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ISOPARAMETRIC HYPERSURFACES WITH FOUR PRINCIPAL CURVATURES, III

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Abstract

The classification work [**5**], [**8**] left unsettled only those anomalous isoparametric hypersurfaces with four principal curvatures and multiplicity pair $\{4,5\}, \{6,9\}$, or $\{7,8\}$ in the sphere.

By systematically exploring the ideal theory in commutative algebra in conjunction with the geometry of isoparametric hypersurfaces, we show that an isoparametric hypersurface with four principal curvatures and multiplicities $\{4,5\}$ in S^{19} is homogeneous, and, moreover, an isoparametric hypersurface with four principal curvatures and multiplicities $\{6,9\}$ in S^{31} is either the inhomogeneous one constructed by Ferus, Karcher, and Münzner, or the one that is homogeneous.

This classification reveals the striking resemblance between these two rather different types of isoparametric hypersurfaces in the homogeneous category, even though the one with multiplicities $\{6,9\}$ is of the type constructed by Ferus, Karcher, and Münzner and the one with multiplicities $\{4,5\}$ stands alone. The quaternion and the octonion algebras play a fundamental role in their geometric structures.

A unifying theme in [**5**], [**8**], and the present sequel to them is Serre's criterion of normal varieties. Its technical side pertinent to our situation that we developed in [**5**], [**8**] and extend in this sequel is instrumental.

The classification leaves only the case of multiplicity pair $\{7,8\}$ open.

1. Introduction

An isoparametric hypersurface M in the sphere is one whose principal curvatures and their multiplicities are fixed constants. The classification of such hypersurfaces has been an outstanding problem in submanifold geometry, listed as Problem 34 in [**27**], as can be witnessed by its long history. Through Münzner's work [**23**], we know the number g of principal curvatures is 1,2,3,4 or 6, and there are at most two multiplicities $\{m_1, m_2\}$ of the principal curvatures, occurring alternately when the

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principal curvatures are ordered, associated with M ($m_1 = m_2$ if g is odd). Over the ambient Euclidean space in which M sits there is a homogeneous polynomial F, called the Cartan-Münzner polynomial, of degree g that satisfies

$$|\nabla F|^2(x) = g^2|x|^{2g-2}, \quad (\Delta F)(x) = (m_2 - m_1)g^2|x|^{g-2}/2$$

whose restriction f to the sphere has image in $[-1, 1]$ with ± 1 the only critical values. For any $c \in (-1, 1)$, the preimage $f^{-1}(c)$ is an isoparametric hypersurface with $f^{-1}(0) = M$. This 1-parameter of isoparametric hypersurfaces degenerates to the two submanifolds $f^{-1}(\pm 1)$ of codimension $m_1 + 1$ and $m_2 + 1$ in the sphere.

The isoparametric hypersurfaces with $g = 1, 2, 3$ were classified by Cartan to be homogeneous [**3**], [**4**]. For $g = 6$, it is known that $m_1 = m_2 = 1$ or 2 by Abresch [**1**]. Dorfmeister and Neher [**13**] showed that the isoparametric hypersurface is homogeneous in the former case and Miyaoka [**22**] settled the latter.

For $g = 4$, there are infinite classes of inhomogeneous examples of isoparametric hypersurfaces, two of which were first constructed by Ozeki and Tackeuchi [**24**, I] to be generalized later by Ferus, Karcher, and Münzner [**15**], referred to collectively as isoparametric hypersurfaces of OT-FKM type subsequently. We remark that the OT-FKM type includes all the homogeneous examples, barring the two with multiplicities $\{2, 2\}$ and $\{4, 5\}$. To construct the OT-FKM type, let $P_0, \cdots, P_{m_1}$ be a Clifford system on $\mathbb{R}^{2l}$, which are orthogonal symmetric operators on $\mathbb{R}^{2l}$ satisfying

$$P_iP_j + P_jP_i = 2\delta_{ij}I, \;\; i, j = 0, \cdots, m.$$

The 4th degree homogeneous polynomial

$$F(x) = |x|^4 - 2\sum_{i=0}^{m}(\langle P_i(x), x\rangle)^2$$

is the Cartan-Münzner polynomial. The two multiplicities of the OT-FKM type are m and $k\delta(m) - 1$ for any $k = 1, 2, 3, \cdots$, where $\delta(m)$ is the dimension of an irreducible module of the Clifford algebra C_{m-1} ($l = k\delta(m)$). Stolz [**26**] showed that these multiplicity pairs and $\{2, 2\}$ and $\{4, 5\}$ are exactly the possible multiplicities of isoparametric hypersurfaces with four principal curvatures in the sphere.

The recent study of n-Sasakian manifolds [**10**]; Hamiltonian stability of the Gauss images of isoparametric hypersurfaces in complex hyperquadrics as Lagrangian submanifolds [**19**], [**20**]; isoparametric functions on exotic spheres [**17**]; and the realization of the Cartan-Münzner polynomial of an isoparametric hypersurface with four principal curvatures as the moment map of a Spin-action on the ambient Euclidean space, regarded as a cotangent bundle with the standard symplectic structure [**16**], [**21**], represent several new directions in the study of such hypersurfaces.

Through [**5**] (see also [**6**], [**7**]) and [**8**] it has been clear by now that isoparametric hypersurfaces with four principal curvatures and multiplicities $\{m_1, m_2\}, m_1 \leq m_2$, fall into two categories: namely, the general category where $m_2 \geq 2m_1 - 1$, and the anomalous category where the multiplicities are $\{2,2\}, \{3,4\}, \{4,5\}, \{6,9\}$, or $\{7,8\}$. The former category enjoys a rich connection with the theory of *reduced* ideals in commutative algebra, and is exactly of OT-FKM type [**5**], [**8**]. The latter is peculiar, in that all known examples of such hypersurfaces with multiplicities $\{3,4\}, \{6,9\}$, or $\{7,8\}$ are of the OT-FKM type and have the property that incongruent isoparametric hypersurfaces with the same multiplicity pair occur in the same ambient sphere, which is not the case in the former category; in contrast, those with multiplicities $\{2,2\}$ or $\{4,5\}$ can never be of OT-FKM type. The theory of reduced ideals breaks down in the anomalous category. Yet, in [**8**], we were still able to utilize more commutative algebra, in connection with the notion of Condition A introduced by Ozeki and Takeuchi [**24**, I], to prove that those hypersurfaces with multiplicities $\{3,4\}$ are of OT-FKM type. This left unsettled only the anomalous isoparametric hypersurfaces with multiplicities $\{4,5\}, \{6,9\}$, or $\{7,8\}$.

Of all known examples of isoparametric hypersurfaces with four principal curvatures in the sphere, the homogeneous one ($= SU(5)/Spin(4)$) with multiplicities $\{4,5\}$ in S^{19} is perhaps one of the most intriguing. First off, it stands alone (together with the (classified) one with multiplicities $\{2,2\}$) as it does not belong to the OT-FKM type. More remarkably, through the work in [**10**], one knows that there is a contact CR structure of dimension 8 on its focal manifold of dimension 14 in S^{19}, giving rise to the notion of 13-dimensional 5-Sasakian manifolds fibered over $\mathbb{C}P^4$ that generalizes the 3-Sasakian ones. The 5-Sasakian manifold constructed from the focal manifold carries a metric of positive sectional curvature [**2**].

Intuitively, it seems remote that the homogeneous example $Spin(10) \cdot T^1/SU(4) \cdot T^1$ of multiplicities $\{6,9\}$ in S^{31}, which is of OT-FKM type, would share any common feature with the above one of multiplicities $\{4,5\}$. We will, however, show through the classification in this paper the striking resemblance between them.

In this paper, we will systematically employ the ideal theory, in conjunction with the geometry of isoparametric hypersurfaces, to prove that an isoparametric hypersurface with four principal curvatures and multiplicities $\{4,5\}$ is the homogeneous one, and moreover, an isoparametric hypersurface with four principal curvatures and multiplicities $\{6,9\}$ is either the homogeneous one mentioned above, or the inhomogeneous one constructed by Ferus, Karcher, and Münzner [**15**]. Serre's criterion of normal varieties, whose technical side pertinent to our situation we developed in [**5**], [**8**], is instrumental. It turns out the quaternion and octonion algebras also play a fundamental role in the structures of these hypersurfaces.

The classification leaves open the only case when the multiplicity pair is $\{7, 8\}$.

I would like to thank Josef Dorfmeister for many conversations, during our visit to Tôhoku University in the summer of 2010, to share the isoparametric triple system approach he and Erhard Neher introduced [**11**].

2. Preliminaries

2.1. The basics. Let M be an isoparametric hypersurface with four principal curvatures in the sphere, and let F be its Cartan-Münzner polynomial. To fix notation, we make the convention, by changing F to $-F$ if necessary, that its two focal manifolds are $M_+ := F^{-1}(1)$ and $M_- := F^{-1}(-1)$ with respective codimensions $m_1 + 1 \leq m_2 + 1$ in the ambient sphere $S^{2(m_1+m_1)+1}$. The principal curvatures of the shape operator S_n of M_+ (vs. M_-) with respect to any unit normal n are $0, 1$ and -1, whose multiplicities are, respectively, m_1, m_2 and m_2 (vs. m_2, m_1 and m_1).

On the unit normal sphere bundle UN_+ of M_+, let $(x, n_0) \in UN_+$ be points in a small open set; here $x \in M_+$ and n_0 is normal to the tangents of M_+ at x. We define a smooth orthonormal frame $n_a, e_p, e_\alpha, e_\mu$, where $1 \leq a, p \leq m_1$ and $1 \leq \alpha, \mu \leq m_2$, in such a way that n_a are tangent to the unit normal sphere at n_0, and e_p, e_α and e_μ, respectively, are basis vectors of the eigenspaces E_0, E_1 and E_{-1} of the shape operator S_{n_0}.

Convention 1. We will sometimes also use b, q, β, and ν in place of a, p, α, and μ, respectively. Henceforth, a, p, α, μ are specifically reserved for indexing the indicated normal and tangential subspaces.

Each of the frame vectors can be regarded as a smooth function from UN_+ to $\mathbb{R}^{2(m_1+m_2)}$. We have [**5**, p 14], in Einstein summation convention,

$$
\begin{aligned}
dx &= \omega^p e_p + \omega^\alpha e_\alpha + \omega^\mu e_\mu, \qquad & dn_0 &= \omega^a n_a - \omega^\alpha e_\alpha + \omega^\mu e_\mu \\
dn_a &= -\omega^a n_0 + \theta_a^t e_t, \qquad & de_p &= -\omega^p x + \theta_p^t e_t \\
de_\alpha &= -\omega^\alpha x + \omega^\alpha n_0 + \theta_\alpha^t e_t, \qquad & de_\mu &= -\omega^\mu x - \omega^\mu n_0 + \theta_\mu^t e_t
\end{aligned} \tag{1}
$$

where the index t runs through the p, α, and μ ranges, and

$$
\begin{aligned}
\theta_a^p &= -S_{p\alpha}^a \omega^\alpha - S_{p\mu}^a \omega^\mu, \qquad & \theta_a^\alpha &= -S_{p\alpha}^a \omega^p - S_{\alpha\mu}^a \omega^\mu \\
\theta_p^\alpha &= -S_{p\alpha}^a \omega^a - S_{\alpha\mu}^p \omega^\mu, \qquad & \theta_a^\mu &= -S_{p\mu}^a \omega^p - S_{\alpha\mu}^a \omega^\alpha \\
\theta_p^\mu &= S_{p\mu}^a \omega^a + S_{\alpha\mu}^p \omega^\alpha, \qquad & \theta_\alpha^\mu &= (S_{\alpha\mu}^a/2)\omega^a + (S_{\alpha\mu}^p/2)\omega^p
\end{aligned} \tag{2}
$$

where $S^a_{ij} := \langle S(e_i, e_j), n_a\rangle$ are the components of the second fundamental form S of M_+ at x, and $S^p_{\alpha\mu}$ are the $\alpha\mu$-components of S at the "mirror" point $n_0 \in M_+$ where the normal $x, e_p, 1 \le p \le m_1$, and the tangent $n_a, 1 \le a \le 4, e_\alpha, e_\mu, 1 \le \alpha, \mu \le 5$, form an adapted frame. Knowing S at x does not necessarily mean knowing S at n_0. This is fundamentally the reason the classification of isoparametric hypersurfaces can be rather entangling. In any event, there are two identities connecting $S^a_{\alpha\mu}$ and $S^p_{\alpha\mu}$ as follows [**5**, p 16]:

$$\begin{aligned}\sum_a S^a_{p\alpha}S^a_{q\beta} + \sum_a S^a_{q\alpha}S^a_{p\beta}\\ + 1/2\sum_\mu (S^p_{\alpha\mu}S^q_{\beta\mu} + S^q_{\alpha\mu}S^p_{\beta\mu}) \quad &= \delta_{pq}\delta_{\alpha\beta}.\end{aligned} \tag{3}$$

The other is entirely symmetric, obtained by interchanging the α and μ ranges.

The third fundamental form of M_+ is the symmetric tensor

$$q(X, Y, Z) := (\nabla^\perp_X S)(Y, Z)/3$$

where $\nabla^\perp$ is the normal connection. Write $p_a(X, Y) := \langle S(X, Y), n_a\rangle$ and $q^a(X, Y, Z) = \langle q(X, Y, Z), n_a\rangle, 0 \le a \le m_1$. The Cartan-Münzner polynomial F is related to p_a and q^a by the expansion formula of Ozeki and Takeuchi [**24**, I, p 523]

$$\begin{aligned} F(tx + y + w) &= t^4 + (2|y|^2 - 6|w|^2)t^2 + 8(\sum_{i=0}^m p_i w_i)t\\ &+ |y|^4 - 6|y|^2|w|^2 + |w|^4 - 2\sum_{i=0}^m p_i^2 - 8\sum_{i=0}^m q^i w_i\\ &+ 2\sum_{i,j=0}^m \langle \nabla p_i, \nabla p_j\rangle w_i w_j\end{aligned} \tag{4}$$

where $w := \sum_{i=0}^{m_1} w_i n_i$, y is tangential to M_+ at x, $p_i := p_i(y, y)$ and $q^i := q^i(y, y, y)$. Note that our definition of q^i differs from that of Ozeki and Takeuchi [**24**, I] by a sign.

