkl}} \\ & \quad \left[(-(k-j)\xi_{k-j} + k\xi_{i-j} + l\xi_{j-i} + l\delta_{ij}\delta_{kl})\,\widetilde{N}_{\alpha,-\beta}\widetilde{N}_{\gamma,-\delta}.\right] \\ =: & R_1(\alpha, \bar{\beta}, \gamma, \bar{\delta}) + R_2(\alpha, \bar{\beta}, \gamma, \bar{\delta}). \end{aligned}$$

*Proof.* (1) follows from Lemma 2.2, and the fact that $K(E_\alpha, E_\beta) = 0$ unless $\alpha + \beta = 0$.

(2) Note that $\bar{e}_\alpha = -e_{-\alpha}$, etc. First assume that $\alpha + \gamma$ and $\alpha - \beta$ are both roots. By Lemma 2.3,

$$\begin{aligned} [e_\alpha, e_\gamma] = & \frac{1}{\sqrt{ik}}[E_\alpha, E_\gamma] \\ = & n_{\alpha,\gamma}\frac{1}{\sqrt{ik}}E_{\alpha+\gamma} \\ = & N_{\alpha,\gamma}\frac{|\alpha||\gamma|}{\sqrt{2ik}}E_{\alpha+\gamma}. \end{aligned}$$

Similarly,

$$[e_{-\beta}, e_{-\delta}] = N_{\beta,\delta}\frac{|\gamma||\delta|}{\sqrt{2jl}}E_{-\beta-\delta},$$

We may assume that $\alpha-\beta$ is a positive root, then $\gamma-\delta$ is a negative root.

$$\begin{aligned}[e_\alpha, e_{-\beta}] =& N_{\alpha,-\beta}\frac{|\alpha||\beta|}{\sqrt{2ij}}E_{\alpha-\beta},\\ [e_\gamma, e_{-\delta}] =& -N_{\gamma,-\delta}\frac{|\gamma||\delta|}{\sqrt{2kl}}E_{\gamma-\delta}.\end{aligned}$$

Since $\alpha+\gamma=\beta+\delta$ and $K(E_\sigma, E_{-\sigma})=-1$, we see that (2) is true by Proposition 2.1. The cases where $\alpha+\gamma$ or $\alpha-\beta$ is not a root can be proved similarly.

q.e.d.

**2.2. The condition $QB\ge 0$.** We first discuss the condition $QB\ge 0$ on a Kähler manifold $(M,\omega)$ with Kähler form $\omega$. We will also consider the condition $QB>0$ at $p$, which we define as: $QB\ge 0$ at $p$ with strict inequality in (1.1) provided not all $\xi_i's$ are the same. Now define the following bilinear forms on the space $\Omega^{1,1}_{\mathbb{R}}(M)$ of real $(1,1)$ forms on $M$:

$$F(\eta,\sigma)=\sum_{i,j,k,l}R_{i\bar{j}k\bar{l}}\rho^{i\bar{l}}\sigma^{k\bar{j}}=\sum_{i,j,k,l}R_{i\bar{l}k\bar{j}}\rho^{i\bar{l}}\sigma^{k\bar{j}}$$

$$G(\eta,\sigma)=\frac{1}{2}(R_{i\bar{j}}g_{k\bar{l}}+R_{k\bar{l}}g_{i\bar{j}})\rho^{i\bar{l}}\sigma^{k\bar{j}}$$

where $\rho^{i\bar{l}},\sigma^{k\bar{j}}$ are the local components of $\rho,\sigma$ with indices raised. Clearly, $G$ and $F$ are well defined real symmetric bilinear forms on $\Omega^{1,1}_{\mathbb{R}}(p)$ for any $p$. Now let $\theta_A$ be a unitary frame at any $p$ with co-frame $\eta_A$ and let $a_A$ be real numbers. Take $X=\sum_A\sqrt{-1}a_A\eta_A\wedge\overline{\eta_A}\Omega^{1,1}_{\mathbb{R}}(p)$. Then a simple calculation gives

$$\begin{aligned}G(X,X)-F(X,X)&=\sum_A R_{A\bar{A}}a_A^2-\sum_{A,B}R_{A\bar{A}B\bar{B}}a_Aa_B\\ &=\frac{1}{2}\sum_{A,B}R_{A\bar{A}B\bar{B}}(a_A-a_B)^2.\end{aligned}\tag{2.13}$$

The following was observed by Yau [**26**].

**Lemma 2.6.** *At any point $p$ we have*
(a) *$QB\ge 0$ if and only if $G-F\ge 0$.*
(b) *$QB>0$ if and only if $G-F>0$ on $\Omega^{1,1}_{\mathbb{R}}(p)\setminus\mathbb{R}\omega(p)$.*

*Here $\Omega^{1,1}_{\mathbb{R}}(p)\setminus\mathbb{R}\omega(p)$ are the real $(1,1)$ forms at $p$ which are not multiples of the Kähler form.*

*Proof.* We first prove (a). The fact that $G-F\ge 0$ implies $QB\ge 0$ follows immediately from (2.13) and the fact that $\theta_A$ and $a_A$ are arbitrary. Conversely, suppose $QB\ge 0$ and let $X$ be any real $(1,1)$ form at $p$. Then we can always diagonalize $X$. Namely, there exists a unitary frame $e_A$ with co-frame $\eta_A$ such that $X=\sum_A\sqrt{-1}a_A\eta_A\wedge\overline{\eta_A}$. Now

(2.13), and the assumption $QB \geq 0$, immediately implies $G(X,X) - F(X,X) \geq 0$.

Now we prove (b). The proof is basically the same in part (a) once we observe that $X \in \mathbb{R}\omega(p)$ if and only if: for every unitary frame $e_A$ at $p$ with co-frame $\eta_A$ we have $X = c\sum_A \sqrt{-1}\eta_A \wedge \overline{\eta_A}$ for some real constant $c$. The fact that $G - F > 0$ on $\Omega^{1,1}_{\mathbb{R}}(p) \setminus \mathbb{R}\omega(p)$ implies $QB > 0$ now follows immediately from (2.13) and the fact that $\theta_A$ and $a_A$ are arbitrary. Conversely, suppose $QB > 0$ and let $X \in \Omega^{1,1}_{\mathbb{R}}(p) \setminus \mathbb{R}\omega(p)$. Then there exists a unitary frame $e_A$ with co-frame $\eta_A$ such that $X = \sum_A \sqrt{-1}a_A\eta_A \wedge \overline{\eta_A}$ with $a'_A s$ not all the same. Now (2.13) and the assumption $QB > 0$ immediately implies $G(X,X) - F(X,X) > 0$.

This concludes the proof of the lemma. q.e.d.

**Remark 2.1.** Thus $QB > 0$ if and only if $G - F$ is positive in the orthogonal complement of $\mathbb{R}\omega$. In particular, if $(M, g)$ is a compact Kähler manifold with $QB > 0$, then a Kähler metric which is a small perturbation of $g$ will also satisfy $QB > 0$.

**Remark 2.2.** Viewed as an endomorphism on $\Omega^{1,1}_{\mathbb{R}}(M)$, $G - F$ is in fact the curvature term in the Weitzenböck identity for real $(1,1)$ forms: $\Delta_g - \Delta$ is given by $G - F$ up to a positive constant multiple where $\Delta_g$ is the Bochner Laplacian with respect to $g$ and $\Delta$ is the Laplace-Beltrami operator. The standard Bochner technique and Lemma 2.6 then give: *all real harmonic (1,1) forms on $M$ are parallel provided* $QB \geq 0$; moreover, $dim(H^{1,1}_{\mathbb{R}}(M)) = 1$ *provided* $QB > 0$ where $H^{1,1}_{\mathbb{R}}(M)$ is the space of real harmonic (1,1) forms on $M$. See §1 for a reference to these facts and their implicit appearance in earlier works.

By Lemma 2.6, to check whether $QB \geq 0$ (or $QB > 0$), it is sufficient to consider $G - F \geq 0$ in a unitary frame of our choice. In the case of Kähler $C$-spaces, the natural choice is a Weyl frame. By Lemmas 2.2 and 2.6, we have:

**Corollary 2.1.** *On a Kähler $C$-space, let* $\mathrm{Ric} = \mu g$ *and let $e_A$ be a Weyl frame. Then $QB \geq 0$ if and only if the largest eigenvalues of the quadratic forms $\sum_{A,B} R_{A\bar{A}B\bar{B}}x_Ax_B$, with $x_A$'s real, and $\sum_{\substack{A,B,C,D;\\ A\neq B, C\neq D}} R_{A\bar{B}C\bar{D}}x_{AB}x_{CD}$, with $\overline{x_{AB}} = x_{BA}$, are at most $\mu$. $QB > 0$ if $QB \geq 0$ and the eigenvalue $\mu$ of $\sum_{A,B} R_{A\bar{A}B\bar{B}}x_Ax_B$ is simple and the largest eigenvalue of $\sum_{\substack{A,B,C,D;\\ A\neq B, C\neq D}} R_{A\bar{B}C\bar{D}}x_{AB}x_{CD}$ is less than $\mu$.*

The following simple fact will be used throughout the paper to estimate the largest eigenvalue of a quadratic form.

**Lemma 2.7.** *[row sums] Let $x_1, \ldots, x_n$, $a_1, \ldots, a_n$, and $\lambda$ be real or complex numbers. Suppose $|x_k| = \max\{|x_i|\ 1 \leq i \leq n\} > 0$ and*

$$\lambda x_k = \sum_{i=1}^{n} a_i x_i.$$

*Then*

$$|\lambda| \leq \sum_{j=1}^{n} |a_i|.$$

*In particular, if $\lambda$ is an eigenvalue of an $n \times n$ matrix $(a_{ij})$, then*

$$|\lambda| \leq \max_i \left( \sum_{j=1}^{n} |a_{ij}| \right).$$

We also note the following modification of Lemma 2.7 which will only be needed in a few exceptional cases.

**Lemma 2.8.** *[weighted row sums] Let $\lambda$ be an eigenvalue of an $n \times n$ matrix $A = (a_{ij})$ such that $|\lambda| > 0$. Let $\mu > 0$ be a positive number. Define $b_j^s$ inductively: $b_j^{(0)} = 1$, and*

$$b_j^{(s+1)} = \min(1, \sum_l |a_{jl}| \frac{b_l^{(s)}}{\mu}).$$

*Then for all $s \geq 0$,*

$$|\lambda| \leq \max\{\max_i \left( \sum_{j=1}^{n} |a_{ij}| b_j^{(s)} \right), \mu\}. \tag{2.14}$$

*In particular, if for some $s \geq 0$,*

$$\max_i \left( \sum_{j=1}^{n} |a_{ij}| b_j^{(s)} \right) < \mu, \tag{2.15}$$

*then $|\lambda| < \mu$.*

*Proof.* First we show that $b_j^{(s+1)} \leq b_j^{(s)}$ for all $j$. Note that by definition $1 \geq b_j^{(s)} \geq 0$. It is obviously true that $b_j^{(1)} \leq 1 = b_j^{(0)}$. Suppose $b_j^{(s+1)} \leq b_j^{(s)}$ for all $j$; then

$$b_j^{(s+2)} = \min(1, \sum_l |a_{jl}| b_l^{(s+1)}/\mu) \leq \min(1, \sum_l |a_{jl}| b_l^{(s)}/\mu) = b_j^{(s+1)}.$$

To prove the lemma: If $|\lambda| \leq \mu$, then the lemma is true. Suppose $|\lambda| > \mu$. Let $x_i$ be the components of an eigenvector of $A$ with eigenvalue $\lambda$. Suppose, without loss of generality, that $\max_i |x_i| = 1$. We claim that for all $s \geq 0$,

$$|x_i| \leq b_i^{(s)}$$

for all $i \geq 1$. For $s = 1$, then, for any $j$,

$$\lambda x_j = \sum_l a_{jl} x_l$$

and

$$|x_j| \le \frac{1}{|\lambda|}\sum_l |a_{jl}x_l| \le \frac{1}{|\lambda|}\sum_l |a_{jl}|.$$

So $|x_j| \le b_j^{(1)}$ because $|\lambda| > \mu$ and $|x_j| \le 1$. Now suppose $|x_j| \le b_j^{(s)}$ for all $j$. Then as before,

$$|\lambda||x_j| \le \sum_l |a_{jl}||x_l| \le \sum_l |a_{jl}|b_l^{(s)}$$

and

$$|x_j| \le \sum_l |a_{jl}|\frac{b_l^{(s)}}{\mu}.$$

Hence $|x_j| \le b_j^{(s+1)}$. Hence the claim is true.

Now we may assume without loss of generality that $|x_1| = 1$. (2.14) is true for $s = 0$ by the previous lemma. For $s \ge 1$,

$$|\lambda| = |\lambda x_1| \le \sum_l |a_{1l}||x_l| \le \sum_l |a_{1l}b_l^{(s)}.$$

Hence (2.14) is also true in this case.

If (2.15) is true, then it is still true if $\mu$ is replaced by $\mu - \epsilon$ for $\epsilon > 0$ small enough. Then by (2.14), $|\lambda| \le \mu - \epsilon < \mu$.

q.e.d.

## 3. Kähler $C$-spaces of classical type

According to [**15**], the Kähler $C$-spaces with $b_2 = 1$ of classical type which are not Hermitian symmetric spaces are $(B_n, \alpha_p)$, with $n \ge 3$, $1 < p < n$, $(C_n, \alpha_p)$, with $n \ge 3$, $1 < p < n$, and $(D_n, \alpha_p)$, with $n \ge 4$, $1 < p < n - 1$. For each Lie algebra $B_n, C_n, D_n$ below, we assume an identification has been made between $\mathfrak{h}^*$, the dual Cartan subalgebra, and $V = \mathbb{R}^n$ so that the induced Killing form corresponds to the Euclidean inner product $(\cdot,\cdot)$. We will then present the corresponding root system $\Delta$ as a set of vectors in $V = \mathbb{R}^n$. We refer to [**7**] for details.

**3.1. The spaces $(B_n, \alpha_p)$.** We first consider the space $(B_n, \alpha_p)$, with $n \ge 3$, $1 < p < n$. Let $V = \mathbb{R}^n$ and let $\varepsilon_i$ be the standard basis on $V$. The root system for $B_n$ is

$$\Delta = \{\pm\varepsilon_i \pm \varepsilon_j | 1 \le i, j \le n, i \ne j\} \cup \{\pm\varepsilon_i |\ 1 \le i \le n\} \tag{3.1}$$

Simple positive roots are

$$\alpha_1 = \varepsilon_1 - \varepsilon_2, \alpha_2 = \varepsilon_2 - \varepsilon_3, \ldots, \alpha_{n-1} = \varepsilon_{n-1} - \varepsilon_n, \alpha_n = \varepsilon_n. \tag{3.2}$$

Positive roots are

$$\Delta^+ = \{\varepsilon_i + \varepsilon_j\}_{i<j} \cup \{\varepsilon_i - \varepsilon_j\}_{i<j} \cup \{\varepsilon_i\}. \tag{3.3}$$

In terms of the $\alpha_i$'s, the positive roots are

$$\begin{aligned}\varepsilon_i =&\alpha_i + \cdots + \alpha_n\\ \varepsilon_i + \varepsilon_j =&\alpha_i + \cdots + \alpha_{j-1} + 2\alpha_j + \cdots + 2\alpha_n,\ i < j\\ \varepsilon_i - \varepsilon_j =&\alpha_i + \cdots + \alpha_{j-1},\ i < j.\end{aligned} \tag{3.4}$$

Let $1 < p < n$. Recall that

$$\Delta_p^+(k) = \left\{ \alpha \in \Delta^+ |\ \alpha = k\alpha_p + \sum_{i \neq p} m_i\alpha_i, m_i \geq 0, m_i \in Z \right\}.$$

By (3.3) and (3.4), we have

$$\begin{aligned}\Delta_p^+(1) =&\{\varepsilon_a |\ 1 \leq a \leq p\} \bigcup \{\varepsilon_a + \varepsilon_i | 1 \leq a \leq p, p+1 \leq i \leq n\}\\ &\bigcup \{\varepsilon_a - \varepsilon_i | 1 \leq a \leq p, p+1 \leq i \leq n\},\end{aligned} \tag{3.5}$$

$$\Delta_p^+(2) =\{\varepsilon_a + \varepsilon_b | 1 \leq a < b \leq p\}. \tag{3.6}$$

$$\Delta_p^+(k) = \emptyset \tag{3.7}$$

for $k \geq 3$. The dimension of $(B_m, \alpha_p)$ is $\frac{1}{2}p(4n - 3p + 1)$. We denote the elements of the $\Delta_p^+(k)$'s by: $X_{ai} = \varepsilon_a - \varepsilon_i$, $Y_{ai} = \varepsilon_a + \varepsilon_i$, $1 \leq a \leq p$, $p + 1 \leq i \leq n$; $U_a = \varepsilon_a$, $W_{ab} = \varepsilon_a + \varepsilon_b$, $1 \leq a, b \leq p$. In the following $a, b, \ldots$ will range from 1 to $p$ and $i, j, \ldots$ will range from $p + 1$ to $n$. Thus

$$\begin{aligned}&\Delta_p^+(1) =\{X_{ai}\}_{1\leq a\leq p; p+1\leq i\leq n} \bigcup \{Y_{ai}\}_{1\leq a\leq p; p+1\leq i\leq n} \bigcup \{U_a\}_{1\leq a\leq p},\\ &\Delta_p^+(2) =\{W_{ab}\}_{1\leq a<b\leq p}.\end{aligned}$$

Now recall that $N_{\alpha,\beta} = \pm(p + 1)$ where $p$ is the largest integer, so that $\beta - p\alpha$ is a root, and also the definition of $\widetilde{N}_{\alpha,\beta}$ in (2.10).

**Lemma 3.1.** *Let $\alpha, \beta$ be positive roots in $(B_n, \alpha_p)$; then $\widetilde{N}_{\alpha,\beta} = \sqrt{2}\,\mathrm{sgn}(N_{\alpha,\beta})$. If $\alpha - \beta \neq 0$, then $\widetilde{N}_{\alpha,-\beta} = \sqrt{2}\,\mathrm{sgn}(N_{\alpha,-\beta})$.*

*Proof.* Note that if $\sigma$ is a root, then either $|\sigma|^2 = 1$ or $|\sigma|^2 = 2$. We begin by proving the first part of the lemma. Let $\alpha, \beta$ be positive roots. We may assume $\alpha + \beta$ is a root; otherwise the first part of the lemma is obviously true.

Suppose $|\alpha|^2 = |\beta|^2 = 1$ and suppose $|\alpha + \beta|^2 = 1$; then $(\alpha, \beta) = -\frac{1}{2}$, and this is impossible because one can see that $(\alpha, \beta)$ is an integer. Hence $|\alpha + \beta|^2 = 2$ and $(\alpha, \beta) = 0$. So $\alpha - \beta$ is also a root [**11**, p. 324]. $\alpha - 2\beta$ is not a root because $|\alpha - 2\beta|^2 = 5$. Hence $N_{\alpha,\beta} = \pm 2$. Therefore, by the definition of $\widetilde{N}_{\alpha,\beta}$ in (2.10), $\widetilde{N}_{\alpha,\beta} = \sqrt{2}\,\mathrm{sgn}(N_{\alpha,\beta})$.

Suppose $|\alpha|^2 = 1$ and $|\beta|^2 = 2$. As before, one can prove that $(\alpha, \beta) = -1$ and $N_{\alpha,\beta} = \pm 1$. Hence $\widetilde{N}_{\alpha,\beta} = \sqrt{2}\,\mathrm{sgn}(N_{\alpha,\beta})$.

Suppose $|\alpha|^2 = |\beta|^2 = 2$. As before, one can prove that $(\alpha, \beta) = -1$, $N_{\alpha,\beta} = \pm 1$, and hence $\widetilde{N}_{\alpha,\beta} = \sqrt{2}\,\mathrm{sgn}(N_{\alpha,\beta})$.

The case for $\widetilde{N}_{\alpha,-\beta}$ can be proved similarly. q.e.d.

