Commun. Math. Phys. 225, 191 – 217 (2002)

Communications in
Mathematical Physics

Weak Hopf Algebras and Singular Solutions of Quantum Yang–Baxter Equation

Fang Li[1,3,⋆], **Steven Duplij**[2]

[1] Department of Mathematics, Zhejiang University (Xixi Campus), Hangzhou, Zhejiang 310028, P.R. China. E-mail: fangli@mail.hz.zj.cn; fli63@yahoo.com
[2] Theoretical Physics, Kharkov National University, Kharkov 61077, Ukraine. E-mail: Steven.A.Duplij@univer.kharkov.ua
[3] Institute of Mathematics, Chinese Academy of Sciences, Beijing 100080, P.R. China

Received: 1 May 2001 / Accepted: 1 September 2001

Abstract: We investigate a generalization of Hopf algebra $\mathfrak{sl}_q(2)$ by weakening the invertibility of the generator K, i.e. exchanging its invertibility $KK^{-1} = 1$ to the regularity $K\overline{K}K = K$. This leads to a weak Hopf algebra $w\mathfrak{sl}_q(2)$ and a J-weak Hopf algebra $v\mathfrak{sl}_q(2)$ which are studied in detail. It is shown that the monoids of group-like elements of $w\mathfrak{sl}_q(2)$ and $v\mathfrak{sl}_q(2)$ are regular monoids, which supports the general conjucture on the connection betweek weak Hopf algebras and regular monoids. Moreover, from $w\mathfrak{sl}_q(2)$ a quasi-braided weak Hopf algebra $\overline{U}_q^w$ is constructed and it is shown that the corresponding quasi-R-matrix is regular $R^w \hat{R}^w R^w = R^w$.

1. Introduction

The concept of a weak Hopf algebra as a generalization of a Hopf algebra [29, 1] was introduced in [18] and its characterizations and applications were studied in [20]. A k-bialgebra[1] $H = (H, \mu, \eta, \Delta, \varepsilon)$ is called a *weak Hopf algebra* if there exists $T \in \mathrm{Hom}_k(H, H)$ such that $id * T * id = id$ and $T * id * T = T$, where T is called a *weak antipode* of H. This concept also generalizes the notion of the left and right Hopf algebras [24, 12].

The first aim of this concept is to give a new sub-class of bialgebras which includes all of Hopf algebras such that it is possible to characterize this sub-class through their monoids of all group-like elements [18, 20]. It was known that for every regular monoid S, its semigroup algebra kS over k is a weak Hopf algebra as the generalization of a group algebra [19].

The second aim is to construct some singular solutions of the quantum Yang-Baxter equation (QYBE) and research QYBE in a larger scope. On this hand, in [20] a quantum quasi-double $D(H)$ for a finite dimensional cocommutative perfect weak Hopf algebra

⋆ Project (No. 19971074) supported by the National Natural Science Foundation of China.

[1] In this paper, k always denotes a field.

with invertible weak antipode was built and it was verified that its quasi-R-matrix is a regular solution of the QYBE. In particular, the quantum quasi-double of a finite Clifford monoid as a generalization of the quantum double of a finite group was derived [20].

In this paper, we will construct two weak Hopf algebras in the other direction as a generalization of the quantum algebra $\mathfrak{sl}_q(2)$ [22,2]. We show that $w\mathfrak{sl}_2(q)$ possesses a quasi-R-matrix which becomes a singular (in fact, regular) solution of the QYBE, with a parameter q. In this reason, we want to treat the meaning of $w\mathfrak{sl}_q(2)$ and its quasi-R-matrix just as $\mathfrak{sl}_q(2)$ [28,16]. It is interesting to note that $w\mathfrak{sl}_q(2)$ is a natural and non-trivial example of weak Hopf algebras.

2. Weak Quantum Algebras

For completeness and consistency we remind the definition of the enveloping algebra $U_q = U_q\left(\mathfrak{sl}(2)\right)$ (see e.g. [16]). Let $q \in \mathbb{C}$ and $q \neq \pm 1, 0$. The algebra U_q is generated by four variables (Chevalley generators) E, F, K, K^{-1} with the relations

$$K^{-1}K = KK^{-1} = 1, \tag{1}$$

$$KEK^{-1} = q^2 E, \tag{2}$$

$$KFK^{-1} = q^{-2} F, \tag{3}$$

$$EF - FE = \frac{K - K^{-1}}{q - q^{-1}}. \tag{4}$$

Now we try to generalize the invertibility condition (1). The first thought is weaken the invertibility to regularity, as it is usually made in semigroup theory [17] (see also [10, 6,7] for higher regularity). So we will consider such weakening the algebra $U_q\left(\mathfrak{sl}_q(2)\right)$, in which instead of the set $\left\{K, K^{-1}\right\}$ we introduce a pair $\left\{K_w, \overline{K}_w\right\}$ by means of the regularity relations

$$K_w \overline{K}_w K_w = K_w, \; \overline{K}_w K_w \overline{K}_w = \overline{K}_w. \tag{5}$$

If $\overline{K}_w$ satisfying (5) is unique for a given K_w, then it is called *inverse of* K_w (see e.g. [27,11]). The regularity relations (5) imply that one can introduce the variables

$$J_w = K_w \overline{K}_w, \qquad \overline{J}_w = \overline{K}_w K_w. \tag{6}$$

In terms of J_w the regularity conditions (5) are

$$J_w K_w = K_w, \qquad \overline{K}_w J_w = \overline{K}_w, \tag{7}$$

$$\overline{J}_w \overline{K}_w = \overline{K}_w, \qquad K_w \overline{J}_w = K_w. \tag{8}$$

Since the noncommutativity of generators K_w and $\overline{K}_w$ very much complexifies the generalized construction[2], we first consider the commutative case and imply in what follow that

$$J_w = \overline{J}_w. \tag{9}$$

[2] This case will be considered elsewhere.

Let us list some useful properties of J_w which will be needed below. First we note that commutativity of K_w and $\overline{K}_w$ leads to idempotency condition

$$J_w^2 = J_w, \tag{10}$$

which means that J_w is a projector (see e.g. [15]).

Conjecture 1. In algebras satisfying the regularity conditions (5) there exists as minimum one zero divisor $J_w - 1$.

Remark 1. In addition with unity 1 we have an idempotent analog of unity J_w which makes the structure of weak algebras more complicated, but simultaneously more interesting.

For any variable X we will define "J-conjugation" as

$$X_{J_w} \stackrel{\text{def}}{=} J_w X J_w \tag{11}$$

and the corresponding mapping will be written as $\mathbf{e}_w(X) : X \to X_{J_w}$. Note that the mapping $\mathbf{e}_w(X)$ is idempotent

$$\mathbf{e}_w^2(X) = \mathbf{e}_w(X). \tag{12}$$

Remark 2. In the invertible case $K_w = K$, $\overline{K}_w = K^{-1}$ we have $J_w = 1$ and $\mathbf{e}_w(X) = X = \text{id}(X)$ for any X, so $\mathbf{e}_w = \text{id}$.

It is seen from (5) that the generators K_w and $\overline{K}_w$ are stable under "J_w-conjugation"

$$K_{J_w} = J_w K_w J_w = K_w, \qquad \overline{K}_{J_w} = J_w \overline{K}_w J_w = \overline{K}_w. \tag{13}$$

Obviously, for any X

$$K_w X \overline{K}_w = K_w X_{J_w} \overline{K}_w, \tag{14}$$

and for any X and Y

$$K_w X \overline{K}_w = Y \Rightarrow K_w X_{J_w} \overline{K}_w = Y_{J_w}. \tag{15}$$

Another definition connected with the idempotent analog of unity J_w is the "J_w-product" for any two elements X and Y, viz.

$$X \odot_{J_w} Y \stackrel{\text{def}}{=} X J_w Y. \tag{16}$$

Remark 3. From (7) it follows that the "J_w-product" coincides with the usual product, if X ends with generators K_w and $\overline{K}_w$ on right side or Y starts with them on left side.

Let $J^{(ij)} = K_w^i \overline{K}_w^j$ then we will need a formula

$$J_w^{(ij)} = K_w^i \overline{K}_w^j = \begin{cases} K_w^{i-j}, & i > j, \\ J_w, & i = j, \\ \overline{K}_w^{j-i}, & i < j, \end{cases} \tag{17}$$

which follows from the regularity conditions (7). The variables $J^{(ij)}$ satisfy the regularity conditions

$$J_w^{(ij)} J_w^{(ji)} J_w^{(ij)} = J_w^{(ij)} \tag{18}$$

and stable under "J-conjugation" (11) $J_{wJ_w}^{(ij)} = J_w^{(ij)}$.

The regularity conditions (7) lead to the noncancellativity: for any two elements X and Y the following relations hold valid:

$$X = Y \Rightarrow K_w X = K_w Y, \tag{19}$$
$$K_w X = K_w Y \nRightarrow X = Y, \tag{20}$$
$$X = Y \Rightarrow \overline{K}_w X = \overline{K}_w Y, \tag{21}$$
$$\overline{K}_w X = \overline{K}_w Y \nRightarrow X = Y, \tag{22}$$
$$X = Y \Rightarrow X_{J_w} = Y_{J_w}, \tag{23}$$
$$X_{J_w} = Y_{J_w} \nRightarrow X = Y. \tag{24}$$

The generalization of $U_q\left(\mathfrak{sl}_q(2)\right)$ by exploiting regularity (5) instead of invertibility (1) can be done in two different ways.

Definition 1. *Define $U_q^w = w\mathfrak{sl}_q(2)$ as the algebra generated by the four variables E_w, F_w, K_w, $\overline{K}_w$ with the relations:*

$$K_w \overline{K}_w = \overline{K}_w K_w, \tag{25}$$
$$K_w \overline{K}_w K_w = K_w, \quad \overline{K}_w K_w \overline{K}_w = \overline{K}_w, \tag{26}$$
$$K_w E_w = q^2 E_w K_w, \quad \overline{K}_w E_w = q^{-2} E_w \overline{K}_w, \tag{27}$$
$$K_w F_w = q^{-2} F_w K_w, \quad \overline{K}_w F_w = q^2 F_w \overline{K}_w, \tag{28}$$
$$E_w F_w - F_w E_w = \frac{K_w - \overline{K}_w}{q - q^{-1}}. \tag{29}$$

We call $w\mathfrak{sl}_q(2)$ a **weak quantum algebra**.

Definition 2. *Define $U_q^v = v\mathfrak{sl}_q(2)$ as the algebra generated by the four variables E_v, F_w, K_v, $\overline{K}_v$ with the relations ($J_v = K_v \overline{K}_v$):*

$$K_v \overline{K}_v = \overline{K}_v K_v, \tag{30}$$
$$K_v \overline{K}_v K_v = K_v, \quad \overline{K}_v K_v \overline{K}_v = \overline{K}_v, \tag{31}$$
$$K_v E_v \overline{K}_v = q^2 E_v, \tag{32}$$
$$K_v F_v \overline{K}_v = q^{-2} F_v, \tag{33}$$
$$E_v J_v F_v - F_v J_v E_v = \frac{K_v - \overline{K}_v}{q - q^{-1}}. \tag{34}$$

We call $v\mathfrak{sl}_q(2)$ a **J-weak quantum algebra**.

In these definitions indeed the first two lines (25)–(26) and (30)–(31) are called to generalize the invertibility $KK^{-1} = K^{-1}K = 1$. Each next line (27)–(29) and (32)–(34) generalizes the corresponding line (2)–(4) in two different ways respectively. In the first almost quantum algebra $w\mathfrak{sl}_q(2)$ the last relation (29) between E and F generators remains unchanged from $\mathfrak{sl}_q(2)$, while two EK and FK relations are extended to four ones (27)–(28). In $v\mathfrak{sl}_q(2)$, oppositely, two EK and FK relations remain unchanged from $\mathfrak{sl}_q(2)$ (with $K^{-1} \to \overline{K}$ substitution only), while the last relation (34) between E and F generators has the additional multiplier J_v which role will be clear later. Note that the EK and FK relations (32)–(33) can be written in the following form close to (27)–(28):

$$K_v E_v J_v = q^2 J_v E_v K_v, \quad \overline{K}_v E_v J_v = q^{-2} J_v E_v \overline{K}_v, \tag{35}$$

$$K_v F_v J_v = q^{-2} J_v F_v K_v, \quad \overline{K}_v F_v J_v = q^2 J_v F_v \overline{K}_v. \tag{36}$$

Using (16) and (7) in the case of J_v we can also present the $v\mathfrak{sl}_q(2)$ algebra as an algebra with the "J_v-product"

$$K_v \odot_{J_v} \overline{K}_v = \overline{K}_v \odot_{J_v} K_v, \tag{37}$$

$$K_v \odot_{J_v} \overline{K}_v \odot_{J_v} K_v = K_v, \quad \overline{K}_v \odot_{J_v} K_v \odot_{J_v} \overline{K}_v = \overline{K}_v, \tag{38}$$

$$K_v \odot_{J_v} E_v \odot_{J_v} \overline{K}_v = q^2 E_v, \tag{39}$$

$$K_v \odot_{J_v} F_v \odot_{J_v} \overline{K}_v = q^{-2} F_v, \tag{40}$$

$$E_v \odot_{J_v} F_v - F_v \odot_{J_v} E_v = \frac{K_v - \overline{K}_v}{q - q^{-1}}. \tag{41}$$

Remark 4. Due to (7) the only relation where the "J_w-product" really plays its role is the last relation (41).

From the following proposition, one can find the connection between $U_q^w = w\mathfrak{sl}_q(2)$, $U_q^v = v\mathfrak{sl}_q(2)$ and the quantum algebra $\mathfrak{sl}_q(2)$.

Proposition 1. $w\mathfrak{sl}_q(2)/(J_w - 1) \cong \mathfrak{sl}_q(2)$; $v\mathfrak{sl}_q(2)/(J_v - 1) \cong \mathfrak{sl}_q(2)$.

