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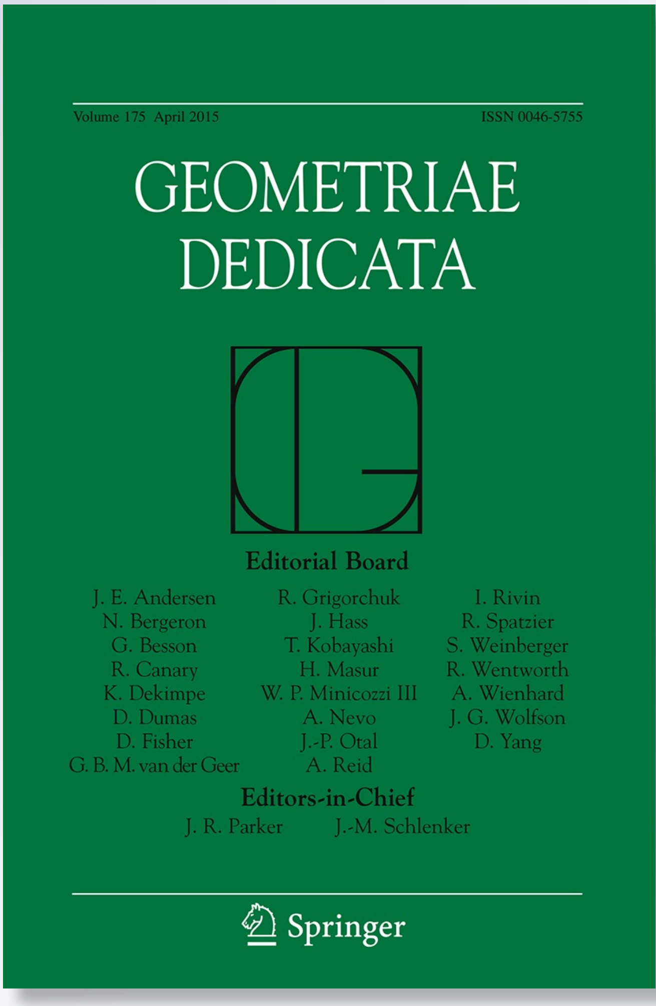

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ORIGINAL PAPER

Connectedness of planar self-affine sets associated with non-collinear digit sets

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Abstract We study a class of planar self-affine sets $T(A, \mathcal{D})$ generated by the integer expanding matrices A with $|\det A| = 3$ and the non-collinear digit sets $\mathcal{D} = \{0, v, kAv\}$ where $k \in \mathbb{Z} \setminus \{0\}$ and $v \in \mathbb{R}^2$ such that $\{v, Av\}$ is linearly independent. By examining the characteristic polynomials of A carefully, we prove that $T(A, \mathcal{D})$ is connected if and only if the parameter $k = \pm 1$.

Keywords Connectedness · Self-affine set · Digit set · $\mathcal{E}$-Connected

Mathematics Subject Classification (2010) Primary 28A80 · Secondary 52C20 · 52C45

1 Introduction

Let $M_n(\mathbb{Z})$ denote the set of $n \times n$ matrices with integer entries, and $A \in M_n(\mathbb{Z})$ be expanding, i.e., all eigenvalues of A have moduli strictly larger than 1. Let $\mathcal{D} = \{d_1, \ldots, d_q\} \subset \mathbb{R}^n$ be a finite set of q distinct vectors, we call it a *q-digit set*. It is well known that there exists a unique nonempty compact set $T := T(A, \mathcal{D})$ [13] satisfying the set-valued equation

$$T = A^{-1}(T + \mathcal{D}).$$

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The set T is called *self-affine set*, and which can often be written as the form of radix expansions:

$$T = \left\{ \sum_{i=1}^{\infty} A^{-i} d_{j_i} : d_{j_i} \in \mathcal{D} \right\}.$$

If $|\det A| = q$ and T has a nonvoid interior, we call such T a *self-affine tile*.