By Lemmas 2.4, 2.5, and 3.1 and the fact that $R(\alpha, \bar\beta, \gamma, \bar\delta) = R(\alpha, \bar\delta, \gamma, \bar\beta)$, we have:

**Corollary 3.1.** *Let $\alpha \in \Delta^+(i)$, $\beta \in \Delta^+(j)$, $\gamma \in \Delta^+(k)$, and $\delta \in \Delta^+(l)$.*

1)

$$R_{\alpha\bar\alpha\beta\bar\beta} = \begin{cases} (\alpha,\beta) + \frac{1}{2}(\mathrm{sgn}(N_{\alpha,\beta})^2), & i = j = 1, \\ \frac{1}{2}(\alpha,\beta), & i = 1, j = 2; \\ \frac{1}{2}(\alpha,\beta), & i = j = 2, \end{cases}$$

2) *If $\alpha - \beta \neq \delta - \gamma$, then $R(\alpha, \bar\beta, \gamma, \bar\delta) = 0$.*

3) *If $\alpha - \beta = \delta - \gamma \neq 0$, then for $(i,j,k,l) = (1,1,1,1)$,*

$$R(\alpha, \bar\beta, \gamma, \bar\delta) = \begin{cases} \frac{1}{2}\mathrm{sgn}(N_{\alpha,\gamma})\mathrm{sgn}(N_{\beta,\delta}), & \text{if } \alpha - \beta \text{ is not a root,} \\ \mathrm{sgn}(N_{\alpha,-\beta})\mathrm{sgn}(N_{\gamma,-\delta}), & \text{if } \alpha + \gamma \text{ is not a root,} \\ -\frac{1}{2}\mathrm{sgn}(N_{\alpha,\gamma})\mathrm{sgn}(N_{\delta,\beta}), & \text{if } \beta - \gamma \neq 0 \text{ is not a root.} \end{cases}$$

*For other cases,*

$$R(\alpha, \bar\beta, \gamma, \bar\delta) = \begin{cases} \frac{1}{2}\mathrm{sgn}(N_{\alpha,-\beta})\mathrm{sgn}(N_{\gamma,-\delta}), & \text{if } (i,j,k,l) = (1,1,2,2), \\ \frac{1}{2}\mathrm{sgn}(N_{\alpha,-\beta})\mathrm{sgn}(N_{\gamma,-\delta}), & \text{if } (i,j,k,l) = (2,2,2,2), \\ \frac{1}{2}\mathrm{sgn}(N_{\alpha,-\beta})\mathrm{sgn}(N_{\gamma,-\delta}) & \text{if } (i,j,k,l) = (1,2,2,1). \end{cases}$$

To compute the Ricci curvature, we know that $\mathrm{Ric} = \mu g$ and thus

$$\begin{aligned} \mu &= \mathrm{Ric}(W_{12}, \bar W_{12}) \\ &= \sum_{a,i} \left[ R(W_{12}, \bar W_{12}, X_{ai}, \bar X_{ai}) + R(W_{12}, \bar W_{12}, Y_{ai}, \bar Y_{ai}) \right] \\ &\quad + \sum_a R(W_{12}, \bar W_{12}, U_a, \bar U_a) + \sum_{a<b} R(W_{12}, \bar W_{12}, W_{ab}, \bar W_{ab}) \\ &= \frac{1}{2}\left[2(n-p) + 2(n-p)\right] + 1 + \frac{1}{2}\left(p + (p-2)\right) \\ &= 2n - p. \end{aligned}$$

**Lemma 3.2.** *Let $\lambda$ be the largest eigenvalue of the quadratic form*

$$\sum_{A,B} R_{A\bar A B\bar B} x_A x_B$$

*in the Weyl frame, where $x_A$ are real.*

(a) *$\lambda \leq 2n - p$ if and only if $5p + 1 \leq 4n$.*

(b) *If $5p + 1 < 4n$, then $\lambda = (2n - p)$ iff the corresponding eigenvector satisfies $x_A = x_B$ for all $A, B$.*

(c) *If $5p+1 = 4n$, then there is an eigenvector with eigenvalue $(2n-p)$ such that $x_A \neq x_B$ for some $A \neq B$.*

*Proof.* We begin with the proof of $(a)$. Let $v = (x_A)$ be an eigenvector corresponding to the largest eigenvalue $\lambda$ for the quadratic form. Assume the components satisfy $max_A|x_A| = 1$. Let us denote the components $x_A$ more specifically by $x_{ai}, y_{ai}, a \leq p < i; u_a, a \leq p; t_{ab}, a < b \leq p$, and let us denote $R(X_{ai}, \bar{X}_{ai}, X_{bj}, \bar{X}_{bj})$ by $R(X_{ai}, X_{bj})$ etc. Then $P(v) = \sum_{A,B} R_{A\bar{A}B\bar{B}} x_A x_B$ is equal to:

(3.8)

$$\begin{aligned} P(v) = &\sum_{a,b\leq p<i,j} R(X_{ai}, X_{bj}) x_{ai} x_{bj} + \sum_{a,b\leq p<i,j} R(Y_{ai}, Y_{bj}) y_{ai} y_{bj} \\ &+ \sum_{a,b\leq p<i,j} R(X_{ai}, Y_{bj}) x_{ai} y_{bj} + \sum_{a,b\leq p<i,j} R(Y_{ai}, X_{bj}) y_{ai} x_{bj} \\ &+ 2\sum_{a,c\leq p<i} R(X_{ai}, U_c) x_{ai} u_c + 2\sum_{a,c\leq p<i} R(Y_{ai}, U_c) y_{ai} u_c \\ &+ 2\sum_{a<b,c\leq p<i} R(w_{ab}, X_{ci}) t_{ab} x_{ci} + 2\sum_{a<b,c\leq p<i} R(w_{ab}, Y_{ci}) t_{ab} y_{ci} \\ &+ \sum_{a,b\leq p} R(U_a, U_b) u_a u_b + 2\sum_{a<b,c\leq p} R(w_{ab}, U_c) t_{ab} u_c \\ &+ \sum_{a<b\leq p,c<d\leq p} R(W_{ab}, W_{cd}) t_{ab} t_{cd}. \end{aligned}$$

From Corollary 3.1, it is easy to see that $R(X_{ai}, X_{bj}) = R(Y_{ai}, Y_{bj})$, $R(X_{ai}, Y_{bj}) = R(Y_{ai}, X_{bj})$, $R(X_{ai}, U_b) = R(Y_{ai}, U_b)$, $R(X_{ai}, W_{bc}) = R(Y_{ai}, W_{bc})$. We see that if we interchange $x_{ai}$ and $y_{ai}$ for all $a, i$ and obtain a vector $w$, then $P(v) = P(w)$ and $|v| = |w|$. We may then assume that either $x_{ai} = y_{ai}$ for all $a, i$, or by considering $v - w$, that $x_{ai} = -y_{ai}$ and $u_a = t_{ab} = 0$ for all $a, b$.

Suppose $|u_a| = 1$ for some $a$. We may assume that $u_a = 1$. By Corollary 3.1, $R(U_a, \bar{U}_a, x, \bar{x}) \geq 0$ because $(U_a, x) \geq 0$ for all $x \in \Delta_p^+(k), k = 1, 2$.

(3.9)

$$\begin{aligned} \lambda u_a = &\sum_{b\leq p} R(U_a, U_b) u_b + \sum_{b\leq p<i} R(X_{bi}, U_a) x_{bi} + \sum_{b\leq p<i} R(Y_{bi}, U_a) y_{bi} \\ &+ \sum_{c<d\leq p} R(w_{cd}, U_a) t_{cd}. \end{aligned}$$

Notice that the coefficients are all nonnegative and the sum is just $\mathrm{Ric}(U_a, \bar{U}_a) = 2n - p$. Hence $\lambda \leq 2n - p$. Moreover, if $\lambda = 2n - p$, then we must in fact have

$$x_{a,i} = y_{a,i} = u_b = t_{cd} = 1 \tag{3.10}$$

for all $a, b \leq p < i$ and $c < d \leq p$.

Since $(W_{ab}, x) \geq 0$ for all $x \in \Delta_p^+(k), k = 1, 2$. We have a similar result when $|t_{ab}| = 1$.

Suppose $x_{ai} = 1$ for some $a, i$.

**Case 1** ($x_{bj} = y_{bj}$ for all $b, j$.): As above, we have

$$\begin{aligned}\lambda x_{ai} =& \sum_{b,j} R(X_{ai}, X_{bj}) x_{bj} + \sum_{b,j} R(X_{ai}, Y_{bj}) y_{bj} + \sum_{b} R(X_{ai}, U_b) u_b \\ &+ \sum_{c<d} R(X_{ai}, W_{cd}) t_{cd}.\end{aligned} \tag{3.11}$$

Since $x_{bj} = y_{bj}$, this equation is the same as:

$$\begin{aligned}\lambda x_{ai} =& \sum_{b,j} \frac{1}{2}(R(X_{ai}, X_{bj}) + R(X_{ai}, Y_{bj})) x_{bj} + \sum_{b,j} \frac{1}{2}(R(X_{ai}, X_{bj}) \\ &+ R(X_{ai}, Y_{bj})) y_{bj} + \sum_{b} R(X_{ai}, U_b) u_b + \sum_{c<d} R(X_{ai}, W_{cd}) t_{cd}.\end{aligned} \tag{3.12}$$

By Corollary 3.1, $R(X_{ai}, X_{bj}) + R(X_{ai}, Y_{bj}) \geq 0$ since $(X_{ai}, X_{bj} + Y_{bj}) = 2\delta_{ab} \geq 0$. Hence the coefficients are all nonnegative. Also, the sum of the coefficients is still the Ricci curvature $2n - p$. Hence we have $\lambda \leq 2n - p$ as before, and if equality holds, then (3.10) is true.

**Case 2** ($x_{bj} = -y_{bj}$ and $u_c = w_{cd} = 0$ for all $b, c, d, j$.): Then

$$\begin{aligned}\lambda x_{ai} &= \sum_{b,j} R(X_{ai}, X_{bj}) x_{bj} + \sum_{b,j} R(X_{ai}, Y_{bj}) y_{bj} \\ &= \sum_{b,j} \left(R(X_{ai}, X_{bj}) - R(X_{ai}, Y_{bj})\right) x_{bj}.\end{aligned} \tag{3.13}$$

By Corollary 3.1, $R(X_{ai}, X_{bj}) - R(X_{ai}, Y_{bj}) \geq 0$ because $(X_{ai}, X_{bj} - Y_{bj}) = 2\delta_{ij}$. Hence the coefficients are all nonnegative. The sum of the coefficients is:

$$\begin{aligned}\sum_{b,j} \left((\delta_{ab} + \delta_{ij}) - \left(\delta_{ab} - \delta_{ij} + \frac{1}{2}\delta_{ij}(1 - \delta_{ab})\right)\right) &= p + \frac{1}{2}(p+1) \\ &= \frac{1}{2}(3p+1).\end{aligned} \tag{3.14}$$

Here we have used the fact that $X_{ai} + X_{bj}$ is not a root, and $X_{ai} + Y_{bj}$ is a root if and only if $b \neq a$ and $j = i$. Hence if $5p + 1 \leq 4n$, then $\lambda \leq 2n - p$. Moreover, if $5p + 1 < 4n$ then $\lambda < 2n - p$

Now suppose $5p + 1 > 4n$. Let $v$ be such that $x_{ai} = -y_{ai} = 1$, $u_a = w_{ab} = 0$ for all $a, b$. Then

$$P(v) = 2 \sum_{a,b \leq p < i,j} (R(X_{ai}, X_{bj}) - R(X_{ai}, Y_{bj})) = p(n-p)(3p+1).$$

On the other hand, $|v|^2 = 2p(n-p)$. Hence $P(v) > (2n-p)|v|^2$ because $5p+1 > 4n$.

The case that $y_{ai} = 1$ for some $a, i$ is similar. This completes the proof of (a).

To prove (b), suppose $5p+1 < 4n$. Then $\lambda \leq 2n-p$, and as $(2n-p)$ is always an eigenvalue, we have $\lambda = 2n-p$. Let $v$ be the corresponding eigenvector with components $x_{ai}, y_{ai}, u_a, t_{cd}$. Thus $P(v) = \lambda|v|^2$. The above proof then shows that if $x_{ai} = y_{ai}$ for all $a, i$, then (3.10) must be true, while if $x_{ai} \neq y_{ai}$ for some $a, i$, then we must have $\lambda < 2n-p$, which is impossible by our assumption. Hence (b) is true.

To prove (c), suppose $5p+1 = 4n$. Then $\lambda = 2n-p$ in this case too. Let $v$ be such that $x_{ai} = -y_{ai} = 1$, $u_a = w_{ab} = 0$ for all $a, b$. Then the computations above give $P(v) = \lambda|v|^2$. Since $x_{ai} \neq y_{ai}$, (c) is true. q.e.d.

**Lemma 3.3.** *Let $\lambda$ be the largest eigenvalue of the quadratic form*

$$\sum_{A,B,C,D;A\neq B,C\neq D} R_{A\bar{B}C\bar{D}} x_{AB} x_{CD}$$

*in the Weyl frame, where $x_{AB} = \overline{x_{BA}}$.*

(a) *If $5p+1 \leq 4n$, then $\lambda \leq 2n-p$.*
(b) *If $5p+1 < 4n$, then $\lambda < 2n-p$.*

*Proof.* We want to estimate

$$S_{AB} = \sum_{x\neq y} |R_{A\bar{B}y\bar{x}}| \tag{3.15}$$

for each case of $A, B$. Note that $S_{AB} = S_{BA}$. Recall the following properties of the curvature from Corollary 3.1, which we repeat here for convenience of reference:

(C1) If $A - B \neq x - y$, then $R_{A\bar{B}y\bar{x}} = 0$.
(C2) If neither $A - B$ nor $A + y$ are roots, then $R_{A\bar{B}y\bar{x}} = 0$.

In each case we will use these to reduce the terms in (3.15) as much as possible. Then Corollary 3.1 will be used to calculate the absolute values of the remaining curvature terms.

**Case (i)** $A = X_{ai}, B = X_{bj}$ with $(a, i) \neq (b, j)$.

Note that $A, B \in \Delta_p^+(1)$. By (C1) we may assume that $x, y \in \Delta_p^+(1)$ or $x, y \in \Delta_p^+(2)$. Note that the sum of the coordinates of $X$'s is 0, that the sum of the coordinates of $Y$'s is 2, that the sum of the coordinates of $U$'s is 1, and that the sum of the coordinates of $W$'s is 2. Thus by

(C1), (3.15) reduces to:

(3.16)
$$\begin{aligned} S_{AB} = &\sum_{c,k,d,l} |R(X_{ai}, \bar{X}_{bj}, X_{ck}, \bar{X}_{dl})| + \sum_{c,k,d,l} |R(X_{ai}, \bar{X}_{bj}, Y_{ck}, \bar{Y}_{dl})| \\ &+ \sum_{c,d} |R(X_{ai}, \bar{X}_{bj}, U_c, \bar{U}_d)| + \sum_{c,d,e,f} |R(X_{ai}, \bar{X}_{bj}, W_{cd}, \bar{W}_{ef})| \\ =: &I + II + III + IV. \end{aligned}$$

If $a \neq b$, $i \neq j$, then: All terms in $III, IV$ are zero by (C1). All terms in $I$ are zero by (C1) except for $|R(X_{ai}, \bar{X}_{bj}, X_{bj}, \bar{X}_{ai})|$, which is zero by $(C2)$. By (C1), the only non-zero term in $II$ is $|R(X_{ai}, \bar{X}_{bj}, Y_{bi}, \bar{Y}_{aj})| = \frac{1}{2}$. Hence $S_{AB} = 1/2 < 2n - p$ because $p < n$.

If $a = b$, $i \neq j$, then: All terms in $III, IV$ are zero by (C1). By (C1), the only non-zero terms in $I$ are $|R(X_{ai}, \bar{X}_{aj}, X_{cj}, \bar{X}_{ci})|$ for any $c$, leaving $I = p$. By (C1), the only non-zero terms in $II$ are $|R(X_{ai}, \bar{X}_{aj}, Y_{cj}, \bar{Y}_{ci})|$ for any $c$, giving a contribution of 1 from the case $c = a$ and $1/2(p-1)$ from the cases $c \neq a$. Hence $S_{AB} = p+1+\frac{1}{2}(p-1) = 3p/2+1/2 \leq 2n-p$ if and only if $5p+1 \leq 4n$, and $S_{AB} < 2n-p$ if and only if $5p+1 < 4n$.

If $a \neq b$, $i = j$, we may assume that $a < b$, then: By (C1), the only non-zero terms in $I$ are $|R(X_{ai}, \bar{X}_{bi}, X_{bk}, \bar{X}_{ak})|$ for any $k$, leaving $I = n-p$. By (C1), the only non-zero terms in $II$ are $|R(X_{ai}, \bar{X}_{bi}, Y_{bk}, \bar{Y}_{ak})|$, leaving a contribution of $1/2$ when $k = i$ and a contribution of $n-p-1$ for the cases when $k \neq i$. By (C1), the only non-zero term in $III$ is $|R(X_{ai}, \bar{X}_{bi}, U_b, \bar{U}_a)|$, leaving $III = 1$. By (C1), the only non-zero terms in $IV$ are $|R(X_{ai}, \bar{X}_{bi}, W_{bc}, \bar{W}_{ac})|$ for $c > b$, or $|R(X_{ai}, \bar{X}_{bi}, W_{cb}, \bar{W}_{ca})|$ for $c < a$, or $|R(X_{ai}, \bar{X}_{bi}, W_{bc}, \bar{W}_{ac})|$ for $a < c < b$, in which cases the contributions to $IV$ are $1/2(p-b), 1/2(a-1), 1/2(b-a-1)$ respectively. Hence $S_{AB} = (n-p)+\frac{1}{2}+(n-p-1)+1+\frac{1}{2}(p-b)+\frac{1}{2}(a-1)+\frac{1}{2}(b-a-1) = 2n - \frac{3}{2}p - \frac{1}{2} < 2n - p$.

**Case (ii)** $A = X_{ai}, B = Y_{bj}$.

Note that $A, B \in \Delta_p^+(1)$. By (C1), $x, y \in \Delta_p^+(1)$ or $x, y \in \Delta_p^+(2)$ and (3.15) reduces to:

$$S_{AB} = \sum_{c,k,d,l} |R(X_{ai}, \bar{Y}_{bj}, Y_{ck}, \bar{X}_{dl})|. \tag{3.17}$$

If $a \neq b$, $i \neq j$: then by (C1), the only non-zero term in (3.17) is given by $|R(X_{ai}, \bar{Y}_{bj}, Y_{bi}, \bar{X}_{aj})| = \frac{1}{2}$. Thus $S_{AB} = \frac{1}{2} < 2n - p$.

If $a = b$, $i \neq j$, then by (C1), the only non-zero terms in (3.17) are $|R(X_{ai}, \bar{Y}_{aj}, Y_{cj}, \bar{X}_{ci})|$ or $|R(X_{ai}, \bar{Y}_{aj}, Y_{ci}, \bar{X}_{cj})|$, for any $c$. In the first case the contribution to (3.17) is $p$, and in the second case the contribution to (3.17) is 1 when $c = a$ and $\frac{1}{2}(p-1)$ from the cases $c \neq a$. Thus $S_{AB} = p + 1 + \frac{1}{2}(p-1) = \frac{3}{2}p + \frac{1}{2} \leq 2n - p$ if and only if $5p + 1 \leq 4n$, and $S_{AB} < 2n - p$ if and only if $5p + 1 < 4n$.

If $a \neq b$, $i = j$, then by (C1), the only non-zero term in (3.17) is given by $|R(X_{ai}, \bar{Y}_{bi}, Y_{bi}, \bar{X}_{bi})| = \frac{1}{2}$. Thus $S_{AB} = \frac{1}{2} < 2n - p$.

If $a = b$, $i = j$, then by (C1), the only non-zero terms in (3.17) are $|R(X_{ai}, \bar{Y}_{ai}, Y_{ci}, \bar{X}_{ci})|$ for any $c$. Thus $S_{AB} = \frac{1}{2}(p-1) < 2n - p$.

**Case (iii)** $A = X_{ai}, B = U_b$.

Note that $A, B \in \Delta_p^+(1)$. By (C1), $x, y \in \Delta_p^+(1)$ or $x, y \in \Delta_p^+(2)$ and (3.15) reduces to:

(3.18)
$$S_{AB} = \sum_{c,d,j} |R(X_{ai}, \bar{U}_b, U_c, \bar{X}_{dj})| + \sum_{c,d,j} |R(X_{ai}, \bar{U}_b, Y_{cj}, \bar{U}_d|) =: I + II$$

If $a \neq b$, then: All terms in $I$ are zero by (C2). By (C1), the only non-zero term in $II$ is $|R(X_{ai}, \bar{U}_b, Y_{bi}, \bar{U}_a)|$ leaving $II = 1$ . Thus $S_{AB} = 1 < 2n - p$.

If $a = b$ then: By (C1), the only non-zero terms in $I$ are $|R(X_{ai}, \bar{U}_a, U_c, \bar{X}_{ci})|$ for any $c$, leaving $I = p$. By (C1), the only non-zero terms in $II$ are given by $|R(X_{ai}, \bar{U}_a, Y_{ci}, \bar{U}_c)|$ for any $c$, and the contribution to $II$ is 1 when $c = a$ and is $\frac{1}{2}(p-1)$ from the cases $c \neq a$. Thus $S_{AB} = \frac{3}{2}p + \frac{1}{2} < 2n - p$.

**Case (iv)** $A = X_{ai}, B = W_{bc}$.

Note that $A \in \Delta_p^+(1), B \in \Delta_p^+(2)$. By (C1), $x \in \Delta_p^+(1)$, and $y \in \Delta_p^+(2)$ and (3.15) reduces to:

$$S_{AB} = \sum_{d,j,e,f} |R(X_{ai}, \bar{W}_{bc}, W_{ef}, \bar{X}_{dj})|. \tag{3.19}$$

If $a = b$, then by (C1), the only non-zero terms in (3.19) are given by $|R(X_{ai}, \bar{W}_{ac}, W_{dc}, \bar{X}_{di})|$ for $d < c$, or $|R(X_{ai}, \bar{W}_{ac}, W_{cf}, \bar{X}_{fi})|$ for $f > c$. In the first case the contribution is $c - 1$, and in the second case, is $p - 1 - c$. Thus $S_{AB} = p - 1 < 2n - p$.

If $a \neq b$ then by (C1) and (C2), the only non-zero terms in (3.19) are when $a = c$, in which case we get, as above, that $S_{AB} = p - 1 < 2n - p$.

