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ON THE NONCOMMUTATIVE RESIDUE FOR PROJECTIVE PSEUDODIFFERENTIAL OPERATORS

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Abstract

A well-known result on pseudodifferential operators states that the noncommutative residue (Wodzicki residue) of a zero-order pseudodifferential projection on a closed manifold vanishes. This statement is non-local and implies the regularity of the eta invariant at zero of Dirac type operators. We prove that in the odd dimensional case an analogous statement holds for the algebra of projective pseudodifferential operators, i.e., the noncommutative residue of a projective pseudodifferential projection vanishes. Our strategy of proof also simplifies the argument in the classical setting. We show that the noncommutative residue factors to a map from the twisted K-theory of the co-sphere bundle and then use arguments from twisted K-theory to reduce this question to a known model case. It can then be verified by local computations that this map vanishes in odd dimensions.

1. Introduction

If $\Psi\mathrm{DO}_{\mathrm{cl}}(X,E)$ denotes the algebra of classical pseudodifferential operators on a closed manifold X acting on sections of a vector bundle E, the corresponding algebra of symbols can be defined as the quotient of $\Psi\mathrm{DO}_{\mathrm{cl}}(X,E)$ by the ideal of smoothing operators. Since pseudodifferential operators are smooth off the diagonal, the symbol algebra is localized on the diagonal and it therefore can also be defined locally, using the product expansion formula and the change of charts formula for pseudodifferential operators. That the local heat kernel coefficients and the index of elliptic pseudodifferential operators are locally computable relies on the fact that the index and asymptotic spectral properties of pseudodifferential operators depend only on their class in the symbol algebra. Note that the principal symbol of a pseudodifferential operator is a section of the bundle of endomorphisms of π^*E, where $\pi: T^*X \to X$ is the canonical projection.

The bundle of endomorphisms of a complex hermitian vector bundle is a bundle of simple matrix algebras with $*$-structure. However, not

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all bundles of simple matrix algebras with $*$-structure, so-called Azumaya bundles, are isomorphic to endomorphism bundles of hermitian vector bundles. The obstruction is the so-called Dixmier-Douady class in $H^3(X,\mathbb{Z})$. Given an Azumaya bundle $\mathcal{A}$, it is possible to construct algebras of symbols whose principal symbols take values in the space of sections of the Azumaya bundle $\pi^*\mathcal{A}$ (see for example [**MMS05**] and the discussion in [**MMS06**]). Following [**MMS05**], we refer to such a symbol algebra as the algebra of symbols of projective pseudodifferential operators. For such symbol algebras, one can define an index and Mathai, Melrose, and Singer [**MMS06**] proved an index formula for projective pseudodifferential operators, analogous to the Atiyah-Singer index formula. The topological index in this case is a map from twisted K-theory to $\mathbb{R}$. It has also been shown in [**MMS06**] that any oriented manifold admits a projective Dirac operator even if the manifold does not admit a spin structure. In this case, its index may fail to be an integer.

Another important quantity that depends only on the class of the symbol of a pseudodifferential operator is the so-called Wodzicki residue or noncommutative residue. Up to a factor, it is the unique trace on the algebra of pseudodifferential operators. The Wodzicki residue appeared first as a residue of a zeta function measuring spectral asymmetry ([**APS76**, **Wo84**]). Wodzicki showed that the regularity of the η-function of a Dirac type operator at zero—a necessary ingredient to define the η-invariant—follows as a special case from the vanishing of the Wodzicki residue on pseudodifferential projections (as remarked by Brüning and Lesch [**BL99**], the regularity of the η-function at zero for any Dirac type operator and the vanishing of the Wodzicki residue on pseudodifferential projections are actually equivalent). The regularity of the η-function was proved by Atiyah, Patodi, and Singer in [**APS76**] in the case when X is odd dimensional and later by Gilkey ([**Gi81**]) in the general case using K-theoretic arguments. Note that whereas the Wodzicki residue can be locally computed, its vanishing on pseudodifferential projections is not a local phenomenon. Gilkey [**Gi79**] constructed a pseudodifferential projection whose residue density is non-vanishing but integrates to zero.

In our paper we show that the Wodzicki residue can also be defined for projective pseudodifferential operators (as already has been observed in [**MMS06**]) and show that it vanishes on zero-order projections in case the dimension of the manifold is odd. Our proof is based on the Leray-Hirsch theorem in twisted K-theory and well-known facts from algebraic K-theory. These results can be applied to the Wodzicki residue, showing that it descends to a map from twisted K-theory $K^0(S^*X,\pi^*\mathcal{A})$ to $\mathbb{C}$. We then use the Leray-Hirsch theorem to show that this map vanishes by reducing the problem to positive spectral projections of generalized

Dirac operators for which it is known [**BG92**] that the residue density vanishes.

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2. Convolution bundles and Azumaya bundles

Pseudodifferential operators on a smooth closed Riemannian manifold X acting on sections of a vector bundle E can be understood as conormal distributional sections in the vector bundle $E \boxtimes E^*$ over the space $X \times X$ (i.e., the external tensor product of E and its dual bundle E^*, having fibre $E_x \otimes E_y^*$ over a point (x, y)), by identifying the operators with their distributional kernel. The bundle $E \boxtimes E^*$ has the following structure that allows us to define the convolution of integral operators: any element in the fibre over (x, y) may be multiplied by an element in the fibre over (y, z) to give an element in the fibre over (x, z). Moreover, this multiplication satisfies natural conditions such as associativity. In order to define projective pseudodifferential operators, it is convenient to formalize this structure, as we shall do in this section.