Proof. For cancellative K_w and K_v it is obvious. □

Proposition 2. *Quantum algebras $w\mathfrak{sl}_q(2)$ and $v\mathfrak{sl}_q(2)$ possess zero divisors, one of which is*[3] $\left(J_{w,v} - 1\right)$ *which annihilates all generators.*

Proof. From regularity (26) and (31) it follows $K_{w,v}\left(J_{w,v} - 1\right) = 0$ (see also (1)). Multiplying (27) on J_w gives $K_w E_w J_w = q^2 E_w K_w J_w \Rightarrow K_w \left(E_w \overline{K}_w\right) K_w = q^2 E_w K_w$. Using the second equation in (27) for the term in the bracket we obtain $K_w \left(q^2 \overline{K}_w E_w\right) K_w = q^2 E_w K_w \Rightarrow (J_w - 1) E_w K_w = 0$. For F_w similarly, but we use Eq. (28). By analogy, multiplying (32) on J_v we have $K_v E_v \overline{K}_v K_v \overline{K}_v = q^2 E_v J_v \Rightarrow K_v E_v \overline{K}_v = q^2 E_v J_v \Rightarrow q^2 E_v = q^2 E_v J_v$, and so $E_v (J_v - 1) = 0$. For F_v similarly, but we use Eq. (33). □

Remark 5. Since $\mathfrak{sl}_q(2)$ is an algebra without zero divisors, some properties of $\mathfrak{sl}_q(2)$ cannot be upgraded to $w\mathfrak{sl}_q(2)$ and $v\mathfrak{sl}_q(2)$, e.g. the standard theorem of Ore extensions and its proof (see Theorem I.7.1 in [16]).

---

[3] We denote by $X_{w,v}$ one of the variables X_w or X_v.

Remark 6. We conjecture that in U_q^w and U_q^v there are no other than $\left(J_{w,v}-1\right)$ zero divisors which annihilate *all* generators. In other case thorough analysis of them will be much more complicated and very different from the standard case of non-weak algebras.

We can get some properties of U_q^w and U_q^v as follows.

Lemma 1. *The idempotent J_w is in the center of $w\mathfrak{sl}_q(2)$.*

Proof. For K_w it follows from (13). Multiplying the first equation in (27) on $\overline{K}_w$ we derive $K_w\left(E_w\overline{K}_w\right)=q^2E_wJ_w$, and applying the second equation in (27) we obtain $E_wJ_w=J_wE_w$. For F_w similarly, but we use Eq. (28). □

Lemma 2. *There are unique algebra automorphisms ω_w and ω_v of U_q^w and U_q^v respectively such that*

$$\begin{aligned}\omega_{w,v}(K_{w,v})=\overline{K}_{w,v},\ \omega_{w,v}(\overline{K}_{w,v})=K_{w,v},\\ \omega_{w,v}(E_{w,v})=F_{w,v},\ \ \omega_{w,v}(F_{w,v})=E_{w,v}.\end{aligned} \tag{42}$$

Proof. The proof is obvious, if we note that $\omega_w^2=\mathrm{id}$ and $\omega_v^2=\mathrm{id}$. □

As in the case of the automorphism ω for $\mathfrak{sl}_q(2)$ [16], the mappings ω_w and ω_v can be called the *weak Cartan automorphisms*.

Remark 7. Note that $\omega_w\neq\omega$ and $\omega_v\neq\omega$ in general case.

The connection between the algebras $w\mathfrak{sl}_q(2)$ and $v\mathfrak{sl}_q(2)$ can be seen from the following

Proposition 3. *There exist the following partial algebra morphism $\chi : v\mathfrak{sl}_q(2)\to w\mathfrak{sl}_q(2)$ such that*

$$\chi\left(X\right)=\mathbf{e}_v\left(X\right) \tag{43}$$

or more exactly: generators $X_w^{(v)}=J_vX_vJ_v=X_{vJ_v}$ for all $X_v=K_v,\overline{K}_v,E_v,F_v$ satisfy the same relations as X_w (25)–(29).

Proof. Multiplying Eq. (32) on K_v we have $K_vE_v\overline{K}_vK_v=q^2E_vK_v$, and using (7) we obtain $K_vE_vJ_v=q^2E_vJ_vK_v\Rightarrow K_vJ_vE_vJ_v=q^2J_vE_vJ_vK_v$, and so

$$K_{vJ_v}E_{vJ_v}=q^2E_{vJ_v}K_{vJ_v},$$

which has the shape of the first equation in (27). For F_v similarly using Eq. (33) we obtain

$$K_{vJ_v}F_{vJ_v}=q^{-2}F_{vJ_v}K_{vJ_v}.$$

Equation (34) can be modified using (7) and then applying (11), then we obtain

$$E_{vJ_v}F_{vJ_v}-F_{vJ_v}E_{vJ_v}=\frac{K_{vJ_v}-\overline{K}_{vJ_v}}{q-q^{-1}}$$

which coincides with (29).

For conjugated equations (the second ones in (27)–(28)) after multiplication of (32) on $\overline{K}_v$ we have $\overline{K}_v K_v E_v \overline{K}_v = q^2 \overline{K}_v E_v \Rightarrow J_v E_v J_v \overline{K}_v = q^2 \overline{K}_v J_v E_v J_v$ or using definition (11) and (7)

$$\overline{K}_{vJ_v} E_{vJ_v} = q^{-2} E_{vJ_v} \overline{K}_{vJ_v}.$$

By analogy from (33) it follows

$$\overline{K}_{vJ_v} F_{vJ_v} = q^2 F_{vJ_v} \overline{K}_{vJ_v}.$$

□

Note that the generators $X_w^{(v)}$ coincide with X_w if $J_v = 1$ only. Therefore, some (but not all) properties of $w\mathfrak{sl}_q(2)$ can be extended on $v\mathfrak{sl}_q(2)$ as well, and below we mostly will consider $w\mathfrak{sl}_q(2)$ in detail.

Lemma 3. *Let $m \geq 0$ and $n \in \mathbb{Z}$. The following relations hold in U_q^w:*

$$E_w^m K_w^n = q^{-2mn} K_w^n E_w^m, \qquad F_w^m K_w^n = q^{2mn} K_w^n F_w^m, \tag{44}$$

$$E_w^m \overline{K}_w^n = q^{2mn} \overline{K}_w^n E_w^m, \qquad F_w^m \overline{K}_w^n = q^{-2mn} \overline{K}_w^n F_w^m, \tag{45}$$

$$[E_w, F_w^m] = [m] F_w^{m-1} \frac{q^{-(m-1)} K_w - q^{m-1} \overline{K}_w}{q - q^{-1}} = [m] \frac{q^{m-1} K_w - q^{-(m-1)} \overline{K}_w}{q - q^{-1}} F_w^{m-1}, \tag{46}$$

$$[E_w^m, F_w] = [m] \frac{q^{-(m-1)} K_w - q^{m-1} \overline{K}_w}{q - q^{-1}} E_w^{m-1} = [m] E_w^{m-1} \frac{q^{m-1} K_w - q^{-(m-1)} \overline{K}_w}{q - q^{-1}}. \tag{47}$$

Proof. The first two relations result easily from Definition 1. The third one follows by induction using Definition 1 and

$$[E_w, F_w^m] = [E_w, F_w^{m-1}] F_w + F_w^{m-1} [E_w, F_w] = [E_w, F_w^{m-1}] F_w + F_w^{m-1} \frac{K_w - \overline{K}_w}{q - q^{-1}}.$$

Applying the automorphism ω_w (42) to (46), one gets (47). □

Note that the commutation relations (44)–(47) coincide with the $\mathfrak{sl}_q(2)$ case. For $v\mathfrak{sl}_q(2)$ the situation is more complicated, because Eqs. (32)–(33) cannot be solved under $\overline{K}_v$ due to noncancellativity (see also (19)–(24)). Nevertheless, some analogous relations can be derived. Using the morphism (43) one can conclude that the similar relations (44)–(47) hold for $X_w^{(v)} = J_v X_v J_v$, from which we obtain for $v\mathfrak{sl}_q(2)$,

$$J_v E_v^m K_v^n = q^{-2mn} K_v^n E_v^m J_v, \qquad J_v F_v^m K_v^n = q^{2mn} K_v^n F_v^m J_v, \tag{48}$$

$$J_v E_v^m \overline{K}_v^n = q^{2mn} \overline{K}_v^n E_v^m J_v, \qquad J_v F_v^m \overline{K}_v^n = q^{-2mn} \overline{K}_v^n F_v^m J_v, \tag{49}$$

$$J_vE_vJ_vF_v^mJ_v - J_vF_v^mJ_vE_vJ_v = [m]J_vF_v^{m-1}\frac{q^{-(m-1)}K_v - q^{m-1}\overline{K}_v}{q-q^{-1}} \quad (50)$$
$$= [m]\frac{q^{m-1}K_v - q^{-(m-1)}\overline{K}_v}{q-q^{-1}}F_v^{m-1}J_v,$$
$$J_vE_v^mJ_vF_vJ_v - J_vF_vJ_vE_v^mJ_v = [m]\frac{q^{-(m-1)}K_v - q^{m-1}\overline{K}_v}{q-q^{-1}}E_v^{m-1}J_v \quad (51)$$
$$= [m]J_vE_v^{m-1}\frac{q^{m-1}K_v - q^{-(m-1)}\overline{K}_v}{q-q^{-1}}.$$

It is important to stress that due to noncancellativity of weak algebras we cannot cancel these relations on J_v (see (19)–(24)).

In order to discuss the basis of $U_q^w = w\mathfrak{sl}_q(2)$, we need to generalize some properties of Ore extensions (see [16]).

3. Weak Ore Extensions

Let R be an algebra over k and $\mathsf{R}[t]$ be the free left R-module consisting of all polynomials of the form $P = \sum_{i=0}^{n} a_i t^i$ with coefficients in R. If $a_n \neq 0$, define $\deg(P) = n$; say $\deg(0) = -\infty$. Let α be an algebra morphism of R. An *α-derivation* of R is a k-linear endomorphism δ of R such that $\delta(ab) = \alpha(a)\delta(b) + \delta(a)b$ for all $a, b \in \mathsf{R}$. It follows that $\delta(1) = 0$.

Theorem 1. (i) *Assume that $\mathsf{R}[t]$ has an algebra structure such that the natural inclusion of R into $\mathsf{R}[t]$ is a morphism of algebras and $\deg(PQ) \leq \deg(P) + \deg(Q)$ for any pair (P, Q) of elements of $\mathsf{R}[t]$. Then there exists a unique injective algebra endomorphism α of R and a unique α-derivation δ of R such that $ta = \alpha(a)t + \delta(a)$ for all $a \in \mathsf{R}$;*

(ii) *Conversely, given an algebra endomorphism α of R and an α-derivation δ of R, there exists a unique algebra structure on $\mathsf{R}[t]$ such that the inclusion of R into $\mathsf{R}[t]$ is an algebra morphism and $ta = \alpha(a)t + \delta(a)$ for all $a \in \mathsf{R}$.*

Proof. (i) Take any $0 \neq a \in \mathsf{R}$ and consider the product ta. We have $\deg(ta) \leq \deg(t) + \deg(a) = 1$. By the definition of $\mathsf{R}[t]$, there exists uniquely determined elements $\alpha(a)$ and $\delta(a)$ of R such that $ta = \alpha(a)t + \delta(a)$. This defines maps α and δ in a unique fashion. The left multiplication by t being linear, so are α and δ. Expanding both sides of the equality $(ta)b = t(ab)$ in $\mathsf{R}[t]$ using $ta = \alpha(a)t + \delta(a)$ for $a, b \in \mathsf{R}$, we get

$$\alpha(a)\alpha(b)t + \alpha(a)\delta(b) + \delta(a)b = \alpha(ab)t + \delta(ab).$$

It follows that $\alpha(ab) = \alpha(a)\alpha(b)$ and $\delta(ab) = \alpha(a)\delta(b) + \delta(a)b$, and, $\alpha(1)t + \delta(1) = t1 = t$. So, $\alpha(1) = 1$, $\delta(1) = 0$. Therefore, we know that α is an algebra endomorphism and δ is an α-derivation. The uniqueness of α and δ follows from the freeness of $\mathsf{R}[t]$ over R.

(ii) We need to construct the multiplication on $\mathsf{R}[t]$ as an extension of that on R such that $ta = \alpha(a)t + \delta(a)$. For this, it needs only to determine the multiplication ta for any $a \in \mathsf{R}$.

Let $M = \{(f_{ij})_{i,j\geq 1} : f_{ij} \in \mathrm{End}_k(\mathsf{R})$ and each row and each column has only finitely many $f_{ij} \neq 0\}$ and $I = \begin{pmatrix} 1 & & \\ & 1 & \\ & & \ddots \end{pmatrix}$ is the identity of M.

For $a \in \mathsf{R}$, let $\widehat{a} : \mathsf{R} \to \mathsf{R}$ satisfying $\widehat{a}(r) = ar$. Then $\widehat{a} \in \mathrm{End}_k(\mathsf{R})$; and for $r \in \mathsf{R}$, $(\alpha\widehat{a})(r) = \alpha(ar) = \alpha(a)\alpha(r) = (\widehat{\alpha(a)}\alpha)(r)$, $(\delta\widehat{a})(r) = \delta(ar) = \alpha(a)\delta(r) + \delta(a)r = (\widehat{\alpha(a)}\delta + \widehat{\delta(a)})(r)$, thus $\alpha\widehat{a} = \widehat{\alpha(a)}\alpha$, $\delta\widehat{a} = \widehat{\alpha(a)}\delta + \widehat{\delta(a)}$ in $\mathrm{End}_k(\mathsf{R})$, and, obviously, for $a, b \in \mathsf{R}$, $\widehat{ab} = \widehat{a}\widehat{b}$; $\widehat{a+b} = \widehat{a} + \widehat{b}$. □

Let $T = \begin{pmatrix} \delta & & & \\ \alpha & \delta & & \\ & \alpha & \ddots & \\ & & \ddots & \end{pmatrix} \in M$ and define $\Phi : \mathsf{R}[t] \to M$ satisfying $\Phi(\sum_{i=0}^n a_i t^i) = \sum_{i=0}^n (\widehat{a_i} I) T^i$. It is seen that Φ is a k-linear map.