Lemma 1. *$q^0(y, y, y) = -\sum_{p\alpha\mu} S^p_{\alpha\mu} X_\alpha Y_\mu Z_p$, where $y = \sum_\alpha X_\alpha e_\alpha + \sum_\mu Y_\mu e_\mu + \sum_p Z_p e_p$.*

Proof. One uses (4) and observes that at $n_0 \in M_+$, by (1), the normal space is $\mathbb{R}x \oplus E_0$, the 0-eigenspace of the shape operator S_x is spanned by $n_1, \cdots, n_{m_1}$, and the ± 1-eigenspaces of S_x are identical with E_1 and E_{-1}, respectively. q.e.d.

We remark that the symmetric matrices S_a of the components $p_a, 0 \leq a \leq m_1$, relative to E_1, E_{-1}, and E_0, are

$$S_0 = \begin{pmatrix} Id & 0 & 0 \\ 0 & -Id & 0 \\ 0 & 0 & 0 \end{pmatrix}, S_a = \begin{pmatrix} 0 & A_a & B_a \\ A_a^{tr} & 0 & C_a \\ B_a^{tr} & C_a^{tr} & 0 \end{pmatrix}, 1 \leq a \leq m_1, \tag{5}$$

where $A_a : E_{-1} \to E_1$, $B_a : E_0 \to E_1$, and $C_a : E_0 \to E_{-1}$.

2.2. The duality between M_+ and M_-. Let UN_+ and UN_- be respectively the unit normal bundles of M_+ and M_-. The map

$$(x, n_0) \to (x^* := (x + n_0)/\sqrt{2}, n_0^* := (x - n_0)/\sqrt{2})$$

is a diffeomorphism from UN_+ to UN_-. Finding dx^* by (1), we see that the normal space at x^* is $\mathbb{R}n_0^* \oplus E_+$. Finding $-dn_0^*$ by (1), we obtain that E_1^*, the $+1$-eigenspace of the shape operator $S_{n_0^*}$, is spanned by $n_1, \cdots, n_{m_1}$; E_{-1}^*, the -1-eigenspace, is E_0; and E_0^*, the 0-eigenspace, is E_{-1}. We leave it to the reader as a simple exercise to verify the following duality property by exploring (1) and (2) on both M_+ and M_- at x and x^*.

Lemma 2. *Referring to* (5)*, let the counterpart matrices at x^* and their blocks be denoted by the same notation with an additional *. Then*

$$\begin{aligned} A_\alpha^* &= -\sqrt{2}\left(S_{p\alpha}^a\right), 1 \leq \alpha \leq m_2, \\ B_\alpha^* &= -1/\sqrt{2}\left(S_{\alpha\mu}^a\right), 1 \leq \alpha \leq m_2, \\ C_\alpha^* &= -1/\sqrt{2}\left(S_{\alpha\mu}^p\right), 1 \leq \alpha \leq m_2, \end{aligned} \tag{6}$$

where the upper scripts denote rows.

2.3. The homogeneous example of multiplicities $\{4, 5\}$. Consider the complex Lie algebra $so(5, \mathbb{C})$. The unitary group $U(5)$ acts on it by

$$g \cdot Z = \overline{g} Z g^{-1}$$

for $g \in U(5)$ and $Z \in so(5, \mathbb{C})$. The principal orbits of the action form the homogeneous 1-parameter family of isoparametric hypersurfaces with multiplicities $(m_1, m_2) = (4, 5)$. Let the (i, j)-entry of Z be denoted by a_{ij}, and let $a_{ij} = x_{ij} + \sqrt{-1} y_{ij}$ in which x_{ij} and y_{ij} are real. The Euclidean space is $so(5, \mathbb{C})$ coordinatized by x_{ij} and y_{ij}, and the Cartan-Münzner polynomial is [**24**, II, p 27]

$$F(Z) = -5/4 \sum_i |Z_i|^4 + 3/2 \sum_{i<j} |Z_i|^2 |Z_j|^2 - 4 \sum_{i<j} |\langle Z_i, Z_j \rangle|^2,$$

where $Z_1, \ldots, Z_5$ are the row vectors of Z. It is readily seen that the point x with coordinates $x_{12} = x_{34} = 1/\sqrt{2}$ and zero otherwise satisfies

$F(x) = 1$, so that $x \in M_+ = SU(5)/Sp(2)$. Let us introduce new coordinates

$$\begin{aligned} x_{12} &:= (t+w_0)/\sqrt{2}, & x_{34} &:= (t-w_0)/\sqrt{2},\\ x_{13} &:= (w_3-z_4)/\sqrt{2}, & x_{24} &:= (w_3+z_4)/\sqrt{2},\\ y_{13} &:= (-z_3-w_4)/\sqrt{2}, & y_{24} &:= (-z_3+w_4)/\sqrt{2},\\ x_{14} &:= (z_2-w_1)/\sqrt{2}, & x_{23} &:= (z_2+w_1)/\sqrt{2},\\ y_{14} &:= (w_2+z_1)/\sqrt{2}, & y_{23} &:= (w_2-z_1)/\sqrt{2}. \end{aligned}$$

Then $w_0, \ldots, w_4$ are the normal coordinates, $z_1, \ldots, z_4$ the E_0-coordinates, and

$$\begin{aligned} &x_1 := x_{35}, x_2 := y_{35}, x_3 := x_{45}, x_4 := y_{45}, x_5 := y_{34},\\ &y_1 := x_{15}, y_2 := y_{35}, y_3 := x_{25}, y_4 := y_{25}, y_5 := y_{12} \end{aligned}$$

are the five E_1 and five E_{-1} coordinates, in order. In fact, the components of the second fundamental form of M_+ at x are, by (4),

$$\begin{aligned} p_0 &= (x_1)^2 + \cdots + (x_5)^2 - (y_1)^2 - \cdots - (y_5)^2,\\ p_1 &= 2(x_1y_1 + x_2y_2 + \cdots + x_4y_4) + \sqrt{2}(x_5+y_5)z_1,\\ p_2 &= 2(x_2y_1 - x_1y_2) + 2(x_3y_4 - x_4y_3) + \sqrt{2}(x_5+y_5)z_2,\\ p_3 &= 2(x_3y_1 - x_1y_3) + 2(x_4y_2 - x_2y_4) + \sqrt{2}(x_5+y_5)z_3,\\ p_4 &= 2(x_2y_3 - x_3y_2) + 2(x_4y_1 - x_1y_4) + \sqrt{2}(x_5+y_5)z_4. \end{aligned} \tag{7}$$

Note that the 5-by-5 matrices A_i of $p_i, 1 \le i \le 4$, given in (5) are

$$\begin{aligned} A_1 &:= \begin{pmatrix} I & 0 & 0\\ 0 & I & 0\\ 0 & 0 & 0 \end{pmatrix}, & A_2 &:= \begin{pmatrix} J & 0 & 0\\ 0 & -J & 0\\ 0 & 0 & 0 \end{pmatrix},\\ A_3 &:= \begin{pmatrix} 0 & -I & 0\\ I & 0 & 0\\ 0 & 0 & 0 \end{pmatrix}, & A_4 &:= \begin{pmatrix} 0 & J & 0\\ J & 0 & 0\\ 0 & 0 & 0 \end{pmatrix}, \end{aligned} \tag{8}$$

where I is the 2-by-2 identity matrix and J is the 2-by-2 matrix

$$J := \begin{pmatrix} 0 & -1\\ 1 & 0 \end{pmatrix}. \tag{9}$$

It is readily checked that the upper 4-by-4 blocks of $A_1, \ldots, A_4$, still denoted by $A_1, \ldots, A_4$ for notational convenience, satisfy

$$A_jA_k + A_kA_j = -2\delta_{jk}I$$

with

$$A_2A_3 = -A_4. \tag{10}$$

Note that $A_1, \ldots, A_4$ are exactly the matrix representations of the multiplications by $1, i, j, k$, respectively, on the right over $\mathbb{H}$. The 5-by-4 matrices $B_i = C_i$ of $p_i, 1 \leq i \leq 4$, given in (5) are

$$
\begin{aligned}
&B_1 := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1/\sqrt{2} & 0 & 0 & 0 \end{pmatrix}, \qquad B_2 := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1\sqrt{2} & 0 & 0 \end{pmatrix}, \\
&B_3 := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1\sqrt{2} & 0 \end{pmatrix}, \qquad B_4 := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/\sqrt{2} \end{pmatrix},
\end{aligned} \tag{11}
$$

where the first zero row in each matrix is of size 4-by-4.

Note that it follows from (11) that all nontrivial linear combinations of $B_1, \ldots, B_4$ are of rank 1, which will play a decisive role later.

A calculation with the expansion formula (4) gives the components of the third fundamental form $\tilde{q}$ of the homogeneous example. We will only display $\tilde{q}^0$ for later purposes.

$$
\begin{aligned}
\tilde{q}^0 = &-2z_4(x_1y_3 + x_3y_1 + x_2y_4 + x_4y_2) \\
&- 2z_3(-x_1y_4 - x_4y_1 + x_2y_3 + x_3y_2) \\
&- 2z_2(x_1y_1 + x_2y_2 - x_3y_3 - x_4y_4) \\
&- 2z_1(x_1y_2 - x_2y_1 + x_3y_4 - x_4y_3)
\end{aligned} \tag{12}
$$

2.4. The homogeneous example of multiplicities $\{6, 9\}$. This is the example of OT-FKM type with multiplicity pair $(m_1, m_2) = (6, 9)$ whose Clifford action is on M_- of codimension $9 + 1 = 10$ in S^{31}, given as follows.

Let $\check{J}_1, \ldots, \check{J}_8$ be the unique (up to equivalence) irreducible representation of the (anti-symmetric) Clifford algebra C_8 on $\mathbb{R}^{16}$. Set

$$
\begin{aligned}
P_0 &: (c, d) \mapsto (c, -d), \\
P_1 &: (c, d) \mapsto (d, c), \\
P_{1+i} &: (c, d) \mapsto (\check{J}_i(d), -\check{J}_i(c)), 1 \leq i \leq 8,
\end{aligned}
$$

over $\mathbb{R}^{32} = \mathbb{R}^{16} \oplus \mathbb{R}^{16}$. $P_0, P_1, \ldots, P_9$ form a representation of the (symmetric) Clifford algebra C'_{10} on $\mathbb{R}^{32}$.

We know that M_- with the Clifford action on it can be realized as the Clifford-Stiefel manifold [**15**]. Namely,

$$
\begin{aligned}
M_- = \{(\zeta, \eta) \in S^{31} \subset \mathbb{R}^{16} \times \mathbb{R}^{16} :& \\
|\zeta| = |\eta| = 1/\sqrt{2}, \zeta \perp \eta, \check{J}_i(\zeta) \perp \eta, i = 1, \ldots, 8\}&.
\end{aligned}
$$

At $(\zeta, \eta) \in M_-$, the normal space is

$$N = \operatorname{span}\langle f_0 := P_0((\zeta, \eta)), \ldots, f_9 := P_9((\zeta, \eta))\rangle.$$

E_0, the 0-eigenspace of the shape operator $S_0 := S_{f_0}$, is

$$E_0 = \operatorname{span}\langle g_1 := P_1P_0((\zeta, \eta)), \ldots, g_9 := P_9P_0((\zeta, \eta))\rangle.$$

$E_\pm$, the ± 1-eigenspaces of S_0, are

$$E_\pm := \{X : P_0(X) = \mp X, X \perp N\}.$$

Since E_+ (vs. E_-) consists of $(0,d) \in \mathbb{R}^{32}$ (vs. $(e,0) \in \mathbb{R}^{32}$), we obtain

$$\begin{aligned} E_+ &:= \{(0,d) : d \perp \zeta, d \perp \eta, d \perp \check{J}_i(\zeta), \forall i\}, \\ E_- &:= \{(e,0) : e \perp \zeta, e \perp \eta, e \perp \check{J}_i(\eta), \forall i\}. \end{aligned} \tag{13}$$

The second fundamental form $S_a := S_{f_a}$ at (ζ, η) is

$$S_a(X,Y) = -\langle P_a(X), Y\rangle,$$

The representation $\check{J}_1, \ldots, \check{J}_8$ can be constructed out of the octonion algebra as follows. Let $e_1, e_2, \ldots, e_8$ be the standard basis of the octonion algebra $\mathbb{O}$ with e_1 the multiplicative unit. Let $J_1, J_2, \ldots, J_7$ be the matrix representations of the octonion multiplications by $e_1, e_2, \ldots, e_8$ on the right over $\mathbb{O}$. Then

$$\check{J}_i = \begin{pmatrix} J_i & 0 \\ 0 & -J_i \end{pmatrix}, 1 \le i \le 7, \qquad \check{J}_8 = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}. \tag{14}$$

We may set

$$\eta = (0, e_1/\sqrt{2}), \qquad \zeta = (e_2/\sqrt{2}, 0)$$

(in fact any purely imaginary e in place of e_2 is fine). Then it is easily checked that $(\zeta, \eta) \in M_-$. Moreover,

$$\begin{aligned} E_+ &= \{(0,d) \in \mathbb{R}^{16} \times \mathbb{R}^{16} : d = (0, \alpha) \in \mathbb{R}^8 \times \mathbb{R}^8, \alpha \perp e_1, e_2\}, \\ E_- &= \{(e,0) \in \mathbb{R}^{16} \times \mathbb{R}^{16} : e = (\beta, 0) \in \mathbb{R}^8 \times \mathbb{R}^8, \beta \perp e_1, e_2\}. \end{aligned}$$