**Case (v)** $A = Y_{ai}, B = Y_{bj}$, $(a, i) \neq (b, j)$. Similar to (i).

**Case (vi)** $A = Y_{ai}, B = U_b$. Similar to (iii).

**Case (vii)** $A = Y_{ai}, B = W_{bc}$. Similar to (iv).

**Case (viii)** $A = U_a$, $B = U_b$, $a < b$.

From (C1) it is not hard to see here that (3.15) reduces to:

$$\begin{aligned} S_{AB} = &\sum_{c,d,k,l} |R(U_a, \bar{U}_b, X_{ck}, \bar{X}_{dl})| + \sum_{c,d,k,l} |R(U_a, \bar{U}_b, Y_{ck}, \bar{Y}_{dl})| \\ &+ \sum_{c,d} |R(U_a, \bar{U}_b, U_c, \bar{U}_d)| + \sum_{c,d,e,f} |R(U_a, \bar{U}_b, W_{cd}, \bar{W}_{ef})| \\ = &\, I + II + III + IV. \end{aligned}$$

Now by (C1) the only non-zero terms in $I$ are $|R(U_a, \bar{U}_b, X_{bi}, \bar{X}_{ai})|$ for any $i$, leaving $I = n-p$. We similarly get $II = n-p$. By (C1), the only non-zero term in $III$ is $|R(U_a, \bar{U}_b, U_b, \bar{U}_a)|$, leaving $III = \frac{1}{2}$. By (C1), the only non-zero terms in $IV$ are $|R(U_a, \bar{U}_b, W_{bc}, \bar{W}_{ac})|$ for $c > b$, or $|R(U_a, \bar{U}_b, W_{cb}, \bar{W}_{ca})|$ for $c < a$, or $|R(U_a, \bar{U}_b, W_{cb}, \bar{W}_{ac})|$ for $a < c < b$, in which cases the respective contributions to $IV$ are $1/2(p-b), 1/2(a-1), 1/2(b-a-1)$ respectively. Thus $S_{AB} = (n-p) + (n-p) + \frac{1}{2} + \frac{1}{2}\left[(p-b) + (a-1) + (b-a-1)\right] = 2n - \frac{3}{2}p - \frac{1}{2} < 2n - p$.

**Case (ix)** $A = U_a$, $B = W_{bc}$, $b < c$.

Note that $A \in \Delta_p^+(1)$ and $B \in \Delta_p^+(2)$. By (C1), $x \in \Delta_p^+(1)$ and $y \in \Delta_p^+(2)$ and (3.15) reduces to:

$$S_{AB} = \sum_{d,e,f} |R(U_a, \bar{W}_{bc}, W_{de}, \bar{U}_f)|. \tag{3.20}$$

Note that $A + x$ is never a root, and thus by (C2), the only non-zero terms are when $a = b$ or $a = c$. If $a = b$, then by (C1), the only non-zero terms are $|R(U_a, \bar{W}_{ac}, W_{dc}, \bar{U}_d)|$ for $d < c$, or $|R(U_a, \bar{W}_{ac}, W_{cd}, \bar{U}_d)|$ for $c < d$, in which cases the respective contributions to $S_{AB}$ are $1/2(c-1), 1/2(p-c)$ respectively. Thus when $a = b$, and similarly when $a = c$, we have $S_{AB} = \frac{1}{2}(c-1) + \frac{1}{2}(p-c) = \frac{1}{2}(p-1) < 2n - p$.

**Case (x)** $A = W_{ab}, B = W_{cd}$, $(a,b) \neq (c,d)$. We may assume that $a \leq c$.

Note that $A, B \in \Delta_p^+(2), B \in \Delta_p^+(2)$. By (C1), $x, y \in \Delta_p^+(1)$ or $x, y \in \Delta_p^+(2)$ and (3.20) reduces to:

$$\begin{aligned} S_{AB} = &\sum_{e,f,i,j} (|R(W_{ab}, \bar{W}_{cd}, X_{ei}, \bar{X}_{fj})| + \sum_{e,f,i,j} (|R(W_{ab}, \bar{W}_{cd}, Y_{ei}, \bar{Y}_{fj})| \\ &+ \sum_{e,f} |R(W_{ab}, \bar{W}_{cd}, U_e, \bar{U}_f)| + \sum_{e,f,g,h} |R(W_{ab}, \bar{W}_{cd}, W_{ef}, \bar{W}_{gh})| \\ = &I + II + III + IV. \end{aligned}$$

Note that $A + y$ is never a root for any positive root $y$. Thus by (C2), all terms in $I, II, III, IV$ are zero unless either $a = c$ or $b = c$ or $b = d$. Assume that $a = c$, and without loss of generality that $b < d$. By (C1), the only non-zero terms in $I$ are $|R(W_{ab}, \bar{W}_{ad}, X_{di}, \bar{X}_{bi})|$ for any $i$, leaving $I = \frac{1}{2}(n-p)$. We similarly get $II = \frac{1}{2}(n-p)$. By (C1), the only non-zero term in $III$ is $|R(W_{ab}, \bar{W}_{cd}, U_d, \bar{U}_b)|$, leaving $III = \frac{1}{2}$. By (C1), the only non-zero terms in $IV$ are $|R(W_{ab}, \bar{W}_{ad}, W_{de}, \bar{W}_{be})|$ for $e > d$, or $|R(W_{ab}, \bar{W}_{ad}, W_{ed}, \bar{W}_{eb})|$ for $e < b$, or $|R(W_{ab}, \bar{W}_{ad}, W_{ed}, \bar{W}_{be})|$ for $b < e < d$, in which cases the respective contributions to $IV$ are $1/2(p-d), 1/2(b-1), 1/2(d-b-1)$. Thus when $a = c$, and similarly when $b = c$ or $b = d$, we have $S_{AB} = \frac{1}{2}(n-p) + \frac{1}{2}(n-p) + \frac{1}{2} + \frac{1}{2}\left[(p-d) + (b-1) + (d-b-1)\right] = n - \frac{1}{2}p - \frac{1}{2} < 2n - p$.

This completes the proof of the lemma. q.e.d.

**Theorem 3.1.** *The Kähler $C$-space $(B_n, \alpha_p)$, $n \geq 3$, $1 < p < n$ satisfies $QB \geq 0$ if and only if $5p + 1 \leq 4n$. Moreover, $QB > 0$ if and only if $5p + 1 < 4n$.*

*Proof.* The first statement follows from part (a) of Lemmas 3.2 and 3.3 and Corollary 2.1. For the second statement, note that $QB > 0$ iff $G - F > 0$ on $\Omega^{1,1}_{\mathbb{R}}(p) \setminus \mathbb{R}\omega(p)$ by Lemma 2.6. On the other hand, here $G = (2n - p)Id$ on $\Omega^{1,1}_{\mathbb{R}}(p) \setminus \mathbb{R}\omega(p)$, and thus by parts (b) and (c) of Lemma 3.2 and part (b) of Lemma 3.3, we have $G - F > 0$ iff $5p + 1 < 4n$. Thus we have $QB > 0$ if and only if $5p + 1 < 4n$. q.e.d.

**3.2. The spaces $(D_n, \alpha_p)$.** In this section we will consider the space $(D_n, \alpha_p)$, with $n \geq 4$, $1 < p < n - 1$. Let $V = \mathbb{R}^n$ and $\varepsilon_i$ be as before. The root system for $D_n$ is

$$\Delta = \{\pm\varepsilon_i \pm \varepsilon_j | 1 \leq i, j \leq n, i \neq j\}. \tag{3.21}$$

Positive roots are

$$\Delta^+ = \{\varepsilon_i + \varepsilon_j\}_{i<j} \cup \{\varepsilon_i - \varepsilon_j\}_{i<j}. \tag{3.22}$$

Simple positive roots are

$$\alpha_1 = \varepsilon_1 - \varepsilon_2, \alpha_2 = \varepsilon_2 - \varepsilon_3, \ldots, \alpha_{n-1} = \varepsilon_{n-1} - \varepsilon_n, \alpha_n = \varepsilon_{n-1} + \varepsilon_n. \tag{3.23}$$

In terms of the $\alpha_i$'s, the positive roots are

$$\begin{aligned} \varepsilon_i + \varepsilon_j &= \alpha_i + \cdots + \alpha_{j-1} + 2\alpha_j + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n, \\ &\quad i < j \leq n - 2 \\ \varepsilon_i + \varepsilon_{n-1} &= \alpha_i + \cdots + \alpha_n, \ i < n - 1 \\ \varepsilon_i + \varepsilon_n &= \alpha_i + \cdots + \alpha_{n-2} + \alpha_n, \ i < n - 1 \\ \varepsilon_{n-1} + \varepsilon_n &= \alpha_n \\ \varepsilon_i - \varepsilon_j &= \alpha_i + \cdots + \alpha_{j-1}, \ i < j. \end{aligned} \tag{3.24}$$

Let $1 < p < n - 1$. By (3.22) and (3.24) we have

$$\begin{aligned} \Delta^+_p(1) =& \{\varepsilon_a - \varepsilon_i | 1 \leq a \leq p, p + 1 \leq i \leq n\} \\ & \bigcup \{\varepsilon_a + \varepsilon_i | 1 \leq a \leq p, p + 1 \leq i \leq n\}, \end{aligned} \tag{3.25}$$

$$\Delta^+_p(2) = \{\varepsilon_a + \varepsilon_b | 1 \leq a < b \leq p\}. \tag{3.26}$$

$$\Delta^+_p(k) = \emptyset \tag{3.27}$$

for $k \geq 0$. The dimension is $\frac{1}{2}p(4n - 3p - 1)$. The structure of the roots is similar to $(B_n, \alpha_p)$, except that $U_a$'s do not appear. Hence the computations are basically the same. In this case $\mathrm{Ric} = (2n - p - 1)$.

**Theorem 3.2.** *The Kähler $C$-space $(D_n, \alpha_p)$, $n \geq 4$, $1 < p < n - 1$ satisfies $QB \geq 0$ if and only if $5p + 3 \leq 4n$. Moreover, $QB > 0$ if and only if $5p + 3 < 4n$.*

**Remark 3.1.** As in the $B$ cases, one can see that $(D_n, \alpha_p)$ does not satisfy $B^\perp \geq 0$.

**3.3. The spaces $(C_n, \alpha_p)$.** We will consider the space $(C_n, \alpha_p)$, with $n \geq 3$, $1 < p < n$. Let $V = \mathbb{R}^n$ and $\varepsilon_i$ be as before. The root system for $C_n$ is

$$\Delta = \{\pm\varepsilon_i \pm \varepsilon_j | 1 \leq i, j \leq n\}. \tag{3.28}$$

Positive roots are

$$\Delta^+ = \{\varepsilon_i + \varepsilon_j\}_{i \leq j} \cup \{\varepsilon_i - \varepsilon_j\}_{i<j}. \tag{3.29}$$

Simple positive roots are

$$\alpha_1 = \varepsilon_1 - \varepsilon_2, \alpha_2 = \varepsilon_2 - \varepsilon_3, \ldots, \alpha_{n-1} = \varepsilon_{n-1} - \varepsilon_n, \alpha_n = 2\varepsilon_n. \tag{3.30}$$

In terms of the $\alpha_i$'s, the positive roots are

(3.31)
$$\begin{aligned} \varepsilon_i - \varepsilon_j &= \alpha_i + \cdots + \alpha_{j-1},\ i < j \leq n, \\ 2\varepsilon_i &= 2(\alpha_i + \cdots + \alpha_{j-1} + \alpha_j + \cdots + \alpha_{n-1}) + \alpha_n, i < n, 2\varepsilon_n = \alpha_n, \\ \varepsilon_i + \varepsilon_j &= \alpha_i + \cdots + \alpha_{j-1} + 2(\alpha_j + \cdots + +\alpha_{n-1}) + \alpha_n,\ i < j \leq n. \end{aligned}$$

Let $1 < p < n$. By (3.29) and (3.31) we have

$$\Delta_p^+(1) = \{\varepsilon_a \pm \varepsilon_i | 1 \leq a \leq p, p+1 \leq i \leq n\} \tag{3.32}$$

$$\Delta_p^+(2) = \{2\varepsilon_a | 1 \leq a \leq p\} \cup \{\varepsilon_a + \varepsilon_b | 1 \leq a < b \leq p\} \tag{3.33}$$

$$\Delta_p^+(k) = \emptyset, \tag{3.34}$$

for $k \geq 3$. The dimension is $\frac{1}{2}p(4n - 3p + 1)$.

As before, $a, b, \ldots$ will range from 1 to $p$, and $i, j, \ldots$ will range from $p+1$ to $n$. Let $X_{ai} = \varepsilon_a - \varepsilon_i$, $Y_{ai} = \varepsilon_a + \varepsilon_i$, $U_a = 2\varepsilon_a$, $W_{ab} = \varepsilon_a + \varepsilon_b$, $a < b$. Then as in Lemma 3.1 and Corollary 3.1, we have the following:

**Lemma 3.4.** 1) *Let $\alpha, \beta$ be positive roots in $\Delta_p^+(k)$, $k = 1, 2$.*

$$\widetilde{N}_{\alpha,\pm\beta} = \begin{cases} 2\,\mathrm{sgn}(N_{\alpha,\pm\beta}), & \text{if } \{\alpha,\beta\} = \{X_{ai}, Y_{ai}\} \text{ for some } a, i, \text{ or} \\ & \text{one of } \alpha,\ \beta \text{ is } U_a \text{ for some } a; \\ \sqrt{2}\,\mathrm{sgn}(N_{\alpha,\pm\beta}), & \text{otherwise.} \end{cases}$$

*Here in the case of $\alpha - \beta$, we assume in addition that $\alpha - \beta \neq 0$.*

2) *$R(X_{ai}, \bar{X}_{aj}, Y_{ai}, \bar{Y}_{aj}) = 0$ for any $a$ if $i \neq j$, and $R(X_{ai}, \bar{X}_{ci}, Y_{ci}, \bar{Y}_{ai}) = \pm\frac{1}{2}$ for any $i$ if $a \neq c$.*

*Proof.* (1) Suppose $\alpha = X_{ai}$, $\beta = Y_{ai}$; then $\alpha + \beta$ and $\alpha - \beta$ are both roots. It is easy to see that $N_{\alpha,\beta}$ and $N_{\alpha,-\beta}$ are equal to $\pm 2$. Moreover, $|\alpha|^2 = |\beta|^2 = 2$ and $|\alpha \pm \beta|^2 = 4$. Hence $\frac{|\alpha||\beta|}{|\alpha\pm\beta|} N_{\alpha,\pm\beta} = 2\mathrm{sgn}(N_{\alpha,\pm\beta})$. If $\alpha = U_a$, say, then $U_a + y$ and $U_a - U_b$ are not roots for any $y \in \Delta_p^+(k)$,

$k = 1, 2$. Moreover, if $\beta = X_{bi}$, then $\alpha - \beta$ is a root if and only if $b = a$. In this case, $N_{\alpha,-\beta} = \pm 1$, $|\alpha|^2 = 4$, $|\beta|^2 = 2$, and $|\alpha - \beta|^2 = 2$. Again $\frac{|\alpha||\beta|}{|\alpha \pm \beta|} N_{\alpha,\pm\beta} = 2\mathrm{sgn}(N_{\alpha,\pm\beta})$.

If $\alpha, \beta$ are not as above, and if $\alpha + \beta$ is a root, then $|\alpha + \beta|^2 = 2$. In this case, one can see that $N_{\alpha,\beta} = \pm 1$. So $\frac{|\alpha||\beta|}{|\alpha \pm \beta|} N_{\alpha,\pm\beta} = \sqrt{2}\mathrm{sgn}(N_{\alpha,\pm\beta})$. The case for $\alpha - \beta$ is similar.

(2) Let $\alpha = X_{ai}, \beta = X_{aj}, \gamma = Y_{ai}, \delta = Y_{aj}$. It is easy to see that

$$\begin{aligned} R(X_{ai}, \bar{X}_{aj}, Y_{ai}, \bar{Y}_{aj}) = & -\frac{1}{2} \cdot \frac{|\alpha||\beta||\gamma||\delta|}{|\alpha + \gamma|^2} N_{\alpha,\gamma} N_{-\beta,-\delta} \\ & + \frac{|\alpha||\beta||\gamma||\delta|}{|\alpha - \beta|^2} N_{\alpha,-\beta} N_{\gamma,-\delta}. \end{aligned}$$

On the other hand, since $\alpha - \beta + \gamma - \delta = 0$,

$$\frac{N_{\gamma,\alpha} N_{-\beta,-\delta}}{|\alpha + \gamma|^2} + \frac{N_{\alpha,-\beta} N_{\gamma,-\delta}}{|\alpha - \beta|^2} + \frac{N_{-\beta,\gamma} N_{\alpha,-\delta}}{|\gamma - \beta|^2} = 0,$$

By (1), we have

$$\frac{N_{\gamma,\alpha} N_{-\beta,-\delta}}{4} + \frac{N_{\alpha,-\beta} N_{\gamma,-\delta}}{2} = \pm\frac{1}{2}$$

because $i \neq j$. Squaring the above equality, noting that $N_{\gamma,\alpha}^2 = N_{-\beta,-\delta}^2 = 4$, $N_{\alpha,-\beta}^2 = N_{\gamma,-\delta}^2 = 1$, we have

$$N_{\gamma,\alpha} N_{-\beta,-\delta} N_{\alpha,-\beta} N_{\gamma,-\delta} = -4.$$

Hence

$$N_{\alpha,\gamma} N_{-\beta,-\delta} N_{\alpha,-\beta} N_{\gamma,-\delta} > 0$$

because $N_{\alpha,\gamma} = -N_{\gamma,\alpha}$. From this it is easy to see that $R(X_{ai}, \bar{X}_{aj}, Y_{ai}, \bar{Y}_{aj}) = 0$. The other part can be proved similarly.

q.e.d.

To compute the Ricci curvature, we know that $\mathrm{Ric} = \mu g$ and thus

$$\begin{aligned} \mu = & \sum_{a,i} \left[ R(U_1, \bar{U}_1, X_{ai}, \bar{X}_{ai}) + R(U_1, \bar{U}_1, Y_{ai}, \bar{Y}_{ai}) \right] + \sum_a R(U_1, \bar{U}_1, U_a, \bar{U}_a) \\ & + \sum_{a<b} R(U_1, \bar{U}_1, W_{ab}, \bar{W}_{ab}) \\ = & \frac{1}{2} \left( 2(n-p) + 2(n-p) \right) + 2 + (p-1) \\ = & 2n - p + 1. \end{aligned}$$

**Lemma 3.5.** *Let $\lambda$ be the largest eigenvalue of the quadratic form*

$$\sum_{A,B} R_{A\bar{A}B\bar{B}} x_A x_B$$

*in the Weyl frame, where $x_A$ are real.*

(a) $\lambda \leq 2n - p + 1$ *if and only if* $5p \leq 4n + 3$.

(b) *If* $5p < 4n+3$*, then* $\lambda = (2n-p+1)$ *iff the corresponding eigenvector has* $x_A = x_B$ *for all* $A, B$*.*
(c) *If* $5p = 4n+3$*, then there is an eigenvector with eigenvalue* $(2n-p+1)$ *such that* $x_A \neq x_B$ *for some* $A \neq B$*.*

*Proof.* Part (a): the argument is identical to the proof of Lemma 3.2 (a) except that: in Case 1 we use that for any $B$ the coefficients in $\sum_A R(A,B)x_A$ must add to $2n-p+1$ (instead of $2n-p$); in Case 2 we use that

$$\sum_{b,j}(R(X_{ai},X_{bj}) - R(X_{ai},Y_{bj})) = \frac{3}{2}p - \frac{1}{2}$$

(instead of $\frac{3}{2}p+\frac{1}{2}$).

Parts (b) and (c): the argument is similar to the corresponding proofs for Lemma 3.2. q.e.d.

**Lemma 3.6.** *Let* $\lambda$ *be the largest eigenvalue of the quadratic form*

$$\sum_{A,B,C,D;A\neq B,C\neq D} R_{A\bar{B}C\bar{D}}x_{AB}x_{CD} \tag{3.35}$$

*in the Weyl frame, where* $x_{AB} = \overline{x_{BA}}$*.*

(a) $\lambda \leq 2n-p+1$ *if and only if* $5p \leq 4n+3$*.*
(b) *If* $5p < 4n+3$*, then* $\lambda < 2n-p+1$*.*

*Proof.* We want to estimate

$$S_{AB} = \sum_{x\neq y}|R_{A\bar{B}y\bar{x}}| \tag{3.36}$$

for each case of $A, B$. Note that $S_{AB} = S_{BA}$. Recall the following properties:

(C1) If $A - B \neq x - y$, then $R_{A\bar{B}y\bar{x}} = 0$.
(C2) If neither $A-B$ nor $A+y$ are roots, then $R_{A\bar{B}y\bar{x}} = 0$.

In each case we will use these to reduce the terms in (3.36) as much as possible. Then Lemmas 2.5 and 3.4 will be used to calculate the absolute values of the remaining curvature terms.

**Case (i)** $A = X_{ai}, B = X_{bj}$ with $(a,i) \neq (b,j)$.

Note that $A, B \in \Delta_p^+(1)$. By (C1) we may assume that $x, y \in \Delta_p^+(1)$ or $x, y \in \Delta_p^+(2)$. Note that the sum of the coordinates of each $X$ is 0, the sum of the coordinates of each $Y$ is 2, the sum of the coordinates of each $U$ is 2, and the sum of the coordinates of each $W$ is 2. Thus by

(C1), (3.36) reduces to:

(3.37)

$$\begin{aligned} S_{AB} = &\sum_{c,k,d,l} |R(X_{ai}, \bar{X}_{bj}, X_{ck}, \bar{X}_{dl})| + \sum_{c,k,d,l} |R(X_{ai}, \bar{X}_{bj}, Y_{ck}, \bar{Y}_{dl})| \\ &+ \sum_{c,d,e,f} |R(X_{ai}, \bar{X}_{bj}, W_{cd}, \bar{W}_{ef})| \\ &+ \sum_{c,d,e} |R(X_{ai}, \bar{X}_{bj}, U_c, W_{de})| + \sum_{c,d,e} |R(X_{ai}, \bar{X}_{bj}, W_{de}, U_c)| \\ =: &I + II + III + IV + V. \end{aligned}$$

If $a \neq b$, $i \neq j$, then: All terms in $III, IV, V$ are zero by (C1). By (C1), the only non-zero term in $I$ is $|R(X_{ai}, \bar{X}_{bj}, X_{bj}, \bar{X}_{ai})|$, leaving $I = \frac{1}{4}$. We get $II = \frac{1}{4}$ in the same way. Hence $S_{AB} = \frac{1}{4} + \frac{1}{4} = \frac{1}{2} < 2n - p + 1$.