Lemma 4. *The map Φ is injective.*

Proof. Let $p = \sum_{i=0}^n a_i t^i$. Assume $\Phi(p) = 0$.

For $e_i = \begin{pmatrix} 0_1 \\ \vdots \\ 0_{i-1} \\ 1_i \\ 0_{i+1} \\ \vdots \\ 0_n \end{pmatrix}$, obviously, $\{e_i\}_{i\geq 1}$ are linear independent. Since $\delta(1) = 0$ and $\alpha(1) = 1$, we have $Te_i = \begin{pmatrix} 0_1 \\ \vdots \\ 0_{i-1} \\ \delta(1)_i \\ \alpha(1)_{i+1} \\ 0_{i+2} \\ \vdots \\ 0_n \end{pmatrix} = e_{i+1}$ and $T^i e_1 = e_{i+1}$ for any $i \geq 0$. Thus, $0 = \Phi(P)e_1 = \sum_{i=0}^n (\widehat{a_i} I) T^i e_1 = \sum_{i=0}^n \widehat{a_i} e_{i+1}$. It means that $\widehat{a_i} = 0$ for all i, then $a_i = a_i 1 = \widehat{a_i} 1 = 0$. Hence $P = 0$. □

Lemma 5. *The following relation holds $T(\widehat{a}I) = (\widehat{\alpha(a)}I)T + \widehat{\delta(a)}I$.*

Proof. We have

$$T(\widehat{a}I) = \begin{pmatrix} \delta & & & \\ \alpha & \delta & & \\ & \alpha & \ddots & \\ & & \ddots & \end{pmatrix} \begin{pmatrix} \widehat{a} & & \\ & \widehat{a} & \\ & & \ddots \end{pmatrix} = \begin{pmatrix} \widehat{\alpha(a)}\delta + \widehat{\delta(a)} & & & \\ \widehat{\alpha(a)}\alpha & \widehat{\alpha(a)}\delta + \widehat{\delta(a)} & & \\ & \widehat{\alpha(a)}\alpha & \ddots & \\ & & \ddots & \end{pmatrix}$$
$$= \widehat{\alpha(a)}T + \widehat{\delta(a)}I = (\widehat{\alpha(a)}I)T + \widehat{\delta(a)}I.$$

Now, we complete the proof of Theorem 1. Let S denote the subalgebra generated by T and $\widehat{a}I$ (all $a \in \mathsf{R}$) in M. From Lemma 5, we see that every element of S can be generated linearly by some elements in the form as $(\widehat{a}I)T^n$ ($a \in \mathsf{R}$, $n \geq 0$).

But $\Phi(at^n) = (\widehat{a}I)T^n$, so $\Phi(\mathsf{R}[t]) = S$, i.e. Φ is surjective. Then by Lemma 4, Φ is bijective. It follows that $\mathsf{R}[t]$ and S are linearly isomorphic.

Define $ta = \Phi^{-1}(T(\widehat{a}I))$, then we can extend this formula to define the multiplication of $\mathsf{R}[t]$ with $fg = \Phi^{-1}(xy)$ for any $f, g \in \mathsf{R}[t]$ and $x = \Phi(f)$, $y = \Phi(g)$. Under this definition, $\mathsf{R}[t]$ becomes an algebra and Φ is an algebra isomorphism from $\mathsf{R}[t]$ to S, and, $ta = \Phi^{-1}(T(\widehat{a}I)) = \Phi^{-1}((\widehat{\alpha(a)})I)T + \widehat{\delta(a)}I) = \alpha(a)t + \delta(a)$ for all $a \in \mathsf{R}$. Obviously, the inclusion of R into $\mathsf{R}[t]$ is an algebra morphism. □

Remark 8. Note that Theorem 1 can be recognized as a generalization of Theorem I.7.1 in [16], since R does not need to be without zero divisors, α does not need to be injective and only $\deg(PQ) \leq \deg(P) + \deg(Q)$.

Definition 3. *We call the algebra constructed from α and δ a* **weak Ore extension** *of* R, *denoted as* $\mathsf{R}_w[t, \alpha, \delta]$.

Let $S_{n,k}$ be the linear endomorphism of R defined as the sum of all $\binom{n}{k}$ possible compositions of k copies of δ and of $n-k$ copies of α. By induction n, from $ta = \alpha(a)t+\delta(a)$ under the condition of Theorem 1(ii), we get $t^n a = \sum_{k=0}^{n} S_{n,k}(a)t^{n-k}$ and moreover, $\left(\sum_{i=0}^{n} a_i t^i\right)\left(\sum_{i=0}^{m} b_i t^i\right) = \sum_{i=0}^{n+m} c_i t^i$, where $c_i = \sum_{p=0}^{i} a_p \sum_{k=0}^{p} S_{p,k}(b_{i-p+k})$.

Corollary 1. *Under the condition of Theorem 1(ii), the following statements hold:*

(i) *As a left* R*-module,* $\mathsf{R}_w[t, \alpha, \delta]$ *is free with basis* $\{t^i\}_{i\geq 0}$*;*
(ii) *If α is an automorphism, then* $\mathsf{R}_w[t, \alpha, \delta]$ *is also a right free* R*-module with the same basis* $\{t^i\}_{i\geq 0}$.

Proof. (i) It follows from the fact that $\mathsf{R}_w[t, \alpha, \delta]$ is just $\mathsf{R}[t]$ as a left R-module.

(ii) Firstly, we can show that $\mathsf{R}_w[t, \alpha, \delta] = \sum_{i\geq 0} t^i \mathsf{R}$, i.e. for any $p \in \mathsf{R}_w[t, \alpha, \delta]$, there are $a_0, a_1, \cdots, a_n \in \mathsf{R}$ such that $p = \sum_{i=0}^{n} t^i a_i$. Equivalently, we show by induction on n that for any $b \in \mathsf{R}$, bt^n can be in the form $\sum_{i=0}^{n} t^i a_i$ for some a_i.

When $n = 0$, it is obvious. Suppose that for $n \leq k-1$ the result holds. Consider the case $n = k$. Since α is surjective, there is $a \in \mathsf{R}$ such that $b = \alpha^n(a) = S_{n,0}(a)$. But $t^n a = \sum_{k=0}^{n} S_{n,k}(a)t^{n-k}$, we get $bt^n = t^n a - \sum_{k=1}^{n} S_{n,k}(a)t^{n-k} = \sum_{i=0}^{n} t^i a_i$ by the hypothesis of induction for some a_i with $a_n = a$. For any i and $a, b \in \mathsf{R}$, $(t^i a)b = t^i(ab)$ since $\mathsf{R}_w[t, \alpha, \delta]$ is an algebra. Then $\mathsf{R}_w[t, \alpha, \delta]$ is a right R-module.

Suppose $f(t) = t^n a_n + \cdots + ta_1 + A_0 = 0$ for $a_i \in \mathsf{R}$ and $a_n \neq 0$. Then $f(t)$ can be written as an element of $\mathsf{R}[t]$ by the formula $t^n a = \sum_{k=0}^{n} S_{n,k}(a)t^{n-k}$ whose highest degree term is just that of $t^n a_n = \sum_{k=0}^{n} S_{n,k}(a_n)t^{n-k}$, i.e. $\alpha^n(a_n)t^n$. From (i), we get $\alpha^n(a_n) = 0$. It implies $a_n = 0$. It is a contradiction. Hence $\mathsf{R}_w[t, \alpha, \delta]$ is a free right R-module. □

We will need the following:

Lemma 6. *Let* R *be an algebra, α be an algebra automorphism and δ be an α-derivation of* R. *If* R *is a left (resp. right) Noetherian, then so is the weak Ore extension* $\mathsf{R}_w[t, \alpha, \delta]$.

The proof can be made similarly as for Theorem I.8.3 in [16].

Theorem 2. *The algebra $w\mathfrak{sl}_q(2)$ is Noetherian with the basis*

$$\mathsf{P}_w = \{E_w^i F_w^j K_w^l,\ E_w^i F_w^j \overline{K}_w^m,\ E_w^i F_w^j J_w\}, \tag{52}$$

where i, j, l are any non-negative integers, m is any positive integer.

Proof. As is well known, the two-variable polynomial algebra $k[K_w, \overline{K}_w]$ is Noetherian (see e.g. [15]). Then $A_0 = k[K_w, \overline{K}_w]/(J_w K_w - K_w, \overline{K}_w J_w - \overline{K}_w)$ is also Noetherian. For any $i, j \geq 0$ and $a, b, c \in k$, if at least one element of a, b, c does not equal 0, $aK_w^i + b\overline{K}_w^j + cJ_w$ is not in the ideal $(J_w K_w - K_w, \overline{K}_w J_w - \overline{K}_w)$ of $k[K_w, \overline{K}_w]$. So, in A_0, $aK_w^i + b\overline{K}_w^j + cJ_w \neq 0$. It follows that $\{K_w^i, \overline{K}_w^j, J_w : i, j \geq 0\}$ is a basis of A_0.

Let α_1 satisfy $\alpha_1(K_w) = q^2 K_w$ and $\alpha_1(\overline{K}_w) = q^{-2}\overline{K}_w$. Then α_1 can be extended to an algebra automorphism on A_0 and $A_1 = A_0[F_w, \alpha_1, 0]$ is a weak Ore extension of A_0 from $\alpha = \alpha_1$ and $\delta = 0$. By Corollary 1, A_1 is a free left A_0-module with basis $\{F_w^j\}_{i \geq 0}$. Thus, A_1 is a k-algebra with basis $\{K_w^l F_w^j, \overline{K}_w^m F_w^j, J_w F_w^j : l$ and j run respectively over all non-negative integers, m runs over all positive integers$\}$. But, from the definition of the weak Ore extension, we have $K_w^l F_w^j = q^{-2lj} F_w^j K_w^l$, $\overline{K}_w^m F_w^j = q^{2mj} F_w^j \overline{K}_w^m$, $J_w F_w^j = F_w^j J_w$. So, we conclude that $\{F_w^j K_w^l, F_w^j \overline{K}_w^m, F_w^j J_w : l$ and j run respectively over all non-negative integers, m runs over all positive integers$\}$ is a basis of A_1.

Let α_2 satisfy $\alpha_2(F_w^j K_w^l) = q^{-2l} F_w^j K_w^l$, $\alpha_2(F_w^j \overline{K}_w^m) = q^{2m} F_w^j \overline{K}_w^m$, $\alpha_2(F_w^j J_w) = F_w^j J_w$. Then α_2 can be extended to an algebra automorphism on A_1. Let δ satisfy

$$\delta(1) = \delta(K_w) = \delta(\overline{K}_w) = 0,$$

$$\delta(F_w^j K_w^l) = \sum_{i=0}^{j-1} F_w^{j-1} \frac{q^{-2i} K_w - q^{2i}\overline{K}_w}{q - q^{-1}} K_w^l,$$

$$\delta(F_w^j \overline{K}_w^l) = \sum_{i=0}^{j-1} F_w^{j-1} \frac{q^{-2i} K_w - q^{2i}\overline{K}_w}{q - q^{-1}} \overline{K}_w^l,$$

$$\delta\left(F_w^j J_w\right) = \sum_{i=0}^{j-1} F_w^{j-1} \frac{q^{-2i} K_w - q^{2i}\overline{K}_w}{q - q^{-1}} J_w$$

for $j > 0$ and $l \geq 0$. Then just as in the proof of Lemma VI.1.5 in [16], it can be shown that δ can be extended to an α_2-derivation of A_1 such that $A_2 = A_1[E_w, \alpha_2, \delta]$ is a weak Ore extension of A_1. Then in A_2,

$$E_w K_w = \alpha_2(K_w)E_w + \delta(K_w) = q^{-2} K_w E_w, \quad E_w \overline{K}_w = q^2 \overline{K}_w E_w,$$

$$E_w F_w = \alpha_2(F_w)E_w + \delta(F_w) = F_w E_w + \frac{K_w - \overline{K}_w}{q - q^{-1}}.$$

From these, we conclude that $A_2 \cong U_q^w$ as algebras. Thus, from Lemma 6, U_q^w is Noetherian. By Corollary 1, U_q^w is free with basis $\{E_w^i\}_{i \geq 0}$ as a left A_1-module. Thus, as a k-linear space, U_q^w has the basis $\mathsf{Q}_w = \{F_w^j K_w^l E_w^i, F_w^j \overline{K}_w^m E_w^i, F_w^j J_w E_w^i : i, j, l$ run over all non-negative integers, m runs over all positive integers$\}$. By Lemma 3 any $x \in \mathsf{P}_w$ (resp. Q_w) can be k-linearly generated by some elements of Q_w (resp. P_w), and therefore P_w and Q_w generate the same space U_q^w. □

The similar theorem can be proved for $v\mathfrak{sl}_q(2)$ as well.

Theorem 3. *The algebra $v\mathfrak{sl}_q(2)$ is Noetherian with the basis*

$$\mathsf{P}_v = \left\{ J_v E_v^i J_v F_v^j K_v^l,\ J_v E_v^i J_v F_v^j \overline{K}_v^m,\ J_v E_v^i J_v F_v^j J_v \right\}, \tag{53}$$

where i, j, l are any non-negative integers, m is any positive integer.

Proof. The two-variable polynomial algebra $k[K_v, \overline{K}_v]$ is Noetherian (see e.g. [15]). Then $A_0 = k[K_v, \overline{K}_v]/(J_v K_v - K_v, \overline{K}_v J_v - \overline{K}_v)$ is also Noetherian. For any $i, j \geq 0$ and $a, b, c \in k$, if at least one element of a, b, c does not equal 0, $aK_v^i + b\overline{K}_v^j + cJ_v$ is not in the ideal $(J_v K_v - K_v, \overline{K}_v J_v - \overline{K}_v)$ of $k[K_v, \overline{K}_v]$. So, in A_0, $aK_v^i + b\overline{K}_v^j + cJ_v \neq 0$. It follows that $\{K_v^i, \overline{K}_v^j, J_v : i, j \geq 0\}$ is a basis of A_0.