For $h_\alpha = (0, e_\alpha) \in E_+$ and $k_\mu = (e_\mu, 0) \in E_-$, $3 \le \alpha, \mu \le 8$, we calculate to see

$$\begin{aligned} \langle P_1(h_\alpha), k_\mu\rangle = 0, \qquad \langle P_9(h_\alpha), k_\mu\rangle = -\langle e_\alpha, e_\mu\rangle, \\ \langle P_{1+i}(h_\alpha), k_\mu\rangle = 0, \quad 1 \le i \le 7. \end{aligned} \tag{15}$$

The point is that what we are after is the second fundamental form of M_+ of codimension $6+1=7$ in S^{31}. Observe that

$$((e_2, 0), 0) = ((\zeta, \eta) + P_0((\zeta, \eta)))/\sqrt{2} \in M_+,$$

where by (6) the six 9-by-9 matrices $A_3, \ldots, A_8$ (to be compatible with the octonion setup, we do not denote them by $A_1, A_2, \ldots, A_6$), similar to the ones in (8), are given by, for $3 \le \alpha \le 8, 1 \le a, p \le 9$,

$$A_\alpha = \left(\sqrt{2}\langle P_a(h_\alpha), g_p\rangle\right), \tag{16}$$

where A_α is skew-symmetric with the (i,j)-entry $= \langle e_\alpha, e_j e_i\rangle$ for $1 \le i < j \le 8$, and the ninth row and column $= 0$. That is, the upper 8-by-8 block of A_α is the matrix representation of the multiplication of $-e_\alpha$ on the right over $\mathbb{O}$. Explicitly,

$$A_3=\begin{pmatrix}0&I&0&0&0\\-I&0&0&0&0\\0&0&0&-I&0\\0&0&I&0&0\\0&0&0&0&0\end{pmatrix},\quad A_4=\begin{pmatrix}0&-J&0&0&0\\-J&0&0&0&0\\0&0&0&J&0\\0&0&J&0&0\\0&0&0&0&0\end{pmatrix},$$

$$A_5=\begin{pmatrix}0&0&I&0&0\\0&0&0&I&0\\-I&0&0&0&0\\0&-I&0&0&0\\0&0&0&0&0\end{pmatrix},\quad A_6=\begin{pmatrix}0&0&-J&0&0\\0&0&0&-J&0\\-J&0&0&0&0\\0&-J&0&0&0\\0&0&0&0&0\end{pmatrix},$$

$$A_7=\begin{pmatrix}0&0&0&K&0\\0&0&-K&0&0\\0&K&0&0&0\\-K&0&0&0&0\\0&0&0&0&0\end{pmatrix},\quad A_8=\begin{pmatrix}0&0&0&L&0\\0&0&-L&0&0\\0&L&0&0&0\\-L&0&0&0&0\\0&0&0&0&0\end{pmatrix},$$

where J is given in (9) and

$$K:=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\qquad L:=\begin{pmatrix}0&1\\1&0\end{pmatrix}. \tag{17}$$

The upper 8-by-8 blocks of $A_3,\dots,A_8$, still denoted by the same symbols for notational convenience, satisfy

$$A_\alpha A_\beta+A_\beta A_\alpha=-2\delta_{\alpha\beta}I;$$

this is the unique (up to equivalence) Clifford representation of C_6 on $\mathbb{R}^8$. We will employ later the five matrices

$$\alpha_j=-A_3A_j, 4\leq j\leq 8, \tag{18}$$

which generate the unique (up to equivalence) representation of C_5 on $\mathbb{R}^8$. Note that $I,\alpha_4,\dots,\alpha_8$ are compatible with (8). Meanwhile, $B_3,\dots,B_8$, similar to the ones in (11), are given, in view of (6), by

$$B_\alpha=\left(\langle P_a(h_\alpha),k_\mu\rangle/\sqrt{2}\right), 1\leq a\leq 9, 3\leq\alpha,\mu\leq 8, \tag{19}$$

whose $(9,\alpha)$-entry is $1/\sqrt{2}$ and is zero elsewhere, in complete agreement with (11).

We remark that the third fundamental form of M_- is

$$\langle q(X,Y,Z),W\rangle=\mathcal{C}\sum_{b=0}^{9}(\langle S_b(X,Y)P_b(Z),W\rangle)/3,$$

where $\mathcal{C}$ denotes the cyclic sum over X,Y,Z. In particular,

$$q^0(h_\alpha,k_\mu,g_p)=\mathcal{C}\sum_{b=0}^{9}\langle S_b(h_\alpha,k_\mu)P_b(g_p),f_0\rangle=S_{b=p}(h_\alpha,k_\mu),$$

which implies, by Lemma 1, $B_\alpha = C_\alpha$ for $3 \leq \alpha \leq 8$, as in the $(4,5)$ case.

3. Normal varieties and codimension 2 estimates

This section gives a brief account of the background commutative algebra and algebraic geometry needed for the subsequent development. Though we can proceed in an algebraic way as done in [**25**], we choose to present it in an analytic way as done in [**18**] for more geometric intuition.

Let V be an affine variety in $\mathbb{C}^n$ defined by the zeros of $m+1$ polynomials $p_0, p_1, \ldots, p_m$, and let $\mathcal{S}$ be its singular set. A function f is weakly holomorphic in an open set O of V if it is holomorphic on $O \setminus \mathcal{S}$ and is locally bounded in O. Passing to the limit as O shrinks to a point p, we can talk about the germs of weakly holomorphic functions at p. The variety is said to be normal at p if the germs of weakly holomorphic functions at p coincide with the germs of holomorphic functions at p. That is, the Riemann extension theorem holds true in the germs of neighborhoods around p. V is said to be normal if it is normal at all its points.

If V is normal, then its irreducible components are disconnected [**18**]; or else a constant function with different values on different local irreducible branches, which is not even continuous, would give rise to a weakly holomorphic function that could be extended to a holomorphic function, a piece of absurdity. Each irreducible component is normal whose singularity set is of codimension ≥ 2. The key point to this is that if we realize an irreducible normal variety X of dimension l locally as a finite branched covering $\pi : X \to \mathbb{C}^l$, then the local irreducibility of X gives that the branch locus B of X and $\pi(B)$ are both of dimension $l-1$, and so the singular set $\mathcal{S}\pi(B)$ of $\pi(B)$ is of codimension at least 2 in $\mathbb{C}^l$. Then observe that the singular set of X is contained in the preimage of $\mathcal{S}\pi(B)$.

In particular, if V is normal and connected, then V is irreducible with the singular set of codimension ≥ 2.

Corollary 1. *If $p_0, p_1, \ldots, p_m$ are homogeneous polynomials whose zeros define a normal variety V, then V is irreducible and the singular set of V is of codimension ≥ 2.*

The corollary holds because V defined by the zeros of homogeneous polynomials is a cone, which is clearly connected.

Conversely, if V is defined by the zeros of homogeneous polynomials $p_0, \ldots, p_m$, what are the conditions that guarantee that V is normal? A necessary condition is that the singular set of V is of codimension ≥ 2. The other crucial condition is that $p_0, \ldots, p_m$ form a *regular sequence* in the polynomial ring of $\mathbb{C}^n$.

Definition 1. A regular sequence in a commutative ring R with identity is a sequence $a_1, \dots, a_k$ in R such that the ideal $(a_1, \dots, a_k)$ is not R, and moreover, a_1 is not a zero divisor in R and a_{i+1} is not a zero divisor in the quotient ring $R/(a_1, \dots, a_i)$ for $1 \leq i \leq k-1$.

We have the criterion of normality of Serre [**14**, p 457].

Theorem 1. *(Special case) Let $V \subset \mathbb{C}^n$ be defined by the zeros of homogeneous polynomials $p_0, \dots, p_m$ that form a regular sequence in the polynomial ring of $\mathbb{C}^n$. Let J be the subvariety of V where the Jacobian matrix of $p_0, \dots, p_m$ is of rank $< m+1$. Then V is an irreducible normal variety if the codimension of J is at least 2 in V, in which case the ideal $(p_0, p_1, \dots, p_m)$ is prime.*

The criterion provides a scheme for checking whether a sequence of homogeneous polynomials $p_0, \dots, p_m$ of the same degree ≥ 1 in the polynomial ring of $\mathbb{C}^n$ is a regular sequence [**5**, p 57].

Proposition 1. *Let $p_0, \dots, p_m$ be a sequence of linearly independent homogeneous polynomials of the same degree ≥ 1 in the polynomial ring of $\mathbb{C}^n$. For each $0 \leq k \leq m-1$, let V_k be the variety defined by the zeros of $p_0, \dots, p_k$ and let J_k be the subvariety of V_k where the Jacobian of $p_0, \dots, p_k$ is of rank $< k+1$. Then $p_0, p_1, \dots, p_m$ form a regular sequence if J_k is of codimension at least 2 in V_k for $0 \leq k \leq m-1$.*

In fact, repeated applications of Theorem 1 establish that the ideals $(p_0, \dots, p_k)$ are all prime for $0 \leq k \leq m-1$. The linear independence of $p_0, \dots, p_m$ of equal degree then demands that p_{k+1} cannot be a zero divisor in the quotient ring $P[n]/(p_0, \dots, p_k)$ by Nullstellensatz, where $P[n]$ stands for the polynomial ring of $\mathbb{C}^n$. The homogeneity of $p_0, \dots, p_{m_1}$ of degree ≥ 1 shows that $(p_0, \dots, p_{m_1})$ is a proper ideal.

The components $p_0, \dots, p_{m_1}$ of the second fundamental form of M_+ of an isoparametric hypersurface with four principal curvatures are linearly independent homogeneous polynomials of second degree, which fits perfectly in Proposition 1. By exploring more commutative algebra (the algebraic independence of a regular sequence) and investigating the codimension 2 condition in Proposition 1, it is established in [**8**]:

Theorem 2. *Let M be an isoparametric hypersurface with four principal curvatures and multiplicities $(m_1, m_2), m_1 < m_2$. Assume the components $p_0, p_1, \dots, p_{m_1}$ of the second fundamental form of the focal submanifold M_+ form a regular sequence in the ring of polynomials of $\mathbb{C}^{m_1+2m_2}$. Then M is of OT-FKM type.*

Corollary 2. *$p_0, \dots, p_{m_1}$ of M_+ do not form a regular sequence in general when $(m_1, m_2) = (4,5), (3,4), (7,8)$, or $(6,9)$.*

Proof. For $(m_1, m_2) = (3,4), (7,8), (6,9)$, consider an OT-FKM type hypersurface whose Clifford action is on M_-. If $p_0, \dots, p_{m_1}$ formed a regular sequence, then the isoparametric hypersurface would be of OT-FKM type with the Clifford action on M_+; this is impossible because such an OT-FKM type hypersurface whose Clifford action is on M_- is incongruent to one whose Clifford action is on M_+. On the other hand, a hypersurface with $(m_1, m_2) = (4, 5)$ can never be of OT-FKM type. q.e.d.

It is shown in [**8**] that $p_0, \ldots, p_{m_1}$ do form a regular sequence when $m_2 \geq 2m_1 - 1$ so that the isoparametric hypersurface is of OT-FKM type. This leaves open only $(m_1, m_2) = (4, 5), (3, 4), (6, 9)$, and $(7, 8)$. On the other hand, though $p_0, \ldots, p_3$ no longer form a regular sequence in general for $(m_1, m_2) = (3, 4)$, an argument in [**8**] that explores Proposition 1 and the notion of Condition A [**24**, I] shows that the isoparametric hypersurface with $(m_1, m_2) = (3, 4)$ is of OT-FKM type. We will carry this scheme one step further in the next section when $(m_1, m_2) = (4, 5)$ or $(6, 9)$.

4. The second fundamental form

We show in this section that the second fundamental form of M_+ of an isoparametric hypersurface with multiplicities $(m_1, m_2) = (4, 5)$ in S^{19} is, up to an orthonormal frame change, identical with that of the homogeneous example given in Section 2.3. Furthermore, in the case $(m_1, m_2) = (6, 9)$ in S^{31}, either the isoparametric hypersurface is the inhomogeneous example constructed by Ferus, Karcher, and Münzner, or, after an orthonormal frame change, the second fundamental form of M_+ is identical with that of the homogeneous example.

Let us first recall the codimension 2 estimates in [**8**] that are crucial for the classification of isoparametric hypersurfaces with four principal curvatures when the multiplicity pair (m_1, m_2) is either where $m_2 \geq 2m_1 - 1$, or is $(3, 4)$.

Let $p_0, p_1, \ldots, p_{m_1}$ be the components of the second fundamental form of M_+. We agree that $\mathbb{C}^{2m_2+m_1}$ consists of points (u, v, w) with coordinates u_α, v_μ and w_p, where $1 \leq \alpha, \mu \leq m_2$ and $1 \leq p \leq m_1$. For $0 \leq k \leq m_1$, let

$$W_k := \{(u, v, w) \in \mathbb{C}^{2m_2+m_1} : p_0(u, v, w) = \cdots = p_k(u, v, w) = 0\}.$$

We want to estimate the dimension of the subvariety U_k of $\mathbb{C}^{2m_2+m_1}$, where

$$U_k := \{(u, v, w) \in \mathbb{C}^{2m_2+m_1} : \text{rank of the Jacobian of } p_0, \ldots, p_k < k+1\}.$$

$p_0, \ldots, p_k$ give rise to a linear system of cones $\mathcal{C}_\lambda$ defined by

$$c_0 p_0 + \cdots + c_k p_k = 0$$

with

$$\lambda := [c_0 : \cdots : c_k] \in \mathbb{C}P^k. \tag{20}$$

The singular subvariety of $\mathcal{C}_\lambda$ is

$$\mathscr{S}_\lambda := \{(u,v,w) \in \mathbb{C}^{2m_2+m_1} : (c_0 S_{n_0} + \cdots + c_k S_{n_k}) \cdot (u,v,w)^{tr} = 0\},$$

where $\langle S_{n_i}(X), Y\rangle = \langle S(X,Y), n_i\rangle$ is the shape operator of the focal manifold M_+ in the normal direction n_i; we have

$$U_k = \cup_\lambda \mathscr{S}_\lambda. \tag{21}$$

We wish to establish

$$\dim(W_k \cap U_k) \le \dim(W_k) - 2 \tag{22}$$

for $k \le m_1 - 1$ to verify that $p_0, p_1, \ldots, p_{m_1}$ form a regular sequence.