If $a = b$, $i \neq j$, then: All terms in $III, IV, V$ are zero by (C1). By (C1), the only non-zero term in $I$ is $|R(X_{ai}, \bar{X}_{aj}, X_{cj}, \bar{X}_{ci})|$ for any $c$, leaving $I = p$. By (C1), the only non-zero term in $II$ is $|R(X_{ai}, \bar{X}_{aj}, Y_{cj}, \bar{Y}_{ci})|$, which is 0 by Lemma 3.4 if $a = c$, leaving $II = \frac{1}{2}(p-1)$. Hence $S_{AB} = p + 0 + \frac{1}{2}(p-1) = 3p/2 - 1/2 \leq 2n - p + 1$ if and only if $5p \leq 4n + 3$, and $S_{AB} < 2n - p + 1$ if and only if $5p < 4n + 3$.

If $a \neq b$, $i = j$, we may assume that $a < b$, then: By (C1), the only non-zero terms in $I$ are $|R(X_{ai}, \bar{X}_{bi}, X_{bk}, \bar{X}_{ak})|$ for any $k$, leaving $I = n - p$. By (C1), the only non-zero terms in $II$ are $|R(X_{ai}, \bar{X}_{bi}, Y_{bk}, \bar{Y}_{ak})|$ for any $k$, leaving a contribution of $1/2$ when $k = i$ by Lemma 3.4 and a contribution of $n - p - 1$ for the cases when $k \neq i$. By (C1), the only non-zero terms in $III$ are $|R(X_{ai}, \bar{X}_{bi}, W_{bc}, \bar{W}_{ac})|$ for $c > b$, or $|R(X_{ai}, \bar{X}_{bi}, W_{cb}, \bar{W}_{ca})|$ for $c < a$, or $|R(X_{ai}, \bar{X}_{bi}, W_{cb}, \bar{W}_{ac})|$ for $a < c < b$, in which cases the respective contributions to $III$ are $1/2(p - b), 1/2(a-1), 1/2(b-a-1)$. By (C1), the only non-zero term in $IV$ is $|R(X_{ai}, \bar{X}_{bi}, U_b, W_{ab})|$, leaving $IV = \sqrt{2}/2$. By (C1), the only non-zero term in $V$ is $|R(X_{ai}, \bar{X}_{bi}, W_{ab}, U_a)|$, leaving $V = \sqrt{2}/2$. Hence $S_{AB} = (n-p) + \frac{1}{2} + (n-p-1) + \sqrt{2} + \frac{1}{2}\left[(p-b) + (a-1) + (b-a-1)\right] = 2n - \frac{3}{2}p - \frac{1}{2} + \sqrt{2} < 2n - p + 1$.

**Case (ii)** $A = X_{ai}$, $Y_{bj}$.

Note that $A, B \in \Delta_p^+(1)$. By (C1), $x, y \in \Delta_p^+(1)$ or $x, y \in \Delta_p^+(2)$ and (3.36) reduces to:

$$S_{AB} = \sum_{c,k,d,l} |R(X_{ai}, \bar{Y}_{bj}, Y_{ck}, \bar{X}_{dl})|. \tag{3.38}$$

If $a \neq b$, $i \neq j$, then by (C1), the only possible non-zero terms in (3.38) are $|R(X_{ai}, \bar{Y}_{bj}, Y_{bj}, \bar{X}_{ai})|$, which is zero by (C2), and $|R(X_{ai}, \bar{Y}_{bj}, Y_{bi}, \bar{X}_{aj})|$, which is $\frac{1}{2}$. Hence $S_{AB} = \frac{1}{2} < 2n - p + 1$.

If $a = b$, $i \neq j$, then by (C1), the only non-zero terms in (3.38) are $|R(X_{ai}, \bar{Y}_{aj}, Y_{cj}, \bar{X}_{ci})|$ or $|R(X_{ai}, \bar{Y}_{aj}, Y_{ci}, \bar{X}_{cj})|$, for any $c$. In the first case the contribution to (3.38) is $p$, and in the second case the contribution to (3.38) is 0 when $c = a$ and $\frac{1}{2}$ when $c \neq a$. Thus $S_{AB} = p + \frac{1}{2}(p-1) = \frac{3}{2}p - \frac{1}{2} < 2n - p + 1$.

If $a \neq b$, $i = j$, then by (C1), the only non-zero term in (3.38) is given by $|R(X_{ai}, \bar{Y}_{bi}, Y_{bi}, \bar{X}_{bi})| = \frac{1}{2}$. Thus $S_{AB} = \frac{1}{2} < 2n - p + 1$.

If $a = b$, $i = j$, then by (C1), the only non-zero terms in (3.38) are $|R(X_{ai}, \bar{Y}_{ai}, Y_{ci}, \bar{X}_{ci})|$ and the contribution to (3.38) is 1 when $c = a$ and $\frac{1}{2}(p-1)$ from the cases $c \neq a$ by Lemma 3.4. Thus $S_{AB} = 1 + \frac{1}{2}(p-1) < 2n - p + 1$.

**Case (iii)** $A = X_{ai}, B = U_b$.

Note that $A \in \Delta_p^+(1)$ and $B \in \Delta_p^+(2)$. By (C1), $x \in \Delta_p^+(1)$ and $y \in \Delta_p^+(2)$ and (3.36) reduces to:

(3.39)
$$S_{AB} = \sum_{c,d,j} |R(X_{ai}, \bar{U}_b, U_c, \bar{X}_{dj})| + \sum_{c,d,e,j} |R(X_{ai}, \bar{U}_b, W_{cd}, \bar{X}_{ej})| =: I + II.$$

If $a \neq b$, then: All terms in $II$ are zero by (C1). By (C1), the only possible non-zero term in $I$ is $|R(X_{ai}, \bar{U}_b, U_b, \bar{X}_{ai})|$, which in turn is zero by (C2).

If $a = b$, then: By (C1), the only non-zero term in $I$ is $|R(X_{ai}, \bar{U}_a, U_a, \bar{X}_{ai})|$, leaving $I = 1$. By (C1), the only non-zero terms in $II$ are $|R(X_{ai}, \bar{U}_a, W_{ac}, \bar{X}_{ci})|$ for $c > a$, and $|R(X_{ai}, \bar{U}_a, W_{ca}, \bar{X}_{ci})|$ for $c < a$, in which cases the respective contributions to $II$ are $\frac{\sqrt{2}}{2}(p-a)$, $\frac{\sqrt{2}}{2}(a-1)$. Thus $S_{AB} = 1 + \frac{\sqrt{2}}{2}\left((p-a) + (a-1)\right) = 1 + \frac{\sqrt{2}}{2}(p-1) < 2n - p + 1$.

**Case (iv)** $A = X_{ai}, B = W_{bc}$.

Note that $A \in \Delta_p^+(1), B \in \Delta_p^+(2)$. By (C1), $x \in \Delta_p^+(1)$ and $y \in \Delta_p^+(2)$ and (3.36) reduces to:

$$S_{AB} = \sum_{d,j,e,f} |R(X_{ai}, \bar{W}_{bc}, W_{ef}, \bar{X}_{dj})| + \sum_{d,j,e} |R(X_{ai}, \bar{W}_{bc}, U_e, \bar{X}_{dj})| := I + II.$$

If $a = b$, then by (C1), the only non-zero terms in $I$ are $|R(X_{ai}, \bar{W}_{ac}, W_{dc}, \bar{X}_{di})|$ for $d < c$, or $|R(X_{ai}, \bar{W}_{ac}, W_{cf}, \bar{X}_{fi})|$ for $f > c$. In the first case the contribution is $\frac{1}{2}(c-1)$, and the contribution in the second case is $\frac{1}{2}(p-c)$. By (C1), the only non-zero term in $II$ is $|R(X_{ai}, \bar{W}_{ac}, U_c, \bar{X}_{ci})|$, which is $\sqrt{2}$. Thus $S_{AB} = \frac{1}{2}(p-1) + \sqrt{2} < 2n - p + 1$.

If $a \neq b$, then by (C1) and (C2), the terms in $I, II$ are zero unless $a = c$, in which case we get, as above, that $S_{AB} = \frac{1}{2}(p-1) + \sqrt{2} < 2n - p + 1$.

**Case (v)** $A = Y_{ai}, B = Y_{bj}$, $(a,i) \neq (b,j)$. Similar to (i).

**Case (vi)** $A = Y_{ai}, B = U_b$. Similar to (iii).

**Case (vii)** $A = Y_{ai}, B = W_{bc}$. Similar to (iv).

**Case (viii)** $A = U_a$, $B = U_b$, $a < b$.

From (C1) it is not hard to see that (3.36) reduces to:

$$S_{AB} = \sum_{c,d} |R(U_a, \bar{U}_b, U_b, \bar{U}_a)| = 0$$

where the last equality follows by (C2).

**Case (ix)** $A = U_a$, $B = W_{bc}$, $b < c$.

Note that $A, B \in \Delta_p^+(2)$. By (C1), $x, y \in \Delta_p^+(1)$ or $A, B \in \Delta_p^+(2)$ and (3.15) reduces to:

$$\begin{aligned} S_{AB} =& \sum_{d,e,i} |R(U_a, \bar{W}_{bc}, X_{di}, \bar{X}_{ei})| + \sum_{d,e,i} |R(U_a, \bar{W}_{bc}, Y_{di}, \bar{Y}_{ei})| \\ &+ \sum_{d,e,f,g} |R(U_a, \bar{W}_{bc}, W_{de}, \bar{W}_{fg})| + \sum_{d,e,f} |R(U_a, \bar{W}_{bc}, U_d, \bar{W}_{ef})| \\ &+ \sum_{d,e,f} |R(U_a, \bar{W}_{bc}, W_{de}, \bar{U}_f)| \\ =& I + II + III + IV + V. \end{aligned}$$

Note that $A + x$ is never a root and thus, by (C2), the only non-zero terms are when $a = b$ or $a = c$. If $a = b$, then by (C1), the only non-zero terms in $I$ are $|R(U_a, \bar{W}_{ac}, X_{ci}, \bar{X}_{ai})|$ for any $i$, leaving $I = \frac{\sqrt{2}}{2}(n - p)$. Similarly, we get $II = \frac{\sqrt{2}}{2}(n - p)$. By (C1), the only non-zero terms in $III$ are $|R(U_a, \bar{W}_{ac}, W_{ce}, \bar{W}_{ae})|$ for $e > c$, and $|R(U_a, \bar{W}_{ac}, W_{dc}, \bar{W}_{da})|$ for $d < a$, and $|R(U_a, \bar{W}_{ac}, W_{dc}, \bar{W}_{ad})|$ for $a < d < a$, in which cases the respective contributions to $III$ are $\frac{\sqrt{2}}{2}(p - c)$, $\frac{\sqrt{2}}{2}(a - 1)$, $\frac{\sqrt{2}}{2}(c - a - 1)$. By (C1), the only non-zero term in $IV$ is $|R(U_a, \bar{W}_{ac}, U_c, \bar{W}_{ac})|$, leaving $IV = 1$. By (C1), the only non-zero term in $V$ is $|R(U_a, \bar{W}_{ac}, W_{ac}, \bar{U}_a)|$, leaving $V = 1$. Thus when $a = b$, and similarly when $a = c$, $S_{AB} = \sqrt{2}(n - p) + \frac{\sqrt{2}}{2}(p - 2) + 2 < 2n - p + 1$.

**Case (x)** $A = W_{ab}, B = W_{cd}$, $(a, b) \neq (c, d)$. We may assume that $a \leq c$.

Note that $A, B \in \Delta_p^+(2)$. By (C1), $x, y \in \Delta_p^+(1)$ or $x, y \in \Delta_p^+(2)$ and (3.36) reduces to:

$$\begin{aligned} S_{AB} =& \sum_{e,f,i} |R(W_{ab}, \bar{W}_{cd}, X_{ei}, \bar{X}_{fi})| + \sum_{e,f,i} |R(W_{ab}, \bar{W}_{cd}, Y_{ei}, \bar{Y}_{fi})| \\ &+ \sum_{e,f,g,h} |R(W_{ab}, \bar{W}_{cd}, W_{ef}, \bar{W}_{gh})| + \sum_{e,f,g} |R(W_{ab}, \bar{W}_{cd}, W_{ef}, \bar{U}_g)| \\ &+ \sum_{e,f,g} |R(W_{ab}, \bar{W}_{cd}, U_e, \bar{W}_{fg})| \\ =& I + II + III + IV + V. \end{aligned}$$

Note that $A + y$ is never a root for any positive root $y$. Thus by (C2), all terms in $I, II, III, IV, V$ are zero unless either $a = c$ or $b = c$ or $b = d$. Assume that $a = c$, and without loss of generality that $b < d$. By (C1), the only non-zero terms in $I$ are $|R(W_{ab}, \bar{W}_{ad}, X_{di}, \bar{X}_{bi})|$, leaving $I = \frac{1}{2}(n - p)$. We similarly get $II = \frac{1}{2}(n - p)$. By (C1), the only non-zero terms in $III$ are $|R(W_{ab}, \bar{W}_{ad}, W_{de}, \bar{W}_{be})|$ for $e > d$, or $|R(W_{ab}, \bar{W}_{ad}, W_{ed}, \bar{W}_{eb})|$ for $e < b$, or $|R(W_{ab}, \bar{W}_{ad}, W_{ed}, \bar{W}_{be})|$ for $b < e < d$, in which cases the respective contributions to $III$ are $1/2(p - d), 1/2(b - 1), 1/2(d - b - 1)$. By (C1), the only non-zero term in $IV$ is $|R(U_a, \bar{W}_{ac}, W_{bd}, \bar{U}_b)|$, leaving $V = \frac{\sqrt{2}}{2}$. Similarly, we get $IV = \frac{\sqrt{2}}{2}$. Thus when $a = c$, and similarly when $b = c$ or $b = d$, $S_{AB} = (n-p)+\frac{1}{2}(n-p)+\frac{1}{2}\left[(p - d) + (b - 1) + (d - b - 1)\right] + \sqrt{2} = n - 1 - \frac{1}{2}p + \sqrt{2} < 2n - p + 1$.

This completes the proof of the lemma. q.e.d.

By Lemmas 3.5 and 3.6, we can proceed as in the $B$ cases to obtain:

**Theorem 3.3.** *The Kähler $C$-space $(C_n, \alpha_p)$, $n \geq 3$, $1 < p < n$ satisfies $QB \geq 0$ if and only if $5p \leq 4n + 3$. Moreover, $QB > 0$ if and only if $5p < 4n + 3$.*

**Remark 3.2.** As in the $B$ cases, one can see that $(C_n, \alpha_p)$ does not satisfy $B^{\perp} \geq 0$.

## 4. Kähler $C$-spaces of exceptional type

For each of the exceptional Lie algebras $G_2, F_4, E_6, E_7, E_8$, we will establish whether or not the corresponding Kähler C-spaces with $b_2 = 1$ have $QB \geq 0$ or not. For each case, we define the following quadratic forms with respect to the Weyl frames:

$$M_1 := \sum_{A,B} R_{A\bar{A}B\bar{B}} x_A x_B$$

$$M_2 := \sum_{\substack{A,B,C,D;\\ A\neq B, C\neq D}} R_{A\bar{B}C\bar{D}} x_{AB} x_{CD},$$

where the $x_A$'s are real and $\overline{x_{AB}} = x_{BA}$. We will study the largest eigenvalues of these two quadratic forms. By Corollary 2.1, these will tell us whether the space satisfies $QB \geq 0$, or $QB > 0$.

For each exceptional Lie algebra $\mathfrak{g}$, we will first present an explicit root system (in some Euclidean space $\mathbb{R}^n$) and fundamental set of roots $\{\alpha_1, ..\}$. Then for each corresponding Kähler $C$-space $(\mathfrak{g}, \alpha_k)$, we present a Weyl frame. Lemma 2.4 then allows explicit calculation of the matrix for $M_1$. The main point here is to determine $\widetilde{N}_{\alpha,\pm\beta}$. From this point, while eigenvalue estimates are possible by row sum and symmetry arguments, as in $M_1$ in the classical cases, we compute the eigenvalues and Ricci curvature (row sum) of $M_1$ directly using MAPLE.

To estimate the largest eigenvalue of $M_2$, we will use Lemmas 2.4 and 2.5 to compute the curvature tensor. However, in this case the lemmas allow only an upper estimate for the absolute value of the entries of $M_2$, since we can only calculate the $N_{\alpha\beta}$'s appearing there up to a sign. By the same reason, this can only be estimated from above by $|R_1|+|R_2|$ in the formula Lemma 2.5(2). In some cases, this becomes too large, and we cannot get a good estimate. Hence, for $A, B, C, D$ corresponding to positive roots $\alpha \in \Delta^+(i), \beta \in \Delta^+(j), \gamma \in \Delta^+(k), \delta \in \Delta^+(l)$ where $A \neq B$, define $\widetilde{R}(A, \bar{B}, C, \bar{D})$ as follows:
(4.1)

$$\widetilde{R}(A, \bar{B}, C, \bar{D}) = \begin{cases} 0, & \text{if } \alpha - \beta \neq \delta - \gamma, \\ |R(A, \bar{A}, B, \bar{B})|, & \text{if } B = C \text{ (i.e. if } \beta = \gamma), \\ m, & \text{if } \alpha - \beta = \delta - \gamma, \text{ and } \beta - \gamma \neq 0, \end{cases}$$

where $m = \min\{|R_1(A, \bar{B}, C, \bar{D})| + |R_2(A, \bar{B}, C, \bar{D})|, |R_1(C, \bar{B}, A, \bar{D})| + |R_2(C, \bar{B}, A, \bar{D}))|\}$.

Note that in the last case, we also have $\alpha - \delta \neq 0$. Since $R(A, \bar{B}, C, \bar{D}) = R(C, \bar{B}, A, \bar{D})$ by symmetries of the curvature tensor, we have $|R(A, \bar{B}, C, \bar{D})| \leq \widetilde{R}(A, \bar{B}, C, \bar{D})$, for all $A \neq B, C \neq D$.

**Remark 4.1.** In many cases, $|R(A, \bar{B}, C, \bar{D})|$ is exactly equal to the quantity $m$ in (4.1). For example, this is the case if one of $\alpha + \gamma, \alpha - \beta, \beta - \gamma$ is not a root.

Consider the following matrix $Z_{AB,CD}$, which is defined for all pairs $AB, CD$:

$$Z_{AB,CD} = \begin{cases} 0, & \text{if } A = B \text{ or } C = D, \\ \widetilde{R}(A, \bar{B}, C, \bar{D}), & \text{otherwise.} \end{cases} \tag{4.2}$$

Recall that $(M_2)_{AB,CD} = R(A, \bar{B}, C\bar{D})$ is only defined for pairs with $A \neq B$, $C \neq D$. For any $A$ the $AA$th row and column of $Z$ has zero in every entry, and removing these rows and columns leaves a symmetric matrix with the same dimension as $M_2$, bounding $M_2$ from above, entry-wise in absolute values. The following simple lemma justifies estimating the largest eigenvalue of $M_2$ by the largest absolute eigenvalue of $Z$.

**Lemma 4.1.** *Let $N, M$ be real symmetric $n \times n$ matrices such that $N_{ij} \geq |M_{ij}|$ for all $i, j$. Then spectral radius (the maximal absolute value of eigenvalues) $\lambda_N$ of $N$ is greater than or equal to the spectral radius $\lambda_M$ of $M$.*

*Proof.* Let $x = (x_1, \ldots, x_n)$ be a unit eigenvector of $M$ for which $|Mx| = \lambda_M$. Note that $|x| = (|x_1|, \ldots, |x_n|)$ is also a unit vector. Then we have $\lambda_M = |Mx| = \sqrt{|\sum_j M_{ij}x_i|^2} \leq \sqrt{|\sum_j N_{ij}|x_i||^2} = N|x| \leq \lambda_N$. q.e.d.

We will calculate $Z$ in each case using MAPLE. From this point, we can of course compute the eigenvalues of $Z$ directly using MAPLE, thus obtaining an eigenvalue estimate for $M_2$. However, we will use Lemmas 2.7 and 2.8 here, as they are elementary and similar to our methods for $M_2$ in the classical case. In fact, most of the terms in $Z$ are zero and one may be able to decompose $Z$ into quadratic forms of much smaller size so that Lemmas 2.7 and 2.8 can be applied without using a computer. In all cases other than $(G_2, \alpha_2)$, $(E_7, \alpha_5)$, and $(F_4, \alpha_2)$, the estimate provided by Lemma 2.7 will be sufficient, while in the cases of $(G_2, \alpha_2)$, $(E_7, \alpha_5)$, and $(F_4, \alpha_2)$, Lemma 2.8 is used to estimate the eigenvalue of $Z$.