2.1. Convolution bundles. In the sequel, we shall denote by $M_k(R)$ the $k \times k$-matrices with entries from R. Let $\mathcal{U}$ denote an open neighborhood of the diagonal $\Delta(X)$ in $X \times X$ which is symmetric under the reflection map $s : (x, y) \mapsto (y, x)$. Let $p_{ik} : X \times X \times X \to X \times X$ be defined by $p_{ik}(x_1, x_2, x_3) = (x_i, x_k)$ and set $\widetilde{\mathcal{U}} := p_{12}^{-1}(\mathcal{U}) \cap p_{23}^{-1}(\mathcal{U}) \cap p_{13}^{-1}(\mathcal{U})$. Denote by $\widetilde{p}_{ik}$ the restriction of the map p_{ik} to $\widetilde{\mathcal{U}}$.

Definition 2.1. Let $\pi : F \to \mathcal{U}$ be a locally trivial vector bundle with typical fibre $M_k(\mathbb{C})$. We call F a *convolution bundle* if there exists a homomorphism of vector bundles $m : \widetilde{p}_{12}^* F \otimes \widetilde{p}_{23}^* F \to F$ such that the following conditions are satisfied:

(i) The following diagram is commutative:

$$\begin{array}{ccc} \widetilde{p}_{12}^* F \otimes \widetilde{p}_{23}^* F & \xrightarrow{m} & F \\ \downarrow & & \downarrow \\ \widetilde{\mathcal{U}} & \xrightarrow{\widetilde{p}_{13}} & \mathcal{U} \end{array}$$

(ii) m is associative, i.e., whenever f_{ij} belong to the fibre $F_{(x_i, x_j)}$, then

$$m\left(m(f_{12} \otimes f_{23}) \otimes f_{34}\right) = m\left(f_{12} \otimes m(f_{23} \otimes f_{34})\right)$$

(we implicitly assume that (x_1, x_2, x_3), (x_1, x_3, x_4), and (x_1, x_2, x_4) belong to $\widetilde{\mathcal{U}}$).

(iii) There is an atlas $\{\mathcal{O}_\alpha\}$ of $\mathcal{U}$ together with local trivializations

$$\phi_\alpha : \pi^{-1}\mathcal{O}_\alpha \to \mathcal{O}_\alpha \times M_k(\mathbb{C}),$$

such that

$$\phi_\alpha(m(f_{12} \otimes f_{23})) = \phi_\alpha(f_{12}) \cdot \phi_\alpha(f_{23})$$

whenever $f_{ij} \in F_{(x_i,x_j)}$ with $(x_1, x_2, x_3) \in \widetilde{p}_{12}^{-1}(\mathcal{O}_\alpha) \cap \widetilde{p}_{23}^{-1}(\mathcal{O}_\alpha)$.

Definition 2.2. A $*$-*structure* on F is a conjugate linear map $* : F \to F$ of vector bundles such that

$$\begin{array}{ccc} F & \xrightarrow{\ *\ } & F \\ \downarrow & & \downarrow \\ \mathcal{U} & \xrightarrow{\ *\ } & \mathcal{U} \end{array}$$

commutes, such that $(m(f \otimes g))^* = m(g^* \otimes f^*)$, and such that the above local trivializations additionally satisfy

$$\forall f \in \pi^{-1}(\mathcal{O}_\alpha \cap s(\mathcal{O}_\alpha)) : \qquad \phi_\alpha(f^*) = \phi_\alpha(f)^*,$$

where the star on the right hand side denotes the hermitian conjugation of matrices. We will refer to a convolution bundle with $*$-structure as a $*$-*convolution bundle.*

Note that $E \boxtimes E^*$ is a particular example for a $*$-convolution bundle; in this case we can choose $\mathcal{U} = X \times X$. The restriction of a $*$-convolution bundle F to the diagonal in $X \times X$ is a bundle $\mathcal{A}$ of finite dimensional simple C^*-algebras. Following the literature, we refer to such bundles of matrix algebras as Azumaya bundles.

As shown in [**MMS06**], any Azumaya bundle $\mathcal{A}$ on X gives rise to a convolution bundle near the diagonal in the following way, using an atlas of local trivializations with respect to a good cover $\{U_\alpha\}$ of X (recall that a cover is good if finite intersections of elements therein are either empty or contractible): The transition functions $\sigma_{\alpha\beta}$ are smooth functions on $U_{\alpha\beta} = U_\alpha \cap U_\beta$ with values in the automorphisms of $M_k(\mathbb{C})$. Since all automorphisms are inner, we can choose local functions $\varphi_{\alpha\beta} : U_{\alpha\beta} \to SU(k)$ that implement $\sigma_{\alpha\beta}$, i.e., $\sigma_{\alpha\beta}(x)(A) = \varphi_{\alpha\beta}(x) A \varphi_{\alpha\beta}^{-1}(x)$. In general, the functions $\varphi_{\alpha\beta}$ may violate the co-cycle condition and therefore are not the transition functions of a vector bundle. The cocycle condition for the $\sigma_{\alpha\beta}$ together with the condition that the $\varphi_{\alpha\beta}$ are chosen in $SU(k)$ show that any $\varphi_{\alpha\beta}\varphi_{\beta\gamma}\varphi_{\gamma\alpha}$ must be a constant function on $U_\alpha \cap U_\beta \cap U_\gamma$, equal to a k-th root of unity times the identity matrix (note that on different triple intersections the resulting unit-root can be different; this induces a torsion element in $H^3(X, \mathbb{Z})$, the Dixmier-Douady class). Then we obtain a convolution bundle F with typical fibre

$M_k(\mathbb{C})$ on a neighborhood of the diagonal by choosing the transition functions

$$\phi_{\alpha\beta}(x,y)(A) = \varphi_{\alpha\beta}(x)A\,\varphi_{\alpha\beta}(y)^{-1}, \qquad A \in M_k(\mathbb{C}),$$

on $U_{\alpha\beta}\times U_{\alpha\beta}$. There are also other possible extensions of $\mathcal{A}$, cf. [**MMS06**], and Proposition 2.4, below.