We first estimate the dimension of $\mathscr{S}_\lambda$. We break it into two cases. If $c_0, \ldots, c_k$ are constant multiples of either all real or all purely imaginary numbers, then

$$\dim(\mathscr{S}_\lambda) = m_1,$$

since $c_0 S_{n_0} + \cdots + c_k S_{n_k} = c S_n$ for some unit normal vector n and some nonzero constant c, and we know that the null space of S_n is of dimension m_1. Otherwise, after a normal basis change we can assume that $\mathscr{S}_\lambda$ consists of elements (u,v,w) of the form $(S_{n_1^*} - \tau_\lambda S_{n_0^*}) \cdot (u,v,w)^{tr} = 0$ for some nonzero complex number τ_λ, relative to a new orthonormal normal basis $n_0^*, n_1^*, \ldots, n_k^*$ in the linear span of $n_0, n_1, \ldots, n_k$. That is, in matrix form,

$$\begin{pmatrix} 0 & A & B \\ A^{tr} & 0 & C \\ B^{tr} & C^{tr} & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \tau_\lambda \begin{pmatrix} I & 0 & 0 \\ 0 & -I & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \tag{23}$$

where x, y, and z are (complex) eigenvectors of (real) $S_{n_0^*}$ with eigenvalues $1, -1$, and 0, respectively.

Remark 1. We agree to choose n_0^* and n_1^* as follows. Decompose $n := c_0 n_0 + \cdots + c_k n_k$ into its real and imaginary parts $n = \alpha + \sqrt{-1}\beta$. Define n_0^* and n_1^* by performing the Gram-Schmidt process on α and β.

Lemma 49 [**5**, p 64] ensures that we can assume

$$B = C = \begin{pmatrix} 0 & 0 \\ 0 & \sigma \end{pmatrix}, \tag{24}$$

where σ is a nonsingular diagonal matrix of size r_λ-by-r_λ with r_λ the rank of B, and A is of the form

$$A = \begin{pmatrix} I & 0 \\ 0 & \Delta \end{pmatrix}, \tag{25}$$

where $\Delta = \mathrm{diag}(\Delta_1, \Delta_2, \Delta_3, \ldots)$ is of size r_λ-by-r_λ, in which $\Delta_1 = 0$ and $\Delta_i, i \ge 2$, are nonzero skew-symmetric matrices expressed in the block form $\Delta_i = \mathrm{diag}(\Theta_i, \Theta_i, \Theta_i, \ldots)$ with Θ_i a 2-by-2 matrix of the form

$$\begin{pmatrix} 0 & f_i \\ -f_i & 0 \end{pmatrix}$$

for some $0 < f_i < 1$. We decompose x, y, z into $x = (x_1, x_2), y = (y_1, y_2), z = (z_1, z_2)$ with $x_2, y_2, z_2 \in \mathbb{C}^{r_\lambda}$. Equation (23) is

$$\begin{gathered} x_1 = -\tau_\lambda y_1, \qquad y_1 = \tau_\lambda x_1, \\ -\Delta x_2 + \sigma z_2 = -\tau_\lambda y_2, \qquad \Delta y_2 + \sigma z_2 = \tau_\lambda x_2, \\ \Delta(x_2 + y_2) = 0. \end{gathered} \tag{26}$$

This can be solved explicitly to obtain that $x_2 = -y_2$ and z_2 can be solved (linearly) in terms of x_2. Conversely, $x_2 = -y_2$ can be solved in terms of z_2 when $\tau_\lambda \neq \pm f_i\sqrt{-1}$ for all i, so that z can be chosen to be a free variable in this case. So, either $x_1 = y_1 = 0$, in which case

$$\dim(\mathscr{S}_\lambda) = m_1,$$

or both x_1 and y_1 are nonzero, in which case $y_1 = \pm\sqrt{-1}x_1$ and so

$$\dim(\mathscr{S}_\lambda) = m_1 + m_2 - r_\lambda. \tag{27}$$

Since eventually we must estimate the dimension of $W_k \cap U_k$, let us cut $\mathscr{S}_\lambda$ by

$$0 = p_0^* = \sum_\alpha (x_\alpha)^2 - \sum_\mu (y_\mu)^2.$$

Case 1. x_1 and y_1 are both nonzero. This is the case of nongeneric $\lambda \in \mathbb{C}P^k$. We substitute $y_1 = \pm\sqrt{-1}x_1$ and x_2 and y_2 in terms of z_2 into $p_0^* = 0$ to deduce

$$0 = p_0^* = (x_1)^2 + \cdots + (x_{m_2 - r_\lambda})^2 + z \text{ terms};$$

hence $p_0^* = 0$ cuts $\mathscr{S}_\lambda$ to reduce the dimension by 1, i.e., by (27),

$$\dim(W_k \cap \mathscr{S}_\lambda) \leq m_1 + m_2 - r_\lambda - 1, \tag{28}$$

noting that W_k is also cut out by $p_0^*, p_1^*, \ldots, p_k^*$. Meanwhile, only a subvariety of λ of dimension $k-1$ in $\mathbb{C}P^k$ assumes $\tau_\lambda = \pm\sqrt{-1}$. (In fact, the subvariety is the hyperquadric $\mathcal{Q}$. See Remark 2 below.) Therefore, if we stratify $\mathcal{Q}$ into subvarieties $\mathcal{L}_j$ over which $r_\lambda = j$, then by (28) an irreducible component $\mathcal{W}_j$ of $W_k \cap (\cup_{\lambda \in \mathcal{L}_j} \mathscr{S}_\lambda)$ will satisfy

$$\dim(\mathcal{W}_j) \leq \dim(W_k \cap \mathscr{S}_\lambda) + k - 1 \leq m_1 + m_2 + k - 2 - j. \tag{29}$$

Case 2. $x_1 = y_1 = 0$. This is the case of generic λ, where $\dim(\mathscr{S}_\lambda) = m_1$, so that an irreducible component $\mathcal{V}$ of $W_k \cap (\cup_{\lambda \in \mathcal{G}} \mathscr{S}_\lambda)$, where $\mathcal{G}$ is the Zariski open set of $\mathbb{C}P^k$ of generic λ, will satisfy

$$\dim(\mathcal{V}) \leq m_1 + k. \tag{30}$$

On the other hand, since W_k is cut out by $k+1$ equations, we have

$$\dim(W_k) \geq m_1 + 2m_2 - k - 1. \tag{31}$$

Lemma 3. *When* $(m_1, m_2) = (4, 5)$ (*respectively,* $(m_1, m_2) = (6, 9)$) *and* $j \geq 2$*, there holds in equation* (29) *the estimate*

$$\dim(\mathcal{W}_j) \leq \dim(W_k) - 2 \tag{32}$$

for $k \leq m_1 - 1 = 3$ (*respectively,* $k \leq 5$).

Proof. For (32) to be true, we must have both

$$\begin{aligned} m_1 + m_2 + k - 2 - j &\leq m_1 + 2m_2 - k - 3, \\ m_1 + k &\leq m_1 + 2m_2 - k - 3 \end{aligned}$$

by (29), (30), and (31). The second inequality is $2m_2 \geq 2k + 3$, which is always true, while the first is $m_2 \geq 2k + 1 - j$, which is true if $j \geq 2$. q.e.d.

Remark 2. In view of the proof of Lemma 3, the codimension 2 estimate for the case of generic $\lambda \in \mathcal{G}$ always holds true. Henceforth, we may ignore this case and consider only the nongeneric case where $\tau_\lambda = \pm\sqrt{-1}$.

Observe that if we write $(c_0, \ldots, c_k) = \alpha + \sqrt{-1}\beta$ where α and β are real vectors, then $\tau_\lambda = \pm\sqrt{-1}$ is equivalent to the conditions that $\langle \alpha, \beta \rangle = 0$ and $|\alpha|^2 = |\beta|^2$. That is, the nongeneric λ in (20) is the hyperquadric $\mathcal{Q}$ in $\mathbb{C}P^k$.

Lemma 4. *Suppose* $(m_1, m_2) = (4, 5)$ *or* $(6, 9)$*, and in the latter case suppose the isoparametric hypersurface is not the inhomogeneous one constructed by Ferus, Karcher, and Münzner. Then* $r_\lambda \leq 1$ *for all* λ *in* $\mathcal{Q}$.

Proof. Suppose the contrary. Generic λ in $\mathcal{Q}$ would have $r_\lambda \geq 2$.

We will only consider the $(4, 5)$ case; the other case is verbatim. The multiplicity pair $(4, 5)$ cannot allow any points of Condition A on M_+. Hence, one of the four pairs of matrices $(B_1, C_1), (B_2, C_2), (B_3, C_3)$, and (B_4, C_4) of the shape operators $S_{n_1}, S_{n_2}, S_{n_3}$, and S_{n_4}, similar to the one given in (23), must be nonzero; we may assume one of $(B_1, C_1), (B_2, C_2)$, and (B_3, C_3) is nonzero in the neighborhood of a given point, over which generic $\lambda \in \mathcal{Q}$ have $r_\lambda \geq 2$.

Firstly, Lemma 3 would reduce the proof to considering $r_\lambda \leq 1$.

Case 1. On $\mathcal{L}_1$ where $r_\lambda = 1$: The codimension 2 estimate would still go through. This is because (29) is now replaced by $(j = 1)$

$$\dim(\mathcal{W}_j) \leq m_1 + m_2 + k - 3 - j = m_1 + m_2 + k - 4 \tag{33}$$

due to the fact that such nongeneric λ in $\mathcal{Q}$ constitute a subvariety of $\mathcal{Q}$ of dimension at most $k - 2$.

Case 2. On $\mathcal{L}_0$ where $r_\lambda = 0$: Now

$$\dim(\mathcal{W}_j) \leq m_1 + m_2 + k - 3$$

with $j=0$. We need to cut back one more dimension to make (33) valid. Since $r_\lambda = 0$, we see $B_1^* = C_1^* = 0$ and $A^* = I$ in (23) for $S_{n_1^*}$. It follows that $p_0^* = 0$ and $p_1^* = 0$ cut $\mathscr{S}_\lambda$ in the variety

$$\{(x, \pm\sqrt{-1}x, z) : \sum_\alpha (x_\alpha)^2 = 0\}. \tag{34}$$

(B_2^*, C_2^*) or (B_3^*, C_3^*) of $S_{n_2^*}$ or $S_{n_3^*}$ must be nonzero now; we may assume it is the former. Since z is a free variable in (34), $p_2^* = 0$ will have nontrivial z-terms,

$$\begin{aligned} 0 = p_2^* &= \sum_{\alpha p} S_{\alpha p} x_\alpha z_p + \sum_{\mu p} T_{\mu p} y_\mu z_p + \sum_{\alpha\mu} U_{\alpha\mu} x_\alpha y_\mu \\ &= \sum_{\alpha p} (S_{\alpha p} \pm \sqrt{-1} T_{\alpha p}) x_\alpha z_p, \end{aligned} \tag{35}$$

taking $y = \pm\sqrt{-1}x$ into account and remarking that the $x_\alpha x_\mu$ terms are gone because $U_{\alpha\mu} = -U_{\mu\alpha}$ (see (41) below), where $U_{\alpha\mu} := \langle S(X_\alpha^*, X_\mu^*), n_2^*\rangle$, $S_{\alpha p} := \langle S(X_\alpha^*, Z_p^*), n_2^*\rangle$, and $T_{\mu p} := \langle S(Y_\mu^*, Z_p^*), n_2^*\rangle$ are (real) entries of A_2^*, B_2^*, and C_2^*, respectively, and $X_\alpha^*, 1 \le \alpha \le m_2$, $Y_\mu^*, 1 \le \mu \le m_2$, and $Z_p^*, 1 \le p \le m_1$, are orthonormal eigenvectors for the eigenspaces of $S_{n_0^*}$ with eigenvalues $1, -1$, and 0, respectively. Hence the dimension of $\mathscr{S}_\lambda$ will be cut down by 2 by $p_0^*, p_1^*, p_2^* = 0$, so that again

$$\dim(W_2 \cap \mathscr{S}_\lambda) \le m_1 + m_2 - 2, \tag{36}$$

noting that $p_0^*, p_1^*, p_2^* = 0$ also cut out W_2. In conclusion, we deduce

$$\dim(\mathcal{W}_j) \le \dim(W_k \cap \mathscr{S}_\lambda) + k - 2 \le m_1 + m_2 + k - 4, \tag{37}$$

so that the codimension 2 estimate would also go through. In conclusion, we obtain that (22) holds true.

However, the validity of (22) would imply that the isoparametric hypersurface is of OT-FKM type by Proposition 1 and Theorem 2, which is absurd in the $(4,5)$ case.

In the $(6,9)$ case, the same arguments as above imply that the isoparametric hypersurface is the inhomogeneous one constructed by Ferus, Karcher, and Münzner since the Clifford action is on M_+, contradicting the assumption. The lemma is proven. q.e.d.

Lemma 5. *Suppose* $(m_1, m_2) = (4,5)$ *or* $(6,9)$*, and in the latter case suppose the isoparametric hypersurface is not the inhomogeneous one constructed by Ferus, Karcher, and Münzner. Then* $r_\lambda = 1$ *for generic* λ *in* $\mathcal{Q}$.