In the subsections below, we just present the results of the MAPLE calculations, and we indicate the algorithms used in the appendix. For each Lie algebra below, the dual Cartan subalgebra $\mathfrak{h}^*$ is associated to some Euclidean subspace $V$, and the root system is given as a set of vectors in $V$. We refer to [**7**] for details. We will use $\xi_1, \dots, \xi_n$ to denote the standard coordinates on $\mathbb{R}^n$ and $\varepsilon_1, \dots, \varepsilon_n$ to denote the standard basis vectors of $\mathbb{R}^n$.

**4.1. The space $(G_2, \alpha_2)$.** Let $V$ be the hyperplane in $\mathbb{R}^3$ with $\xi_1 + \xi_2 + \xi_3 = 0$. The positive roots in $V$ are

$$\varepsilon_1 - \varepsilon_2, -2\varepsilon_1 + \varepsilon_2 + \varepsilon_3, -\varepsilon_1 + \varepsilon_3, -\varepsilon_2 + \varepsilon_3, \varepsilon_1 - 2\varepsilon_2 + \varepsilon_3, -\varepsilon_1 - \varepsilon_2 + 2\varepsilon_3.$$

Simple positive roots are $\alpha_1 = \varepsilon_1 - \varepsilon_2, \alpha_2 = -2\varepsilon_1 + \varepsilon_2 + \varepsilon_3$ with respect to which the positive roots are

$$\alpha_1, \alpha_2, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2.$$

Now $(G_2, \alpha_1)$ is Hermitian symmetric, so we only consider $(G_2, \alpha_2)$ for which we have

$$\begin{aligned} \Delta_2^+(1) =& \{\alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, \alpha_2, 3\alpha_1 + \alpha_2\} \\ \Delta_2^+(2) =& \{3\alpha_1 + 2\alpha_2\}. \end{aligned} \tag{4.3}$$

$\Delta_2^+(k) = \emptyset$, for $k \geq 3$.

$$\begin{cases} \text{dim} = 5, \text{Ric} = 9g, \\ \text{4 largest eigenvalues of } M_1 \text{ are } 1.5000, 1.5000,\ 8.5000,\ 9.000, \\ \text{eigenvalues of } M_2 \text{ are less than } 9 \end{cases}$$

(the estimate for $M_2$ is obtained by using $\mu = 9$ and $s = 1$ in Lemma 2.8, in which case the maximum weighted row sum is 8.6309). Thus the space has $QB > 0$.

**4.2. The spaces $(F_4, \alpha_i)$.**

**4.2.1. Root system.** Let $V = \mathbb{R}^4$. The positive roots in $V$ are

$$\{\varepsilon_i\}_{1\le i\le 4} \cup \{\varepsilon_i + \varepsilon_j\}_{1\le i<j\le 4} \cup \{\varepsilon_i - \varepsilon_j\}_{1\le i<j\le 4} \cup \{\frac{1}{2}(\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4)\}.$$

There are a total of $4+6+6+8 = 24$ positive roots. Let

(4.4)
$$A = (a_i)_{i=1}^{12} = \begin{pmatrix} \frac{1}{2}(\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4) \\ \frac{1}{2}(\varepsilon_1+\varepsilon_2-\varepsilon_3+\varepsilon_4) \\ \frac{1}{2}(\varepsilon_1+\varepsilon_2+\varepsilon_3-\varepsilon_4) \\ \frac{1}{2}(\varepsilon_1+\varepsilon_2-\varepsilon_3-\varepsilon_4) \\ \frac{1}{2}(\varepsilon_1-\varepsilon_2+\varepsilon_3+\varepsilon_4) \\ \frac{1}{2}(\varepsilon_1-\varepsilon_2-\varepsilon_3+\varepsilon_4) \\ \frac{1}{2}(\varepsilon_1-\varepsilon_2+\varepsilon_3-\varepsilon_4) \\ \frac{1}{2}(\varepsilon_1-\varepsilon_2-\varepsilon_3-\varepsilon_4) \\ \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \end{pmatrix}, B = (b_i)_{i=1}^{12} = \begin{pmatrix} \varepsilon_1+\varepsilon_2 \\ \varepsilon_1+\varepsilon_3 \\ \varepsilon_1+\varepsilon_4 \\ \varepsilon_2+\varepsilon_3 \\ \varepsilon_2+\varepsilon_4 \\ \varepsilon_3+\varepsilon_4 \\ \varepsilon_1-\varepsilon_2 \\ \varepsilon_1-\varepsilon_3 \\ \varepsilon_1-\varepsilon_4 \\ \varepsilon_2-\varepsilon_3 \\ \varepsilon_2-\varepsilon_4 \\ \varepsilon_3-\varepsilon_4 \end{pmatrix}$$

The simple positive roots are $\alpha_1 = b_{10}, \alpha_2 = b_{12}, \alpha_3 = a_{12}, \alpha_4 = a_8$. The matrix for $(\alpha_i)$ is

$$g = \begin{pmatrix} 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \\ 1/2 & -1/2 & -1/2 & -1/2 \end{pmatrix}.$$

The coordinates of $(a_i)$ with respect to the ordered basis $\{\alpha_1, \dots, \alpha_4\}$ are given by the columns of

$$(4.5) \qquad (gg^t)^{-1} g A^t = \begin{pmatrix} 1&1&1&1&0&0&0&0&1&1&0&0 \\ 2&1&2&1&1&0&1&0&2&1&1&0 \\ 3&2&2&1&2&1&1&0&3&1&1&1 \\ 1&1&1&1&1&1&1&1&2&0&0&0 \end{pmatrix}$$

The coordinates of $(b_i)$ with respect to the ordered basis $\{\alpha_1, \dots, \alpha_4\}$ are given by the columns of

$$(4.6) \qquad (gg^t)^{-1} g B^t = \begin{pmatrix} 2&1&1&1&1&0&0&1&1&1&1&0 \\ 3&3&2&2&1&1&1&1&2&0&1&1 \\ 4&4&4&2&2&2&2&2&2&0&0&0 \\ 2&2&2&0&0&0&2&2&2&0&0&0 \end{pmatrix}$$

That is, $a_1 = \alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4$, etc. From this it is easy to write down the $\Delta^+(k)'s$ for $(F_4, \alpha_i)$ for $1 \le i \le 4$. Next, let us determine $\widetilde{N}_{\alpha,\pm\beta}$.

**Lemma 4.2.** *Let $\alpha$, $\beta$ be positive roots.*

$$\widetilde{N}_{\alpha,\beta} = \begin{cases} \operatorname{sgn}(N_{\alpha,\beta}), & \text{if } \alpha,\beta \in A, \text{ and } |\alpha+\beta|^2 = 1, \\ \sqrt{2}\operatorname{sgn}(N_{\alpha,\beta}), & \text{otherwise.} \end{cases} \tag{4.7}$$

*If $\alpha - \beta \neq 0$, then*

$$\widetilde{N}_{\alpha,-\beta} = \begin{cases} \operatorname{sgn}(N_{\alpha,-\beta}), & \text{if } \alpha,\beta \in A, \text{ and } |\alpha-\beta|^2 = 1, \\ \sqrt{2}\operatorname{sgn}(N_{\alpha,-\beta}), & \text{otherwise.} \end{cases} \tag{4.8}$$

*Proof.* First note that a root $\alpha$ is in $A$ or $-\alpha$ is in $A$ if and only if $||\alpha||^2 = 1$, and it is in $B$ or $-\alpha$ is in $B$ if and only if $||\alpha||^2 = 2$.

To prove (4.7), it is sufficient to consider the case that $\alpha+\beta$ is a root. Suppose $\alpha, \beta \in A$, and $||\alpha+\beta||^2 = 1$, then $(\alpha, \beta) = -\frac{1}{2}$. Suppose $\alpha - \beta$ is also a root then $||\alpha - \beta||^2 = 1$ or 2, and $(\alpha, \beta) = \frac{1}{2}$ or 0, which is impossible. Hence $N_{\alpha,\beta} = \pm 1$. In this case,

$$\widetilde{N}_{\alpha,\beta} = \frac{|\alpha||\beta|}{|\alpha+\beta|} N_{\alpha,\beta} = \operatorname{sgn}(N_{\alpha,\beta}).$$

Suppose $||\alpha+\beta||^2 = 2$; then $(\alpha, \beta) = 0$, and $\alpha-\beta$ is also a root (see [**11**, p. 324]). Moreover, $||\alpha - 2\beta||^2 = 5$ and so $\alpha - 2\beta$ is not a root. Hence $N_{\alpha,\beta} = \pm 2$. Then $\widetilde{N}_{\alpha,\beta} = \sqrt{2}\operatorname{sgn}(N_{\alpha,\beta})$.

Suppose $\alpha \in A, \beta \in B$, and $||\alpha+\beta||^2 = 1$; then $(\alpha, \beta) = -1$. Suppose $\alpha - \beta$ is also a root; then $||\alpha - \beta||^2 = 1$, or 2, and $(\alpha, \beta) = 1$ or $\frac{1}{2}$. Hence $\alpha - \beta$ is not a root. In this case, $\widetilde{N}_{\alpha,\beta} = \sqrt{2}\operatorname{sgn}(N_{\alpha,\beta})$. Suppose $||\alpha + \beta||^2 = 2$, then $(\alpha, \beta) = -\frac{1}{2}$. That $(\alpha, \beta)$ is an integer. Hence this is impossible.

Suppose $\alpha, \beta \in B$, and $||\alpha + \beta||^2 = 1$; then $(\alpha, \beta) = -\frac{3}{2}$. This is impossible. Hence $||\alpha + \beta||^2 = 2$ and $(\alpha, \beta) = -1$. As before, we can prove that $\alpha - \beta$ is not a root. So $N_{\alpha,\beta} = \pm 1$ and $\widetilde{N}_{\alpha,\beta} = \sqrt{2}\operatorname{sgn}(N_{\alpha,\beta})$. This completes the proof of (4.7). The proof of (4.8) is similar.

q.e.d.

#### 4.2.2. The space $(F_4, \alpha_1)$.

$$\begin{aligned} \Delta_1^+(1) =& \{a_1, a_2, a_3, a_4, a_9, a_{10}, b_2, b_3, b_4, b_5, b_8, b_9, b_{10}, b_{11}\} \\ \Delta_1^+(2) =& \{b_1\} \end{aligned} \tag{4.9}$$

$\Delta_1^+(k) = \emptyset$, for $k \geq 3$.

$$\begin{cases} \dim = 15, \operatorname{Ric} = 8g, \\ \text{4 largest eigenvalues of } M_1 \text{ are 8, 4.5, 4.5, 4.5,} \\ \text{eigenvalues of } M_2 \text{ are at most 4.9142 (using Lemma 2.7).} \end{cases}$$

Thus the space has $QB > 0$.

**4.2.3. The space $(F_4, \alpha_2)$.**

$$\begin{aligned} \Delta_2^+(1) =& \{a_2, a_4, a_5, a_7, a_{10}, a_{11}; b_5, b_6, b_7, b_8, b_{11}, b_{12}\} \\ \Delta_2^+(2) =& \{a_1, a_3, a_9, b_3, b_4, b_9\} \\ \Delta_2^+(3) =& \{b_1, b_2\}. \end{aligned} \tag{4.10}$$

$\Delta_2^+(k) = \emptyset$, for $k \geq 4$.

$$\begin{cases} \dim = 20, \mathrm{Ric} = 5g, \\ \text{4 largest eigenvalues of } M_1 \text{ are 5, 4.8941, 4.8941, 4.6543,} \\ \text{eigenvalues of } M_2 \text{ are less than 5.} \end{cases}$$

(The estimate for $M_2$ is obtained by using $\mu = 5$ and $s = 4$ in Lemma 2.8, in which case the maximum weighted row sum is 4.9822. When we take $s = 10$, then the maximal weighted row sum actually becomes 4.8070.)

Thus the space has $QB > 0$.

**4.2.4. The space $(F_4, \alpha_3)$.**

$$\begin{aligned} \Delta_3^+(1) =& \{a_4, a_6, a_7, a_{10}, a_{11}, a_{12}\} \\ \Delta_3^+(2) =& \{a_2, a_3, a_5, b_4, \dots, b_9\} \\ \Delta_3^+(3) =& \{a_1, a_9\} \\ \Delta_3^+(4) =& \{b_1, b_2, b_3\}. \end{aligned} \tag{4.11}$$

$\Delta_3^+(k) = \emptyset$, for $k \geq 5$.

$$\begin{cases} \dim = 20, \mathrm{Ric} = 7/2g, \\ \text{4 largest eigenvalues of } M_1 \text{ are 3.6888, 3.5, 2.4137, 2.4137.} \end{cases}$$

Thus the space does not have $QB \geq 0$.

**4.2.5. The space $(F_4, \alpha_4)$.**

$$\begin{aligned} \Delta_4^+(1) =& \{a_1, .., a_8\} \\ \Delta_4^+(2) =& \{a_9; b_1, b_2, b_3, b_7, b_8, b_9\}. \end{aligned} \tag{4.12}$$

$\Delta_4^+(k) = \emptyset$, for $k \geq 3$.

$$\begin{cases} \dim = 15, \mathrm{Ric} = 11/2g, \\ \text{4 largest eigenvalues of } M_1 \text{ are 5.5, 2.1328, 2.1328, 2.1328,} \\ \text{eigenvalues of } M_2 \text{ are at most 3.9571 (using Lemma 2.7).} \end{cases}$$

Thus the space has $QB > 0$.

**4.3. The spaces $(E_6, \alpha_i)$.**

**4.3.1. Root system.** Consider the subspace $V$ of $\mathbb{R}^8$ such that $\xi_6 = \xi_7 = -\xi_8$. The positive roots in $V$ are $\pm\varepsilon_i + \varepsilon_j, 1 \le i < j \le 5$ (total 20), and

$$\frac{1}{2}(\varepsilon_8 - \varepsilon_7 - \varepsilon_6 + \sum_{i=1}^{5}(-1)^{\nu(i)}\varepsilon_i)$$

so that $\sum_1^5 \nu(i)$ is even, i.e. the number of minus signs is even (total 16). Let

$$A = (a_i)_{i=1}^{10} = \begin{pmatrix} \varepsilon_1+\varepsilon_2 \\ \varepsilon_1+\varepsilon_3 \\ \varepsilon_1+\varepsilon_4 \\ \varepsilon_1+\varepsilon_5 \\ \varepsilon_2+\varepsilon_3 \\ \varepsilon_2+\varepsilon_4 \\ \varepsilon_2+\varepsilon_5 \\ \varepsilon_3+\varepsilon_4 \\ \varepsilon_3+\varepsilon_5 \\ \varepsilon_4+\varepsilon_5 \end{pmatrix}, B = (b_i)_{i=1}^{10} = \begin{pmatrix} -\varepsilon_1+\varepsilon_2 \\ -\varepsilon_1+\varepsilon_3 \\ -\varepsilon_1+\varepsilon_4 \\ -\varepsilon_1+\varepsilon_5 \\ -\varepsilon_2+\varepsilon_3 \\ -\varepsilon_2+\varepsilon_4 \\ -\varepsilon_2+\varepsilon_5 \\ -\varepsilon_3+\varepsilon_4 \\ -\varepsilon_3+\varepsilon_5 \\ -\varepsilon_4+\varepsilon_5 \end{pmatrix} \tag{4.13}$$

$$C = (c_i)_{i=1}^{10} = \frac{1}{2}\begin{pmatrix} \varepsilon_8-\varepsilon_7-\varepsilon_6-\varepsilon_1-\varepsilon_2+\varepsilon_3+\varepsilon_4+\varepsilon_5 \\ \varepsilon_8-\varepsilon_7-\varepsilon_6-\varepsilon_1+\varepsilon_2-\varepsilon_3+\varepsilon_4+\varepsilon_5 \\ \varepsilon_8-\varepsilon_7-\varepsilon_6-\varepsilon_1+\varepsilon_2+\varepsilon_3-\varepsilon_4+\varepsilon_5 \\ \varepsilon_8-\varepsilon_7-\varepsilon_6-\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4-\varepsilon_5 \\ \varepsilon_8-\varepsilon_7-\varepsilon_6+\varepsilon_1-\varepsilon_2-\varepsilon_3+\varepsilon_4+\varepsilon_5 \\ \varepsilon_8-\varepsilon_7-\varepsilon_6+\varepsilon_1-\varepsilon_2+\varepsilon_3-\varepsilon_4+\varepsilon_5 \\ \varepsilon_8-\varepsilon_7-\varepsilon_6+\varepsilon_1-\varepsilon_2+\varepsilon_3+\varepsilon_4-\varepsilon_5 \\ \varepsilon_8-\varepsilon_7-\varepsilon_6+\varepsilon_1+\varepsilon_2-\varepsilon_3-\varepsilon_4+\varepsilon_5 \\ \varepsilon_8-\varepsilon_7-\varepsilon_6+\varepsilon_1+\varepsilon_2-\varepsilon_3+\varepsilon_4-\varepsilon_5 \\ \varepsilon_8-\varepsilon_7-\varepsilon_6+\varepsilon_1+\varepsilon_2+\varepsilon_3-\varepsilon_4-\varepsilon_5 \end{pmatrix} \tag{4.14}$$

$$D = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \varepsilon_8-\varepsilon_7-\varepsilon_6+\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4+\varepsilon_5 \\ \varepsilon_8-\varepsilon_7-\varepsilon_6+\varepsilon_1-\varepsilon_2-\varepsilon_3-\varepsilon_4-\varepsilon_5 \\ \varepsilon_8-\varepsilon_7-\varepsilon_6-\varepsilon_1+\varepsilon_2-\varepsilon_3-\varepsilon_4-\varepsilon_5 \\ \varepsilon_8-\varepsilon_7-\varepsilon_6-\varepsilon_1-\varepsilon_2+\varepsilon_3-\varepsilon_4-\varepsilon_5 \\ \varepsilon_8-\varepsilon_7-\varepsilon_6-\varepsilon_1-\varepsilon_2-\varepsilon_3+\varepsilon_4-\varepsilon_5 \\ \varepsilon_8-\varepsilon_7-\varepsilon_6-\varepsilon_1-\varepsilon_2-\varepsilon_3-\varepsilon_4+\varepsilon_5 \end{pmatrix} \tag{4.15}$$

Simple positive roots are: $\alpha_1 = d_2, \alpha_2 = a_1, \alpha_3 = b_1, \alpha_4 = b_5, \alpha_5 = b_8, \alpha_6 = b_{10}$. The matrix for $(\alpha_i)$ is

$$g = \begin{pmatrix} 1/2 & -1/2 & -1/2 & -1/2 & -1/2 & -1/2 & -1/2 & 1/2 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \end{pmatrix} \tag{4.16}$$

The coordinates of $a_i$ relative to the ordered basis $\{\alpha_1, \dots, \alpha_6\}$ are the columns of

$$(gg^t)^{-1}gA^t = \begin{pmatrix} 0&0&0&0&0&0&0&0&0&0 \\ 1&1&1&1&1&1&1&1&1&1 \\ 0&0&0&0&1&1&1&1&1&1 \\ 0&1&1&1&1&1&1&2&2&2 \\ 0&0&1&1&0&1&1&1&1&2 \\ 0&0&0&1&0&0&1&0&1&1 \end{pmatrix} \tag{4.17}$$

The coordinates of $b_i$ relative to the ordered basis $\{\alpha_1, \dots, \alpha_6\}$ are the columns of

$$(gg^t)^{-1}gB^t = \begin{pmatrix} 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 1&1&1&1&0&0&0&0&0&0 \\ 0&1&1&1&1&1&1&0&0&0 \\ 0&0&1&1&0&1&1&1&1&0 \\ 0&0&0&1&0&0&1&0&1&1 \end{pmatrix} \tag{4.18}$$

The coordinates of $c_i$ relative to the ordered basis $\{\alpha_1, \dots, \alpha_6\}$ are the columns of

$$(gg^t)^{-1}gC^t = \begin{pmatrix} 1&1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1&1 \\ 2&2&2&2&1&1&1&1&1&1 \\ 3&2&2&2&2&2&2&1&1&1 \\ 2&2&1&1&2&1&1&1&1&0 \\ 1&1&1&0&1&1&0&1&0&0 \end{pmatrix} \tag{4.19}$$

The coordinates of $d_i$ relative to the ordered basis $\{\alpha_1, \dots, \alpha_6\}$ are the columns of

$$(gg^t)^{-1}gD^t = \begin{pmatrix} 1&1&1&1&1&1 \\ 2&0&0&0&0&0 \\ 2&0&1&1&1&1 \\ 3&0&0&1&1&1 \\ 2&0&0&0&1&1 \\ 1&0&0&0&0&1 \end{pmatrix} \tag{4.20}$$

We can determine $\widetilde{N}_{\alpha,\beta}$ as before.

**Lemma 4.3.** *Let $\alpha, \beta$ be positive roots; then*

$$\widetilde{N}_{\alpha,\beta} = \sqrt{2}\,\mathrm{sgn}(N_{\alpha,\beta}). \tag{4.21}$$

*If $\alpha - \beta \neq 0$, then*

$$\widetilde{N}_{\alpha,-\beta} = \sqrt{2}\,\mathrm{sgn}(N_{\alpha,-\beta}). \tag{4.22}$$

*Proof.* The proof is similar to the proof of Lemma 4.2, using the fact that if $\alpha$ is a root, then $|\alpha|^2 = 2$. q.e.d.

Since $(E_6, \alpha_1)$ and $(E_6, \alpha_6)$ are Hermitian symmetric spaces, we only consider $(E_6, \alpha_i)$, $2 \le i \le 5$ below.