Remark 2.3. In the sequel it will be occasionally convenient to choose an atlas for F consisting of sets $\mathcal{O}_\alpha := U_\alpha \times U_\alpha$, where $\{U_\alpha\}$ is a good cover of X; the corresponding trivializations we shall denote by ϕ_α (so we use the same notation as in Definition 2.1(iii) above, but possibly have changed the atlas).

2.2. Transition functions. In the previous section we have seen how an Azumaya bundle leads to a convolution bundle by choosing certain transition functions. Let us now have a closer look at the transition functions of an arbitrary $*$-convolution bundle. Fix an atlas as explained in Remark 2.3 and let $\phi_{\alpha\beta} : \mathcal{O}_{\alpha\beta} \longrightarrow GL(M_k(\mathbb{C}))$ with

$$\mathcal{O}_{\alpha\beta} := \mathcal{O}_\alpha \cap \mathcal{O}_\beta = (U_\alpha \times U_\alpha) \cap (U_\beta \times U_\beta)$$

be the transition functions defined by

$$\phi_\beta \circ \phi_\alpha^{-1}\big((x,y),A\big) = \big((x,y),\phi_{\alpha\beta}(x,y)(A)\big).$$

Then condition (iii) of Definition 2.1 is equivalent to

$$\phi_{\alpha\beta}(x,y)(A)\phi_{\alpha\beta}(y,z)(B) = \phi_{\alpha\beta}(x,z)(AB).$$

In particular,

$$(x,x) \mapsto \phi_{\alpha\beta}(x,x) : \mathcal{O}_{\alpha\beta} \cap \Delta(X) \longrightarrow \mathrm{Aut}(M_k(\mathbb{C})). \tag{2.1}$$

Moreover, Definition 2.2 on the level of the transition functions means that

$$\phi_{\alpha\beta}(x,y)(A^*) = \phi_{\alpha\beta}(y,x)(A)^*. \tag{2.2}$$

Proposition 2.4. *Let F be a $*$-convolution bundle with transition functions $\phi_{\alpha\beta}$ as described above. Then*

$$\phi_{\alpha\beta}(x,y)(A) = \lambda_{\alpha\beta}(x,y)\varphi_{\alpha\beta}(x)A\,\varphi_{\alpha\beta}(y)^{-1} \tag{2.3}$$

with mappings

$$\varphi_{\alpha\beta} : \mathcal{O}_{\alpha\beta} \longrightarrow SU(k), \qquad \lambda_{\alpha\beta} : \mathcal{O}_{\alpha\beta} \longrightarrow \mathbb{C},$$

satisfying

$$\lambda_{\alpha\beta}(x,x) = 1,\ \lambda_{\alpha\beta}(x,y)\lambda_{\alpha\beta}(y,z) = \lambda_{\alpha\beta}(x,z),\ \lambda_{\alpha\beta}(x,y) = \overline{\lambda_{\alpha\beta}(y,x)},$$

and such that all $\varphi_{\alpha\beta}\varphi_{\beta\gamma}\varphi_{\gamma\alpha}$ are constant functions on their domain of definition, equal to a k-th root of unity times the identity matrix.

Proof. Combining (2.1) with (2.2), we find $\varphi_{\alpha\beta}$ with

$$\phi_{\alpha\beta}(x,x)(A) = \varphi_{\alpha\beta}(x)A\,\varphi_{\alpha\beta}(x)^{-1},$$

since all automorphisms of $M_k(\mathbb{C})$ are inner. Now let us define

$$\phi'_{\alpha\beta}(x,y)(A) = \varphi_{\alpha\beta}(x)^{-1}\phi_{\alpha\beta}(x,y)(A)\varphi_{\alpha\beta}(y).$$

We then have

$$\phi'_{\alpha\beta}(x,x)(A) = A, \qquad \phi'_{\alpha\beta}(A)(x,y)\phi'_{\alpha\beta}(y,z)(B) = \phi'_{\alpha\beta}(x,z)(AB).$$

It follows that

$$\phi'_{\alpha\beta}(x,y)(AB) = \phi'_{\alpha\beta}(x,y)(A)\phi'_{\alpha\beta}(y,y)(B) = \phi'_{\alpha\beta}(x,y)(A)B$$

and, analogously, $\phi'_{\alpha\beta}(x,y)(AB) = A\phi'_{\alpha\beta}(x,y)(B)$. Thus, for any matrix A,

$$\phi'_{\alpha\beta}(x,y)(\mathbf{1})A = \phi'_{\alpha\beta}(x,y)(A) = A\phi'_{\alpha\beta}(x,y)(\mathbf{1}),$$

where $\mathbf{1}$ is the identity matrix. This shows $\phi'_{\alpha\beta}(x,y)(\mathbf{1})$ is a multiple of the identity matrix. Denoting the corresponding factor by $\lambda_{\alpha\beta}(x,y)$, the claim follows. q.e.d.

3. Projective Pseudodifferential Operators

Projective pseudodifferential operators were defined in [**MMS05**]. We adapt this definition to fit in our setting of convolution bundles.

3.1. Pseudodifferential operators. To clarify notation, let us briefly recall the definition of classical (or polyhomogeneous) pseudodifferential operators on an open subset Ω of $\mathbb{R}^n$. Further details the reader may find, for example, in [**Ku82**], [**Sh87**], or other textbooks on pseudodifferential analysis. Let $V \cong \mathbb{C}^k$ be a k-dimensional vector space.