Proof. We consider the $(4,5)$ case; the other case is verbatim. Suppose the contrary. Then $r_\lambda = 0$ for all λ in $\mathcal{Q}$. It would follow that B_1 of S_{n_1} is identically zero by considering $\lambda = [1 : \sqrt{-1} : 0 : 0 : 0]$, because then B_0^* and B_1^* associated with $S_{n_0^*}$ and $S_{n_1^*}$ are zero. Likewise, $B_a = 0$

for all $1 \leq a \leq 4$. However, this would imply that the isoparametric hypersurface is of Condition A. This is impossible. q.e.d.

Lemma 6. *Suppose $(m_1, m_2) = (4,5)$ or $(6,9)$, and in the latter case suppose the isoparametric hypersurface is not the inhomogeneous one constructed by Ferus, Karcher, and Münzner. Then $r_\lambda = 1$ for all λ in $\mathcal{Q}$.*

Proof. For a λ with $r_\lambda = 0$, we have that A in (23) is the identity matrix by (25), so that its rank is full ($= 5$ or 9). It follows that generic λ in $\mathcal{Q}$ will have the same full rank property. However, for a λ with $r_\lambda = 1$, the structure of A in (25) implies that $\Delta = 0$ so that such A, which are also generic, will be of rank 4 or 8. This is a contradiction. q.e.d.

Lemma 7. *Suppose $(m_1, m_2) = (4,5)$ or $(6,9)$, and in the latter case suppose the isoparametric hypersurface is not the inhomogeneous one constructed by Ferus, Karcher, and Münzner. Then up to an orthonormal frame change, the only nonzero row of the 5-by-4 (vs. 9-by-6) matrices $B_a, 1 \leq a \leq 4$ (vs. $3 \leq a \leq 8$), of S_{n_a} is the last row.*

Proof. We will prove the $(4,5)$ case. The other case is verbatim with obvious changes on index ranges. For λ in $\mathcal{Q}$, we construct n_0^* and n_1^* as given in Remark 1 and extend them to a smooth local orthonormal frame $n_0^*, n_1^*, \ldots, n_{m_1}^*$ such that $S_{n_0^*}$ and $S_{n_1}^*$ assume the matrix form in (23), (24), and (25). Note that $\Delta = 0\,(= \Delta_1)$ in (25) because $r_\lambda = 1$; it follows that $\sigma = 1/\sqrt{2}$ in (24) [**5**, p 67]. Suppose there is a λ_0 at which $S_{n_2^*}$ in matrix form is such that the matrix B_2^* associated with $S_{n_2^*}$ has a nonzero row other than the last one; this property will continue to be true in a neighborhood of λ_0. Modifying (34), $p_0^* = 0$ and $p_1^* = 0$ now cut $\mathscr{S}_\lambda$ in the variety

(38)
$$\left\{(x_1, \ldots, x_4, \frac{t}{\sqrt{2}\tau_\lambda}, \tau_\lambda x_1, \ldots, \tau_\lambda x_4, -\frac{t}{\sqrt{2}\tau_\lambda}, z_1, \ldots, z_3, t) : \sum_{j=1}^{4}(x_j)^2 = 0\right\}$$

where

$$\begin{aligned}
&x = (x_1, x_2, x_3, x_4, x_5 = t/\sqrt{2}\tau_\lambda),\\
&y = (y_1, \ldots, y_5) = (\tau_\lambda x_1, \tau_\lambda x_2, \tau_\lambda x_3, \tau_\lambda x_4, -t/\sqrt{2}\tau_\lambda)\\
&z = (z_1, z_2, z_3, z_4 = t).
\end{aligned}$$

Meanwhile, (35) becomes

$$\begin{aligned}
0 = &\sum_{\alpha=1,p=1}^{7,7}(S_{\alpha p} \pm \sqrt{-1}T_{\alpha p})x_\alpha z_p + \sum_{p=1}^{7}(S_{8p} - T_{8p})tz_p/(\sqrt{2}\tau_\lambda)\\
&- \sum_{\alpha \leq 7} U_{\alpha 8} t x_\alpha/(\sqrt{2}\tau_\lambda) + \sum_{\mu \leq 7} U_{8\mu} t x_\mu/\sqrt{2} + U_{88}t^2/2.
\end{aligned} \tag{39}$$

The assumption that B_2^* (or C_2^*) assumes an extra nonzero row other than the last one implies that one more dimension cut can be achieved since $x_1, \dots, x_4, z_1, \dots, z_4$ are independent variables and (39) is now nontrivial (see the remark below). It follows that once more

$$\dim(W_k \cap U_k) \leq m_1 + m_2 + k - 4$$

for $k \leq 3$, so that (22) goes through in the neighborhood of λ_0, which is absurd as the hypersurface would be of OT-FKM type (respectively, would be the inhomogeneous one constructed by Ferus, Karcher, and Münzner in the $(6,9)$ case). We therefore conclude that no such λ_0 exist and so the only nonzero entry of B_2^* (or C_2^*) is the last one. Since any unit normal vector perpendicular to n_0^* and n_1^* can be n_2^*, the conclusion follows. q.e.d.

Proposition 2. *Suppose* $(m_1, m_2) = (4,5)$ *or* $(6,9)$*, and in the latter case suppose the isoparametric hypersurface is not the inhomogeneous one constructed by Ferus, Karcher, and Münzner. Then we can choose an orthonormal frame such that the second fundamental form of* M_+ *is exactly that of the homogeneous example.*

Proof. We will prove the $(4,5)$ case and remark on the $(6,9)$ case at the end. $S_{n_0^*}$ is the square matrix on the right hand side of (23) while $S_{n_1^*}$ is the square one on the left hand side, where the 1-by-1 matrix $\sigma = 1/\sqrt{2}$ in (24) and the 1-by-1 matrix $\Delta = 0$ in (25). We proceed to understand $S_{n_j^*}$ with the associated matrices A_j, B_j, and C_j for $2 \leq j \leq 4$ similar to what is given in (23). We know by Lemma 7 the 5-by-4 matrices B_j and C_j are of the form

$$B_j = \begin{pmatrix} 0 & 0 \\ b_j & c \end{pmatrix}, \qquad C_j = \begin{pmatrix} 0 & 0 \\ e_j & f \end{pmatrix}$$

for some real numbers c and f. Write the 5-by-5 matrix A_j as

$$A_j = \begin{pmatrix} \alpha_j & \beta \\ \gamma & \delta \end{pmatrix}$$

with δ a real number. Then the identities [**24**, II, p 45]

$$\begin{aligned} A_j A^{tr} + A A_j^{tr} + 2B_j B^{tr} + 2B B_j^{tr} &= 0, \\ A_j A^{tr} + A A_j^{tr} + 2C_j C^{tr} + 2C C_j^{tr} &= 0 \end{aligned} \tag{40}$$

result in

$$\alpha_j = -\alpha_j^{tr}, \qquad \gamma = 0, \qquad c = f = 0. \tag{41}$$

On the other hand, the matrix

$$A_j C B^{tr} + B_j C^{tr} A^{tr} + A C_j B^{tr} \tag{42}$$

being skew-symmetric [**24**, II, p 45] implies

$$\beta = 0, \qquad \delta = 0.$$

Meanwhile, the identity [**24**, II, p 45]

$$A_jA_j^{tr} + 2B_jB_j^{tr} = I \tag{43}$$

derives

$$\alpha_j\alpha_j^{tr} = I, \qquad b_jb_j^{tr} = 1/2.$$

Next, the identity [**24**, II, p 45]

$$A_jA_k^{tr} + A_kA_j^{tr} + 2B_jB_k^{tr} + 2B_kB_j^{tr} = 0 \tag{44}$$

with $j \neq k$ arrives at

$$\alpha_j\alpha_k = -\alpha_k\alpha_j, \qquad b_ib_k^{tr} = 0.$$

Lastly, the identity [**24**, II, p 45]

$$B_j^{tr}B + B^{tr}B_j = C_j^{tr}C + C^{tr}C_j \tag{45}$$

yields

$$b_j = e_j.$$

The upshot is that

$$A_1 = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \quad A_j = \begin{pmatrix} \alpha_j & 0 \\ 0 & 0 \end{pmatrix}, j = 2, 3, 4, \quad B_j = C_j = \begin{pmatrix} 0 & 0 \\ b_j & 0 \end{pmatrix}$$

of the same block sizes with

$$\alpha_j\alpha_k + \alpha_k\alpha_j = -2\delta_{jk}I, \qquad < b_j, b_k >= \delta_{jk}/2.$$

As a consequence, first of all we can perform an orthonormal basis change on n_2^*, n_3^*, n_4^* so that the resulting new b_j is $1/\sqrt{2}$ at the jth slot and is zero elsewhere. Meanwhile, we can perform an orthonormal basis change of the E_1 and E_{-1} spaces so that I and $\alpha_j, 2 \leq j \leq 4$, are exactly the matrix representations of the right multiplication of $1, i, j, k$ on $\mathbb{H}$ without affecting the row vectors $b_j, 2 \leq j \leq 4$. This is precisely the second fundamental form of the homogeneous example.

In the $(6, 9)$ case, $I, \alpha_4, \ldots, \alpha_8$ can be chosen to be the ones in (18) by a frame change; multiplying them through by A_3 on the left, which amounts to changing the E_1-frame, will arrive at (16). q.e.d.

Remark 3. More generally, one can show that if all $(B_j, C_j), j \geq 2$, are of the form

$$B_j = \begin{pmatrix} 0 & d_j \\ b_j & c_j \end{pmatrix}, \quad C_j = \begin{pmatrix} 0 & g_j \\ e_j & f_j \end{pmatrix},$$

for some real numbers c_j and f_j, then $d_j = g_j = c_j = f_j = 0$. We will be explicit in the $(4, 5)$ case and remark on the similar $(6, 9)$ case.

Indeed, with

$$A_j = \begin{pmatrix} \alpha_j & \beta_j \\ \gamma_j & \delta_j \end{pmatrix}, \quad A_1 = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \quad B_1 = C_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1/\sqrt{2} \end{pmatrix},$$

one derives easily (we suppress the index for notational ease), by setting $i = 1, j \geq 2$, in (40) and (42) through (45), that $c = f = \delta = 0$, and

$$\beta = -\sqrt{2}g,\ \gamma = -\sqrt{2}d^{tr},\ \alpha\gamma^{tr} = 0,\ |d| = |g|, \\ \alpha\alpha^{tr} + \beta\beta^{tr} + 2dd^{tr} = I,\ b = e,\ |\gamma|^2 + 2|b|^2 = 1. \tag{46}$$

Suppose $d \neq 0$. By a basis change we may assume $d = (t, 0, 0, \dots, 0)^{tr}$ for some positive number t. The skew-symmetry of α and the second and third identities of (46) ensure that the first row and column of α are zero. If the first entry of β is not zero, then the fifth identity derives that all the other entries of β are zero. Hence the fifth identity again implies, ignoring the trivial first row and column of α, that the remainder of α is a 3-by-3 (vs. 7-by-7) matrix in the $(4, 5)$ (vs. $(6, 9)$) case that acts on $\mathbb{R}^3$ (vs. $\mathbb{R}^7$) as a Clifford C_1-module, so that an odd number (3 or 7) is divisible by 2, a contradiction. It follows that the first entry of β is zero, i.e., $\langle d, g\rangle = 0$. So now the fourth and fifth identities assert that $t = 1/\sqrt{2} = |d| = |g|$, so that the second and the last identity arrive at $b = 0 = e$ in view of the sixth identity. As a consequence, if $d_2 \neq 0$ and $d_3 = 0$, then $\beta_3 = \gamma_3 = 0$ and α_3 is invertible by the preceding proposition. However, (44) applied to $j = 3, k = 2$ results in $\alpha_3\gamma_2 = 0$, so that $\gamma_2 = 0 = d_2$, a contradiction. That is, we may assume now that $b_j = 0$ for all j. Then the last identity of (46) gives that $|d_j| = 1/\sqrt{2}$ for all $j \geq 2$.

Applying (40) and (42) through (45), for distinct $i, j \geq 2$, we derive

$$\langle d_i, d_j\rangle = \langle g_i, g_j\rangle = 0, \quad \langle d_i, g_j\rangle = -\langle d_j, g_i\rangle. \tag{47}$$

In the $(4, 5)$ case, performing a basis change, we may assume d_2, d_3, d_4 constitute the last three standard basis elements of $\mathbb{R}^4$. Then (47) implies that the lower 3-by-3 block M of the 4-by-3 matrix $G := \begin{pmatrix}\beta_2 & \beta_3 & \beta_4\end{pmatrix}$ is skew-symmetric, and so M can be brought into the eigen-decomposition block form. That is, upon making a basis change, if necessary, in the span of $\beta_2, \beta_3, \beta_4$, we may assume G is of the form

$$G = \begin{pmatrix} a_2 & a_3 & a_4 \\ 0 & 0 & 0 \\ 0 & 0 & u \\ 0 & -u & 0 \end{pmatrix} = \begin{pmatrix}\beta_2 & \beta_3 & \beta_4\end{pmatrix} \tag{48}$$

for some positive number u. However, the orthogonality of $\beta_2, \beta_3, \beta_4$ deduces that we may choose $a_2 = -1$ and $a_3 = a_4 = 0$ and $u = 1$. Thus we obtain

$$\gamma_3 := (0, 0, -1, 0), \gamma_4 = (0, 0, 0, -1), \beta_3 = (0, 0, 0, -1)^{tr}, \beta_4 = (0, 0, 1, 0)^{tr}.$$

Ignoring the third and fourth rows and columns that the nonzero entries of $\beta_3, \beta_4, \gamma_3, \gamma_4$ occupy, let A_3' and A_4' be the remainder, of size 2-by-2, of the 4-by-4 α-block of A_3 and A_4. Then the identity

$$\alpha_i\alpha_j^{tr} + \alpha_j\alpha_i^{tr} + \beta_i\beta_j^{tr} + \beta_j\beta_i^{tr} + \gamma_i^{tr}\gamma_j + \gamma_j^{tr}\gamma_i = 0$$

for $i \neq j$ establishes that A_2' and A_3' act on $\mathbb{R}^2$ as a Clifford C_2-module, so that 2 is divisible by 4, a contradiction. Therefore, $d_j = g_j = 0$ for all $j \geq 2$.