Let α_1 satisfy $\alpha_1(K_v) = q^2 K_v$ and $\alpha_1(\overline{K}_v) = q^{-2}\overline{K}_v$. Then α_1 can be extended to an algebra automorphism on A_0 and $A_1 = A_0[J_v F_v J_v, \alpha_1, 0]$ is a weak Ore extension of A_0 from $\alpha = \alpha_1$ and $\delta = 0$. By Corollary 7, A_1 is a free left A_0-module with basis $\{J_v F_v^j J_v\}_{i\geq 0}$. Thus, A_1 is a k-algebra with basis $\{K_v^l F_v^j J_v, \overline{K}_v^m F_v^j J_v, J_v F_v^j J_v : l$ and j run respectively over all non-negative integers, m runs over all positive integers$\}$. From the definition of the weak Ore extension, we have $K_v^l F_v^j J_v = q^{-2lj} J_v F_v^j K_v^l$, $\overline{K}_v^m F_v^j J_v = q^{2mj} J_v F_v^j \overline{K}_v^m$, $J_v F_v^j = F_v^j J_v$. So, we conclude that $\{F_v^j K_v^l J_v, F_v^j \overline{K}_v^m J_v, J_v F_v^j J_v : l$ and j run respectively over all non-negative integers, m runs over all positive integers$\}$ is a basis of A_1.

Let α_2 satisfy $\alpha_2(J_v F_v^j K_v^l) = q^{-2l} J_v F_v^j K_v^l$, $\alpha_2(J_v F_v^j \overline{K}_v^m) = q^{2m} J_v F_v^j \overline{K}_v^m$, $\alpha_2(J_v F_v^j J_v) = J_v F_v^j J_v$. Then α_2 can be extended to an algebra automorphism on A_1. Let δ satisfy

$$\delta(1) = \delta(K_v) = \delta(\overline{K}_v) = 0,$$

$$\delta(J_v F_v^j K_v^l) = \sum_{i=0}^{j-1} J_v F_v^{j-1} \frac{q^{-2i} K_v - q^{2i}\overline{K}_v}{q - q^{-1}} K_v^l,$$

$$\delta(J_v F_v^j \overline{K}_v^l) = \sum_{i=0}^{j-1} J_v F_v^{j-1} \frac{q^{-2i} K_v - q^{2i}\overline{K}_v}{q - q^{-1}} \overline{K}_v^l,$$

$$\delta\left(J_v F_v^j J_v\right) = \sum_{i=0}^{j-1} J_v F_v^{j-1} \frac{q^{-2i} K_v - q^{2i}\overline{K}_v}{q - q^{-1}} J_v$$

for $j > 0$ and $l \geq 0$. Then just as in the proof of Lemma VI.1.5 in [16], it can be shown that δ can be extended to an α_2-derivation of A_1 such that $A_2 = A_1[J_v E_v J_v, \alpha_2, \delta]$ is a weak Ore extension of A_1. Then in A_2,

$$J_v E_v K_v = \alpha_2(K_v) J_v E_v J_v + \delta(K_v) = q^{-2} K_v E_v J_v,\ \ J_v E_v \overline{K}_v = q^2 \overline{K}_v E_v J_v,$$

$$J_v E_v J_v F_v J_v = \alpha_2(F_v) J_v E_v J_v + \delta(J_v F_v J_v) = J_v F_v J_v E_v J_v + \frac{K_v - \overline{K}_v}{q - q^{-1}}.$$

From these, we conclude that $A_2 \cong U_q^v$ as algebras. Thus, from Lemma 6, U_q^v is Noetherian. By Corollary 1, U_q^v is free with basis $\{J_v E_v^i J_v\}_{i\geq 0}$ as a left A_1-module. Thus, as a k-linear space, U_q^v has the basis

$$\mathsf{Q}_v = \{J_v F_v^j K_v^l E_v^i J_v,\ J_v F_v^j \overline{K}_v^m E_v^i J_v,\ J_v F_v^j J_v E_v^i J_v\},$$

where i, j, l run over all non-negative integers, m runs over all positive integers. By (48)–(51) any $x \in \mathsf{P}_v$ (resp. Q_v) can be k-linearly generated by some elements of Q_v (resp. P_v), and therefore P_v and Q_v generate the same space U_q^v. □

4. Extension to the $q = 1$ Case

Let us discuss the relation between $U_q^w = w\mathfrak{sl}_q(2)$ and $U(\mathfrak{sl}_q(2))$. Just like the quantum algebra $\mathfrak{sl}_q(2)$, we first have to give another presentation for U_q^w.

Let $q \in \mathbb{C}$ and $q \neq \pm 1,0$. Define $U_q^{w\prime}$ as the algebra generated by the five variables $E_w, F_w, K_w, \overline{K}_w, L_v$ with the relations (for $U_q^{v\prime}$ Eqs. (56) and (57) should be exchanged with (32) and (33) respectively):

$$K_w\overline{K}_w = \overline{K}_wK_w, \tag{54}$$

$$K_w\overline{K}_wK_w = K_w, \quad \overline{K}_wK_w\overline{K}_w = \overline{K}_w, \tag{55}$$

$$K_wE_w = q^2E_wK_w, \quad \overline{K}_wE_w = q^{-2}E_w\overline{K}_w, \tag{56}$$

$$K_wF_w = q^{-2}F_wK_w, \quad \overline{K}_wF_w = q^2F_w\overline{K}_w, \tag{57}$$

$$[L_w, E_w] = q(E_wK_w + \overline{K}_wE_w), \tag{58}$$

$$[L_w, F_w] = -q^{-1}(F_wK_w + \overline{K}_wF_w), \tag{59}$$

$$E_wF_w - F_wE_w = L_w, \quad (q - q^{-1})L_w = (K_w - \overline{K}_w). \tag{60}$$

For $v\mathfrak{sl}_q(2)$ we can similarly define the algebra $U_q^{v\prime}$,

$$K_v\overline{K}_v = \overline{K}_vK_v, \tag{61}$$

$$K_v\overline{K}_vK_v = K_v, \quad \overline{K}_vK_v\overline{K}_v = \overline{K}_v, \tag{62}$$

$$K_vE_v\overline{K}_v = q^2E_v, \tag{63}$$

$$K_vF_v\overline{K}_v = q^{-2}F_v, \tag{64}$$

$$L_vJ_vE_v - E_vJ_vL_v = q(E_vK_v + \overline{K}_vE_v), \tag{65}$$

$$L_vJ_vF_v - F_vJ_vL_v = -q^{-1}(F_vK_v + \overline{K}_vF_v), \tag{66}$$

$$E_vJ_vF_v - F_vJ_vE_v = L_v, \ (q - q^{-1})L_v = (K_v - \overline{K}_v). \tag{67}$$

Note that contrary to U_q^w and U_q^v, the algebras $U_q^{w\prime}$ and $U_q^{w\prime}$ are defined for all *invertible* values of the parameter q, in particular for $q = 1$.

Proposition 4. *The algebra U_q^w is isomorphic to the algebra $U_q^{w\prime}$ with φ_w satisfying $\varphi_w(E_w) = E_w, \varphi_w(F_w) = F_w, \varphi_w(K_w) = K_w, \varphi_w(\overline{K}_w) = \overline{K}_w$.*

Proof. The proof is similar to that of Proposition VI.2.1 in [16] for $\mathfrak{sl}_q(2)$. It suffices to check that φ_w and the map $\psi_w : U_q^{w\prime} \to U_q^w$ satisfying $\psi_w(E_w) = E_w, \psi_w(F_w) = F_w$, $\psi_w(K_w) = K_w, \psi_w(L_w) = [E_w, F_w]$ are reciprocal algebra morphisms. □

On the other hand, we can give the following relationship between $U_q^{w\prime}$ and $U(\mathfrak{sl}(2))$ whose proof is easy.

Proposition 5. *For* $q = 1$

(i) *the algebra isomorphism* $U(\mathfrak{sl}(2)) \cong U_1^{w\prime}/(K_w - 1)$ *holds;*
(ii) *there exists an injective algebra morphism* π *from* U_1^w *to* $U(\mathfrak{sl}(2))[K_w]/(K_w^3 - K_w)$ *satisfying* $\pi(E_w) = XK_w, \pi(F_w) = Y, \pi(K_w) = K_w, \pi(L) = HK_w$.

Remark 9. In Proposition 5(ii), π is only injective, but not surjective since $K^2 \neq 1$ in $U(\mathfrak{sl}(2))[K]/(K^3 - K)$ and then X does not lie in the image of π.

5. Weak Hopf Algebras Structure

Here we define weak analogs in $w\mathfrak{sl}_q(2)$ and $v\mathfrak{sl}_q(2)$ for the standard Hopf algebra structures Δ, ε, S – comultiplication, counit and antipod, which should be algebra morphisms.

For the weak quantum algebra $w\mathfrak{sl}_q(2)$ we define the maps $\Delta_w : w\mathfrak{sl}_q(2) \to w\mathfrak{sl}_q(2) \otimes w\mathfrak{sl}_q(2)$, $\varepsilon_w : w\mathfrak{sl}_q(2) \to k$ and $T_w : w\mathfrak{sl}_q(2) \to w\mathfrak{sl}_q(2)$ satisfying respectively

$$\Delta_w(E_w) = 1 \otimes E_w + E_w \otimes K_w,\ \Delta(F_w) = F_w \otimes 1 + \overline{K}_w \otimes F_w, \tag{68}$$

$$\Delta_w(K_w) = K_w \otimes K_w,\ \Delta_w(\overline{K}_w) = \overline{K}_w \otimes \overline{K}_w, \tag{69}$$

$$\varepsilon_w(E_w) = \varepsilon_w(F_w) = 0,\ \varepsilon_w(K_w) = \varepsilon_w(\overline{K}_w) = 1, \tag{70}$$

$$T_w(E_w) = -E_w\overline{K}_w,\ T_w(F_w) = -K_wF_w,\ T(K_w) = \overline{K}_w,\ T_w(\overline{K}_w) = K_w. \tag{71}$$

The difference with the standard case (we follow notations of [16]) is in substitution of K^{-1} with $\overline{K}_w$ and the last line, where instead of antipod S the weak antipod T_w is introduced [18].

Proposition 6. *The relations (68)–(71) endow* $w\mathfrak{sl}_q(2)$ *with a bialgebra structure.*

Proof. It can be shown by direct calculation that the following relations hold valid:

$$\Delta_w(K_w)\Delta_w(\overline{K}_w) = \Delta_w(\overline{K}_w)\Delta_w(K_w), \tag{72}$$

$$\Delta_w(K_w)\Delta_w(\overline{K}_w)\Delta_w(K_w) = \Delta_w(K_w), \tag{73}$$

$$\Delta_w(\overline{K}_w)\Delta_w(K_w)\Delta_w(\overline{K}_w) = \Delta_w(\overline{K}_w), \tag{74}$$

$$\Delta_w(K_w)\Delta_w(E_w) = q^2\Delta_w(E_w)\Delta_w(K_w), \tag{75}$$

$$\Delta_w(\overline{K}_w)\Delta_w(E_w) = q^{-2}\Delta_w(E_w)\Delta_w(\overline{K}_w), \tag{76}$$

$$\Delta_w(K_w)\Delta_w(F_w) = q^{-2}\Delta_w(F_w)\Delta_w(K_w), \tag{77}$$

$$\Delta_w(\overline{K}_w)\Delta_w(F_w) = q^2\Delta_w(F_w)\Delta_w(\overline{K}_w), \tag{78}$$

$$\Delta_w(E_w)\Delta_w(F_w) - \Delta_w(F_w)\Delta_w(E_w) = \frac{(\Delta_w(K_w) - \Delta_w(\overline{K}_w))}{(q - q^{-1})}; \tag{79}$$

$$\varepsilon_w(K_w)\varepsilon_w(\overline{K}_w) = \varepsilon_w(\overline{K}_w)\varepsilon_w(K_w), \tag{80}$$
$$\varepsilon_w(K_w)\varepsilon_w(\overline{K}_w)\varepsilon_w(K_w) = \varepsilon_w(K_w), \tag{81}$$
$$\varepsilon_w(\overline{K}_w)\varepsilon_w(K_w)\varepsilon_w(\overline{K}_w) = \varepsilon_w(\overline{K}_w), \tag{82}$$
$$\varepsilon_w(K_w)\varepsilon_w(E_w) = q^2\varepsilon_w(E_w)\varepsilon_w(K_w), \tag{83}$$
$$\varepsilon_w(\overline{K}_w)\varepsilon_w(E_w) = q^{-2}\varepsilon_w(E_w)\varepsilon_w(\overline{K}_w), \tag{84}$$
$$\varepsilon_w(K_w)\varepsilon_w(F_w) = q^{-2}\varepsilon_w(F_w)\varepsilon_w(K_w), \tag{85}$$
$$\varepsilon_w(\overline{K}_w)\varepsilon_w(F_w) = q^2\varepsilon_w(F_w)\varepsilon_w(\overline{K}_w), \tag{86}$$
$$\varepsilon_w(E_w)\varepsilon_w(F_w) - \varepsilon_w(F_w)\varepsilon_w(E_w) = \frac{(\varepsilon_w(K_w) - \varepsilon_w(\overline{K}_w))}{(q-q^{-1})}; \tag{87}$$

$$T_w(\overline{K}_w)T_w(K_w) = T_w(K_w)T_w(\overline{K}_w), \tag{88}$$
$$T_w(K_w)T_w(\overline{K}_w)T_w(K_w) = T_w(K_w), \tag{89}$$
$$T_w(\overline{K}_w)T_w(K_w)T_w(\overline{K}_w) = T_w(\overline{K}_w), \tag{90}$$
$$T_w(E_w)T_w(K_w) = q^2T_w(K_w)T_w(E_w), \tag{91}$$
$$T_w(E_w)T_w(\overline{K}_w) = q^{-2}T_w(\overline{K}_w)T_w(K_w), \tag{92}$$
$$T_w(F_w)T_w(K_w) = q^{-2}T_w(K_w)T_w(F_w), \tag{93}$$
$$T_w(F_w)T_w(\overline{K}_w) = q^2T_w(\overline{K}_w)T_w(F_w), \tag{94}$$
$$T_w(F_w)T_w(E_w) - T_w(E_w)T_w(F_w) = \frac{(T_w(K_w) - T_w(\overline{K}_w))}{(q-q^{-1})}. \tag{95}$$

Therefore, through the basis in Theorem 2, Δ and ε_w can be extended to algebra morphisms from $w\mathfrak{sl}_q(2)$ to $w\mathfrak{sl}_q(2) \otimes w\mathfrak{sl}_q(2)$ and from $w\mathfrak{sl}_q(2)$ to k, T_w can be extended to an anti-algebra morphism from $w\mathfrak{sl}_q(2)$ to $w\mathfrak{sl}_q(2)$ respectively.