Recently, the topological structure of $T(A, \mathcal{D})$, especially its connectedness, has attracted a lot of attentions in the literature. It was asked by Gröchenig and Haas [6] that given an expanding integer matrix $A \in M_n(\mathbb{Z})$, whether there exists a digit set $\mathcal{D}$ such that $T(A, \mathcal{D})$ is a connected tile and they partially solved it in $\mathbb{R}^2$. Hacon et al. [7] proved that any self-affine tile $T(A, \mathcal{D})$ with a 2-digit set is always pathwise connected. Lau et al. ([9,11,12,14]) systematically studied the connectedness of self-affine tiles arising from a kind of digit sets of the form $\{0, 1, \ldots, q-1\}v$ where $v \in \mathbb{Z}^n \setminus \{0\}$, which were called *consecutive collinear (CC) digit sets*. They observed a *height reducing property (HRP)* of the characteristic polynomial of A to determine the connectedness of $T(A, \mathcal{D})$, and conjectured that all monic expanding polynomials have HRP, thus all the tiles generated by CC digit sets are connected. Akiyama and Gjini [1] solved it up to degree 4. However it is still open for arbitrary degree. Moreover, Liu et al. [17] classified the connected self-affine sets with CC digit sets in $\mathbb{R}^2$. On the other hand, the disk-likeness (i.e. homeomorphic to a closed unit disk) is also an interesting topic in the planar geometry. Bandt and Gelbrich [2], Bandt and Wang [3], and Leung and Lau [14] investigated the disk-like self-affine tiles by making use of neighbour graphs of T. Along this line, Leung and Luo [16] further studied the boundary structure of disk-like tiles. Deng and Lau [4], as well as Kirat [10] concerned themselves about a class of disk-like self-affine tiles generated by product digit sets.

With regard to other types of digit sets, there are few results about the connectedness of $T(A, \mathcal{D})$ generated by non-consecutive or non-collinear digit sets. In [15], by counting the neighbours of T, the authors made a first attempt to exploit the case of non-consecutive collinear digit set $\mathcal{D} = \{0, v, kv\}$ with $|\det A| = 3$, and obtained a complete characterization for $T(A, \mathcal{D})$ to be connected or not. In this paper, we go further to discuss the non-collinear digit set $\mathcal{D} = \{0, v, kAv\}$ for $k \in \mathbb{Z} \setminus \{0\}$. By examining the characteristic polynomials of A case by case, we can determine the connectedness of $T(A, \mathcal{D})$ based on the parameter k in the following way.

Theorem 1.1 *Let A be a 2×2 expanding integer matrix with $|\det A| = 3$, and let $\mathcal{D} = \{0, v, kAv\}$ be a digit set where $k \in \mathbb{Z} \setminus \{0\}$ and $v \in \mathbb{R}^2$ such that $\{v, Av\}$ is linearly independent. Then $T(A, \mathcal{D})$ is connected if and only if $k = \pm 1$.*

In the consecutive collinear case, to determine the connectedness, it suffices to check whether $v = \sum_{i=1}^{\infty} A^{-i} v_i,\ v_i \in \Delta\mathcal{D}$ [11]. In the non-consecutive collinear case, we also need to check whether $(k-1)v = \sum_{i=1}^{\infty} A^{-i} v_i,\ v_i \in \Delta\mathcal{D}$ [15]. However, in the present case, for the non-collinear digit set, the proof is more complicated, we have to check not only $v = \sum_{i=1}^{\infty} A^{-i} v_i,\ v_i \in \Delta\mathcal{D}$ but also kAv or $kAv - v = \sum_{i=1}^{\infty} A^{-i} v_i,\ v_i \in \Delta\mathcal{D}$.

The rest of the paper is organized as follows: In Sect. 2, we recall several well-known results on self-affine sets. The complete proof of Theorem 1.1 is shown in Sect. 3. In Sect. 4, we give some remarks and open questions on the related studies.

2 Preliminaries

In this section, we give some preparatory results of self-affine sets which will be used frequently in the paper. Let $A \in M_n(\mathbb{Z})$ be expanding and $\mathcal{D} = \{d_1, \ldots, d_q\} \subset \mathbb{R}^n$ be a digit set. Define

$$\mathcal{E} = \{(d_i, d_j) : \ (T + d_i) \cap (T + d_j) \neq \emptyset, \ d_i, \ d_j \in \mathcal{D}\}.$$

We say that d_i and d_j are *$\mathcal{E}$-connected* if there exists a finite sequence $\{d_{j_1}, \ldots, d_{j_k}\} \subset \mathcal{D}$ such that $d_i = d_{j_1}, d_j = d_{j_k}$ and $(d_{j_l}, d_{j_{l+1}}) \in \mathcal{E}, 1 \le l \le k-1$. It is easy to check that $(d_i, d_j) \in \mathcal{E}$ if and only if $d_i - d_j \in T - T$, i.e.,

$$d_i - d_j = \sum_{k=1}^{\infty} A^{-k} v_k \quad \text{where} \quad v_k \in \Delta\mathcal{D} := \mathcal{D} - \mathcal{D}.$$

The following criterion for connectedness of self-affine set $T(A, \mathcal{D})$ was first proved by Hata [8] and rediscovered by Kirat and Lau [11].