In the $(6,9)$ case, we have mutually orthogonal 3-by-1 matrices $a_2, \ldots, a_6$ in (48), where the 8-by-5 matrix G is extended to accommodate one more 2-by-2 eigen-block along the main diagonal. It follows by (47) that either (a_3, a_4) or (a_5, a_6) is zero; assume it is the former. As in the $(4,5)$ case, ignoring the 4th and 5th rows and columns that the nonzero entries of d_3, d_4, g_3, g_4 occupy, we see that the remainder of the 8-by-8 α-block of A_3 and A_4, denoted by A_3' and A_4' of size 6-by-6, act on $\mathbb{R}^6$ as a Clifford C_2-module, so that 6 is divisible by 4, a contradiction.

Lastly, note that, a priori, (39) is made void by (46), in which case, however, our analysis shows that we are led to the preceding proposition. On the other hand, nontriviality of (39) lands us onto the example of Ozeki and Takeuchi.

Corollary 3. *Suppose* $(m_1, m_2) = (4,5)$ *or* $(6,9)$*, and in the latter case suppose the isoparametric hypersurface is not the inhomogeneous one constructed by Ferus, Karcher, and Münzner. Then* $S^p_{\alpha\mu} = 0$ *if* $\alpha = 5$ *or* $\mu = 5$ (*respectively* $\alpha = 9$ *or* $\mu = 9$).

Proof. Setting $\alpha = \beta = 5$ or 9 and $p = q$ in (3), the result follows by (11) and (19). q.e.d.

5. The third fundamental form

In this section we express the third fundamental form of an isoparametric hypersurface with multiplicities $(m_1, m_2) = (4,5)$ or $(6,9)$ in terms of $S^p_{\alpha\mu}$, provided in the latter case the hypersurface is not the inhomogeneous one constructed by Ferus, Karcher, and Münzner. Again for simplicity in exposition, we will only consider the $(4,5)$ case with an obvious modification for the $(6,9)$ case.

Let us recall that if we let $S(X,Y)$ be the second fundamental form, then the third fundamental form is $q(X,Y)Z = (\nabla^{\perp}_X)(Y,Z)/3$ with $\nabla^{\perp}$ the normal connection. Relative to an adapted frame with the normal basis $n_a, 0 \leq a \leq m_1$, and the tangential basis $e_p, 1 \leq p \leq m_1$, $e_\alpha, 1 \leq \alpha \leq m_2$, and $e_\mu, 1 \leq \mu \leq m_2$, spanning E_0, E_+, and E_-, respectively, of M_+, let $S(X,Y) = \sum_a S^a(X,Y)n_a$ and $q(X,Y,Z) = \sum_a q^a(X,Y,Z)n_a$. Then, with the Einstein summation convention,

$$3q^a_{ijk}\omega^k = dS^a_{ij} - \theta^a_t S^t_{ij} + \theta^t_i S^a_{tj} + \theta^t_j S^a_{it},$$

where ω^k are the dual forms and θ^s_t are the normal and space connection forms. By Proposition 2, choose an adapted orthonormal frame such that (8) and (11) hold.

Lemma 8. *$q^0_{ijk} = 0$ when two of the three lower indexes are in the same p, α, or μ range.*

Proof. This was proved in [**24**, I, p 537]. q.e.d.

Lemma 9. *$q^a_{pqk} = 0$ for $1 \le a \le 4$ and all k.*

Proof. $S^a_{pq} = 0$ for $0 \le a \le 4$, $S^p_{\alpha\mu} = 0$ when either $\alpha = 5$ or $\mu = 5$ by Corollary 3, and $S^a_{\alpha p} = S^a_{\mu p} = 0$ when $\alpha, \mu \le 4$ by Proposition 2 and (8). So, in Einstein summation convention,

$$\begin{aligned}3q^a_{pqk} &= 3q^a_{pqk}\omega^k(e_k)\\ &= (\theta_p^{\alpha=5} S^a_{\alpha=5\ q} + \theta_p^{\mu=5} S^a_{\mu=5\ q} + \theta_q^{\alpha=5} S^a_{p\ \alpha=5} + \theta_q^{\mu=5} S^a_{p\ \mu=5})(e_k) = 0\end{aligned}$$

by (2) when k is in either the α or μ range. $q^a_{pqk} = 0$ when k is in the p-range [**24**, I, p 537]. q.e.d.

Lemma 10. *For $1 \le \alpha, \beta \le 4$, there holds $q^a_{\alpha\beta\mu} = 0$, while*

$$3q^a_{\alpha\beta p} = 1/2 \sum_{\mu=1}^{4} (S^p_{\alpha\mu} S^a_{\beta\mu} + S^p_{\beta\mu} S^a_{\alpha\mu}).$$

For $\alpha = 5$, there holds $q^a_{\alpha\beta p} = 0$, while

$$3q^a_{\alpha\beta\mu} = S^{p=a}_{\beta\mu}/\sqrt{2}.$$

Proof. For $1 \le \alpha, \beta \le 4$, similar calculations as above yield

$$3q^a_{\alpha\beta p} = \theta^\mu_\alpha(e_p) S^a_{\beta\mu} + \theta^\mu_\beta(e_p) S^a_{\alpha\mu}$$

which is the desired result by (2). Likewise,

$$3q^a_{\alpha\beta\nu} = \theta^\mu_\alpha(e_\mu) S^a_{\beta\nu} + \theta^\mu_\beta(e_\nu) S^a_{\alpha\mu} = 0.$$

For $\alpha = 5$,

$$3q^a_{\alpha\beta p} = (\theta^q_\alpha S^a_{q\beta} + \theta^q_\beta S^a_{q\alpha} + \theta^\mu_\alpha S^a_{\mu\beta} + \theta^\mu_\beta S^a_{\mu\alpha})(e_p) = 0$$

by (2) and Corollary 3. Likewise,

$$3q^a_{\alpha\beta\mu} = \theta^q_\beta(e_\mu) S^a_{\alpha q} = S^{p=a}_{\beta\mu}/\sqrt{2}$$

by (2), Corollary 3, and (11). q.e.d.

A parallel argument gives the following.

Lemma 11. *For $1 \le \mu, \nu \le 4$, there holds $q^a_{\mu\nu\alpha} = 0$, while*

$$3q^a_{\mu\nu p} = -1/2 \sum_{\alpha=1}^{4} (S^p_{\alpha\mu} S^a_{\alpha\nu} + S^p_{\alpha\nu} S^a_{\alpha\mu}).$$

For $\mu = 5$, there holds $q^a_{\mu\nu p} = 0$, while

$$3q^a_{\mu\nu\alpha} = -S^{p=a}_{\alpha\nu}/\sqrt{2}.$$

Lemma 12. $3q^0_{p\alpha\mu} = -S^p_{\alpha\mu}$.

Proof. This is Lemma 1. q.e.d.

Lemma 13. *For $1 \leq a \leq 4$, suppose either $\alpha \leq 4$ (respectively, $\mu \leq 4$). Then we have $q^a_{p\alpha\mu} = 0$ if $p \neq a$, and*

$$3q^a_{p\alpha\mu} = \theta^5_\alpha(e_\mu)/\sqrt{2} \text{ (respectively } 3q^a_{p\alpha\mu} = \theta^5_\mu(e_\alpha)/\sqrt{2})$$

if $p = a$. Here the superscript 5 is in the α-range (respectively, μ-range).

Proof. Suppose $1 \leq \alpha \leq 4$. Then

$$\begin{aligned} 3q^a_{\alpha p\mu} = 3q^a_{\alpha pk}\omega^k(e_\mu) &= (-\theta^a_t S^t_{\alpha p} + \theta^t_\alpha S^a_{tp} + \theta^t_p S^a_{\alpha t})(e_\mu) \\ &= \theta^{\beta=5}_\alpha S^a_{\beta=5\ p}(e_\mu) + \theta^{\nu=5}_\alpha S^a_{\nu=5\ p}(e_\mu) + \theta^\nu_p S^a_{\alpha\nu}(e_\mu) \\ &= \theta^\nu_p(e_\mu) S^a_{\alpha\nu} = 0 \end{aligned}$$

if $p \neq 5$, because $S^a_{\beta=5\ p} = 0$ by (11) and $\theta^\nu_p(e_\mu) = 0$ by (2).

If $p = 5$, then

$$3q^a_{\alpha p\mu} = (\theta^{\beta=5}_\alpha S^a_{\beta=5\ p} + \theta^\nu_p S^a_{\alpha\nu})(e_\mu).$$

It follows that

$$3q^a_{p\alpha\mu} = \theta^{\beta=5}_\alpha(e_\mu)/\sqrt{2} + \theta^\nu_p(e_\mu) S^a_{\alpha\nu} = \theta^{\beta=5}_\alpha(e_\mu)/\sqrt{2}$$

because $\theta^\nu_p(e_\mu) = 0$. q.e.d.

Lemma 14. *For $\alpha = \mu = 5$, we have $q^a_{p\alpha\mu} = 0$.*

Proof. We have, by (8), that the fifth row and column of A_a is identically zero, so that

$$\begin{aligned} 3q^a_{\mu\alpha p} &= (3q^a_{\mu\alpha k}\omega^k)(e_p) = (-\theta^a_t S^t_{\mu\alpha} + \theta^t_\mu S^a_{t\alpha} + \theta^t_\alpha S^a_{\mu t})(e_p) \\ &= (\theta^q_\mu S^a_{q\alpha} + \theta^q_\alpha S^a_{\mu q})(e_p) = 0 \end{aligned}$$

by (2). q.e.d.

It follows from Lemmas 8 through 14 that the third fundamental form q of M_+ of the isoparametric hypersurface under consideration is, for $1 \leq a \leq 4$,

$$q^0 := -2 \sum_{p,\alpha,\mu=1}^{4} S^p_{\alpha\mu} x_\alpha y_\mu z_p$$

$$\begin{aligned} q^a := {} & F z_a + \sqrt{2}(x_5 - y_5) \sum_{\alpha,\mu=1}^{4} S^{p=a}_{\alpha\mu} x_\alpha y_\mu + \sum_{p,\alpha,\beta=1}^{4} U^a_{\alpha\beta p} x_\alpha x_\beta z_p \\ & + \sum_{p,\mu,\nu=1}^{4} V^a_{\mu\nu p} y_\mu y_\nu z_p \end{aligned}$$

where

$$F := \sum_{(\alpha,\mu)\neq(5,5)} f_{\alpha\mu} x_\alpha y_\mu$$

with $f_{\alpha\mu}$ either $\sqrt{2}\theta^5_\alpha(e_\mu)$ or $\sqrt{2}\theta^5_\mu(e_\alpha)$, and

$$\begin{aligned} U^a_{\alpha\beta p} &:= 1/2\sum_{\mu=1}^{4}(S^p_{\alpha\mu}S^a_{\beta\mu} + S^p_{\beta\mu}S^a_{\alpha\mu}) \\ V^a_{\mu\nu p} &:= -1/2\sum_{\alpha=1}^{4}(S^p_{\alpha\mu}S^a_{\alpha\nu} + S^p_{\alpha\nu}S^a_{\alpha\mu}) \end{aligned} \tag{49}$$

with $S^a_{\alpha\mu}$ the data in (8).

Lemma 15. *$F = 0$.*

Proof. $p_a q_a$ contributes

$$f_{\alpha\mu}x_\alpha x_\beta y_\mu y_\nu z_a$$

for each $1 \leq a \leq 4$, and $1 \leq \beta, \nu \leq 4$, that is not shared by any other terms in the equation [**24**, I, p 530]

$$p_0q^0 + p_1q^1 + \cdots + p_4q^4 = 0. \tag{50}$$

q.e.d.

6. The interplay between the second and third fundamental forms

We show in this section that the third fundamental form of the isoparametric hypersurface under consideration is that of the homogeneous example for the multiplicity pair $(m_1, m_2) = (4, 5)$ or $(6, 9)$, provided in the latter case the hypersurface is not the inhomogeneous one constructed by Ferus, Karcher, and Münzner. We thus arrive at the classification in these two cases.

6.1. The $(4, 5)$ **case.** To set it in the intrinsic quaternionic framework, let us now identify the normal space of M_+ spanned by $n_0, n_1, \ldots, n_4$ with $\mathbb{R}n_0 \oplus \mathbb{H}$, where $n_1, \ldots, n_4$ are identified with $1, i, j, k$, respectively.

Then the second fundamental form in (7) can be written succinctly in the vector form as

$$\begin{aligned} &\langle p, w_0n_0 + W\rangle \\ &= (|X|^2 + (x_5)^2 - |Y|^2 - (y_5)^2)w_0 + 2\langle\overline{Y}X, W\rangle \\ &+ \sqrt{2}(x_5 + y_5)\langle Z, W\rangle \end{aligned} \tag{51}$$

where

$$X := x_1 + x_2i + x_3j + x_4k, \qquad Y := y_1 + y_2i + y_3j + y_4k,$$
$$Z := z_1 + z_2i + z_3j + z_4k, \qquad W := w_1 + w_2i + w_3j + w_4k$$

with normal coordinates $w_0, w_1, \ldots, w_4$ in the respective normal directions $n_0, \ldots, n_4$, and e_α, e_μ, and e_p basis vectors are also identified with

$1, i, j, k$ in the natural way. (Recall X, Y, and Z parametrize respectively the E_1, E_{-1}, and E_0 spaces.) Thus there will be no confusion to set

$$(e_1, e_2, e_3, e_4) := (1, i, j, k)$$

for notational convenience. Let us define

$$X \circ Y := \sum_{p=1}^{4} S^p(X, Y)\, e_p. \tag{52}$$

The vector-valued third fundamental form is now

$$\begin{aligned} &\langle q, w_0 n_0 + W\rangle \\ &= -2\langle X \circ Y, Z\rangle w_0 + \sqrt{2}(x_5 - y_5)\langle X \circ Y, W\rangle \\ &+ \sum_{\mu=1}^{4} \langle X \circ e_\mu, Z\rangle \langle \overline{e_\mu} X, W\rangle \\ &- \sum_{\alpha=1}^{4} \langle e_\alpha \circ Y, Z\rangle \langle \overline{Y} e_\alpha, W\rangle \\ &= -2\langle X \circ Y, Z\rangle w_0 + \sqrt{2}(x_5 - y_5)\langle X \circ Y, W\rangle \\ &+ \langle X \circ (X\overline{W}), Z\rangle - \langle la(YW) \circ Y, Z\rangle \end{aligned} \tag{53}$$

where $Xe_\mu, e_\alpha Y, X\overline{W}$, and YW, etc., are quaternionic products.