A symbol of order $m \in \mathbb{R}$ is a smooth function $a : \Omega \times \Omega \times \mathbb{R}^n \to \mathrm{End}(V) = V \otimes V^*$ satisfying estimates

$$\left\|\partial_\xi^\alpha \partial_{(x,y)}^\beta a(x,y,\xi)\right\| \le C_{\alpha\beta K}(1+|\xi|)^{m-|\alpha|}$$

for any multi-indices α, β and any compact subset K of $\Omega \times \Omega$, and having an asymptotic expansion $a \sim \sum_{j=0}^{\infty} \chi a_{m-j}$ with a zero-excision function $\chi = \chi(\xi)$ and homogeneous components a_{m-j}, i.e.,

$$a_{m-j}(x,y,t\xi) = t^{m-j}a_{m-j}(x,y,\xi)$$

for all (x,ξ) with $\xi \neq 0$ and all $t > 0$. The pseudodifferential operator $\mathrm{op}(a) : C_0^\infty(\Omega, V) \to C^\infty(\Omega, V)$ associated with a is

$$[\mathrm{op}(a)\varphi](x) = (2\pi)^{-n} \iint e^{i(x-y)\xi} a(x,y,\xi)\varphi(y)\,dyd\xi, \quad \varphi \in C_0^\infty(\Omega, V).$$

An operator $R : C_0^\infty(\Omega, V) \to C^\infty(\Omega, V)$ is called smoothing if it has a smooth integral kernel $k \in C^\infty(\Omega \times \Omega, \mathrm{End}(V))$, i.e.,

$$(R\varphi)(x) = \int_\Omega k(x,y)\varphi(y)\, dy, \qquad \varphi \in C_0^\infty(\Omega, V).$$

A pseudodifferential operator of order $m \in \mathbb{R}$ on Ω is an operator of the form $A = \mathrm{op}(a) + R$, where a is a symbol of order m and R is smoothing.

Any pseudodifferential operator $A = \mathrm{op}(a) + R$ of order m can be represented in the form $\mathrm{op}(a_L) + R'$, where $a_L(x, \xi)$ is a y-independent 'left-symbol' of order m; up to order $-\infty$ the left-symbol is uniquely determined by the asymptotic expansion

$$a_L(x,\xi) \sim \sum_{|\alpha|=0}^{\infty} \frac{1}{\alpha!} \partial_\xi^\alpha D_y^\alpha a(x,y,\xi)\Big|_{x=y}.$$

The homogeneous components of A are by definition those of a_L,

$$\sigma_{m-j}(A)(x,\xi) := (a_L)_{m-j}(x,\xi).$$

By the Schwarz kernel theorem, we can identify A with its distributional kernel

$$K_A \in \mathscr{D}'(\Omega \times \Omega, V \otimes V^*),$$

the topological dual of $C_0^\infty(\Omega \times \Omega; V^* \otimes V)$. It is uniquely defined by the relation

$$\langle K_A, \psi \otimes \varphi \rangle = \langle \psi, A\varphi \rangle, \qquad \psi \in C_0^\infty(\Omega, V^*), \quad \varphi \in C_0^\infty(\Omega, V).$$

Denoting by $\mathrm{tr} : V^* \otimes V \to \mathbb{C}$ the canonical contraction map, we have explicitly

$$\langle K_A, u \rangle = \int_\Omega \mathrm{tr}[Au(x,\cdot)](x)\, dx, \qquad u \in C_0^\infty(\Omega \times \Omega; V^* \otimes V).$$

By pseudo-locality, $K_A \in C^\infty(\Omega \times \Omega \setminus \Delta(\Omega), V \otimes V^*)$.

If $U \subset X$ is a coordinate neighborhood, we can pull back local operators under the coordinate map. The resulting space of operators we shall denote by $\Psi\mathrm{DO}^m_{\mathrm{cl}}(U; \mathrm{End}(V))$, the subspace of smoothing operators by $\Psi\mathrm{DO}^{-\infty}(U; \mathrm{End}(V))$.

3.2. Projective pseudodifferential operators. In the following we choose an atlas as explained in Remark 2.3.

Definition 3.1. Let F be a $*$-convolution bundle over $\mathcal{U}$. A distribution $A \in \mathcal{D}'(\mathcal{U}, F)$ is called a projective pseudodifferential operator of order $m \in \mathbb{R}$ if

(i) A is smooth outside the diagonal.
(ii) For any α the distribution $\left(\phi_\alpha^{-1}\right)^* A\big|_{U_\alpha \times U_\alpha}$ is the distributional kernel of a pseudodifferential operator $A_\alpha \in \Psi\mathrm{DO}^m_{\mathrm{cl}}(U_\alpha; \mathrm{End}(\mathbb{C}^k))$.

We denote the vector space of m-th order projective pseudodifferential operators by $\Psi\mathrm{DO}^m_{\mathrm{cl}}(\mathcal{U};F)$, the subspace of smoothing elements by $\Psi\mathrm{DO}^{-\infty}(\mathcal{U};F)$.

The subspace $\mathrm{Diff}^m(\mathcal{U};F)$ of projective differential operators consists of all projective pseudodifferential operators which are supported on the diagonal.

Remark 3.2. If $\mathcal{U}=X\times X$ and $F=E\boxtimes E^*$ for a bundle E over X, then $\Psi\mathrm{DO}^m_{\mathrm{cl}}(\mathcal{U};F)$ coincides with $\Psi\mathrm{DO}^m_{\mathrm{cl}}(X;E,E)$, the pseudodifferential operators of order m acting on sections of E.