Using (72)–(87) it can be shown that

$$(\Delta_w \otimes \mathrm{id})\Delta_w(X) = (\mathrm{id} \otimes \Delta_w)\Delta_w(X), \tag{96}$$
$$(\varepsilon_w \otimes \mathrm{id})\Delta_w(X) = (\mathrm{id} \otimes \varepsilon_w)\Delta_w(X) = X \tag{97}$$

for any $X = E_w, F_w, K_w$ or $\overline{K}_w$. Let μ_w and η_w be the product and the unit of $w\mathfrak{sl}_q(2)$ respectively. Hence $(w\mathfrak{sl}_q(2), \mu_w, \eta_w, \Delta_w, \varepsilon_w)$ becomes a bialgebra. □ □

Next we introduce the star product in the bialgebra $(w\mathfrak{sl}_q(2), \mu_w, \eta_w, \Delta_w, \varepsilon_w)$ similar to the standard way (see e.g. [16])

$$(A \star_w B)(X) = \mu_w [A \otimes B] \Delta_w(X). \tag{98}$$

Proposition 7. *T_w satisfies the regularity conditions*

$$(\mathrm{id} \star_w T_w \star_w \mathrm{id})(X) = X, \tag{99}$$
$$(T_w \star_w \mathrm{id} \star_w T_w)(X) = T_w(X) \tag{100}$$

for any $X = E_w, F_w, K_w$ or $\overline{K}_w$. It means that T_w is a weak antipode.

Proof. Follows from (72)–(95) by tedious calculations. For $X = K_w,\overline{K}_w$ it is easy, and so we consider $X = E_w$, as an example. We have

$$
\begin{aligned}
&(\mathrm{id} \star_w T_w \star_w \mathrm{id})(E_w) = \mu_w \left[(\mathrm{id} \star_w T_w) \otimes \mathrm{id}\right] \Delta_w(E_w) \\
&\quad = \mu_w \left[(\mathrm{id} \star_w T_w) \otimes \mathrm{id}\right] (1 \otimes E_w + E_w \otimes K_w) \\
&\quad = (\mathrm{id} \star_w T_w)(1)\, \mathrm{id}\,(E_w) + (\mathrm{id} \star_w T_w)(E_w)\, \mathrm{id}\,(K_w) \\
&\quad = \mu_w \left[\mathrm{id} \otimes T_w\right] \Delta_w(1)\, \mathrm{id}\,(E_w) + \mu_w \left[\mathrm{id} \otimes T_w\right] \Delta_w(E_w)\, \mathrm{id}\,(K_w) \\
&\quad = \mu_w \left[\mathrm{id} \otimes T_w\right] (1 \otimes 1)\, \mathrm{id}\,(E_w) + \mu_w \left[\mathrm{id} \otimes T_w\right] (1 \otimes E_w + E_w \otimes K_w)\, \mathrm{id}\,(K_w) \\
&\quad = T_w\,(1)\, \mathrm{id}\,(E_w) + \mathrm{id}\,(1)\, T_w\,(E_w)\, \mathrm{id}\,(K_w) + \mathrm{id}\,(E_w)\, T_w\,(K_w)\, \mathrm{id}\,(K_w) \\
&\quad = E_w - E_w \overline{K}_w \cdot K_w + E_w \cdot \overline{K}_w \cdot K_w = E_w = \mathrm{id}\,(E_w)\,.
\end{aligned}
$$

By analogy, for (100) and $X = E_w$ we obtain

$$
\begin{aligned}
&(T_w \star_w \mathrm{id} \star_w T_w)(E_w) = \mu_w \left[(T_w \star_w \mathrm{id}) \otimes T_w\right] \Delta_w(E_w) \\
&= \mu_w \left[(T_w \star_w \mathrm{id}) \otimes T_w\right] (1 \otimes E_w + E_w \otimes K_w) \\
&= (T_w \star_w \mathrm{id})\,(1) T_w\,(E_w) + (T_w \star_w \mathrm{id})\,(E_w) T_w\,(K_w) \\
&= \mu_w \left[T_w \otimes \mathrm{id}\right] (1 \otimes 1)\, T_w\,(1 E_w 1) + \mu_w \left[T_w \otimes \mathrm{id}\right] (1 \otimes E_w + E_w \otimes K_w)\, T_w\,(K_w) \\
&= T_w\,(1)\, T_w\,(E_w) + T_w\,(1)\, \mathrm{id}\,(E_w)\, T_w\,(K_w) + T_w\,(E_w)\, \mathrm{id}\,(K_w)\, T_w\,(K_w) \\
&= -E_w \overline{K}_w + E_w \overline{K}_w - E_w \overline{K}_w K_w \overline{K}_w = -E_w \overline{K}_w = T_w(E_w).
\end{aligned}
$$

□

Corollary 2. *The bialgebra $w\mathfrak{sl}_q(2)$ is a weak Hopf algebra with the weak antipode T_w.*

We can get an inner endomorphism as follows:

Proposition 8. *T_w^2 is an inner endomorphism of the algebra $w\mathfrak{sl}_q(2)$ satisfying for any $X \in w\mathfrak{sl}_q(2)$,*

$$T_w^2\,(X) = K_w X \overline{K}_w, \tag{101}$$

especially

$$T_w^2\,(K_w) = \mathrm{id}\,(K_w)\,, \qquad T_w^2\left(\overline{K}_w\right) = \mathrm{id}\left(\overline{K}_w\right). \tag{102}$$

Proof. Follows from (71). □

Assume that with the operations μ_w, η_w, Δ_w, ε_w the algebra $w\mathfrak{sl}_q(2)$ would possess an antipode S so as to become a Hopf algebra, which should satisfy $(S \star_w \mathrm{id})(K_w) = \eta_w \varepsilon_w(K_w)$, and so it should follow that $S(K_w)K_w = 1$. But, it is not possible to hold since $S(K_w)$ can be written as a linear sum of the basis in Theorem 2. It implies that it is impossible for $w\mathfrak{sl}_q(2)$ to become a Hopf algebra for the operations above.

Corollary 3. *$w\mathfrak{sl}_q(2)$ is an example of a non-commutative and non-cocommutative weak Hopf algebra which is not a Hopf algebra.*

In order for $U_q^{w\prime}$ to become a weak Hopf algebra, it is enough to define $\Delta_w(E_w)$, $\Delta_w(F_w)$, $\Delta_w(K_w)$, $\Delta_w(\overline{K}_w)$, $\varepsilon_w(E_w)$, $\varepsilon_w(F_w)$, $\varepsilon_w(K_w)$, $\varepsilon_w(\overline{K}_w)$, $T_w(E_w)$, $T_w(F_w)$, $T_w(K_w)$, $T_w(\overline{K}_w)$ just as in $w\mathfrak{sl}_q(2)$ and define

$$\Delta_w(L_w) = \frac{1}{q-q^{-1}}(K_w \otimes K_w - \overline{K}_w \otimes \overline{K}_w),\ \varepsilon_w(L_w) = 0,\ T_w(L_w) = \frac{\overline{K}_w - K_w}{q-q^{-1}}.$$

From Proposition 4 we conclude that $w\mathfrak{sl}_q(2)$ is isomorphic to the algebra $U_q^{w\prime}$ with φ_w. Moreover, one can see easily that φ_w is an isomorphism of weak Hopf algebras from $w\mathfrak{sl}_q(2)$ to $U_q^{w\prime}$.

For the J-weak quantum algebra $v\mathfrak{sl}_q(2)$ we suppose that some additional J_v should appear even in the definitions of comultiplication and antipod. A thorough analysis gives the following nontrivial definitions:

$$\Delta_v(E_v) = J_v \otimes J_v E_v J_v + J_v E_v J_v \otimes K_v, \tag{103}$$

$$\Delta_v(F_v) = J_v F_v J_v \otimes J_v + \overline{K}_v \otimes J_v F_v J_v, \tag{104}$$

$$\Delta_v(K_v) = K_v \otimes K_v,\ \Delta_v(\overline{K}_v) = \overline{K}_v \otimes \overline{K}_v, \tag{105}$$

$$\varepsilon_v(E_v) = \varepsilon_v(F_v) = 0,\quad \varepsilon_v(K_v) = \varepsilon_v(\overline{K}_v) = 1, \tag{106}$$

$$T_v(E_v) = -J_v E_v \overline{K}_v,\quad T_v(F_v) = -K_v F_v J_v, \tag{107}$$

$$T_v(K_v) = \overline{K}_v,\quad T_v(\overline{K}_v) = K_v. \tag{108}$$

Note that from (105) it follows that

$$\Delta_v(J_v) = J_v \otimes J_v, \tag{109}$$

and so J_v is a group-like element.

Proposition 9. *The relations (103)–(108) endow* $v\mathfrak{sl}_q(2)$ *with a bialgebra structure.*

Proof. First we should prove that Δ_v defines a morphism of algebras from $v\mathfrak{sl}_q(2) \otimes v\mathfrak{sl}_q(2)$ into $v\mathfrak{sl}_q(2)$. We check that

$$\Delta_v(K_v)\Delta_v\left(\overline{K}_v\right) = \Delta_v\left(\overline{K}_v\right)\Delta_v(K_v), \tag{110}$$

$$\Delta_v(K_v)\Delta_v\left(\overline{K}_v\right)\Delta_v(K_v) = \Delta_v(K_v), \tag{111}$$

$$\Delta_v\left(\overline{K}_v\right)\Delta_v(K_v)\Delta_v\left(\overline{K}_v\right) = \Delta_v\left(\overline{K}_v\right), \tag{112}$$

$$\Delta_v(K_v)\Delta_v(E_v)\Delta_v\left(\overline{K}_v\right) = q^2\Delta_v(E_v), \tag{113}$$

$$\Delta_v(K_v)\Delta_v(F_v)\Delta_v\left(\overline{K}_v\right) = q^{-2}\Delta_v(F_v), \tag{114}$$

$$\Delta_v(E_v)\Delta_v(J_v)\Delta_v(F_v) - \Delta_v(F_v)\Delta_v(J_v)\Delta_v(E_v) = \frac{\Delta_v(K_v) - \Delta_v\left(\overline{K}_v\right)}{q-q^{-1}}. \tag{115}$$

The relations (110)–(112) are clear from (105). For (113) we have

$$\begin{aligned}\Delta_v(K_v)\Delta_v(E_v)\Delta_v\left(\overline{K}_v\right) &= (K_v \otimes K_v)(J_v \otimes J_v E_v J_v + J_v E_v J_v \otimes K_v)\left(\overline{K}_v \otimes \overline{K}_v\right)\\ &= J_v \otimes K_v E_v \overline{K}_v + K_v E_v \overline{K}_v \otimes K_v\\ &= q^2(J_v \otimes J_v E_v J_v + J_v E_v J_v \otimes K_v) = q^2\Delta_v(E_v).\end{aligned}$$

Relation (114) is obtained similarly. Next for (115) exploiting (7), (34) and (35)–(36) we derive

$$\begin{aligned}
&\Delta_v\left(E_v\right)\Delta_v\left(J_v\right)\Delta_v\left(F_v\right)-\Delta_v\left(F_v\right)\Delta_v\left(J_v\right)\Delta_v\left(E_v\right)\\
&=\left(J_v\otimes J_vE_vJ_v+J_vE_vJ_v\otimes K_v\right)\left(J_v\otimes J_v\right)\left(J_vF_vJ_v\otimes J_v+\overline{K}_v\otimes J_vF_vJ_v\right)\\
&\quad-\left(J_vF_vJ_v\otimes J_v+\overline{K}_v\otimes J_vF_vJ_v\right)\left(J_v\otimes J_v\right)\left(J_v\otimes J_vE_vJ_v+J_vE_vJ_v\otimes K_v\right)\\
&=J_vF_vJ_v\otimes J_vE_vJ_v-J_vF_vJ_v\otimes J_vE_vJ_v+J_vE_v\overline{K}_v\otimes K_vF_vJ_v\\
&\quad-\overline{K}_vE_vJ_v\otimes J_vF_vK_v+J_vE_vJ_vF_vJ_v\otimes K_v-J_vF_vJ_vE_vJ_v\otimes K_v\\
&\quad+\overline{K}_v\otimes J_vE_vJ_vF_vJ_v-\overline{K}_v\otimes J_vF_vJ_vE_vJ_v\\
&=J_v\left(E_vJ_vF_v-F_vJ_vE_v\right)J_v\otimes K_v+\overline{K}_v\otimes J_v\left(E_vJ_vF_v-F_vJ_vE_v\right)J_v\\
&=J_v\frac{K_v-\overline{K}_v}{q-q^{-1}}J_v\otimes K_v+\overline{K}_v\otimes J_v\frac{K_v-\overline{K}_v}{q-q^{-1}}J_v=\frac{K_v\otimes K_v-\overline{K}_v\otimes\overline{K}_v}{q-q^{-1}}\\
&=\frac{\Delta_v\left(K_v\right)-\Delta_v\left(\overline{K}_v\right)}{q-q^{-1}}.
\end{aligned}$$

Then we show that $\Delta_v\left(X\right)$ is coassociative

$$\left(\Delta_v\otimes\mathrm{id}\right)\Delta_v\left(X\right)=\left(\mathrm{id}\otimes\Delta_v\right)\Delta_v\left(X\right). \tag{116}$$

Take E as an example. On the one hand

$$\begin{aligned}
\left(\Delta_v\otimes\mathrm{id}\right)\Delta_v\left(E\right)&=\left(\Delta_v\otimes\mathrm{id}\right)\left(J_v\otimes J_vE_vJ_v+J_vE_vJ_v\otimes K_v\right)\\
&=\Delta_v\left(J_v\right)\otimes J_vE_vJ_v+\Delta_v\left(J_v\right)\Delta_v\left(E\right)\Delta_v\left(J_v\right)\otimes K_v\\
&=J_v\otimes J_v\otimes J_vE_vJ_v+J_v\otimes J_vE_vJ_v\otimes K_v+J_vE_vJ_v\otimes K_v\otimes K_v.
\end{aligned}$$

On the other hand

$$\begin{aligned}
\left(\mathrm{id}\otimes\Delta_v\right)\Delta_v\left(E\right)&=\left(\mathrm{id}\otimes\Delta_v\right)\left(J_v\otimes J_vE_vJ_v+J_vE_vJ_v\otimes K_v\right)\\
&=J_v\otimes\Delta_v\left(J_v\right)\Delta_v\left(E\right)\Delta_v\left(J_v\right)+J_vE_vJ_v\otimes\Delta_v\left(K_v\right)\\
&=J_v\otimes J_v\otimes J_vE_vJ_v+J_v\otimes J_vE_vJ_v\otimes K_v+J_vE_vJ_v\otimes K_v\otimes K_v,
\end{aligned}$$

which coincides with the previous example.