Define the 4-by-4 matrices

$$T^p := \left(S^p_{\alpha\mu}\right), \quad p = 1, \dots, 4. \tag{54}$$

There holds

$$T^p_{\alpha\mu} = \langle e_\alpha \circ e_\mu, e_p\rangle.$$

We remark that in the homogeneous case these matrices are obtained by collecting half of the coefficients, respectively, of the $z_1, \dots, z_4$ coefficients of $-\tilde{q}^0$ in (12), which are

$$\begin{aligned} \tilde{T}^1 &:= \begin{pmatrix} -J & 0 \\ 0 & -J \end{pmatrix}, & \tilde{T}^2 &:= \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \\ \tilde{T}^3 &:= \begin{pmatrix} 0 & J \\ -J & 0 \end{pmatrix}, & \tilde{T}^4 &:= \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}. \end{aligned} \tag{55}$$

Moreover, T^p are orthogonal by (3) because $S^a_{p\alpha} = 0$ for all $1 \leq \alpha \leq 4$ by (11). Note that

$$\langle X \circ Y, e_p\rangle = \langle T^p(Y), X\rangle. \tag{56}$$

Lemma 16.

$$\langle (YZ) \circ Y, Z\rangle = 0 \tag{57}$$

for all $Y, Z \in \mathbb{H}$.

Proof. Let us set $X = x_5 = 0$ in (51) and (53). Then

$$p_0 = -|Y|^2 - (y_5)^2, \qquad q^0 = 0$$

and for $1 \leq a \leq 4$

$$p_a = \sqrt{2} y_5 \langle Z, e_a \rangle, \qquad q^a = -\langle la(Ye_a) \circ Y, Z \rangle$$

so that (50) is

$$0 = \sum_{a=0}^{4} p_a q^a = -\sqrt{2} y_5 \langle Z, e_a \rangle \langle (Ye_a) \circ Y, Z \rangle = -\sqrt{2} y_5 \langle (YZ) \circ Y, Z \rangle.$$

q.e.d.

Corollary 4. *The matrices given in* (54) *are*

$$T^1 = \begin{pmatrix} 0 & a & b & c \\ -a & 0 & -d & -e \\ -b & d & 0 & f \\ -c & e & -f & 0 \end{pmatrix}, \qquad T^2 = \begin{pmatrix} a & 0 & g & -h \\ 0 & a & -i & -j \\ j & -h & -f & 0 \\ -i & -g & 0 & -f \end{pmatrix}$$

$$T^3 = \begin{pmatrix} b & -g & 0 & k \\ -j & -e & -k & 0 \\ 0 & -l & b & -j \\ l & 0 & -g & -e \end{pmatrix}, \qquad T^4 = \begin{pmatrix} c & h & -k & 0 \\ i & d & 0 & -k \\ -l & 0 & d & -h \\ 0 & -l & -i & c \end{pmatrix}$$

for some twelve unknowns a *to* l.

Proof. Polarizing (57) with respect to Y and Z, respectively, we get

$$\begin{aligned} \langle (Y_1 Z) \circ Y_2, Z \rangle &= -\langle (Y_2 Z) \circ Y_1, Z \rangle, \\ \langle (YZ_1) \circ Y, Z_2 \rangle &= -\langle (YZ_2) \circ Y, Z_1 \rangle. \end{aligned} \tag{58}$$

Setting $Z = 1$ in the first equation of (58), we see $T^1_{\alpha\mu} = -S^1_{\alpha\mu}$ so that T^1 is skew-symmetric. Setting $Z = i$ and letting $Y_1 = Y_2 = 1$, we obtain

$$T^2_{21} = -T^2_{21} = 0,$$

while setting $Y_1 = 1, Y_2 = i$ yields

$$T^2_{22} = T^2_{11}.$$

However, setting $Z_1 = 1, Z_2 = i$, and $Y = 1$ in the second equation of (58), we see

$$T^2_{11} = -T^1_{21} = a.$$

Thus we get the upper left 2-by-2 block of T^2. Continuing this fashion finishes the proof. q.e.d.

Corollary 5. *We may assume* $a = f = 1$ *and the only nonzero entries in the matrices in Corollary* 4 *are* $a, f, k,$ *and* l.

Proof. Recall an automorphism σ of the quaternion algebra maps a quaternion basis to a quaternion basis, and vice versa.

Observe that if we consider the new quaternion basis $l_i := \sigma(e_i), 1 \leq i \leq 4$, to set

$$X = \sigma(X'), Y = \sigma(Y'), Z = \sigma(Z'), W = \sigma(W'),$$

then the second fundamental form in (51) remains the same form since $\sigma(\overline{Y'}X') = \overline{\sigma(Y')}\sigma(X') = \overline{Y}X$. Meanwhile, by comparing the homogeneous types in (53), we conclude that the circle product $\circ$ relative to the standard quaternion basis $1, i, j, k$ is converted to

$$X' \circ' Y' = \sigma^{-1}(\sigma(X') \circ \sigma(Y')) = \sigma^{-1}(X \circ Y) \tag{59}$$

relative to the new quaternion basis $\sigma(1), \sigma(i), \sigma(j), \sigma(k)$. Therefore, to verify the lemma, it suffices to find a quaternion basis $l_1 = e_1, l_2, l_3, l_4$ for which

$$1 = \langle l_2 \circ' l_1, l_1\rangle = \langle l_2 \circ' e_1, e_1\rangle = \langle l_2 \circ e_1, e_1\rangle = \langle T^1(e_1), l_2\rangle, \tag{60}$$

where the last equality is obtained by (56). It is now clear that if we define $l_2 = T^1(e_1)$, then (60) is verified readily by the orthogonality of T^1. Complete l_1, l_2 to a quaternion basis $l_1, \ldots, l_4$ (choose $l_3 \perp l_1, l_2$ and set $l_4 = l_2 l_3$). Now $a = 1$. It follows by the orthogonality of T^1 that $b = c = d = e = 0$ so that $f = \pm 1$. If $f = -1$, change l_3, l_4 to $-l_3, -l_4$ so that we may also assume $f = 1$.

It follows that $g = h = i = j = 0$ by the orthogonality of T^2, etc. The lemma is completed by the orthogonality of $T^p, 1 \leq p \leq 4$. q.e.d.

Lemma 17. $\langle \nabla q^a, \nabla q^b\rangle = \langle \nabla \tilde{q}^a, \nabla \tilde{q}^b\rangle$ *for all* $1 \leq a, b \leq 4$.

Proof. This follows from Proposition 2 and the identities of Ozeki and Takeuchi [**24**, I, p 530]:

$$\begin{aligned}
&8\langle \nabla q^a, \nabla q^a\rangle = 8(\langle \nabla p_a, \nabla p_a\rangle(|X|^2 + |Y|^2 + |Z|^2 + (x_5)^2 + (y_5)^2) - p_a^2)\\
&+ \langle \nabla\langle \nabla p_a, \nabla p_a\rangle, \nabla G\rangle - 24G - 2\sum_{b=0}^{4}\langle \nabla p_a, \nabla p_b\rangle^2, \text{ and}\\
&8\langle \nabla q^a, \nabla q^b\rangle = 8(\langle \nabla p_a, \nabla p_a\rangle(|X|^2 + |Y|^2 + |Z|^2 + (x_5)^2 + (y_5)^2) - p_a p_b)\\
&+ \langle \nabla\langle \nabla p_a, \nabla p_b\rangle, \nabla G\rangle - 2\sum_{c=0}^{4}\langle \nabla p_a, \nabla p_c\rangle\langle \nabla p_b, \nabla p_c\rangle, \quad a \neq b,
\end{aligned}$$

where $G = p_0^2 + \cdots + p_4^2$. Observe that the isoparametric hypersurface under consideration and the homogeneous example have the same second fundamental form. q.e.d.

Let us now calculate $\nabla\langle q, W\rangle$ with respect to the X, Y, Z (i.e., α, μ, p) coordinates. By (53)

$$
\begin{aligned}
&\nabla\langle q, W\rangle \\
&= \sum_{\alpha=1}^{4}(\langle e_\alpha \circ (X\overline{W}), Z\rangle + \langle X \circ (e_\alpha\overline{W}), Z\rangle)e_\alpha \\
&+ \sqrt{2}(x_5 - y_5)\sum_{\alpha=1}^{4}\langle e_\alpha \circ Y, W\rangle e_\alpha + \sqrt{2}\langle X \circ Y, W\rangle\zeta_5 \\
&- \sum_{\mu=1}^{4}(\langle (e_\mu W) \circ Y, Z\rangle - \langle (YW) \circ e_\mu, Z\rangle)e_\mu \\
&+ \sqrt{2}(x_5 - y_5)\sum_{\mu=1}^{4}\langle X \circ e_\mu, W\rangle e_\mu - \sqrt{2}\langle X \circ Y, W\rangle\eta_5 \\
&+ \sum_{p=1}^{4}(\langle X \circ (X\overline{W}), e_p\rangle - \langle (YW) \circ Y, e_p\rangle)e_p,
\end{aligned} \tag{61}
$$

where ζ_5 and η_5 are basis vectors of x_5 and y_5, respectively.

Set

$$\langle X * Y, e_p\rangle := \tilde{T}^p(X, Y)$$

with $\tilde{T}^p(e_\alpha, e_\mu)$ given in (55).

Corollary 6. $k = l = 1$ *in Corollary* 4.

Proof. Setting $p = 1, q = 3, \alpha = 1$, and $\beta = 4$ in (3) with Corollary 3 in mind, we obtain by the structure of T^p in Corollaries 4 and 5 (recall $T^p_{\alpha\mu} := S^p_{\alpha\mu}$) that

$$kf - al = 0$$

so that $k = l$ since $a = f$.

Setting $Z = x_5 = y_5 = 0$ in

$$\langle\nabla\langle q, W_1\rangle, \nabla\langle q, W_2\rangle\rangle$$

via (61) and comparing homogeneous types, we obtain

$$
\begin{aligned}
&4\langle X \circ Y, W_1\rangle\langle X \circ Y, W_2\rangle \\
&- \langle X \circ (X\overline{W_1}), (YW_2) \circ Y\rangle \\
&- \langle X \circ (X\overline{W_2}), (YW_1) \circ Y\rangle \\
&= 4\langle X * Y, W_1\rangle\langle X * Y, W_2\rangle \\
&- \langle X * (X\overline{W_1}), (YW_2) * Y\rangle \\
&- \langle X * (X\overline{W_2}), (YW_1) * Y\rangle.
\end{aligned} \tag{62}
$$

Setting $W_1 = e_1$ and $W_2 = e_3$, we expand the preceding identity to derive that the $x_1^2 y_2 y_4$ coefficient of the second term (on both sides) is

$$-(T_{11}^2 T_{44}^2 - T_{11}^2 T_{22}^2) = af + a^2 = 2,$$

while that of the third term (on both sides) is

$$T_{13}^4 T_{24}^4 + T_{13}^4 T_{42}^4 = k^2 + kl = 2k^2 = 2,$$

so that the $x_1^2 y_2 y_4$ coefficient of the first term satisfies

$$k = ak = T_{12}^1 T_{14}^3 = \tilde{T}_{12}^1 \tilde{T}_{14}^3 = 1,$$

noting that the term $T_{14}^1 T_{12}^3$ in the coefficient is zero. q.e.d.

As a consequence, we deduce that $X \circ Y = X * Y$. That is, the third fundamental form of the isoparametric hypersurface under consideration is that of the homogeneous example. We conclude that the isoparametric hypersurface is precisely the homogeneous one.

6.2. The $(6,9)$ **case.** The necessary modifications are as follows. Let $e_1, e_2, \ldots, e_8$ be the octonion basis with e_1 the multiplicative identity. Then in (51) the positive sign in front of $2\langle \overline{Y} X, W\rangle$ is changed to the negative sign (octonion multiplication is understood now). However, changing Z, W to $-Z, -W$ will convert the sign. So, we will assume (51) from now on. Meanwhile,

$$X := x_1 e_1 + x_2 e_2 + \cdots + x_8 e_8, \qquad Y := y_1 e_1 + y_2 e_2 + \cdots + y_8 e_8,$$
$$Z := z_3 e_3 + z_4 e_4 + \cdots + z_8 e_8, \qquad W := w_3 e_3 + w_4 e_4 + \cdots + w_8 e_8.$$

In (55) for the homogeneous case, the matrices are replaced, in view of (6), by

$$\tilde{T}^\mu = \left(\sqrt{2}\langle P_a(k_\mu), g_p\rangle\right), \tag{63}$$

where $2 \le \mu \le 8$, $\tilde{T}^\mu$ is skew-symmetric with the $(1,j)$-entry $= \langle e_\mu, e_2 e_j\rangle$ for $2 \le j \le 8$, the (i,j)-entry $= \langle e_\mu, (e_2 e_j) e_i\rangle$ for $2 \le i < j \le 8$. Explicitly,

$$\tilde{T}^3 = \begin{pmatrix} 0 & J & 0 & 0 \\ J & 0 & 0 & 0 \\ 0 & 0 & 0 & J \\ 0 & 0 & J & 0 \end{pmatrix}, \quad \tilde{T}^4 = \begin{pmatrix} 0 & I & 0 & 0 \\ -I & 0 & 0 & 0 \\ 0 & 0 & 0 & I \\ 0 & 0 & -I & 0 \end{pmatrix},$$

$$\tilde{T}^5 = \begin{pmatrix} 0 & 0 & J & 0 \\ 0 & 0 & 0 & -J \\ J & 0 & 0 & 0 \\ 0 & -J & 0 & 0 \end{pmatrix}, \quad \tilde{T}^6 = \begin{pmatrix} 0 & 0 & I & 0 \\ 0 & 0 & 0 & -I \\ -I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \end{pmatrix},$$

$$\tilde{T}^7 = \begin{pmatrix} 0 & 0 & 0 & L \\ 0 & 0 & L & 0 \\ 0 & -L & 0 & 0 \\ -L & 0 & 0 & 0 \end{pmatrix}, \quad \tilde{T}^8 = \begin{pmatrix} 0 & 0 & 0 & -K \\ 0 & 0 & -K & 0 \\ 0 & K & 0 & 0 \\ K & 0 & 0 & 0 \end{pmatrix},$$

where J, K, and L are as given in (9) and (17).