**4.3.2. The space $(E_6, \alpha_2)$.**

$$\begin{aligned} \Delta_2^+(1) =& \{a_1, \dots, a_{10}; c_1, \dots, c_{10}\} \\ \Delta_2^+(2) =& \{d_1\} \end{aligned} \tag{4.23}$$

$\Delta_2^+(k) = \emptyset$, for $k \ge 3$.

$$\begin{cases} \dim = 21, \mathrm{Ric} = 11g, \\ \text{4 largest eigenvalues of } M_1 \text{ are } 5.5000,\ 5.5000,\ 5.5000,\ 11.0000, \\ \text{eigenvalues of } M_2 \text{ are at most } 5.5 \text{ (using Lemma 2.7).} \end{cases}$$

Thus the space has $QB > 0$.

**4.4. The space $(E_6, \alpha_3)$.**

$$\begin{aligned} \Delta_3^+(1) =& \{a_5, \dots, a_{10}; b_1, \dots, b_4; c_5, \dots, c_{10}; d_3, \dots, d_6\} \\ \Delta_3^+(2) =& \{c_1, \dots, c_4; d_1\} \end{aligned} \tag{4.24}$$

$\Delta_3^+(k) = \emptyset$, for $k \ge 3$.

$$\begin{cases} \dim = 25, \mathrm{Ric} = 9g, \\ \text{4 largest eigenvalues of } M_1 \text{ are } 5.3117,\ 5.3117,\ 8.5000,\ 9.0000, \\ \text{eigenvalues of } M_2 \text{ are at most } 8.5 \text{ (using Lemma 2.7).} \end{cases}$$

Thus the space has $QB > 0$.

**4.4.1. The space $(E_6, \alpha_4)$.**

$$\begin{aligned} \Delta_4^+(1) =& \{a_2, \dots, a_7; b_2, \dots, b_7; c_8, \dots, c_{10}; d_4, \dots, d_6\} \\ \Delta_4^+(2) =& \{a_8, \dots, a_{10}; c_2, \dots, c_7\} \\ \Delta_4^+(3) =& \{c_1, d_1\}. \end{aligned} \tag{4.25}$$

$\Delta_4^+(k) = \emptyset$, for $k \ge 4$.

$$\begin{cases} \dim = 29, \mathrm{Ric} = 7g, \\ \text{4 largest eigenvalues of } M_1 \text{ are } 5.8226,\ 5.8226,\ 7.0000,\ 7.1468. \end{cases}$$

The space does not have $QB \ge 0$.

**4.4.2. The space $(E_6, \alpha_5)$.**

(4.26)

$$\begin{aligned} \Delta_5^+(1) =& \{a_3, a_4, a_6, \dots, a_9; b_3, b_4, b_6, \dots, b_9; c_3, c_4, c_6, \dots, c_9; d_5, d_6\} \\ \Delta_5^+(2) =& \{a_{10}; c_1, c_2, c_5; d_1\} \end{aligned}$$

$\Delta_5^+(k) = \emptyset$, for $k \ge 3$.

$$\begin{cases} \dim = 25, \mathrm{Ric} = 9g, \\ 4 \text{ largest eigenvalues of } M_1 \text{ are } 5.3117,\ 5.3117,\ 8.5000,\ 9.0000, \\ \text{eigenvalues of } M_2 \text{ are at most } 8.5 \text{ (using Lemma 2.7).} \end{cases}$$

Thus the space has $QB > 0$.

## 4.5. The spaces $(E_7, \alpha_i)$.

**4.5.1. Root system.** Consider the subspace $V$ of $\mathbb{R}^8$, orthogonal to $\varepsilon_7 + \varepsilon_8$. The positive roots in $V$ are $\pm\varepsilon_i + \varepsilon_j, 1 \le i < j \le 6$ (total 30), $-\varepsilon_7 + \varepsilon_8$, and

$$\frac{1}{2}(\varepsilon_8 - \varepsilon_7 + \sum_{i=1}^{6}(-1)^{\nu(i)}\varepsilon_i)$$

so that $\sum_1^6 \nu(i)$ is odd, i.e. the number of minus signs, is odd (total 32). Let

$$A = (a_i)_{i=1}^{15} = \begin{pmatrix} \varepsilon_1+\varepsilon_2 \\ \varepsilon_1+\varepsilon_3 \\ \varepsilon_1+\varepsilon_4 \\ \varepsilon_1+\varepsilon_5 \\ \varepsilon_1+\varepsilon_6 \\ \varepsilon_2+\varepsilon_3 \\ \varepsilon_2+\varepsilon_4 \\ \varepsilon_2+\varepsilon_5 \\ \varepsilon_2+\varepsilon_6 \\ \varepsilon_3+\varepsilon_4 \\ \varepsilon_3+\varepsilon_5 \\ \varepsilon_3+\varepsilon_6 \\ \varepsilon_4+\varepsilon_5 \\ \varepsilon_4+\varepsilon_6 \\ \varepsilon_5+\varepsilon_6 \end{pmatrix}, B = (b_i)_{i=1}^{16} = \begin{pmatrix} -\varepsilon_1+\varepsilon_2 \\ -\varepsilon_1+\varepsilon_3 \\ -\varepsilon_1+\varepsilon_4 \\ -\varepsilon_1+\varepsilon_5 \\ -\varepsilon_1+\varepsilon_6 \\ -\varepsilon_2+\varepsilon_3 \\ -\varepsilon_2+\varepsilon_4 \\ -\varepsilon_2+\varepsilon_5 \\ -\varepsilon_2+\varepsilon_6 \\ -\varepsilon_3+\varepsilon_4 \\ -\varepsilon_3+\varepsilon_5 \\ -\varepsilon_3+\varepsilon_6 \\ -\varepsilon_4+\varepsilon_5 \\ -\varepsilon_4+\varepsilon_6 \\ -\varepsilon_5+\varepsilon_6 \\ -\varepsilon_7+\varepsilon_8 \end{pmatrix} \tag{4.27}$$

$$C = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \varepsilon_8-\varepsilon_7-\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4+\varepsilon_5+\varepsilon_6 \\ \varepsilon_8-\varepsilon_7+\varepsilon_1-\varepsilon_2+\varepsilon_3+\varepsilon_4+\varepsilon_5+\varepsilon_6 \\ \varepsilon_8-\varepsilon_7+\varepsilon_1+\varepsilon_2-\varepsilon_3+\varepsilon_4+\varepsilon_5+\varepsilon_6 \\ \varepsilon_8-\varepsilon_7+\varepsilon_1+\varepsilon_2+\varepsilon_3-\varepsilon_4+\varepsilon_5+\varepsilon_6 \\ \varepsilon_8-\varepsilon_7+\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4-\varepsilon_5+\varepsilon_6 \\ \varepsilon_8-\varepsilon_7+\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4+\varepsilon_5-\varepsilon_6 \end{pmatrix} \tag{4.28}$$

$$D = \begin{pmatrix} d_i \end{pmatrix}_{i=1}^{20} = \frac{1}{2}\begin{pmatrix} \varepsilon_8-\varepsilon_7-\varepsilon_1-\varepsilon_2-\varepsilon_3+\varepsilon_4+\varepsilon_5+\varepsilon_6 \\ \varepsilon_8-\varepsilon_7-\varepsilon_1-\varepsilon_2+\varepsilon_3-\varepsilon_4+\varepsilon_5+\varepsilon_6 \\ \varepsilon_8-\varepsilon_7-\varepsilon_1-\varepsilon_2+\varepsilon_3+\varepsilon_4-\varepsilon_5+\varepsilon_6 \\ \varepsilon_8-\varepsilon_7-\varepsilon_1-\varepsilon_2+\varepsilon_3+\varepsilon_4+\varepsilon_5-\varepsilon_6 \\ \varepsilon_8-\varepsilon_7-\varepsilon_1+\varepsilon_2-\varepsilon_3-\varepsilon_4+\varepsilon_5+\varepsilon_6 \\ \varepsilon_8-\varepsilon_7-\varepsilon_1+\varepsilon_2-\varepsilon_3+\varepsilon_4-\varepsilon_5+\varepsilon_6 \\ \varepsilon_8-\varepsilon_7-\varepsilon_1+\varepsilon_2-\varepsilon_3+\varepsilon_4+\varepsilon_5-\varepsilon_6 \\ \varepsilon_8-\varepsilon_7-\varepsilon_1+\varepsilon_2+\varepsilon_3-\varepsilon_4-\varepsilon_5+\varepsilon_6 \\ \varepsilon_8-\varepsilon_7-\varepsilon_1+\varepsilon_2+\varepsilon_3-\varepsilon_4+\varepsilon_5-\varepsilon_6 \\ \varepsilon_8-\varepsilon_7-\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4-\varepsilon_5-\varepsilon_6 \\ \varepsilon_8-\varepsilon_7+\varepsilon_1-\varepsilon_2-\varepsilon_3-\varepsilon_4+\varepsilon_5+\varepsilon_6 \\ \varepsilon_8-\varepsilon_7+\varepsilon_1-\varepsilon_2-\varepsilon_3+\varepsilon_4-\varepsilon_5+\varepsilon_6 \\ \varepsilon_8-\varepsilon_7+\varepsilon_1-\varepsilon_2-\varepsilon_3+\varepsilon_4+\varepsilon_5-\varepsilon_6 \\ \varepsilon_8-\varepsilon_7+\varepsilon_1-\varepsilon_2+\varepsilon_3-\varepsilon_4-\varepsilon_5+\varepsilon_6 \\ \varepsilon_8-\varepsilon_7+\varepsilon_1-\varepsilon_2+\varepsilon_3-\varepsilon_4+\varepsilon_5-\varepsilon_6 \\ \varepsilon_8-\varepsilon_7+\varepsilon_1-\varepsilon_2+\varepsilon_3+\varepsilon_4-\varepsilon_5-\varepsilon_6 \\ \varepsilon_8-\varepsilon_7+\varepsilon_1+\varepsilon_2-\varepsilon_3-\varepsilon_4-\varepsilon_5+\varepsilon_6 \\ \varepsilon_8-\varepsilon_7+\varepsilon_1+\varepsilon_2-\varepsilon_3-\varepsilon_4+\varepsilon_5-\varepsilon_6 \\ \varepsilon_8-\varepsilon_7+\varepsilon_1+\varepsilon_2-\varepsilon_3+\varepsilon_4-\varepsilon_5-\varepsilon_6 \\ \varepsilon_8-\varepsilon_7+\varepsilon_1+\varepsilon_2+\varepsilon_3-\varepsilon_4-\varepsilon_5-\varepsilon_6 \end{pmatrix} \tag{4.29}$$

$$E = \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \varepsilon_8-\varepsilon_7+\varepsilon_1-\varepsilon_2-\varepsilon_3-\varepsilon_4-\varepsilon_5-\varepsilon_6 \\ \varepsilon_8-\varepsilon_7-\varepsilon_1+\varepsilon_2-\varepsilon_3-\varepsilon_4-\varepsilon_5-\varepsilon_6 \\ \varepsilon_8-\varepsilon_7-\varepsilon_1-\varepsilon_2+\varepsilon_3-\varepsilon_4-\varepsilon_5-\varepsilon_6 \\ \varepsilon_8-\varepsilon_7-\varepsilon_1-\varepsilon_2-\varepsilon_3+\varepsilon_4-\varepsilon_5-\varepsilon_6 \\ \varepsilon_8-\varepsilon_7-\varepsilon_1-\varepsilon_2-\varepsilon_3-\varepsilon_4+\varepsilon_5-\varepsilon_6 \\ \varepsilon_8-\varepsilon_7-\varepsilon_1-\varepsilon_2-\varepsilon_3-\varepsilon_4-\varepsilon_5+\varepsilon_6 \end{pmatrix} \tag{4.30}$$

Simple positive roots are: $\alpha_1 = e_1, \alpha_2 = a_1, \alpha_3 = b_1, \alpha_4 = b_6, \alpha_5 = b_{10}, \alpha_6 = b_{13}, \alpha_7 = b_{15}$. The matrix for $(\alpha_i)$ is:
(4.31)

$$g = \begin{pmatrix} 1/2 & -1/2 & -1/2 & -1/2 & -1/2 & -1/2 & -1/2 & 1/2 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \end{pmatrix}$$

The coordinates of $(a_i)$ with respect to the ordered basis $\{\alpha_1, \dots, \alpha_7\}$ are given by the columns of

(4.32)
$$(gg^t)^{-1}gA^t = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

The coordinates of $(b_i)$ with respect to the ordered basis $\{\alpha_1, \dots, \alpha_7\}$ are given by the columns of

(4.33)
$$(gg^t)^{-1}gB^t = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{pmatrix}$$

The coordinates of $(c_i)$ with respect to the ordered basis $\{\alpha_1, \dots, \alpha_7\}$ are given by the columns of

$$(gg^t)^{-1}gC^t = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & 2 & 2 \\ 3 & 2 & 2 & 2 & 2 & 2 \\ 4 & 4 & 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix} \tag{4.34}$$

The coordinates of $(d_i)$ with respect to the ordered basis $\{\alpha_1, \dots, \alpha_7\}$ are given by the columns of $(gg^t)^{-1}gD^t$, which are:

(4.35)
$$\begin{pmatrix} 1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\ 2&2&2&2&2&2&2&2&2&2&1&1&1&1&1&1&1&1&1&1\\ 3&3&3&3&2&2&2&2&2&2&2&2&2&2&2&2&1&1&1&1\\ 3&2&2&2&2&2&2&1&1&1&2&2&2&1&1&1&1&1&1&0\\ 2&2&1&1&2&1&1&1&1&0&2&1&1&1&1&0&1&1&0&0\\ 1&1&1&0&1&1&0&1&0&0&1&1&0&1&0&0&1&0&0&0 \end{pmatrix}$$

The coordinates of $(e_i)$ with respect to the ordered basis $\{\alpha_1, \dots, \alpha_7\}$ are given by the columns of

$$(gg^t)^{-1}gE^t = \begin{pmatrix} 1&1&1&1&1&1\\ 0&0&0&0&0&0\\ 0&1&1&1&1&1\\ 0&0&1&1&1&1\\ 0&0&0&1&1&1\\ 0&0&0&0&1&1\\ 0&0&0&0&0&1 \end{pmatrix} \tag{4.36}$$

Since $\alpha$ is a root implies $|\alpha|^2 = 2$, it is easy to see that Lemma 4.3 is still true in this case. Note that $(E_7, \alpha_7)$ is Hermitian symmetric.

**4.5.2. The space $(E_7, \alpha_1)$.**

$$\begin{aligned} \Delta_1^+(1) =& \{c_1, \dots, c_6, d_1, \dots, d_{20}, e_1, \dots, e_6\} \\ \Delta_1^+(2) =& \{b_{16}\}, \end{aligned} \tag{4.37}$$

$\Delta_1^+(k) = \emptyset$ for $k \geq 3$.

$$\begin{cases} \text{dim} = 33, \text{Ric} = 17g, \\ \text{4 largest eigenvalues of } M_1 \text{ are 17, 7.5, 7.5, 7.5,} \\ \text{eigenvalues of } M_2 \text{ are at most 7.5 (using Lemma 2.7).} \end{cases}$$

Thus the space has $QB > 0$.

**4.5.3. The space $(E_7, \alpha_2)$.**

$$\begin{aligned} \Delta_2^+(1) =& \{a_1, \dots, a_{15}, d_1, \dots, d_{20}\} \\ \Delta_2^+(2) =& \{c_1, \dots, c_6, b_{16}\}, \end{aligned} \tag{4.38}$$

$\Delta_2^+(k) = \emptyset$ for $k \geq 3$.

$$\begin{cases} \dim = 42, \mathrm{Ric} = 14g, \\ \text{4 largest eigenvalues of } M_1 \text{ are 14, 8.012, 8.012, 8.012,} \\ \text{eigenvalues of } M_2 \text{ are at most 9 (using Lemma 2.7).} \end{cases}$$

Thus the space has $QB > 0$.

**4.5.4. The space $(E_7, \alpha_3)$.**

$$\begin{aligned} \Delta_3^+(1) =& \{a_6, \dots, a_{15}, b_1, \dots, b_5, d_{11}, \dots, d_{20}, e_2, \dots, e_6\} \\ \Delta_3^+(2) =& \{d_1, \dots, d_{10}, c_2, \dots, c_6\} \\ \Delta_3^+(3) =& \{b_{16}, c_1\} \end{aligned} \tag{4.39}$$

$\Delta_3^+(k) = \emptyset$ for $k \geq 4$.

$$\begin{cases} \dim = 47, \mathrm{Ric} = 11g, \\ \text{4 largest eigenvalues of } M_1 \text{ are 12.1411, 11, 7.6829, 7.6829.} \end{cases}$$

Thus the space does not have $QB \geq 0$.

**4.5.5. The space $(E_7, \alpha_4)$.**

$$\begin{aligned} \Delta_4^+(1) =& \{a_2, \dots, a_9, b_2, \dots, b_9, d_{17}, \dots, d_{20}, e_3, \dots, e_6\} \\ \Delta_4^+(2) =& \{a_{10}, \dots, a_{15}, d_5, \dots, d_{16}\} \\ \Delta_4^+(3) =& \{c_3, \dots, c_6, d_1, \dots, d_4\} \\ \Delta_4^+(4) =& \{b_{16}, c_1, c_2\} \end{aligned} \tag{4.40}$$

$\Delta_4^+(k) = \emptyset$ for $k \geq 5$.

$$\begin{cases} \dim = 53, \ \mathrm{Ric} = 8g, \\ \text{4 largest eigenvalues of } M_1 \text{ are 9.5692, 8.1727, 8.1727, 8.} \end{cases}$$

Thus the space does not have $QB \geq 0$

**4.5.6. $(E_7, \alpha_5)$.**

(4.41)

$$\begin{aligned} \Delta_5^+(1) =& \{a_3, \dots, a_5, a_7, \dots, a_{12}, b_3, \dots, b_5, b_7, \dots, b_{12}, d_8, \dots, d_{10}, d_{14}, \dots, \\ & d_{19}, e_4, \dots, e_6\} \\ \Delta_5^+(2) =& \{a_{13}, \dots, a_{15}, c_4, \dots, c_6, d_2, \dots, d_7, d_{11}, \dots, d_{13}\} \\ \Delta_5^+(3) =& \{b_{16}, c_1, c_2, c_3, d_1\} \end{aligned}$$

$\Delta_5^+(k) = \emptyset$ for $k \geq 4$.

$$\begin{cases} \dim = 50, \mathrm{Ric} = 10g, \\ \text{4 largest eigenvalues of } M_1 \text{ are 10, 9.7882, 9.7882, 8.0097,} \\ \text{eigenvalues of } M_2 \text{ are less than 10.} \end{cases}$$

(The estimate for $M_2$ is obtained by using $\mu = 10$ and $s = 1$ in Lemma 2.8, in which case the maximum weighted row sum is 9.9806.)

Thus the space has $QB > 0$.

**4.5.7. The space $(E_7, \alpha_6)$.**

(4.42)

$$\begin{aligned} \Delta_6^+(1) = &\{a_4, a_5, a_8, a_9, a_{11}, \dots, a_{14}, b_4, b_5, b_8, b_9, b_{11}, \dots, b_{14}, c_5, c_6, d_3, d_4, \\ &d_6, \dots, d_9, d_{12}, \dots, d_{15}, e_5, e_6, d_{17}, d_{18}\} \\ \Delta_6^+(2) = &\{a_{15}, b_{16}, c_1, c_2, c_3, c_4, d_1, d_2, d_5, d_{11}\} \end{aligned}$$

$\Delta_6^+(k) = \emptyset$ for $k \geq 3$.

$$\begin{cases} \dim = 42, \ \mathrm{Ric} = 13g, \\ \text{4 largest eigenvalues of } M_1 \text{ are 13.5, 13, 7.1504, 7.1504.} \end{cases}$$

Thus the space does not have $QB \geq 0$.