Though projective pseudodifferential operators, in general, are not operators in the usual sense (i.e., acting between sections of vector bundles), all elements of the standard calculus can be generalized to this setting. In particular, the $*$-structure gives rise to a conjugation on $\Psi\mathrm{DO}^m_{\mathrm{cl}}(\mathcal{U};F)$, defined by $A^*(x,y):=(A(y,x))^*$ in the distributional sense.

Let A be a projective pseudodifferential operator with local representatives A_α and A_β, cf. Definition 3.1, where $\mathcal{O}_{\alpha\beta}$ is not empty. By passing to local coordinates on $U_\alpha\cap U_\beta$, we can associate with A_α and A_β local left-symbols $a_\alpha(x,\xi)$ and $a_\beta(x,\xi)$, respectively. These symbols are then related by

$$\begin{aligned}a_\beta(x,\xi)&=\sum_{|\gamma|=0}^{\infty}\frac{1}{\gamma!}\partial_\xi^\gamma D_y^\gamma\Big|_{y=x}\phi_{\alpha\beta}(x,y)\big(a_\alpha(x,\xi)\big)\\&=\sum_{|\gamma|=0}^{\infty}\frac{1}{\gamma!}\partial_\xi^\gamma D_y^\gamma\Big|_{y=x}\big[\lambda_{\alpha\beta}(x,y)\varphi_{\alpha\beta}(x)a_\alpha(x,\xi)\varphi_{\alpha\beta}(y)^{-1}\big],\end{aligned}\tag{3.1}$$

where the transition function $\phi_{\alpha\beta}$ is as described in (2.2) and Proposition 2.4. Note that this behaviour, in general, differs from the standard case, due to the factor $\lambda_{\alpha\beta}(x,y)$. However, (3.1) together with $\lambda_{\alpha\beta}(x,x)=1$ shows that with A we can associate a well-defined homogeneous principal symbol

$$\sigma_m(A)(x,\xi)\in C^\infty(S^*X,\pi^*\mathcal{A}),$$

where $\pi:S^*X\to X$ is the canonical co-sphere bundle over X. Vice versa, any given such section can be realized as the principal symbol of a projective pseudodifferential operator.

If the projective pseudodifferential operators A_1 and A_2 are supported in a sufficiently small neighborhood of the diagonal in $\mathcal{U}$, their usual composition

$$(A_1\circ A_2)(x,z)=\int_X m\left(A_1(x,y)\otimes A_2(y,z)\right)\,dy$$

is a distribution. By passing to local coordinates and using the composition theorems for pseudodifferential operators, one can see that $A_1\circ A_2$

is a projective pseudodifferential operator. The homogeneous principal symbol is multiplicative under composition. Of course, any projective pseudodifferential operator can be written as a sum of two operators, where one is smoothing and the other is supported near the diagonal. Summarizing, the coset space

$$L^*_{\mathrm{cl}}(\mathcal{U},F) := \Psi\mathrm{DO}^*_{\mathrm{cl}}(\mathcal{U},F)/\Psi\mathrm{DO}^{-\infty}(\mathcal{U},F) \tag{3.2}$$

is a filtered $*$-algebra. As in the standard case, asymptotic summations of sequences of projective operators of one-step decreasing orders are possible and parametrices (i.e., inverses modulo smoothing remainders) to elliptic elements (i.e., those having a pointwise invertible principal symbol) can be constructed.

The next theorem introduces the *noncommutative residue* or *Wodzicki residue* in the context of projective pseudodifferential operators, extending the well-known constructions for standard pseudodifferential operators, cf. [**FGLS96**]. Due to its uniqueness, it coincides with the one introduced in [**MMS06**].

Theorem 3.3. *Let F be a $*$-convolution bundle and let A be a projective pseudodifferential operator. For $x \in X$, define*

$$\mathrm{WRes}_x(A) := \int_{S^*_x X} \mathrm{tr}\, a_{-n}(x,\xi)\, d\sigma(\xi)\, dx,$$

where $a_{-n}(x,\xi)$, $n = \dim X$, is the homogeneous component of order $-n$ of a symbol of a local representative A_α with $x \in \mathcal{O}_\alpha$, cf. Definition 3.1. *Then* $\mathrm{WRes}_x(A)$ *is well-defined and defines a global density on X. Moreover,*

$$\mathrm{WRes}(A) := \int_X \mathrm{WRes}_x(A)$$

defines a trace functional on the algebra $L^*_{\mathrm{cl}}(\mathcal{U},F)$. Up to a multiplicative constant, WRes *is the unique functional on $L^*_{\mathrm{cl}}(\mathcal{U},F)$ that vanishes both on commutators as well as on elements of order at most $-n$.*

Proof. Let A_β be another local representative and $x \in \mathcal{O}_\beta$. Fixing local coordinates on $\mathcal{O}_\alpha \cap \mathcal{O}_\beta$, the local symbols a_α and a_β are related by the asymptotic expansion (3.1). Following the proof in [**FGLS96**], terms containing a derivative ∂^γ_ξ, $|\gamma| \geq 1$, vanish under integration. We thus obtain the same value for $\mathrm{WRes}_x(A)$ using either $a_\alpha(x,\xi)$ or $a_\beta(x,\xi)$. That $\mathrm{WRes}_x(A)$ transforms as a density under changes of coordinates is seen as in the standard case, cf. [**FGLS96**].