The proof that the counit ε defines a morphism of algebras from $v\mathfrak{sl}_q(2)$ onto k is straightforward and the result has the form

$$\varepsilon_v\left(K_v\right)\varepsilon_v\left(\overline{K}_v\right)=\varepsilon_v\left(\overline{K}_v\right)\varepsilon_v\left(K_v\right), \tag{117}$$

$$\varepsilon_v\left(K_v\right)\varepsilon_v\left(\overline{K}_v\right)\varepsilon_v\left(K_v\right)=\varepsilon_v\left(K_v\right), \tag{118}$$

$$\varepsilon_v\left(\overline{K}_v\right)\varepsilon_v\left(K_v\right)\varepsilon_v\left(\overline{K}_v\right)=\varepsilon_v\left(\overline{K}_v\right), \tag{119}$$

$$\varepsilon_v\left(K_v\right)\varepsilon_v\left(E_v\right)\varepsilon_v\left(\overline{K}_v\right)=q^2\varepsilon_v\left(E_v\right), \tag{120}$$

$$\varepsilon_v\left(K_v\right)\varepsilon_v\left(F_v\right)\varepsilon_v\left(\overline{K}_v\right)=q^{-2}\varepsilon_v\left(F_v\right), \tag{121}$$

$$\varepsilon_v\left(E_v\right)\varepsilon_v\left(J_v\right)\varepsilon_v\left(F_v\right)-\varepsilon_v\left(F_v\right)\varepsilon_v\left(J_v\right)\varepsilon_v\left(E_v\right)=\frac{\varepsilon_v\left(K_v\right)-\varepsilon_v\left(\overline{K}_v\right)}{q-q^{-1}}. \tag{122}$$

Moreover, it can be shown that

$$(\varepsilon_v \otimes \mathrm{id})\Delta_v(X) = (\mathrm{id} \otimes \varepsilon_v)\Delta_v(X) = X$$

for $X = E_v, F_v, K_v, \overline{K}_v$.

Further we check that T_v defines an anti-morphism of algebras from $v\mathfrak{sl}_q(2)$ to $v\mathfrak{sl}_q^{\mathrm{op}}(2)$ as follows:

$$T_v\left(K_v\right) T_v\left(\overline{K}_v\right) = T_v\left(\overline{K}_v\right) T_v\left(K_v\right), \tag{123}$$

$$T_v\left(K_v\right) T_v\left(\overline{K}_v\right) T_v\left(K_v\right) = T_v\left(K_v\right), \tag{124}$$

$$T_v\left(\overline{K}_v\right) T_v\left(K_v\right) T_v\left(\overline{K}_v\right) = T_v\left(\overline{K}_v\right), \tag{125}$$

$$T_v\left(\overline{K}_v\right) T_v\left(E_v\right) T_v\left(K_v\right) = q^2 T_v\left(E_v\right), \tag{126}$$

$$T_v\left(\overline{K}_v\right) T_v\left(F_v\right) T_v\left(K_v\right) = q^{-2} T_v\left(F_v\right), \tag{127}$$

$$T_v\left(F_v\right) T_v\left(J_v\right) T_v\left(E_v\right) - T_v\left(E_v\right) T_v\left(J_v\right) T_v\left(F_v\right) = \frac{T_v\left(K_v\right) - T_v\left(\overline{K}_v\right)}{q - q^{-1}}. \tag{128}$$

The first three relations are obvious. For (126) using (107) and (35) we have

$$\begin{aligned} T_v\left(\overline{K}_v\right) T_v\left(E_v\right) T_v\left(K_v\right) &= K_v\left(-J_v E_v \overline{K}_v\right)\overline{K}_v = -q^2 K_v\left(-\overline{K}_v E_v J_v\right)\overline{K}_v \\ &= -q^2 J_v E_v J_v \overline{K}_v = q^2 J_v E_v \overline{K}_v = q^2 T_v\left(E_v\right). \end{aligned}$$

For the last relation (128), using (35)–(36), we obtain

$$\begin{aligned} & T_v\left(F_v\right) T_v\left(J_v\right) T_v\left(E_v\right) - T_v\left(E_v\right) T_v\left(J_v\right) T_v\left(F_v\right) \\ &= \left(K_v F_v J_v\right) J_v\left(-J_v E_v \overline{K}_v\right) - \left(-J_v E_v \overline{K}_v\right) J_v\left(K_v F_v J_v\right) \\ &= J_v\left(F_v J_v E_v - E_v J_v F_v\right) J_v = J_v \frac{\overline{K}_v - K_v}{q - q^{-1}} J_v = \frac{T_v\left(K_v\right) - T_v\left(\overline{K}_v\right)}{q - q^{-1}}. \end{aligned}$$

Therefore, we conclude that $\left(v\mathfrak{sl}_q(2), \mu_v, \eta_v, \Delta_v, T_v\right)$ has the structure of a bialgebra. □

The following property of T_v is crucial for understanding the structure of the bialgebra $\left(v\mathfrak{sl}_q(2), \mu_v, \eta_v, \Delta_v, T_v\right)$.

Proposition 10. *For any $X \in v\mathfrak{sl}_q(2)$ we have (cf. (101)–(102))*

$$T_v^2\left(K_v\right) = \mathbf{e}_v\left(K_v\right), \; T_v^2\left(\overline{K}_v\right) = \mathbf{e}_v\left(\overline{K}_v\right), \tag{129}$$

$$T_v^2\left(E_v\right) = K_v E_v \overline{K}_v, \; T_v^2\left(F_v\right) = K_v F_v \overline{K}_v, \tag{130}$$

where $\mathbf{e}_v(X)$ is defined in (11).

Proof. Follows from (7) and (107)–(108). As an example for E_v we have $T_v^2\left(E_v\right) = T_v\left(-J_v E_v \overline{K}_v\right) = -T_v\left(\overline{K}_v\right) T_v\left(E_v\right) T_v\left(J_v\right) = K_v\left(J_v E_v \overline{K}_v\right) J_v = K_v E_v \overline{K}_v$. □

The star product in $\left(v\mathfrak{sl}_q(2), \mu_v, \eta_v, \Delta_v, T_v\right)$ has the form

$$\left(A \star_v B\right)(X) = \mu_v\left[A \otimes B\right]\Delta_v(X). \tag{131}$$

Proposition 11. *T_v satisfies the regularity conditions*

$$(\mathbf{e}_v \star_v \ T_v \star_v \ \mathbf{e}_v)(X) = \mathbf{e}_v\,(X)\,, \tag{132}$$

$$(T_v \star_v \ \mathbf{e}_v \star_v \ T_v)(X) = T_v(X) \tag{133}$$

for any $X = E_v, F_v, K_v$ or $\overline{K}_v$.

Proof. Follows from (103)–(108) and (131). For $X = K_v,\overline{K}_v$ it is easy, and so we consider $X = E_v$, as an example. We have

$$\begin{aligned}
&(\mathbf{e}_v \star_v \ T_v \star_v \ \mathbf{e}_v)(E_v) = \mu_v\,[(\mathbf{e}_v \star_v \ T_v) \otimes \mathbf{e}_v]\,\Delta_v(E_v)\\
&\ = \mu_v\,[(\mathbf{e}_v \star_v \ T_v) \otimes \mathbf{e}_v]\,(J_v \otimes J_v E_v J_v + J_v E_v J_v \otimes K_v)\\
&\ = \ (\mathbf{e}_v \star_v \ T_v)\,(J_v)\,\mathbf{e}_v\,(J_v E_v J_v) + (\mathbf{e}_v \star_v \ T_v)\,(J_v E_v J_v)\,\mathbf{e}_v\,(K_v)\\
&\ = \mu_v\,[\mathbf{e}_v \otimes T_v]\,\Delta_v(J_v)\mathbf{e}_v\,(J_v E_v J_v) + \mu_v\,[\mathbf{e}_v \otimes T_v]\,\Delta_v(E_v)\mathbf{e}_v\,(K_v)\\
&\ = \mu_v\,[\mathbf{e}_v \otimes T_v]\,(J_v \otimes J_v)\,\mathbf{e}_v\,(E_v)\\
&\qquad + \mu_v\,[\mathbf{e}_v \otimes T_v]\,(J_v \otimes J_v E_v J_v + J_v E_v J_v \otimes K_v)\,\mathbf{e}_v\,(K_v)\\
&\ = \mathbf{e}_v\,(J_v)\,T_v\,(J_v)\,\mathbf{e}_v\,(E_v) + \mathbf{e}_v\,(J_v)\,T_v\,(J_v E_v J_v)\,\mathbf{e}_v\,(K_v) + \mathbf{e}_v\,(E_v)\,T_v\,(K_v)\,\mathbf{e}_v\,(K_v)\\
&\ = J_v \cdot J_v \cdot J_v E_v J_v - J_v \cdot J_v J_v E_v \overline{K}_v \cdot J_v K_v J_v + J_v E_v J_v \cdot \overline{K}_v \cdot J_v K_v J_v\\
&\ = J_v E_v J_v = \mathbf{e}_v\,(E_v)\,.
\end{aligned}$$

By analogy, for (133) and $X = E_v$ we obtain

$$\begin{aligned}
&(T_v \star_v \ \mathbf{e}_v \star_v \ T_v)(E_v) = \mu_v\,[(T_v \star_v \ \mathbf{e}_v) \otimes T_v]\,\Delta_v(E_v)\\
&\ = \mu_v\,[(T_v \star_v \ \mathbf{e}_v) \otimes T_v]\,(J_v \otimes J_v E_v J_v + J_v E_v J_v \otimes K_v)\\
&\ = \ (T_v \star_v \ \mathbf{e}_v)\,(J_v) T_v\,(J_v E_v J_v) + (T_v \star_v \ \mathbf{e}_v)\,(E_v) T_v\,(K_v)\\
&\ = \mu_v\,[T_v \otimes \mathbf{e}_v]\,(J_v \otimes J_v)\,T_v\,(J_v E_v J_v)\\
&\qquad + \mu_v\,[T_v \otimes \mathbf{e}_v]\,(J_v \otimes J_v E_v J_v + J_v E_v J_v \otimes K_v)\,T_v\,(K_v)\\
&\ = T_v\,(J_v)\,\mathbf{e}_v\,(J_v)\,T_v\,(J_v E_v J_v) + T_v\,(J_v)\,\mathbf{e}_v\,(J_v E_v J_v)\,T_v\,(K_v)\\
&\qquad + T_v\,(J_v E_v J_v)\,\mathbf{e}_v\,(K_v)\,T_v\,(K_v) = -J_v \cdot J_v \cdot J_v \left(J_v E_v \overline{K}_v\right) J_v + J_v \cdot J_v E_v J_v \cdot \overline{K}_v\\
&\qquad - J_v \left(J_v E_v \overline{K}_v\right) J_v \cdot J_v K_v J_v \cdot \overline{K}_v = -J_v E_v \overline{K}_v = T_v(E_v).
\end{aligned}$$

□

From (132)–(133) it follows that $v\mathfrak{sl}_q(2)$ is not a weak Hopf algebra in the definition of [18]. So we will call it a *J-weak Hopf algebra* and T_v a *J-weak antipode*. As it is seen from (99)–(100) and (132)–(133) the difference between them is in the exchange id with $\mathbf{e}_v$.

Remark 10. The variable $\mathbf{e}_v$ can be treated as an $n = 2$ example of the "tower identity" $e_{\alpha\beta}^{(n)}$ introduced for semisupermanifolds in [9,10] or the "obstructor" $\mathbf{e}_X^{(n)}$ for general mappings, categories and the Yang–Baxter equation in [6–8].

Comparing (68)–(71) with (103)–(108) we conclude that the connection of Δ_w, T_w, ε_w and Δ_v, T_v, ε_v can be written in the following way:

$$\Delta_v\,(X) = \Delta_w\,(\mathbf{e}_v\,(X))\,, \tag{134}$$

$$T_v\,(X) = T_w\,(\mathbf{e}_v\,(X))\,, \tag{135}$$

$$\varepsilon_v\,(X) = \varepsilon_w\,(\mathbf{e}_v\,(X))\,, \tag{136}$$

which means that additionally to the partial algebra morphism (43) there exists a partial coalgebra morphism which is described by (134)–(136).