Lemma 18. *$T^p, 3 \le p \le 8$, in* (54) *are all skew-symmetric. The upper left* 2*-by-*2 *block of each of them is zero.*

Proof. Setting $x_9 = y_9 = 0$ in (51) and (53) (note that x_5 and y_5 in the formulae are replaced by x_9 and y_9 in the present case), we compare homogeneous types in (50) and set $X = e_1$ to obtain

$$\begin{aligned}
0 &= |Y|^2\langle e_1 \circ Y, Z\rangle - \sum_{\alpha=3}^{8}\langle \overline{Y}, e_\alpha\rangle\langle (Ye_\alpha)\circ Y, Z\rangle \\
&= |Y|^2\langle e_1 \circ Y, Z\rangle - \langle (Y(\overline{Y} - y_1e_1 + y_2e_2))\circ Y, Z\rangle \\
&= y_1\langle Y\circ Y, Z\rangle - y_2\langle (Ye_2)\circ Y, Z\rangle,
\end{aligned}$$

of which the coefficient of $y_1y_iy_j$, for $3 \le i, j \le 8$, is

$$0 = \langle e_i\circ e_j + e_j \circ e_i, Z\rangle,$$

so that $T^p_{ij} = -T^p_{ji}$. This is also true for $(i,j) = (1,j), j \ge 3$, or $(i,j) = (2,j), j \ge 3$. For $(i,j) = (1,2)$, the coefficients of $(y_1)^3$ and $(y_2)^3$ result in the $T^p_{11} = T^p_{22} = 0$, while the coefficient of $(y_1)^2y_2$ gives

$$2(T^p_{12} + T^p_{21}) - T^p_{21} = 0$$

and the coefficient of $y_1(y_2)^2$ gives

$$-T^p_{22} + T^p_{21} + T^p_{11} = 0.$$

From this we see $T^p_{12} = T^p_{21} = 0$. q.e.d.

Lemma 19. *Suppose $\langle e_2\circ Z, Z\rangle = 0$ for all $Z \perp e_1, e_2$. Then there is an octonion orthonormal pair of purely imaginary vectors (X,Y) in $\mathbb{O}$ such that $X, Y \perp e_2$ and $\langle Y\circ X, X\rangle \ne 0$.*

Proof. Suppose the contrary. For any such pair (X,Y), consider

$$T^X := \sum_{p=3}^{8} x_pT^p : \mathbb{O} \to \mathbb{O}.$$

Now $\langle Y\circ X, X\rangle = 0$ is equivalent to $\langle T^X(X), Y\rangle = 0$ for all purely imaginary $Y \perp X, e_2$, and hence in fact for all purely imaginary $Y \perp e_2$ because

$$\langle T^X(X), X\rangle = \sum_{p=3,\alpha=3,\mu=3}^{8} T^p_{\alpha\mu}x_\alpha x_\mu x_p = 0$$

by the skew-symmetry of T^p. Moreover, the assumption $\langle e_2\circ X, X\rangle = 0$ is equivalent to $\langle T^X(X), e_2\rangle = 0$. We thus conclude that $T^X(X) = \pm e_1$. Homogenizing $\langle T^X(X), e_1\rangle = \pm 1$, we obtain

$$\sum_{p=3,\mu=3}^{8} T^p_{1\mu}x_\mu x_p = \pm|X|^2$$

for all purely imaginary octonion vectors X. Hence we conclude that $T^p_{1p} = \pm 1$ for $3 \leq p \leq 8$. However, the first identity of (58) with $Z = e_p, Y_1 = Y_2 = e_1$ gives $T^p_{1p} = 0$, which is a contradiction. q.e.d.

Lemma 20. *We may assume $T^3 = \tilde{T}^3$ and $T^4 = \tilde{T}^4$.*

Proof. We first show that, in view of (59), we can choose an octonion basis $l_1 = e_1, l_2, \ldots, l_8$ relative to which $T^3_{41} = 1$, i.e.,

$$1 = \langle l_4 \circ l_1, l_3 \rangle = \langle l_2 \circ l_3, l_3 \rangle, \tag{64}$$

in which the second equality is obtained by the first identity of (58) with $Y_1 = l_2, Z = l_3$, and $Y_2 = l_1$ and the skew-symmetry of T^p. To this end, note that if there is a $Z \perp e_1, e_2$ such that $\langle e_2 \circ Z, Z \rangle \neq 0$, we are done. For then the orthogonal operator

$$U : z \perp\ (\mathrm{span}\langle e_1, e_2 \rangle)^\perp \to e_2 \circ z \in (\mathrm{span}\langle e_1, e_2 \rangle)^\perp$$

is not skew-symmetric and so the structure of an orthogonal matrix tells us that U has an eigenvector $v \perp e_1, e_2$ with eigenvalue ± 1. We may assume it is 1 by changing e_2 to $-e_2$ and construct a new octonion basis in which $l_2 = -e_2, v = l_3$, etc., so that (64) holds. Otherwise, Lemma 19 gives rise to a pair (X, Y) with $X, Y \perp e_1, e_2$. In a similar vein to U, the orthogonal operator

$$R : z \perp\ (\mathrm{span}\langle e_1, Y \rangle)^\perp \to Y \circ z \in (\mathrm{span}\langle e_1, Y \rangle)^\perp$$

is not skew-symmetric because X is in $(\mathrm{span}\langle e_1, Y \rangle)^\perp$. Therefore, we can find an eigenvector w with eigenvalue 1, without loss of generality, for R. Construct an octonion basis in which $l_1 := 1, l_2 := Y, l_3 := w, l_4 := l_2 l_3$, etc. This choice will leave the second fundamental form unchanged while making $T^3_{41} = 1$.

With $T^3_{41} = 1$, the first identity in (58) with $Z = l_3, Y_1 = l_1$, and $Y_2 = l_2$ gives $T^3_{32} = -1$. By skew-symmetry of T^3, its upper left 4-by-4 block is determined to be identical with that of $\tilde{T}^3$. The orthogonality of T^3 then implies that the upper right 4-by-4 and the lower left 4-by-4 blocks of T^3 are zero.

Now a calculation using the first identity of (58) establishes that the lower right 4-by-4 block of T^3 is of the form

$$\begin{pmatrix} 0 & -a & 0 & -b \\ a & 0 & b & 0 \\ 0 & -b & 0 & a \\ b & 0 & -a & 0 \end{pmatrix}.$$

On the other hand, setting $W_1 = W_2 = l_3$, the coefficient of $(x_6)^2(y_5)^2$ of the first term on the left in (62) is

$$4(T^3_{65})^2 = 4a^2$$

and is 0 on the right. The coefficient of $(x_6)^2(y_5)^2$ of the second and third terms on the left is

$$\sum_{i=3}^{8} T^i_{68}T^i_{75} = T^4_{68}T^4_{75} = -b^2$$

because the second identity in (58) derives that $T^4_{68} = -T^3_{58} = b$, $T^4_{75} = -T^3_{85} = -b$, $T^5_{68} = T^3_{48} = 0$, $T^7_{75} = T^3_{35} = 0$, and $T^8_{68} = T^3_{18} = 0$; it is -1 on the right hand side. Therefore, we obtain

$$4a^2 - 2b^2 = -2, \qquad a^2 + b^2 = 1,$$

where the second identity is obtained by the orthogonality of T^3. It follows that $a = 0$ and $b = \pm 1$. We may assume $b = 1$; otherwise, changing l_5 to $-l_5$ does the job. In other words, $T^3 = \tilde{T}^3$ now.

That $T^4 = \tilde{T}^4$ follows from the second identity of (58) and that $T^3 = \tilde{T}^3$. For instance, choosing $Z_1 = e_3, Z_2 = e_4, Y_1 = e_1$, and $Y_2 = e_2$, we obtain $T^4_{42} = T^3_{32} = -1$, etc. q.e.d.

Lemma 21. *The upper left and lower right* 4*-by-*4 *blocks of* T^5, T^6, T^7, T^8 *are all zero.*

Proof. Applying the second identity of (58) to $Z_1 = e_5, Z_2 = e_3$, and $Y = e_1$, we obtain $T^5_{31} = T^3_{51} = 0$ by Lemma 20. Applying the first identity of (58) to $Z = e_5, Y_1 = e_1$, and $Y_2 = e_7$, we see $T^5_{57} = T^5_{31} = 0$. Continuing in this fashion, we can verify that all the upper left 4-by-4 and lower right 4-by-4 entries of T^5 are zero except for $T^5_{34} = -T^5_{43} = T^5_{78} = -T^5_{87}$.

To show $T^5_{43} = 0$, we let $p = 5, q = 3, \alpha = 4$, and $\beta = 2$ in (3). The matrix entries in (16) give $S^a_{\alpha\mu} = A^a_{\alpha\mu}$, and recall we set $S^p_{\alpha\mu} = T^p_{\alpha\mu}$. We derive

$$T^5_{43} = T^5_{43}T^3_{23} + T^3_{41}T^5_{21} = -2\sum_a (A^a_{54}A^a_{32} + A^a_{34}A^a_{52}) = 0.$$

The same goes through for T^6, T^7, T^8 with p replaced by $6, 7, 8$. q.e.d.

Lemma 22. *The lower left* 4*-by-*4 *blocks of* T^5, T^6, T^7, T^8 *are*

$$T^5 : \begin{pmatrix} 0 & -a & -b & -c \\ a & 0 & -d & -e \\ b & d & 0 & -f \\ c & e & f & 0 \end{pmatrix}, \qquad T^6 : \begin{pmatrix} -a & 0 & j & -i \\ 0 & -a & -h & g \\ g & i & k & 0 \\ h & j & 0 & k \end{pmatrix},$$

$$T^7 : \begin{pmatrix} -b & -j & 0 & m \\ -g & e & m & 0 \\ 0 & l & -b & g \\ l & 0 & j & e \end{pmatrix}, \qquad T^8 : \begin{pmatrix} -c & i & -m & 0 \\ -h & -d & 0 & m \\ -l & 0 & -d & i \\ 0 & l & -h & -c \end{pmatrix}$$

a priori for some thirteen unknowns a through m.

Proof. Assuming the unknowns a through f for the lower triangular block of the lower 4-by-4 block of T^5 and setting $T^6_{71} := g, T^6_{81} := h, T^6_{72} := i, T^6_{82} := j, T^6_{73} = k, T^7_{81} = l$, and $T^7_{54} = m$, one uses the two identities in (58) repeatedly to obtain all other entries in terms of these thirteen unknowns. q.e.d.

Lemma 23. *The only nonzero entries in the above matrices are a, f, k, l, m of magnitude 1 with the property that $a = -f = k$ and $l = m$.*

Proof. We know $T^iT^j = -T^jT^i$ when $i \neq j$ by (3), (19), Corollary 3, and Lemma 18.

Now, $(i, j) = (3, 5)$ or $(4, 5)$ gives $a = -f$ and $b = c = d = e = 0$. $(i, j) = (3, 6)$ or $(4, 6)$ gives $a = k$ and $g = h = i = j = 0$. Lastly, $(i, j) = (3, 7)$ gives $l = m$. q.e.d.

Corollary 7. *$a = 1$ and $l = -1$. In particular, $T^5 = \tilde{T}^5, T^6 = \tilde{T}^6, T^7 = \tilde{T}^7, T^8 = \tilde{T}^8$.*

Proof. The proof is similar to the one in Corollary 6. Choosing $W_1 = e_5$ and $W_2 = e_3$, the $x_4x_6(y_1)^2$ coefficient of the second term (on both sides) is

$$T^4_{68}T^4_{31} - T^4_{42}T^4_{31} = -2,$$

while that of the third term (on both sides) is

$$-T^6_{62}T^6_{51} - T^6_{48}T^6_{51} = -a^2 - ka = -2.$$

Therefore, the $x_4x_6(y_1)^2$ coefficient of the first term satisfies

$$a = T^5_{61}T^3_{41} = \tilde{T}^5_{61}\tilde{T}^3_{41} = 1,$$

so that $k = 1$. In particular, $T^5 = \tilde{T}^5$ and $T^6 = \tilde{T}^6$. Choosing $W_1 = e_7$ and $W_2 = e_3$, the $x_1x_5(y_4)^2$ coefficient of the second term (on both sides) is

$$-T^4_{57}T^4_{24} - T^4_{13}T^4_{24} = -2,$$

while that of the third term (on both sides) is

$$T^8_{53}T^8_{64} - T^8_{17}T^8_{64} = -m^2 - lm = -2.$$

Therefore, the $x_1x_5(y_4)^2$ coefficient of the first term is

$$-m = T^7_{54}T^3_{14} = \tilde{T}^7_{54}\tilde{T}^3_{14} = 1.$$

In particular, $T^7 = \tilde{T}^7$. It follows that $T^8 = \tilde{T}^8$. q.e.d.

As a consequence, the isoparametric hypersurface is precisely the homogeneous one.

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