To see that the integral of the residue density defines a trace functional, we need to show that it vanishes on commutators $[A,B]$. To this end, fix a cover $\{U'_\sigma\}$ of X by coordinate maps together with a subordinate partition of unity, such that $U'_\sigma \cup U'_\rho$ is contained in some U_α whenever $U'_\sigma \cap U'_\rho$ is not empty. We then can write $A = \sum_\sigma A_\sigma$ and

$B = \sum_\sigma B_\sigma$ modulo smoothing operators, where the A_σ and B_σ are supported in $\mathcal{O}'_\sigma := U'_\sigma \times U'_\sigma$. Then the commutator $[A, B]$ can be written as a sum of terms $[A_\sigma, B_\rho]$. Such a commutator is smoothing if $\mathcal{O}'_\sigma \cap \mathcal{O}'_\rho$ is empty. Otherwise it is contained in some set $\mathcal{O}_\alpha$. Therefore the calculation reduces to a local one, which is not different from the one for usual pseudodifferential operators that can be found in [**FGLS96**]. Also the uniqueness on the class of projective pseudodifferential operators supported in some set $\mathcal{O}_\alpha$ is verified as in this paper. Using a partition of unity, the uniqueness of the noncommutative residue follows. q.e.d.

4. The noncommutative residue in twisted K-theory

4.1. Twisted K-theory. Suppose that $\mathcal{A}$ is an Azumaya bundle over a compact manifold X. The twisted K-theory is defined to be the K-theory of the C^*-algebra of continuous sections $C(X; \mathcal{A})$ of $\mathcal{A}$.

If $Y \subset X$ is a closed subset, then the set of sections $C(X, Y; \mathcal{A})$ vanishing on Y is a closed two-sided ideal in $C(X, Y; \mathcal{A})$ and the quotient by this ideal can be identified with the space of continuous sections $C(Y; \mathcal{A})$ of the Azumaya bundle $\mathcal{A}|_Y$. We therefore have the six term exact sequence as a consequence of the six term exact sequence in the theory of C^*-algebras,

$$\begin{array}{ccccc} K^0(X, Y; \mathcal{A}) & \longrightarrow & K^0(X; \mathcal{A}) & \longrightarrow & K^0(Y; \mathcal{A}) \\ \uparrow & & & & \downarrow \\ K^1(Y; \mathcal{A}) & \longrightarrow & K^1(X; \mathcal{A}) & \longrightarrow & K^1(X, Y; \mathcal{A}) \end{array}$$

where the relative K-groups $K^*(X, Y; \mathcal{A})$ are defined as $K_*(C(X, Y; \mathcal{A}))$. There is a natural map

$$K_*(C(X, Y; \mathcal{A})) \otimes_{\mathbb{Z}} K_*(C(X)) \mapsto K_*(C(X, Y; \mathcal{A}) \mathbin{\widehat{\otimes}} C(X)).$$

Here $\widehat{\otimes}$ is the tensor product of C^*-algebras, which is well defined in this case as $C(X)$ is nuclear. The usual multiplication

$$C(X, Y; \mathcal{A}) \mathbin{\widehat{\otimes}} C(X) \to C(X, Y; \mathcal{A})$$

induces a map $K_*(C(X, \mathcal{A}) \mathbin{\widehat{\otimes}} C(X)) \to K_*(C(X, \mathcal{A}))$. The composition of these two maps makes $K^*(X, Y; \mathcal{A})$ a module over the $\mathbb{Z}_2$-graded ring $K^*(X)$. Choosing $Y = \emptyset$ defines a $K^*(X)$ module structure on $K^*(X; \mathcal{A})$. Note that the morphisms in the six term exact sequence are module homomorphisms.

These observations can be used to prove the following Leray-Hirsch theorem:

Theorem 4.1. *Let R be a commutative torsion-free ring. Suppose that $\pi : M \xrightarrow{F} X$ is a compact smooth fibre bundle with fibre F over X and let $\mathcal{A}$ be an Azumaya bundle over X. Assume that $K^*(F) \otimes_{\mathbb{Z}} R$ is a* *free R-module and suppose there exist elements $c_1, \dots, c_N \in K^*(M) \otimes_{\mathbb{Z}} R$ such that the $c_j|_{M_x}$ form a basis for $K^*(M_x) \otimes_{\mathbb{Z}} R$ for every $x \in X$. Then the following map is an isomorphism*:

$$K^*(X;\mathcal{A}) \otimes_{\mathbb{Z}} R^N \longrightarrow K^*(M, \pi^*(\mathcal{A})) \otimes_{\mathbb{Z}} R, \quad (p,\alpha) \mapsto \sum_{j=1}^{N} \alpha_j \pi^*(p) \cdot c_j.$$

Indeed, the usual proof of the Leray-Hirsch theorem in topological K-theory (see e.g. [**H09**], Theorem 2.25) can be adapted to our setting. If $Y \subset X$ is a closed subset of X, we have the following diagram (where, for notational convenience, we shall use abbreviations $K^*_{R^N}(\dots) = K^*(\dots) \otimes_{\mathbb{Z}} R^N$):

$$\begin{array}{ccccccc}
\longrightarrow K^*_{R^N}(X,Y;\mathcal{A}) & \longrightarrow & K^*_{R^N}(X;\mathcal{A}) & \longrightarrow & K^*_{R^N}(Y;\mathcal{A}) \longrightarrow \\
\downarrow \Phi & & \downarrow \Phi & & \downarrow \Phi \\
\longrightarrow K^*_R(\pi^{-1}X, \pi^{-1}Y;\mathcal{A}) & \longrightarrow & K^*_R(\pi^{-1}X;\mathcal{A}) & \longrightarrow & K^*_R(\pi^{-1}Y;\mathcal{A}) \longrightarrow
\end{array}$$

with Φ being defined as in the theorem, $\Phi(p,\alpha) = \sum\limits_{j=1}^{N} \alpha_j \pi^*(p) \cdot c_j$. The rows of this diagram are exact since tensoring with R^N and R is an exact functor. All maps in the six term exact sequence are natural and therefore the pull back π^* commutes with them. Moreover, the maps in the six term exact sequence for the pair $(\pi^{-1}X, \pi^{-1}Y)$ are $K^*(M)$ module homomorphisms. Thus, the diagram commutes. Since X is a finite cell complex, one can proceed in the usual way using the 5-lemma and induction in the number of cells and the dimension to prove the theorem.