**4.6. The spaces $(E_8, \alpha_i)$.**

**4.6.1. Root system.** Let $V = \mathbb{R}^8$. The positive roots in $V$ are $\pm\varepsilon_i + \varepsilon_j, 1 \leq i < j \leq 8$ (total 56), and

$$\frac{1}{2}(\varepsilon_8 + \sum_{i=1}^{7} (-1)^{\nu(i)} \varepsilon_i)$$

so that $\sum_{i=1}^{7} \nu(i)$ is even, i.e. the number of minus signs is even (total $21 + 35 + 8 = 64$). Let

$$(4.43)\qquad A = \left(\ a_i\ \right)_{i=1}^{28} = \begin{pmatrix} \varepsilon_1+\varepsilon_2 \\ \varepsilon_1+\varepsilon_3 \\ \varepsilon_1+\varepsilon_4 \\ \varepsilon_1+\varepsilon_5 \\ \varepsilon_1+\varepsilon_6 \\ \varepsilon_1+\varepsilon_7 \\ \varepsilon_1+\varepsilon_8 \\ \varepsilon_2+\varepsilon_3 \\ \varepsilon_2+\varepsilon_4 \\ \varepsilon_2+\varepsilon_5 \\ \varepsilon_2+\varepsilon_6 \\ \varepsilon_2+\varepsilon_7 \\ \varepsilon_2+\varepsilon_8 \\ \varepsilon_3+\varepsilon_4 \\ \varepsilon_3+\varepsilon_5 \\ \varepsilon_3+\varepsilon_6 \\ \varepsilon_3+\varepsilon_7 \\ \varepsilon_3+\varepsilon_8 \\ \varepsilon_4+\varepsilon_5 \\ \varepsilon_4+\varepsilon_6 \\ \varepsilon_4+\varepsilon_7 \\ \varepsilon_4+\varepsilon_8 \\ \varepsilon_5+\varepsilon_6 \\ \varepsilon_5+\varepsilon_7 \\ \varepsilon_5+\varepsilon_8 \\ \varepsilon_6+\varepsilon_7 \\ \varepsilon_6+\varepsilon_8 \\ \varepsilon_7+\varepsilon_8 \end{pmatrix}, B = \left(\ b_i\ \right)_{i=1}^{28} = \begin{pmatrix} -\varepsilon_1+\varepsilon_2 \\ -\varepsilon_1+\varepsilon_3 \\ -\varepsilon_1+\varepsilon_4 \\ -\varepsilon_1+\varepsilon_5 \\ -\varepsilon_1+\varepsilon_6 \\ -\varepsilon_1+\varepsilon_7 \\ -\varepsilon_1+\varepsilon_8 \\ -\varepsilon_2+\varepsilon_3 \\ -\varepsilon_2+\varepsilon_4 \\ -\varepsilon_2+\varepsilon_5 \\ -\varepsilon_2+\varepsilon_6 \\ -\varepsilon_2+\varepsilon_7 \\ -\varepsilon_2+\varepsilon_8 \\ -\varepsilon_3+\varepsilon_4 \\ -\varepsilon_3+\varepsilon_5 \\ -\varepsilon_3+\varepsilon_6 \\ -\varepsilon_3+\varepsilon_7 \\ -\varepsilon_3+\varepsilon_8 \\ -\varepsilon_4+\varepsilon_5 \\ -\varepsilon_4+\varepsilon_6 \\ -\varepsilon_4+\varepsilon_7 \\ -\varepsilon_4+\varepsilon_8 \\ -\varepsilon_5+\varepsilon_6 \\ -\varepsilon_5+\varepsilon_7 \\ -\varepsilon_5+\varepsilon_8 \\ -\varepsilon_6+\varepsilon_7 \\ -\varepsilon_6+\varepsilon_8 \\ -\varepsilon_7+\varepsilon_8 \end{pmatrix}$$

$$(4.44)\qquad C = \left(\ c_i\ \right)_{i=1}^{21} = \begin{pmatrix} \varepsilon_8-\varepsilon_1-\varepsilon_2+\varepsilon_3+\varepsilon_4+\varepsilon_5+\varepsilon_6+\varepsilon_7 \\ \varepsilon_8-\varepsilon_1+\varepsilon_2-\varepsilon_3+\varepsilon_4+\varepsilon_5+\varepsilon_6+\varepsilon_7 \\ \varepsilon_8-\varepsilon_1+\varepsilon_2+\varepsilon_3-\varepsilon_4+\varepsilon_5+\varepsilon_6+\varepsilon_7 \\ \varepsilon_8-\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4-\varepsilon_5+\varepsilon_6+\varepsilon_7 \\ \varepsilon_8-\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4+\varepsilon_5-\varepsilon_6+\varepsilon_7 \\ \varepsilon_8-\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4+\varepsilon_5+\varepsilon_6-\varepsilon_7 \\ \varepsilon_8+\varepsilon_1-\varepsilon_2-\varepsilon_3+\varepsilon_4+\varepsilon_5+\varepsilon_6+\varepsilon_7 \\ \varepsilon_8+\varepsilon_1-\varepsilon_2+\varepsilon_3-\varepsilon_4+\varepsilon_5+\varepsilon_6+\varepsilon_7 \\ \varepsilon_8+\varepsilon_1-\varepsilon_2+\varepsilon_3+\varepsilon_4-\varepsilon_5+\varepsilon_6+\varepsilon_7 \\ \varepsilon_8+\varepsilon_1-\varepsilon_2+\varepsilon_3+\varepsilon_4+\varepsilon_5-\varepsilon_6+\varepsilon_7 \\ \varepsilon_8+\varepsilon_1-\varepsilon_2+\varepsilon_3+\varepsilon_4+\varepsilon_5+\varepsilon_6-\varepsilon_7 \\ \varepsilon_8+\varepsilon_1+\varepsilon_2-\varepsilon_3-\varepsilon_4+\varepsilon_5+\varepsilon_6+\varepsilon_7 \\ \varepsilon_8+\varepsilon_1+\varepsilon_2-\varepsilon_3+\varepsilon_4-\varepsilon_5+\varepsilon_6+\varepsilon_7 \\ \varepsilon_8+\varepsilon_1+\varepsilon_2-\varepsilon_3+\varepsilon_4+\varepsilon_5-\varepsilon_6+\varepsilon_7 \\ \varepsilon_8+\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4+\varepsilon_5+\varepsilon_6-\varepsilon_7 \\ \varepsilon_8+\varepsilon_1+\varepsilon_2+\varepsilon_3-\varepsilon_4-\varepsilon_5+\varepsilon_6+\varepsilon_7 \\ \varepsilon_8+\varepsilon_1+\varepsilon_2+\varepsilon_3-\varepsilon_4+\varepsilon_5-\varepsilon_6+\varepsilon_7 \\ \varepsilon_8+\varepsilon_1+\varepsilon_2+\varepsilon_3-\varepsilon_4+\varepsilon_5+\varepsilon_6-\varepsilon_7 \\ \varepsilon_8+\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4-\varepsilon_5-\varepsilon_6+\varepsilon_7 \\ \varepsilon_8+\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4-\varepsilon_5+\varepsilon_6-\varepsilon_7 \\ \varepsilon_8+\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4+\varepsilon_5-\varepsilon_6-\varepsilon_7 \end{pmatrix}$$

$$D = \begin{pmatrix} d_i \end{pmatrix}_{i=1}^{35} = \frac{1}{2}\begin{pmatrix}
\varepsilon_8 - \varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4 + \varepsilon_5 + \varepsilon_6 + \varepsilon_7 \\
\varepsilon_8 - \varepsilon_1 - \varepsilon_2 - \varepsilon_3 + \varepsilon_4 - \varepsilon_5 + \varepsilon_6 + \varepsilon_7 \\
\varepsilon_8 - \varepsilon_1 - \varepsilon_2 - \varepsilon_3 + \varepsilon_4 + \varepsilon_5 - \varepsilon_6 + \varepsilon_7 \\
\varepsilon_8 - \varepsilon_1 - \varepsilon_2 - \varepsilon_3 + \varepsilon_4 + \varepsilon_5 + \varepsilon_6 - \varepsilon_7 \\
\varepsilon_8 - \varepsilon_1 - \varepsilon_2 + \varepsilon_3 - \varepsilon_4 - \varepsilon_5 + \varepsilon_6 + \varepsilon_7 \\
\varepsilon_8 - \varepsilon_1 - \varepsilon_2 + \varepsilon_3 - \varepsilon_4 + \varepsilon_5 - \varepsilon_6 + \varepsilon_7 \\
\varepsilon_8 - \varepsilon_1 - \varepsilon_2 + \varepsilon_3 - \varepsilon_4 + \varepsilon_5 + \varepsilon_6 - \varepsilon_7 \\
\varepsilon_8 - \varepsilon_1 - \varepsilon_2 + \varepsilon_3 + \varepsilon_4 - \varepsilon_5 - \varepsilon_6 + \varepsilon_7 \\
\varepsilon_8 - \varepsilon_1 - \varepsilon_2 + \varepsilon_3 + \varepsilon_4 - \varepsilon_5 + \varepsilon_6 - \varepsilon_7 \\
\varepsilon_8 - \varepsilon_1 - \varepsilon_2 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5 - \varepsilon_6 - \varepsilon_7 \\
\varepsilon_8 - \varepsilon_1 + \varepsilon_2 - \varepsilon_3 - \varepsilon_4 - \varepsilon_5 + \varepsilon_6 + \varepsilon_7 \\
\varepsilon_8 - \varepsilon_1 + \varepsilon_2 - \varepsilon_3 - \varepsilon_4 + \varepsilon_5 - \varepsilon_6 + \varepsilon_7 \\
\varepsilon_8 - \varepsilon_1 + \varepsilon_2 - \varepsilon_3 - \varepsilon_4 + \varepsilon_5 + \varepsilon_6 - \varepsilon_7 \\
\varepsilon_8 - \varepsilon_1 + \varepsilon_2 - \varepsilon_3 + \varepsilon_4 - \varepsilon_5 - \varepsilon_6 + \varepsilon_7 \\
\varepsilon_8 - \varepsilon_1 + \varepsilon_2 - \varepsilon_3 + \varepsilon_4 - \varepsilon_5 + \varepsilon_6 - \varepsilon_7 \\
\varepsilon_8 - \varepsilon_1 + \varepsilon_2 - \varepsilon_3 + \varepsilon_4 + \varepsilon_5 - \varepsilon_6 - \varepsilon_7 \\
\varepsilon_8 - \varepsilon_1 + \varepsilon_2 + \varepsilon_3 - \varepsilon_4 - \varepsilon_5 - \varepsilon_6 + \varepsilon_7 \\
\varepsilon_8 - \varepsilon_1 + \varepsilon_2 + \varepsilon_3 - \varepsilon_4 - \varepsilon_5 + \varepsilon_6 - \varepsilon_7 \\
\varepsilon_8 - \varepsilon_1 + \varepsilon_2 + \varepsilon_3 - \varepsilon_4 + \varepsilon_5 - \varepsilon_6 - \varepsilon_7 \\
\varepsilon_8 - \varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 - \varepsilon_5 - \varepsilon_6 - \varepsilon_7 \\
\varepsilon_8 + \varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4 - \varepsilon_5 + \varepsilon_6 + \varepsilon_7 \\
\varepsilon_8 + \varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4 + \varepsilon_5 - \varepsilon_6 + \varepsilon_7 \\
\varepsilon_8 + \varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4 + \varepsilon_5 + \varepsilon_6 - \varepsilon_7 \\
\varepsilon_8 + \varepsilon_1 - \varepsilon_2 - \varepsilon_3 + \varepsilon_4 - \varepsilon_5 - \varepsilon_6 + \varepsilon_7 \\
\varepsilon_8 + \varepsilon_1 - \varepsilon_2 - \varepsilon_3 + \varepsilon_4 - \varepsilon_5 + \varepsilon_6 - \varepsilon_7 \\
\varepsilon_8 + \varepsilon_1 - \varepsilon_2 - \varepsilon_3 + \varepsilon_4 + \varepsilon_5 - \varepsilon_6 - \varepsilon_7 \\
\varepsilon_8 + \varepsilon_1 - \varepsilon_2 + \varepsilon_3 - \varepsilon_4 - \varepsilon_5 - \varepsilon_6 + \varepsilon_7 \\
\varepsilon_8 + \varepsilon_1 - \varepsilon_2 + \varepsilon_3 - \varepsilon_4 - \varepsilon_5 + \varepsilon_6 - \varepsilon_7 \\
\varepsilon_8 + \varepsilon_1 - \varepsilon_2 + \varepsilon_3 - \varepsilon_4 + \varepsilon_5 - \varepsilon_6 - \varepsilon_7 \\
\varepsilon_8 + \varepsilon_1 - \varepsilon_2 + \varepsilon_3 + \varepsilon_4 - \varepsilon_5 - \varepsilon_6 - \varepsilon_7 \\
\varepsilon_8 + \varepsilon_1 + \varepsilon_2 - \varepsilon_3 - \varepsilon_4 - \varepsilon_5 - \varepsilon_6 + \varepsilon_7 \\
\varepsilon_8 + \varepsilon_1 + \varepsilon_2 - \varepsilon_3 - \varepsilon_4 - \varepsilon_5 + \varepsilon_6 - \varepsilon_7 \\
\varepsilon_8 + \varepsilon_1 + \varepsilon_2 - \varepsilon_3 - \varepsilon_4 + \varepsilon_5 - \varepsilon_6 - \varepsilon_7 \\
\varepsilon_8 + \varepsilon_1 + \varepsilon_2 - \varepsilon_3 + \varepsilon_4 - \varepsilon_5 - \varepsilon_6 - \varepsilon_7 \\
\varepsilon_8 + \varepsilon_1 + \varepsilon_2 + \varepsilon_3 - \varepsilon_4 - \varepsilon_5 - \varepsilon_6 - \varepsilon_7
\end{pmatrix} \tag{4.45}$$

$$E = \begin{pmatrix} e_i \end{pmatrix}_{i=1}^{8} = \frac{1}{2}\begin{pmatrix}
\varepsilon_8 + \varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5 + \varepsilon_6 + \varepsilon_7 \\
\varepsilon_8 + \varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4 - \varepsilon_5 - \varepsilon_6 - \varepsilon_7 \\
\varepsilon_8 - \varepsilon_1 + \varepsilon_2 - \varepsilon_3 - \varepsilon_4 - \varepsilon_5 - \varepsilon_6 - \varepsilon_7 \\
\varepsilon_8 - \varepsilon_1 - \varepsilon_2 + \varepsilon_3 - \varepsilon_4 - \varepsilon_5 - \varepsilon_6 - \varepsilon_7 \\
\varepsilon_8 - \varepsilon_1 - \varepsilon_2 - \varepsilon_3 + \varepsilon_4 - \varepsilon_5 - \varepsilon_6 - \varepsilon_7 \\
\varepsilon_8 - \varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4 + \varepsilon_5 - \varepsilon_6 - \varepsilon_7 \\
\varepsilon_8 - \varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4 - \varepsilon_5 + \varepsilon_6 - \varepsilon_7 \\
\varepsilon_8 - \varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4 - \varepsilon_5 - \varepsilon_6 + \varepsilon_7
\end{pmatrix} \tag{4.46}$$

Simple positive roots are: $\alpha_1 = e_2, \alpha_2 = a_1, \alpha_3 = b_1, \alpha_4 = b_8, \alpha_5 = b_{14}, \alpha_6 = b_{19}, \alpha_7 = b_{23}, \alpha_8 = b_{26}$. The matrix for $(\alpha_i)$ is:

$$g = \begin{pmatrix} 1/2 & -1/2 & -1/2 & -1/2 & -1/2 & -1/2 & -1/2 & 1/2 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \end{pmatrix} \tag{4.47}$$

The coordinates of $(a_i)$ relative to the ordered basis $\{\alpha_1, \ldots, \alpha_8\}$ are given by the rows of

$$\left((gg^t)^{-1} g A^t\right)^t = \begin{pmatrix} 0&1&0&0&0&0&0&0 \\ 0&1&0&1&0&0&0&0 \\ 0&1&0&1&1&0&0&0 \\ 0&1&0&1&1&1&0&0 \\ 0&1&0&1&1&1&1&0 \\ 0&1&0&1&1&1&1&1 \\ 2&3&3&5&4&3&2&1 \\ 0&1&1&1&0&0&0&0 \\ 0&1&1&1&1&0&0&0 \\ 0&1&1&1&1&1&0&0 \\ 0&1&1&1&1&1&1&0 \\ 0&1&1&1&1&1&1&1 \\ 2&3&4&5&4&3&2&1 \\ 0&1&1&2&1&0&0&0 \\ 0&1&1&2&1&1&0&0 \\ 0&1&1&2&1&1&1&0 \\ 0&1&1&2&1&1&1&1 \\ 2&3&4&6&4&3&2&1 \\ 0&1&1&2&2&1&0&0 \\ 0&1&1&2&2&1&1&0 \\ 0&1&1&2&2&1&1&1 \\ 2&3&4&6&5&3&2&1 \\ 0&1&1&2&2&2&1&0 \\ 0&1&1&2&2&2&1&1 \\ 2&3&4&6&5&4&2&1 \\ 0&1&1&2&2&2&2&1 \\ 2&3&4&6&5&4&3&1 \\ 2&3&4&6&5&4&3&2 \end{pmatrix} \tag{4.48}$$

The coordinates of $(b_i)$ relative to the ordered basis $\{\alpha_1, \dots, \alpha_8\}$ are given by the rows of

$$\left((gg^t)^{-1}gB^t\right)^t = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 4 & 5 & 4 & 3 & 2 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 3 & 5 & 4 & 3 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 2 & 2 & 3 & 4 & 4 & 3 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 2 & 2 & 3 & 4 & 3 & 3 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 2 & 2 & 3 & 4 & 3 & 2 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 2 & 2 & 3 & 4 & 3 & 2 & 1 & 1 \\ 2 & 2 & 3 & 4 & 3 & 2 & 1 & 0 \end{pmatrix} \tag{4.49}$$

The coordinates of $(c_i)$ relative to the ordered basis $\{\alpha_1, \dots, \alpha_8\}$ are given by the rows of

$$\left((gg^t)^{-1}gC^t\right)^t = \begin{pmatrix} 1 & 2 & 3 & 5 & 4 & 3 & 2 & 1 \\ 1 & 2 & 3 & 4 & 4 & 3 & 2 & 1 \\ 1 & 2 & 3 & 4 & 3 & 3 & 2 & 1 \\ 1 & 2 & 3 & 4 & 3 & 2 & 2 & 1 \\ 1 & 2 & 3 & 4 & 3 & 2 & 1 & 1 \\ 1 & 2 & 3 & 4 & 3 & 2 & 1 & 0 \\ 1 & 2 & 2 & 4 & 4 & 3 & 2 & 1 \\ 1 & 2 & 2 & 4 & 3 & 3 & 2 & 1 \\ 1 & 2 & 2 & 4 & 3 & 2 & 2 & 1 \\ 1 & 2 & 2 & 4 & 3 & 2 & 1 & 1 \\ 1 & 2 & 2 & 4 & 3 & 2 & 1 & 0 \\ 1 & 2 & 2 & 3 & 3 & 3 & 2 & 1 \\ 1 & 2 & 2 & 3 & 3 & 2 & 2 & 1 \\ 1 & 2 & 2 & 3 & 3 & 2 & 1 & 1 \\ 1 & 2 & 2 & 3 & 3 & 2 & 1 & 0 \\ 1 & 2 & 2 & 3 & 2 & 2 & 2 & 1 \\ 1 & 2 & 2 & 3 & 2 & 2 & 1 & 1 \\ 1 & 2 & 2 & 3 & 2 & 2 & 1 & 0 \\ 1 & 2 & 2 & 3 & 2 & 1 & 1 & 1 \\ 1 & 2 & 2 & 3 & 2 & 1 & 1 & 0 \\ 1 & 2 & 2 & 3 & 2 & 1 & 0 & 0 \end{pmatrix} \tag{4.50}$$

The coordinates of $(d_i)$ relative to the ordered basis $\{\alpha_1, \ldots, \alpha_8\}$ are given by the rows of

$$\left((gg^t)^{-1}gD^t\right)^t = \begin{pmatrix} 1&1&2&3&3&3&2&1\\ 1&1&2&3&3&2&2&1\\ 1&1&2&3&3&2&1&1\\ 1&1&2&3&3&2&1&0\\ 1&1&2&3&2&2&2&1\\ 1&1&2&3&2&2&1&1\\ 1&1&2&3&2&2&1&0\\ 1&1&2&3&2&1&1&1\\ 1&1&2&3&2&1&1&0\\ 1&1&2&3&2&1&0&0\\ 1&1&2&2&2&2&2&1\\ 1&1&2&2&2&2&1&1\\ 1&1&2&2&2&2&1&0\\ 1&1&2&2&2&1&1&1\\ 1&1&2&2&2&1&1&0\\ 1&1&2&2&2&1&0&0\\ 1&1&2&2&1&1&1&1\\ 1&1&2&2&1&1&1&0\\ 1&1&2&2&1&1&0&0\\ 1&1&2&2&1&0&0&0\\ 1&1&1&2&2&2&2&1\\ 1&1&1&2&2&2&1&1\\ 1&1&1&2&2&2&1&0\\ 1&1&1&2&2&1&1&1\\ 1&1&1&2&2&1&1&0\\ 1&1&1&2&2&1&0&0\\ 1&1&1&2&1&1&1&1\\ 1&1&1&2&1&1&1&0\\ 1&1&1&2&1&1&0&0\\ 1&1&1&2&1&0&0&0\\ 1&1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1&0\\ 1&1&1&1&1&1&0&0\\ 1&1&1&1&1&0&0&0\\ 1&1&1&1&0&0&0&0 \end{pmatrix} \tag{4.51}$$

The coordinates of $(e_i)$ relative to the ordered basis $\{\alpha_1, \ldots, \alpha_8\}$ are given by the rows of

$$\left((gg^t)^{-1}gE^t\right)^t = \begin{pmatrix} 1&3&3&5&4&3&2&1\\ 1&0&0&0&0&0&0&0\\ 1&0&1&0&0&0&0&0\\ 1&0&1&1&0&0&0&0\\ 1&0&1&1&1&0&0&0\\ 1&0&1&1&1&1&0&0\\ 1&0&1&1&1&1&1&0\\ 1&0&1&1&1&1&1&1 \end{pmatrix} \tag{4.52}$$

Since $\alpha$ as a root implies that $|\alpha|^2 = 2$, Lemma 4.3 is still true in this case.

**4.6.2. The space** $(E_8, \alpha_1)$**.**

(4.53)
$$\begin{aligned}\Delta_1^+(1) =&\{c_1, \dots, c_{21}; d_1, \dots, d_{35}; e_1, \dots, e_8\}\\ \Delta_1^+(2) =&\{a_7, a_{13}, a_{18}, a_{22}, a_{25}, a_{27}, a_{28}; b_7, b_{13}, b_{18}, b_{22}, b_{25}, b_{27}, b_{28}\}\end{aligned}$$

$\Delta_1^+(k) = \emptyset$ for $k \geq 3$.

$$\begin{cases} \text{dim} = 78, \ \text{Ric} = 23g, \\ \text{4 largest eigenvalues of } M_1 \text{ are 14.1102, 14.1102, 14.1102, 23,} \\ \text{eigenvalues of } M_2 \text{ are at most 14.5 (using Lemma 2.7).} \end{cases}$$

Thus the space has $QB > 0$.