6. Group-Like Elements

Now, we discuss the set $G(w\mathfrak{sl}_q(2))$ of all group-like elements of $w\mathfrak{sl}_q(2)$. As is well-known (see e.g. [14]) a semigroup S is called an inverse semigroup if for every $x \in S$, there exists a unique $y \in S$ such that $xyx = x$ and $yxy = y$, and a monoid is a semigroup with identity. We will show the following

Proposition 12. *The set of all group-like elements* $G(w\mathfrak{sl}_q(2)) = \{J^{(ij)} = K_w^i \overline{K}_w^j : i, j$ *run over all non-negative integers*$\}$, *which forms a regular monoid under the multiplication of* $w\mathfrak{sl}_q(2)$.

Proof. Suppose $x \in w\mathfrak{sl}_q(2)$ is a group-like element, i.e. $\Delta_w(x) = x \otimes x$. By Theorem 2, x can be written as $x = \sum_{i,j,l,m} \alpha_{ijl} E_w^i F_w^j K_w^l + \beta_{ijm} E_w^i F_w^j \overline{K}_w^m + \gamma_{ij} E_w^i F_w^j J_w$. Here and in the sequel, every α, β and γ with subscripts is in the field k and does not equal zero. Then

$$\begin{aligned}\Delta_w(x) &= \sum_{i,j,l,m} [\alpha_{ijl} \Delta_w(E_w^i F_w^j K_w^l) + \Delta_w(\beta_{ijm} E_w^i F_w^j \overline{K}_w^m) + \Delta_w(\gamma_{ij} E_w^i F_w^j J_w)] \\ &= \sum_{i,j,l,m} [\alpha_{ijl}(1 \otimes E_w + E_w \otimes K_w)^i (F_w \otimes 1 + \overline{K}_w \otimes F_w)^j (K_w \otimes K_w)^l \\ &\quad + \beta_{ijm}(1 \otimes E_w + E_w \otimes K_w)^i (F_w \otimes 1 + \overline{K}_w \otimes F_w)^j (\overline{K}_w \otimes \overline{K}_w)^m \\ &\quad + \gamma_{ij}(1 \otimes E_w + E_w \otimes K_w)^i (F_w \otimes 1 + \overline{K}_w \otimes F_w)^j J_w];\end{aligned}$$

and

$$\begin{aligned}x \otimes x = &\Big(\sum_{i,j,l,m} \alpha_{ijl} E_w^i F_w^j K_w^l + \beta_{ijm} E_w^i F_w^j \overline{K}_w^m + \gamma_{ij} E_w^i F_w^j J_w \Big) \\ &\otimes \Big(\sum_{i,j,l,m} \alpha_{ijl} E_w^i F_w^j K_w^l + \beta_{ijm} E_w^i F_w^j \overline{K}_w^m + \gamma_{ij} E_w^i F_w^j J_w \Big).\end{aligned}$$

It is seen that if $i \neq 0$ or $j \neq 0$, $\Delta_w(x)$ is impossible to equal $x \otimes x$. So, $i = 0$ and $j = 0$. We get $x = \sum_{l,m} \alpha_l K_w^l + \beta_m \overline{K}_w^m + J_w$. Then

$$\begin{aligned}\Delta_w(x) &= \sum_{l,m} \big[\alpha_l K_w^l \otimes K_w^l + \beta_m \overline{K}_w^m \otimes \overline{K}_w^m + J_w \otimes J_w\big]; \\ x \otimes x &= \sum_{l,l',m,m'} \big[\alpha_l \alpha_{l'} K_w^l \otimes K_w^{l'} + \alpha_l \beta_{m'} K_w^l \otimes \overline{K}_w^{m'} + \alpha_l K_w^l \otimes J_w \\ &\quad + \alpha_{l'} \beta_m \overline{K}_w^m \otimes K_w^{l'} + \beta_m \beta_{m'} \overline{K}_w^m \otimes \overline{K}_w^{m'} + \beta_m \overline{K}_w^m \otimes J_w \\ &\quad + \alpha_{l'} J_w \otimes K_w^{l'} + \beta_{m'} J_w \otimes \overline{K}_w^{m'} + J_w \otimes J_w\big].\end{aligned}$$

If there exists $l \neq l'$, then $x \otimes x$ possesses the monomial $K_w^l \otimes K_w^{l'}$, which does not appear in $\Delta_w(x)$. It contradicts $\Delta_w(x) = x \otimes x$. Hence we have only a unique l. Similarly, there exists a unique m. Thus $x = \alpha_l K_w^l + \beta_m \overline{K}_w^m + J_w$. Moreover, it is easy to see that $\alpha_l K_w^l$, $\beta_m \overline{K}_w^m$ and J_w can not appear simultaneously in the expression of x. Therefore, we conclude that $x = \alpha_l K_w^l$, $\beta_m \overline{K}_w^m$ or J_w (no summation) and we have

$$\Delta_w(J_w^{(ij)}) = J_w^{(ij)} \otimes J_w^{(ij)}. \tag{137}$$

It follows that $G(w\mathfrak{sl}_q(2)) = \{J_w^{(ij)} = K_w^i \overline{K}_w^j$: i, j run over all non-negative integers$\}$.

For any $J^{(ij)} = K_w^i \overline{K}_w^j \in G(w\mathfrak{sl}_q(2))$, one can find $J^{(ji)} = K_w^j \overline{K}_w^i \in G(w\mathfrak{sl}_q(2))$ such that the regularity (18) takes place $J_w^{(ij)} J_w^{(ji)} J_w^{(ij)} = J_w^{(ij)}$, which means that $G(w\mathfrak{sl}_q(2))$ forms a regular monoid under the multiplication of $w\mathfrak{sl}_q(2)$. □

For $v\mathfrak{sl}_q(2)$ we have a similar statement.

Proposition 13. *The set of all group-like elements* $G(v\mathfrak{sl}_q(2)) = \{J_v^{(ij)} = K_v^i \overline{K}_v^j : i, j$ *run over all non-negative integers*$\}$, *which forms a regular monoid under the multiplication of* $v\mathfrak{sl}_q(2)$.

Proof. Suppose $x \in v\mathfrak{sl}_q(2)$ is a group-like element, i.e. $\Delta_v(x) = x \otimes x$. By Theorem 3, x can be written as $x = \sum_{i,j,l,m} \alpha_{ijl} J_v E_v^i J_v F_v^j K_v^l + \beta_{ijm} J_v E_v^i J_v F_v^j \overline{K}_v^m + \gamma_{ij} J_v E_v^i J_v F_v^j J_v$. Here and in the sequel, every α, β and γ with subscripts is in the field k and does not equal zero. Then

$$\begin{aligned}
\Delta_v(x) = & \sum_{i,j,l,m} [\alpha_{ijl} \Delta_v(J_v E_v^i J_v F_v^j K_v^l) \\
& + \Delta_v(\beta_{ijm} J_v E_v^i J_v F_v^j \overline{K}_v^m) + \Delta_v(\gamma_{ij} J_v E_v^i J_v F_v^j J_v)] \\
= & \sum_{i,j,l,m} [\alpha_{ijl} (J_v \otimes J_v)(J_v \otimes J_v E_v J_v + J_v E_v J_v \otimes K_v)^i \\
& \times (J_v \otimes J_v)(J_v F_v J_v \otimes J_v + \overline{K}_v \otimes J_v F_v J_v)^j (K_v \otimes K_v)^l \\
& + \beta_{ijm} (J_v \otimes J_v)(J_v \otimes J_v E_v J_v + J_v E_v J_v \otimes K_v)^i \\
& \times (J_v \otimes J_v)(J_v F_v J_v \otimes J_v + \overline{K}_v \otimes J_v F_v J_v)^j (\overline{K}_v \otimes \overline{K}_v)^m \\
& + \gamma_{ij} (J_v \otimes J_v)(J_v \otimes J_v E_v J_v + J_v E_v J_v \otimes K_v)^i \\
& \times (J_v \otimes J_v)(J_v F_v J_v \otimes J_v + \overline{K}_v \otimes J_v F_v J_v)^j J_v];
\end{aligned}$$

and

$$\begin{aligned}
x \otimes x = & \Big(\sum_{i,j,l,m} \alpha_{ijl} J_v E_v^i J_v F_v^j K_v^l + \beta_{ijm} J_v E_v^i J_v F_v^j \overline{K}_v^m + \gamma_{ij} J_v E_v^i J_v F_v^j J_v \Big) \\
& \otimes \Big(\sum_{i,j,l,m} \alpha_{ijl} J_v E_v^i J_v F_v^j K_v^l + \beta_{ijm} J_v E_v^i J_v F_v^j \overline{K}_v^m + \gamma_{ij} J_v E_v^i J_v F_v^j J_v \Big).
\end{aligned}$$

It is seen that if $i \neq 0$ or $j \neq 0$, $\Delta_v(x)$ is impossible to equal $x \otimes x$. So, $i = 0$ and $j = 0$. We get $x = \sum_{l,m} \alpha_l K_v^l + \beta_m \overline{K}_v^m + J_v$. Then

$$\Delta_v(x) = \sum_{l,m} \big[\alpha_l K_v^l \otimes K_v^l + \beta_m \overline{K}_v^m \otimes \overline{K}_v^m + J_v \otimes J_v \big];$$

$$\begin{aligned}
x \otimes x = & \sum_{l,l',m,m'} \big[\alpha_l \alpha_{l'} K_v^l \otimes K_v^{l'} + \alpha_l \beta_{m'} K_v^l \otimes \overline{K}_v^{m'} + \alpha_l K_v^l \otimes J_v \\
& + \alpha_{l'} \beta_m \overline{K}_v^m \otimes K_v^{l'} + \beta_m \beta_{m'} \overline{K}_v^m \otimes \overline{K}_v^{m'} + \beta_m \overline{K}_v^m \otimes J_v \\
& + \alpha_{l'} J_v \otimes K_v^{l'} + \beta_{m'} J_v \otimes \overline{K}_v^{m'} + J_v \otimes J_v \big].
\end{aligned}$$

If there exists $l \neq l'$, then $x \otimes x$ possesses the monomial $K_v^l \otimes K_v^{l'}$, which does not appear in $\Delta_v(x)$. It contradicts $\Delta_v(x) = x \otimes x$. Hence we have only a unique l. Similarly, there exists a unique m. Thus $x = \alpha_l K_v^l + \beta_m \overline{K}_v^m + J_v$ Moreover, it is easy to see that $\alpha_l K_v^l$, $\beta_m \overline{K}_v^m$ and J_v can not appear simultaneously in the expression of x. Therefore, we conclude that $x = \alpha_l K_v^l$, $\beta_m \overline{K}_v^m$ or J_v (no summation) and we have

$$\Delta_v(J_v^{(ij)}) = J_v^{(ij)} \otimes J_v^{(ij)}. \tag{138}$$

It follows that $G(v\mathfrak{sl}_q(2)) = \{J_v^{(ij)} = K_v^i \overline{K}_v^j$: i, j run over all non-negative integers$\}$.

For any $J_v^{(ij)} = K_v^i \overline{K}_v^j \in G(v\mathfrak{sl}_q(2))$, one can find $J_v^{(ji)} = K_v^j \overline{K}_v^i \in G(v\mathfrak{sl}_q(2))$ such that the regularity (18) takes place $J_v^{(ij)} J_v^{(ji)} J_v^{(ij)} = J_v^{(ij)}$, which means that $G(v\mathfrak{sl}_q(2))$ forms a regular monoid under the multiplication of $v\mathfrak{sl}_q(2)$. □

These results show that $w\mathfrak{sl}_q(2)$ and $v\mathfrak{sl}_q(2)$ are examples of a weak Hopf algebra whose monoid of all group-like elements is a regular monoid. It incarnates further the corresponding relationship between weak Hopf algebras and regular monoids [19].

7. Regular Quasi-*R*-Matrix

From Proposition 1 we have seen that $w\mathfrak{sl}_q(2)/(J_w - 1) = \mathfrak{sl}_q(2)$. Now, we give another relationship between $w\mathfrak{sl}_q(2)$ and $\mathfrak{sl}_q(2)$ so as to construct a non-invertible universal R^w-matrix from $w\mathfrak{sl}_q(2)$.

Theorem 4. *$w\mathfrak{sl}_q(2)$ possesses an ideal W and a sub-algebra Y satisfying $w\mathfrak{sl}_q(2) = Y \oplus W$ and $W \cong \mathfrak{sl}_q(2)$ as Hopf algebras.*

Proof. Let W be the linear sub-space generated by $\{E_w^i F_w^j K_w^l, E_w^i F_w^j \overline{K}_w^m, E_w^i F_w^j J_w$: for all $i \geq 0, j \geq 0, l > 0$ and $m > 0\}$, and Y is the linear sub-space generated by $\{E_w^i F_w^j : i \geq 0, j \geq 0\}$. It is easy to see that $w\mathfrak{sl}_q(2) = Y \oplus W$; $w\mathfrak{sl}_q(2) W w\mathfrak{sl}_q(2) \subseteq W$, thus, W is an ideal; and, Y is a sub-algebra of $w\mathfrak{sl}_q(2)$. Note that the identity of W is J_w. Moreover, W is a Hopf algebra with the unit J_w, the comultiplication Δ_w^W satisfying

$$\Delta_w^W(E_w) = J_w \otimes E_w + E_w \otimes K_w, \tag{139}$$

$$\Delta_w^W(F_w) = F_w \otimes J_w + \overline{K}_w \otimes F_w, \tag{140}$$

$$\Delta_w^W(K_w) = K_w \otimes K_w, \quad \Delta_w^W(\overline{K}_w) = \overline{K}_w \otimes \overline{K}_w, \tag{141}$$

and the same counit, multiplication and antipode as in $w\mathfrak{sl}_q(2)$. Let ρ be the algebra morphism from $\mathfrak{sl}_q(2)$ to W satisfying $\rho(E) = E_w$, $\rho(F) = F_w$, $\rho(K) = K_w$ and $\rho(K^{-1}) = \overline{K}_w$. Then ρ is, in fact, a Hopf algebra isomorphism since $\{E_w^i F_w^j K_w^l$, $E_w^i F_w^j \overline{K}_w^m$, $E_w^i F_w^j J_w$: for all $i \geq 0, j \geq 0, l > 0$ and $m > 0\}$ is a basis of W by Theorem 2. □

Let us assume here that q is a root of unity of order d in the field k, where d is an odd integer and $d > 1$.