4.2. The noncommutative residue. In this section we show that the value of the Wodzicki residue of a projection in $L^0_{\mathrm{cl}}(\mathcal{U}, F)$ depends only on the class of its principal symbol in $K_0(C(S^*X, \pi^*\mathcal{A}))$. Therefore, the Wodzicki residue induces a map in twisted K-theory.

Proposition 4.2. *Let $\mathcal{A}$ be the Azumaya bundle obtained by restricting a $*$-convolution bundle F to the diagonal. The noncommutative residue from Theorem* 3.3 *descends to a group homomorphism*

$$\mathrm{WRes} : K^0(S^*X, \pi^*\mathcal{A}) \to \mathbb{C}, \tag{4.1}$$

where $\pi : S^*X \to X$ denotes the co-sphere bundle over X.

Proof. We have the usual exact sequence

$$0 \longrightarrow L^{-1}_{\mathrm{cl}}(\mathcal{U}, F) \longrightarrow L^0_{\mathrm{cl}}(\mathcal{U}, F) \longrightarrow C^\infty(S^*X, \pi^*\mathcal{A}) \longrightarrow 0$$

where the last map is the symbol map. Define

$$A := L^0_{\mathrm{cl}}(\mathcal{U}, F)/L^{-n-1}_{\mathrm{cl}}(\mathcal{U}, F), \qquad I := L^{-1}_{\mathrm{cl}}(\mathcal{U}, F)/L^{-n-1}_{\mathrm{cl}}(\mathcal{U}, F).$$

Then A/I is isomorphic to $C^\infty(S^*X, \pi^*\mathcal{A})$ via the symbol map. By definition, the Wodzicki residue is a trace on A and therefore its restriction to projections descends to a map $K_0(A) \to \mathbb{C}$. The ideal I is a nilpotent ideal in A and it is well known that this implies that the natural map $K_0(A) \to K_0(A/I)$ is an isomorphism, cf. [**B68**]. Thus, the Wodzicki residue descends to a map from $K_0(C^\infty(S^*X, \pi^*\mathcal{A}))$ to $\mathbb{C}$. Finally, the natural inclusion of the K-theory of the local C^*-algebra $C^\infty(S^*X, \pi^*\mathcal{A})$ into that of $C(S^*X, \pi^*\mathcal{A})$ is an isomorphism, cf. [**Bl98**]. q.e.d.

5. Twisted Dirac operators and connections

Let $\mathcal{A}$ be the Azumaya bundle obtained by restricting a $*$-convolution bundle F to the diagonal.

Definition 5.1. A projective connection $\nabla = \nabla^F$ on F is a linear map

$$Y \mapsto \nabla_Y : \quad C^\infty(X; TX) \longrightarrow \mathrm{Diff}^1(\mathcal{U}; F)$$

satisfying, for any vector field $Y \in C^\infty(X, TX)$ and any function $f \in C^\infty(X)$,

(1) $\nabla_{fY} = f\nabla_Y$,
(2) $[\nabla_Y, f] = Yf$ for any $f \in C^\infty(X)$.

It is called a hermitian connection if, additionally,

(3) $\nabla_Y^* + \nabla_Y + \operatorname{div} Y = 0$

(here, f and $\operatorname{div} Y$ are considered as elements of $\mathrm{Diff}^0(\mathcal{U}; F)$).

Note that in case $\mathcal{U} = X \times X$ and $F = E \boxtimes E^*$ for a vector bundle E over X, we just recover a usual hermitian connection on E. One can always construct a projective hermitian connection from local hermitian connections by gluing with a partition of unity.

If $\nabla = \nabla^F$ is a projective connection and ϕ_α is a local trivialization of F over $U_\alpha \times U_\alpha$ as described in Remark 2.3, the corresponding local differential operator

$$\nabla_Y^\alpha \in \mathrm{Diff}^1(U_\alpha, \mathrm{End}(\mathbb{C}^k))$$

is of the form

$$\nabla_Y^\alpha = Y + \Gamma_Y^\alpha(x), \qquad \Gamma_Y^\alpha \in C^\infty(U_\alpha, M_k(\mathbb{C})).$$

If we use another trivialization ϕ_β of F on $U_\beta \times U_\beta$, we have the relation

$$\Gamma_Y^\beta(x) = \phi_{\alpha\beta}(x,x)(\Gamma_Y^\alpha(x)) + Y_y \phi_{\alpha\beta}(x,y)(\mathbf{1})\big|_{y=x}, \qquad x \in U_\alpha \cap U_\beta,$$

where $\mathbf{1}$ is the identity matrix. Thus, in analogy to the theory of standard connections, we may describe projective connections by 'connection matrices' Γ_Y^α associated to a covering $X = \cup_\alpha U_\alpha$ satisfying the above compatibility relations. For a hermitian connection, the connection matrices also have to be skew-symmetric, $\Gamma_Y^\alpha(x)^* = -\Gamma_Y^\alpha(x)$.