**4.6.3. The space** $(E_8, \alpha_2)$**.**

(4.54)
$$\begin{aligned}\Delta_2^+(1) =&\{a_1, \dots, a_6, a_8, \dots, a_{12}, a_{14}, \dots, a_{17}, a_{19}, \dots, a_{21}, a_{23}, a_{24}, a_{26};\\ & d_1, \dots, d_{35}\}\\ \Delta_2^+(2) =&\{b_7, b_{13}, b_{18}, b_{22}, b_{25}, b_{27}, b_{28}, c_1, \dots, c_{21}\}\\ \Delta_2^+(3) =&\{a_7, a_{13}, a_{18}, a_{22}, a_{25}, a_{27}, a_{28}; e_1\}\end{aligned}$$

$\Delta_2^+(k) = \emptyset$ for $k \geq 4$.

$$\begin{cases} \text{dim} = 92, \ \text{Ric} = 17g, \\ \text{4 largest eigenvalues of } M_1 \text{ are 13.8336, 13.8336, 13.8336, 17.0000,} \\ \text{eigenvalues of } M_2 \text{ are not more than 15 (using Lemma 2.7).} \end{cases}$$

Thus the space has $QB > 0$.

**4.6.4. The space** $(E_8, \alpha_3)$**.**

(4.55)
$$\begin{aligned}\Delta_3^+(1) =&\{a_8, \dots, a_{12}, a_{14}, \dots, a_{17}, a_{19}, \dots, a_{21}, a_{23}, a_{24}, a_{26}; b_1, \dots, b_6;\\ & d_{21}, \dots, d_{35}; e_3, \dots, e_8\}\\ \Delta_3^+(2) =&\{c_7, \dots, c_{21}; d_1, \dots, d_{20}\}\\ \Delta_3^+(3) =&\{a_7; b_{13}, b_{18}, b_{22}, b_{25}, b_{27}, b_{28}; c_1, \dots, c_6; e_1\}\\ \Delta_3^+(4) =&\{a_{13}, a_{18}, a_{22}, a_{25}, a_{27}, a_{28}; b_7\}.\end{aligned}$$

$\Delta_3^+(k) = \emptyset$ for $k \geq 5$.

$$\begin{cases} \text{dim} = 98, \ \text{Ric} = 13g, \\ \text{4 largest eigenvalues of } M_1 \text{ are 11.3117, 11.3117, 13.0000, 16.9627.} \end{cases}$$

Thus the space does not have $QB \geq 0$.

**4.6.5. The space $(E_8, \alpha_4)$.**

(4.56)

$$\begin{aligned}
\Delta_4^+(1) =&\{a_2,\dots,a_6,a_8,\dots,a_{12};b_2,\dots,b_6;b_8,\dots,b_{12};d_{31},\dots,d_{35};\\
&e_4,\dots,e_8\}\\
\Delta_4^+(2) =&\{a_{14},\dots,a_{17},a_{19},\dots,a_{21},a_{23},a_{24},a_{26};d_{11},\dots,d_{30}\}\\
\Delta_4^+(3) =&\{c_{12},\dots,c_{21};d_1,\dots,d_{10}\}\\
\Delta_4^+(4) =&\{b_{18},b_{22},b_{25},b_{27},b_{28};c_2,\dots,c_{11}\}\\
\Delta_4^+(5) =&\{a_7,a_{13},b_7,b_{13},c_1,e_1\}\\
\Delta_4^+(6) =&\{a_{18},a_{22},a_{25},a_{27},a_{28}\}.
\end{aligned}$$

$\Delta_4^+(k) = \emptyset$ for $k \geq 7$.

$$\begin{cases} \text{dim} = 106,\ \text{Ric} = 9g, \\ \text{4 largest eigenvalues of } M_1 \text{ are } 9.0798,\ 11.2147,\ 11.2147,\ 12.6168. \end{cases}$$

Thus the space does not have $QB \geq 0$.

**4.6.6. The space $(E_8, \alpha_5)$.**

(4.57)

$$\begin{aligned}
\Delta_5^+(1) =&\{a_3,\dots,a_6,a_9,\dots,a_{12},a_{14},\dots,a_{17}\\
&b_3,\dots,b_6,b_9,\dots,b_{12},b_{14},\dots,b_{17};d_{17},\dots,d_{20},d_{27},\dots,d_{34},\\
&e_5,\dots,e_8\}\\
\Delta_5^+(2) =&\{a_{19},\dots,a_{21},a_{23},a_{24},a_{26};c_{16},\dots,c_{21}\\
&d_5,\dots,d_{16},d_{21},\dots,d_{26}\}\\
\Delta_5^+(3) =&\{b_{22},b_{25},b_{27},b_{28};c_3,\dots,c_6,c_8,\dots,c_{15},d_1,d_2,d_3,d_4\}\\
\Delta_5^+(4) =&\{a_7,a_{13},a_{18};b_7,b_{13},b_{18},c_1,c_2,c_7;e_1\}\\
\Delta_5^+(5) =&\{a_{22},a_{25},a_{27},a_{28}\}.
\end{aligned}$$

$\Delta_5^+(k) = \emptyset$ for $k \geq 6$.

$$\begin{cases} \text{dim} = 104,\ \text{Ric} = 11g, \\ \text{4 largest eigenvalues of } M_1 \text{ are } 11.5575,\ 12.0012,\ 12.0012,\ 12.0012. \end{cases}$$

Thus the space does not have $QB \geq 0$.

**4.6.7. The space** $(E_8, \alpha_6)$**.**

(4.58)

$$\begin{aligned}
\Delta_6^+(1) =&\{a_4, \dots, a_6, a_{10}, \dots, a_{12}, a_{15}, \dots, a_{17}, a_{19}, \dots, a_{21};\\
&b_4, \dots, b_6, b_{10}, \dots, b_{12}, b_{15}, \dots, b_{17}, b_{19}, \dots, b_{21}; c_{19}, \dots, c_{21}\\
&d_8, \dots, d_{10}, d_{14}, \dots, d_{19}, d_{24}, \dots, d_{29}, d_{31}, \dots, d_{33}, e_6, \dots, e_8\}\\
\Delta_6^+(2) =&\{a_{23}, a_{24}, a_{26}; b_{25}, b_{27}, b_{28}; c_4, \dots, c_6, c_9, \dots, c_{11}, c_{13}, \dots, c_{18};\\
&d_2, \dots, d_7, d_{11}, \dots, d_{13}, d_{21}, \dots, d_{23}\}\\
\Delta_6^+(3) =&\{a_7, a_{13}, a_{18}, a_{22}; b_7, b_{13}, b_{18}, b_{22}; c_1, \dots, c_3, c_7, c_8, c_{12}, d_1, e_1\}\\
\Delta_6^+(4) =&\{a_{25}, a_{27}, a_{28}\}
\end{aligned}$$

$\Delta_6^+(k) = \emptyset$ for $k \geq 5$.

$$\begin{cases} \text{dim} = 97, \ \text{Ric} = 14g, \\ \text{4 largest eigenvalues of } M_1 \text{ are } 11.4257,\ 14.0000,\ 16.0721,\ 16.0721. \end{cases}$$

Thus the space does not satisfy $QB \geq 0$.

**4.6.8. The space** $(E_8, \alpha_7)$**.**

(4.59)

$$\begin{aligned}
\Delta_7^+(1) =&\{a_5, a_6, a_{11}, a_{12}, a_{16}, a_{17}, a_{20}, a_{21}; a_{23}, a_{24};\\
&b_5, b_6, b_{11}, b_{12}, b_{16}, b_{17}, b_{20}, b_{21}, b_{23}, b_{24}, b_{27}, b_{28};\\
&c_5, c_6, c_{10}, c_{11}, c_{14}, c_{15}, c_{17}, \dots, c_{20};\\
&d_3, d_4, d_6, \dots, d_9, d_{12}, \dots, d_{15}, d_{17}, d_{18}, d_{22}, \dots, d_{25} d_{27} d_{28}, d_{31}, d_{32},\\
&e_7, e_8\}\\
\Delta_7^+(2) =&\{a_7, a_{13}, a_{18}, a_{22}, a_{25}, a_{26}; b_7, b_{13}, b_{18}, b_{22}, b_{25};\\
&c_1, \dots, c_4, , c_7, c_8, c_9, c_{12}, c_{13}, c_{16}; d_1, d_2, d_5, d_{11}, d_{21}, e_1\}\\
\Delta_7^+(3) =&\{a_{27}, a_{28}\}
\end{aligned}$$

$\Delta_7^+(k) = \emptyset$ for $k \geq 4$.

$$\begin{cases} \text{dim} = 83, \ \text{Ric} = 19g, \\ \text{4 largest eigenvalues of } M_1 \text{ are } 11.4093,\ 11.4093,\ 19.0000,\ 22.1376. \end{cases}$$

Thus the space does not satisfy $QB \geq 0$.

**4.6.9. The space** $(E_8, \alpha_8)$**.**

(4.60)

$$\begin{aligned}
\Delta_8^+(1) =&\{a_6, a_7, a_{12}, a_{13}, a_{17}, a_{18}, a_{21}, a_{22}, a_{24}, a_{25}, a_{26}, a_{27};\\
&b_6, b_7, b_{12}, b_{13}, b_{17}, b_{18}, b_{21}, b_{22}, b_{24}, b_{25}, b_{26}, b_{27}\\
&c_1, \dots, c_5, c_7, \dots, c_{10}, c_{12}, \dots, c_{14}, c_{16}, c_{17}, c_{19}\\
&d_1, \dots, d_3, d_5, d_6, d_8, d_{11}, d_{12}, d_{14}, d_{17}, d_{21}, d_{22}, d_{24}, d_{27}, d_{31}, e_1, e_8\}\\
\Delta_8^+(2) =&\{a_{28}\}
\end{aligned}$$

$\Delta_8^+(k) = \emptyset$ for $k \geq 3$.

$$\begin{cases} \dim = 57, \ \mathrm{Ric} = 29g, \\ 4 \text{ largest eigenvalues of } M_1 \text{ are } 11.5000,\ 11.5000,\ 11.5000,\ 29.0000, \\ \text{eigenvalues of } M_2 \text{ are at most } 11.5 \text{ (using Lemma 2.7).} \end{cases}$$

Thus the space has $QB > 0$.

## 5. Appendix

We illustrate how we initialize the Kähler C-space $(G_2, \alpha_2)$, then calculate bisectional curvatures and estimate the eigenvalues of $M_2$ in MAPLE. The main formulas used to calculate the curvatures will apply to all the other cases. Actual MAPLE code below will be italicized.

**5.1. Initializing $(G_2, \alpha_2)$.** We begin by initializing the root system.

1. Begin by defining the positive root system in $\mathbb{R}^3$, then define $S$ as the set of all positive and negative roots:

```
> a1:=[1,-1,0]; a2:=[-1,0,1]; a3:=[0,-1,1]; b1:=[-2,1,1]; b2:=[1,-2,1];
b3:=[-1,-1,2];
> S:={a1,...,b1,...,-a1,...,-b1,...}
```

2. By expressing the positive roots in terms of the simple positive roots, identify the corresponding sets $\Delta_2^+(1), \Delta_2^+(2), \Delta_2^+(3)$ (appearing as $m1, m2, m3$ below). Refer to the ordered elements of $\Delta_2^+$ by either $C(i)$ or $c(i)$ below, depending on how they are used. The function $g(i)$ is $1, 2, 3$ depending on whether $C(i)$ is in $m1, m2$, or $m3$ respectively.

```
> m1:={a2,a3,b1,b2};
> m2 := {b3};
> m3 := {};
> A := Matrix([a2, a3, b1, b2, b3]);
> C:=i -> convert(Row(A,i), list)
> c:=i -> Row(A,i)
> g:=i ->
  if evalb(C(i) in m1) then 1
  elif evalb(C(i) in m2) then 2
  elif evalb(C(i) in m3) then 3 else 0 end if;
```

**5.2. Bisectional curvature formula and matrix.** Here we compute the matrix $M1$ using Lemma 2.4.

1. We first need to define some basic functions appearing in Lemma 2.4. Below $N(i,j)$ calculates $N_{C(i),C(j)}$, while $T(i,j)N(i,j)$ calculates $\tilde{N}_{C(i),C(j)}$. The function $Nm(i,j)$ calculates $N_{C(i),-C(j)}$, while $Tm(i,j)$ $Nm(i,j)$ calculates $\tilde{N}_{C(i),-C(j)}$ and the MAPLE codes for these are similar.

```
> N:= (i,j) ->
  if evalb(C(i)+C(j) in S and C(i)-2*C(j) in S) then 3
```

```
   elif evalb(C(i)+C(j) in S and C(i)-C(j) in S) then 2
   elif evalb(C(i)+C(j) in S) then 1
   else 0 end if;
> T:= (i,j) - >
   if evalb(not(N(i,j)=0))
   then sqrt(c(i).c(i))*sqrt(c(j).c(j))/sqrt((c(i)+c(j)).(c(i)+c(j)))
   else 0 end if;
```

2. The matrix for bisectional curvature is given by

```
> B:= Matrix(5,5, (i,j) - >

if evalb(i ≤ j) then
1/g(j)*(c(i).c(j)+(1/2)*(g(i)/(g(i)+g(j)))*N(i,j)^2 *T(i,j)^2 )

else
1/g(i)*(c(j).c(i)+(1/2)*(g(j)/(g(j)+g(i)))*N(j,i)^2 *T(j,i)^2 ) end if);
```

**5.3. General curvature formula.** First we want to compute general curvatures $R(\alpha, \bar{\beta}, \gamma, \bar{\delta})$ where $\alpha \neq \beta$, $\gamma \neq \delta$ and where $\delta = \alpha - \beta + \gamma$, which we may assume by Lemma 2.5. In the rest of the formulas in this subsection, we identify a triple index $(i, j, k)$ with the quadruple of roots

$$\alpha = C(k), \beta = C(i), \gamma = C(j), \delta = C(k) - C(i) + C(j).$$

In particular, given any $(i, j, k)$, we always have $\delta = \alpha - \beta + \gamma$.

1. The function $allroots(i, j, k)$ returns $true$ or $false$ depending on whether $\delta$ is a root or not. Moreover, $gd(i, j, k) = l$ if $\delta$ is in $ml$ and is zero otherwise, while $D(i, j, k) = l$ if $\delta = C(l)$ and is zero otherwise.

```
> allroots:=(i,j,k) - > evalb(C(k)-C(i)+C(j) in (m1 union m2 union
m3))

> gd:=(i,j,k) - >
   if C(k)-C(i)+C(j) in m1 then 1
   elif C(k)-C(i)+C(j) in m2 then 2
   elif C(k)-C(i)+C(j) in m3 then 3
   else 0;
> D:=(i,j,k) - >
   if C(k)-C(i)+C(j) =C(1) then 1
   elif C(k)-C(i)+C(j) =C(2) then 2
   elif C(k)-C(i)+C(j) =C(3) then 3
   ⋮
   else 0 end if;
```

2. Now we define the coefficient functions used in Lemma 2.5. Below, $xi(j, k)$ for example is 1 if $j < k$, and zero otherwise.

```
> xi := (k,j) - > if j < k then 1 else 0 end if;
```

```
> delta := (k,j) -> if k = j then 1 else 0 end if;
> coeff1 := (i,j,k,l) -> (k-j)*xi(k, j)-k*l/(i+k);
> coeff2 := (i,j,k,l) -> k*xi(i, j)+l*xi(j, i)+l*delta(i, j)*delta(k, l)+(j-
k)*xi(k, j);
```

3. Finally, given $(i, j, k)$ we apply the formulas above and Lemma 2.5 to calculate curvature associated to $\alpha = C(k), \beta = C(i), \gamma = C(j), \delta = C(k) - C(i) + C(j)$. When $\alpha \neq \beta$ and $\delta$ is a root, then $Rest(i, j, k)$ is the upper estimate for $|R(\alpha, \bar{\beta}, \gamma, \bar{\delta})|$, obtained by replacing the $N's$ and their coefficients in Lemma 2.5 by their absolute values. Otherwise $Rest(i, j, k) = 0$.

```
> Rest:=(i,j,k) ->

  if (allroots(i,j,k) and not(C(k) = C(i)) then

  (1/2)*(1/(sqrt(g(k))*sqrt(g(i))*sqrt(g(j))*sqrt(gd(i, j, k)))*

  (|(coeff1(g(k), g(i), g(j), gd(i, j, k))|*
  T(k,j)*N(k,j)*T(i,D(i,j,k))*N(i,D(i,j,k))
  +
  |(coeff2(g(k), g(i), g(j), gd(i, j, k))|
  *Tm(k,i)*Nm(k,i)*Tm(j,D(i,j,k))*Nm(j,D(i,j,k)))
  else 0 end if;
```

**5.4. Matrix of non-bisectional curvatures.** Now we calculate the $25 \times 25$ matrix $Z$ as defined in (4.2).

1. First we ordered all pairs of the form $(C(i), C(j))$ in a list of length 25 using the two commands below. For example, $LIST[1]$ returns the pair $[C(1), C(1)]$, while $LIST[1][2]$ corresponds to the second component, $C(1)$, of $LIST[1]$.

```
> AA:= Matrix(5,5, (i,j) -> [C(i), C(j)])

> LIST:= convert(AA, list)
```

2. Below, $l1(i) = j$ provided $LIST[i][1] = C(j)$. Similarly, $l2(i) = j$ provided $LIST[i][2] = C(j)$, and its MAPLE code is similar.

```
> l1:= i ->
> if LIST[i][1]=C(1) then 1
> if LIST[i][1]=C(2) then 2
  ⋮
```

3. Now we calculate the matrix $Z$ as defined in (4.2). Associate $0 \leq i, j \leq 25$ to the quadruple $[A, B, C, D] = [LIST[i][1], LIST[i][2], LIST[j][1], LIST[j][2]]$. Now if $A \neq B$, $A = D$, and $B = C$, then $Z_{ij} = |R(ABBA)|$. If not in the previous case and $A \neq B$ and $A - B = D - C$, then $Z_{ij}$ is the upper estimate for $|R(A, B, C, D)|$, obtained by replacing

the $N's$ and their coefficients in Lemma 2.5 by their absolute values (the minimum appearing in the formula below is justified by the curvature identity $R(A,B,C,D) = R(C,B,A,D)$). If not in the previous cases, then $Z_{ij}$ is zero.

```
> Z:= Matrix(25,25, (i,j) −>
if (evalb(not(LIST[i][1]=LIST[i][2])) and evalb(LIST[i][2]=LIST[j][1])
   and evalb(LIST[i][1]=LIST[j][2]))
then |B[l2(i), l1(i)]|

elif (evalb(not(LIST[i][1]=LIST[i][2]))
   and evalb(LIST[i][1]-LIST[i][2]=LIST[j][2]-LIST[j][1])
then evalf(min(Rest(l2(i), l1(j), l1(i)),Rest(l2(i), l1(i), l1(j))))

else 0 end if);
```

Now for the matrix $Z$, the weighted row sums in (2.15) of Lemma 2.8 are given by $S0, S1, S2, \ldots$ in the following commands.

```
>b0:=Matrix((25,1), (i,j) −>1):
>S0:=max(Z.b0)

>v1:=1/9*Z.b0
>b1:=Matrix((25,1), (i,j) −> min(1, v1[i,1])):
>S1:=max(Z.b1)

>v2:=1/9* Z.b1
>b2:=Matrix((25,1), (i,j) −> min(1, v2[i,1])):
>S2:=max(Z.b2)
   ⋮
```

**5.5. The matrices $B$, $Z$ for $(G_2, \alpha_2)$.** Below, we give the matrix $B$ of bisectional curvatures and the matrix $Z$ as calculated in MAPLE. For the matrix $Z_{AB,CD}$, the pairs $AB$ are ordered into a list of 25 elements as: $(C(1), C(1)), (C(2), C(1)), (C(3), C(1)), \ldots$ The matrix $Z_1$ gives the first 13 columns of $Z$, while $Z_2$ gives the next 12.

$$B = \begin{bmatrix} 2 & 5/2 & 3 & 0 & 3/2 \\ 5/2 & 2 & 0 & 3 & 3/2 \\ 3 & 0 & 6 & -3/2 & 3/2 \\ 0 & 3 & -3/2 & 6 & 3/2 \\ 3/2 & 3/2 & 3/2 & 3/2 & 3 \end{bmatrix}$$

$$Z_1 = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \sqrt{2}\sqrt{6} & 0 & 0 & 5/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \sqrt{2}\sqrt{6} & 0 & 0 & 0 & 0 & 0 & 0 & 3/2 & 0 & 3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 3/2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 5/2 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2}\sqrt{6} & 0 & \sqrt{2}\sqrt{6} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 3/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 3/2 & 0 & 0 & \sqrt{2}\sqrt{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 3 & 0 & 0 & \sqrt{2}\sqrt{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3/2 & 0 \\
0 & \sqrt{2}\sqrt{6} & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 3/2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 3/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3/2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

$$
Z_2 = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \sqrt{2}\sqrt{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 3/2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3/2 & 0 & 0 & 0 \\
0 & 0 & 0 & 3/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 3/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 3/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3/2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
3/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3/2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 3/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 3/2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

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