Set $I = (E_w^d, F_w^d, K_w^d - J_w)$ the two-sided ideal of U_q^w generated by E_w^d, F_w^d, $K_w^d - J_w$. Define the algebra $\overline{U}_q^w = U_q^w / I$.

Remark 11. Note that $\overline{K}_w^d = J_w$ in $\overline{U}_q^w = U_q^w/I$ since $K_w^d = J_w$.

It is easy to prove that I is also a coideal of U_q and $T_w(I) \subseteq I$. Then I is a weak Hopf ideal. It follows that $\overline{U}_q^w$ has a unique weak Hopf algebra structure such that the natural morphism is a weak Hopf algebra morphism, so the comultiplication , the counit and the weak antipode of $\overline{U}_q^w$ are determined by the same formulas with U_q^w. We will show that $\overline{U}_q^w$ is a quasi-braided weak Hopf algebra. As a generalization of a braided bialgebra and R-matrix we have the following definitions [18].

Definition 4. *Let there be k-linear maps $\mu : H \otimes H \to H, \eta : k \to H, \Delta : H \to H \otimes H, \varepsilon : H \to k$ in a k-linear space H such that (H, μ, η) is a k-algebra and (H, Δ, ε) is a k-coalgebra. We call H* **an almost bialgebra**, *if Δ is a k-algebra morphism, i.e. $\Delta(xy) = \Delta(x)\Delta(y)$ for every $x, y \in H$.*

Definition 5. *An almost bialgebra $H = (H, \mu, \eta, \Delta, \varepsilon)$ is called* **quasi-braided**, *if there exists an element R of the algebra $H \otimes H$ satisfying*

$$\Delta^{op}(x)R = R\Delta(x) \tag{142}$$

for all $x \in H$ and

$$(\Delta \otimes \mathrm{id}_H)(R) = R_{13}R_{23}, \tag{143}$$

$$(\mathrm{id}_H \otimes \Delta)(R) = R_{13}R_{12}. \tag{144}$$

Such R is called a **quasi-R-matrix.**

By Theorem 4, we have $\overline{U}_q^w = U_q^w/I = Y/I \oplus W/I \cong Y/(E_w^d, F_w^d) \oplus \widetilde{U}_q$ where $\widetilde{U}_q = \mathfrak{sl}_q(2)/(E_w^d, F_w^d, K^d - 1)$ is a finite Hopf algebra. We know in [16] that the sub-algebra $\widetilde{B}_q$ of $\widetilde{U}_q$ generated by $\{E_w^m K_w^n : 0 \le m, n \le d-1\}$ is a finite dimensional Hopf sub-algebra and $\widetilde{U}_q$ is a braided Hopf algebra as a quotient of the quantum double of $\widetilde{B}_q$. The R-matrix of $\widetilde{U}_q$ is

$$\widetilde{R} = \frac{1}{d} \sum_{0 \le i,j,k \le d-1} \frac{(q - q^{-1})^k}{[k]!} q^{k(k-1)/2 + 2k(i-j) - 2ij} E_w^k K_w^i \otimes F_w^k K_w^j.$$

Since $\mathfrak{sl}_q(2) \overset{\rho}{\cong} W$ as Hopf algebras and $(E^d, F^d, K^d - 1) \overset{\rho}{\cong} I$, we get $\widetilde{U}_q \cong W/I$ as Hopf algebras under the induced morphism of ρ. Then W/I is a braided Hopf algebra with a R-matrix,

$$R^w = \frac{1}{d} \sum_{0 \le k \le d-1; 1 \le i,j \le d} \frac{(q - q^{-1})^k}{[k]!} q^{k(k-1)/2 + 2k(i-j) - 2ij} E_w^k K_w^i \otimes F_w^k K_w^j.$$

Because the identity of W/I is J_w, there exists the inverse $\hat{R}^w$ of R^w such that $\hat{R}^w R^w = R^w \hat{R}^w = J_w$. Then we have

$$R^w \hat{R}^w R^w = R^w, \tag{145}$$

$$\hat{R}^w R^w \hat{R}^w = \hat{R}^w, \tag{146}$$

which shows that this R-matrix is regular in $\overline{U}_q$. It obeys the following relations:

$$\Delta_w^{op}(x)R^w = R^w\Delta_w(x) \tag{147}$$

for any $x \in W/I$ and

$$(\Delta_w \otimes \mathrm{id})(R^w) = R_{13}^w R_{23},^w \tag{148}$$
$$(\mathrm{id} \otimes \Delta_w)(R^w) = R_{13}^w R_{12}^w, \tag{149}$$

which are also satisfied in $\overline{U}_q$. Therefore R^w is a von Neumann's regular quasi-R-matrix of $\overline{U}_q$. So, we get the following

Theorem 5. *$\overline{U}_q$ is a quasi-braided weak Hopf algebra with*

$$R^w = \frac{1}{d} \sum_{0\le k\le d-1; 1\le i,j\le d} \frac{(q-q^{-1})^k}{[k]!} q^{k(k-1)/2+2k(i-j)-2ij} E_w^k K_w^i \otimes F_w^k K_w^j$$

as its quasi-R-matrix, which is regular.

The quasi-R-matrix from the J-weak Hopf algebra $v\mathfrak{sl}_q(2)$ has a more complicated structure and will be considered elsewhere.

8. Discussion

In conclusion we would like to compare the presented generalization of the Hopf algebra with the existing ones. A weak Hopf algebra in sense of [4,30,26] is a k-linear vector space H that is both an associative algebra (H, μ, η) and a coassociative coalgebra $(H, \Delta_{\text{weak}}, \varepsilon_{\text{weak}})$ related to each other in a certain self-dual way [3,26] and that possesses an antipode S_{weak} satisfying (in Sweedler notations [29])

$$S_{\text{weak}}\left(x_{(1)}\right)x_{(2)} = 1_{(1)}\varepsilon_{\text{weak}}\left(x1_{(2)}\right), \tag{150}$$
$$x_{(1)}S_{\text{weak}}\left(x_{(2)}\right) = \varepsilon_{\text{weak}}\left(x1_{(1)}\right)1_{(2)}, \tag{151}$$

(pre-antipode), and if in addition

$$S_{\text{weak}}\left(x_{(1)}\right)x_{(2)}S_{\text{weak}}\left(x_{(3)}\right) = S_{\text{weak}}(x), \tag{152}$$

then S_{weak} can be called a Nill's antipode. Weak Hopf algebras have "weaker" axioms related to the unit and counit: $\varepsilon_{\text{weak}}(xyz) = \varepsilon_{\text{weak}}(xy_{(1)})\varepsilon_{\text{weak}}(y_{(2)}z)$ and $\Delta_{\text{weak}}^{(2)}(1) = (\Delta_{weak}(1)\otimes 1)(1\otimes\Delta_{\text{weak}}(1))$. So the comultiplication is non-unital $\Delta_{\text{weak}}(1) \neq 1\otimes 1$ (like in weak quasi Hopf algebras [23]) and the counit is only "weakly" multiplicative, $\varepsilon(xy) = \varepsilon(x1)\varepsilon(1_{(2)}y)$. Therefore they can be called *non-unital weak Hopf algebras*. Note that this kind of "weakness" is the "strength" of weak Hopf algebras [3], because it allows (even in the finite dimensional and semisimple cases) the weak Hopf algebra to possess non-integral (quantum) dimensions. The earlier proposals of *face algebras* [13], *quantum groupoids* [25], the (finite dimensional) *generalized Kac algebras* [31] are weak Hopf algebras in this sense [26], not the most general ones, but having an involutive antipode. The weak antipode T introduced in [18] and in this paper (T_w and T_v) is not usually a pre-antipode in the sense (150)–(151). Therefore the class of non-unital Hopf algebras [26, 3] (or quantum groupoids [25]) and the class of weak Hopf algebras [18, 20, 5] are not included in each other. In fact, we have the following relation:

$$\begin{array}{ccccc} A & \longrightarrow & B & \longrightarrow & C \\ \downarrow & & & & \downarrow \\ D & & \longrightarrow & & E \end{array}$$

where A denotes a Hopf algebra, B a non-unital weak Hopf algebra, C a non-unital almost weak Hopf algebra, D a weak Hopf algebra and E an almost weak Hopf algebra. From this, we see easily that just Hopf algebras compose their common subclass.

Nill [26] points out that these algebras have many examples in the theory of quantum chain models. Dissimilarly, our examples come from regular monoid algebras [18–20] and also from this paper, i.e. $wsl_q(2)$, $vsl_q(2)$, etc.

Note that although the weak Hopf algebras in this paper and the non-unital weak Hopf algebras introduced earlier do not include each other usually, their antipodes are defined by a similar method, that is, by using of the regularity of antipodes in the involution algebra of the original algebras. Therefore, we believe that it is possible to characterize certain aspects in similar ways. A further interesting work, which we want to continue, is to study our weak Hopf algebras through similar objects and methods for the non-unital weak Hopf algebras and moreover, to find applications in the theory of quantum chain models and other relative areas.

Acknowledgements. F.L. thanks M. L. Ge, P. Trotter and N. H. Xi for kind help and fruitful discussions during his visits. S.D. is thankful to A. Kelarev, V. Lyubashenko, W. Marcinek and B. Schein for useful remarks. S.D. is grateful to the Zhejiang University for kind hospitality and the National Natural Science Foundation of China for financial support.

References

1. Abe, E.: *Hopf Algebras*. Cambridge: Cambridge Univ. Press, 1980
2. Bernstein, J. and Khovanova, T.: On quantum group $SL_q(2)$. Preprint MIT, hep-th/9412056, Cambridge, 1994
3. Böhm, G., Nill, F., and Szlachányi, K.: Weak Hopf algebras **I**. Integral theory and C^*-structure. J. Algebra **221**, 385–438 (1999)
4. Böhm, G. and Szlachányi, K.: A coassociative C^*-quantum group with nonintegral dimensions. Lett. Math. Phys. **35**, 437–456 (1996)
5. Duplij, S. and Li, F.: On regular solutions of quantum Yang-Baxter equation and weak Hopf algebras. J. Kharkov National University, ser. Nuclei, Particles and Fields **521**, 15–30 (2001)
6. Duplij, S. and Marcinek, W.: Higher regularity properties of mappings and morphisms. Preprint Univ. Wrocław, IFT UWr 931/00, math-ph/0005033, Wrocław, 2000
7. Duplij, S. and Marcinek, W.: On higher regularity and monoidal categories. Kharkov State University Journal (Vestnik KSU), ser. Nuclei, Particles and Fields **481**, 27–30 (2000)
8. Duplij, S. and Marcinek, W.: Noninvertibility, semisupermanifolds and categories regularization. In: *Noncommutative Structures in Mathematics and Physics* (Duplij S. and Wess J., eds.). Dordrecht: Kluwer, 2001, pp. 125–140
9. Duplij, S.: On semi-supermanifolds. Pure Math. Appl. **9**, 283–310 (1998)
10. Duplij, S.: *Semisupermanifolds and semigroups*. Kharkov: Krok, 2000
11. Goodearl, K.: *Von Neumann Regular Rings*. London: Pitman, 1979
12. Green, J.A., Nicols, W.D., and Taft, E.J.: Left Hopf algebras. J. Algebra **65**, 399–411 (1980)
13. Hayashi, T.: An algebra related to the fusion rules of Wess-Zumino-Witten models. Lett. Math. Phys. **22**, 291–296 (1991)
14. Howie, J.M.: *Fundamentals of Semigroup Theory*. Oxford: Clarendon Press, 1995
15. Hungerford, T.W.: *Algebra*. New York: Springer-Verlag, 1980

16. Kassel, C.: *Quantum Groups*. New York: Springer-Verlag, 1995
17. Lawson, M.V.: *Inverse Semigroups: The Theory of Partial Symmetries*. Singapore: World Sci., 1998
18. Li, F.: Weak Hopf algebras and some new solutions of Yang-Baxter equation. J. Algebra **208**, 72–100 (1998)
19. Li, F.: Weak Hopf algebras and regular monoids. J. Math. Research and Exposition **19**, 325–331 (1999)
20. Li, F.: Solutions of Yang-Baxter equation in endomorphism semigroup and quasi-(co)braided almost bialgebras. Comm. Algebra **28**, 2253–2270 (2000)
21. Li, F.: Weaker structures of Hopf algebras and singular solutions of Yang– Baxter equation. In: Symposium on The Frontiers of Physics at Millennium, Singapore: World Scientific Publishing Co. Pte. Ltd, 2001
22. Lustig, G.: On quantum groups. J. Algebra **131**, 466–475 (1990)
23. Mack, G. and Schomerus, V.: Quasi Hopf quantum symmetry in quantum theory. Nucl. Phys. **B370**, 185–191 (1992)
24. Nichols, W.D. and Taft, E.J.: *The Left Antipodes of a Left Hopf Algebra*. Contemp. Math. 13. Providence: Amer. Math. Soc., 1982
25. Nikshych, D. and Vainerman, L.: Finite quantum groupoids and their applications. Preprint Univ. California, math.QA/0006057, Los Angeles, 2000
26. Nill, F.: Axioms for weak bialgebras. Preprint Inst. Theor. Phys. FU, math.QA/9805104, Berlin, 1998
27. Petrich, M.: *Inverse Semigroups*. New York: Wiley, 1984
28. Shnider, S. and Sternberg, S.: *Quantum Groups*. Boston: International Press, 1993
29. Sweedler, M.E.: *Hopf Algebras*. New York: Benjamin, 1969
30. Szlachányi, K.: Weak hopf algebras, Operator algebras and quantum field theory (Rome, 1996), Cambridge, MA: Internat. Press, 1997, pp. 621–632
31. Yamanouchi, T.: Duality for generalized Kac algebras and a characterization of finite groupoid algebras. J. Algebra **163**, 9–50 (1994)

Communicated by A. Connes