Suppose now S is a Clifford module over X and let γ denote the Clifford multiplication. Moreover, let ∇^S be a connection on S which is compatible with the Clifford structure. Writing $\widetilde{F} := S \boxtimes S^*$ it is easy to see that $F \otimes \widetilde{F}$ is a $*$-convolution bundle over $\mathcal{U}$, and we can define the projective hermitian connection

$$\nabla := \nabla^F \otimes 1 + 1 \otimes \nabla^S$$

by choosing the corresponding connection matrices as

$$\Gamma_Y^{F,\alpha}(x) \otimes 1 + 1 \otimes \Gamma_Y^{S,\alpha}(x), \qquad x \in U_\alpha,$$

where the U_α are chosen in such a way that both F and $\widetilde{F}$ are locally trivial over $U_\alpha \times U_\alpha$. Then we can define the twisted Dirac operator

$$D := (1 \otimes \gamma) \circ \nabla \in \mathrm{Diff}^1(\mathcal{U}; F \otimes \widetilde{F});$$

in fact, in each local trivialization ∇ is a usual hermitean connection and we can compose it locally with $1 \otimes \gamma$.

6. Vanishing of the Wodzicki residue

Theorem 6.1. *If X is an odd dimensional oriented manifold, the map* WRes *of* (4.1) *vanishes identically.*

Consequently, $\mathrm{WRes}(P) = 0$ for any projection $P \in L^0_{\mathrm{cl}}(\mathcal{U}, F)$.

Proof of Theorem 6.1. Suppose the dimension of X is $n = 2\ell - 1$. Let us denote by $S = \bigoplus_{k \text{ even}} \Lambda^k(T^*X)$ the bundle of even-degree forms over X, let $* : \Lambda^k(T^*X) \to \Lambda^{n-k}(T^*X)$ be the Hodge star operator, and denote by d and δ the exterior differential and the co-differential, respectively. Define the operator D^S acting on sections of S as

$$D^S = i^\ell * (\delta + (-1)^{k+1} d) \quad \text{on } k\text{-forms.}$$

By Proposition 1.22 and 2.8 in [**BGV04**], this is a generalized Dirac operator, where the Clifford action on S is given by

$$\gamma(\xi) = i^\ell * \Big(\mathrm{int}(\xi) + (-1)^{k+1}\mathrm{ext}(\xi)\Big), \qquad \xi \in T^*X,$$

and the compatible connection is the Levi-Civita connection. The principal symbol of D^S restricted to the co-sphere bundle is a self-adjoint involution and the projection $\sigma_+(D^S) = \frac{1}{2}(\sigma(D^S) + 1)$ onto its $+1$ eigenspace defines an element in $K^0(S^*X)$. It is well known (see for instance [**APS76**]) that restriction of this element to each co-sphere $S^*_x X$ equals 2^ℓ times the Bott element on S^{n-1} which, together with the class of the trivial line bundle, freely generates $K^0(S^{n-1})$.

For notational convenience, denote by $K^*_{\mathbb{R}}(X)$ the groups $K^*(X) \otimes_{\mathbb{Z}} \mathbb{R}$. By Theorem 4.1 applied to the co-sphere bundle of X, any element of $K^0_{\mathbb{R}}(S^*X, \pi^*\mathcal{A})$ can be represented in the form

$$\alpha_0 \pi^*(p) \cdot [\mathbf{1}] + \alpha_1 \pi^*(p) \cdot [\sigma_+(D^S)]$$

for some $\alpha_0, \alpha_1 \in \mathbb{R}$ and some $p \in K^0(X, \mathcal{A})$. Here both the class $[\mathbf{1}]$ of the trivial line bundle and the class $[\sigma_+(D^S)]$ are understood as elements in $K^0_{\mathbb{R}}(S^*X)$. The elements in $\alpha_0 \pi^*(p) \cdot [\mathbf{1}]$ can be represented by projections in $C^\infty(X; M_N(\mathcal{A}))$. Therefore, the noncommutative residue of these elements vanishes. It remains to show that this is also true for the second summand.

To this end, let p be a projection in $M_N(C^\infty(X;\mathcal{A}))$. Let us define the new convolution bundle F_p having fibre $p(x)M_N(F)_{(x,y)}p(y) \subset M_N(F)_{(x,y)}$ in (x, y). We now apply the above construction and build a twisted Dirac operator D_p with respect to $F_p \otimes \widetilde{F}$, $\widetilde{F} = S \boxtimes S^*$. Then $\sigma_+(D_p)$ represents the class $\pi^*([p]) \cdot [\sigma_+(D^S)]$ in $K^0(S^*X, \pi^*\mathcal{A})$.

The projective differential operator D_p can now be used to construct a certain projection $Q \in L^0_{\mathrm{cl}}(\mathcal{U}, F_p)$ whose principal symbol is $\sigma_+(D_p)$ on S^*X. In the case of an invertible Dirac type operator D acting on a vector bundle, the projection would just be the operator $\frac{1}{2}(|D|^{-1}D+1)$. The symbol of this projection can be constructed from a parametrix of D, and this construction is local modulo smoothing operators. That is, the full symbol of $\frac{1}{2}(|D|^{-1}D+1)$ modulo smoothing terms in local coordinates depends only on the full symbol of D in these local coordinates. Thus, the construction can be repeated for the operator D_p to yield an element in $L^0_{\mathrm{cl}}(\mathcal{U}, F_p)$ which we denote by Q or formally $\frac{1}{2}(|D_p|^{-1}D_p+1)$. By construction, $[\sigma(Q)] \in K^0_{\mathbb{R}}(S^*X;\mathcal{A})$ is equal to $\pi^*([p]) \cdot [\sigma_+(D^S)]$.

In [**BG92**] (Theorem 3.4), Branson and Gilkey have used invariant theory to show that the residue *density* of the positive spectral projection for any generalized Dirac operator vanishes identically. Locally, D_p is a generalized Dirac operator and, since the construction of the residue density is local, the residue density of Q vanishes as well. So we can conclude that the noncommutative residue of Q vanishes, which completes our proof. q.e.d.

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