Proposition 2.1 [8,11] *A self-affine set T with a digit set $\mathcal{D}$ is connected if and only if any two $d_i, d_j \in \mathcal{D}$ are $\mathcal{E}$-connected.*

In the paper, we mainly consider the planar self-affine set $T(A, \mathcal{D})$ generated by a 2×2 expanding integer matrix A with $|\det A| = 3$ and a digit set $\mathcal{D} = \{0, v, kAv\}$ such that $\{v, Av\}$ is linearly independent, where $k \in \mathbb{Z} \setminus \{0\}$. Denote the characteristic polynomial of A by $f(x) = x^2 + px + q$, and define α_i, β_i by

$$A^{-i} v = \alpha_i v + \beta_i A v, \quad i = 1, 2, \ldots.$$

By applying the Hamilton-Cayley theorem $f(A) = A^2 + pA + qI = 0$, the following consequence is immediate.

Lemma 2.2 [14] *Let α_i, β_i be defined as the above. Then $q\alpha_{i+2} + p\alpha_{i+1} + \alpha_i = 0$ and $q\beta_{i+2} + p\beta_{i+1} + \beta_i = 0$. Especially, $\alpha_1 = -p/q, \ \alpha_2 = (p^2 - q)/q^2; \ \beta_1 = -1/q, \ \beta_2 = p/q^2$. Moreover for $\Delta = p^2 - 4q \neq 0$, we have*

$$\alpha_i = \frac{q\left(y_1^{i+1} - y_2^{i+1}\right)}{\Delta^{1/2}} \quad \text{and} \quad \beta_i = \frac{-\left(y_1^i - y_2^i\right)}{\Delta^{1/2}}$$

where $y_1 = \frac{-p+\Delta^{1/2}}{2q}, \ y_2 = \frac{-p-\Delta^{1/2}}{2q}$ are the two roots of $qx^2 + px + 1 = 0$.

Let

$$\tilde{\alpha} := \sum_{i=1}^{\infty} |\alpha_i|, \quad \tilde{\beta} := \sum_{i=1}^{\infty} |\beta_i|.$$

Corollary 2.3 *Assume $f(x) = x^2 + px + q$ and $g(x) = x^2 - px + q$ be the characteristic polynomials of expanding matrices A and B, respectively. Let $\alpha_i, \beta_i, \tilde{\alpha}, \tilde{\beta}$ for $f(x)$ be as before; let $\alpha_i', \beta_i', \tilde{\alpha}', \tilde{\beta}'$ be the corresponding terms for $g(x)$. Then*

$$\alpha_{2j}' = \alpha_{2j}, \ \alpha_{2j-1}' = -\alpha_{2j-1}, \ \beta_{2j}' = -\beta_{2j}, \ \beta_{2j-1}' = \beta_{2j-1},$$

and hence $\tilde{\alpha} = \tilde{\alpha}', \ \tilde{\beta} = \tilde{\beta}'$.

When $|\det A| = 3$, it is known by [2] that there are ten eligible characteristic polynomials of A:

$$x^2 \pm 3; \quad x^2 \pm x + 3; \quad x^2 \pm 2x + 3; \quad x^2 \pm 3x + 3; \quad x^2 \pm x - 3.$$

Following [15], together with Corollary 2.3, we obtained the estimates or values of the corresponding $\tilde{\alpha}$ and $\tilde{\beta}$ as follows:

$$f(x) = x^2 \pm x + 3: \quad \tilde{\alpha} < 0.88, \ \tilde{\beta} < 0.63; \tag{2.1}$$
$$f(x) = x^2 \pm 2x + 3: \quad \tilde{\alpha} < 1.17, \ \tilde{\beta} < 0.73; \tag{2.2}$$
$$f(x) = x^2 \pm 3x + 3: \quad \tilde{\alpha} < 2.24, \ \tilde{\beta} < 1.08; \tag{2.3}$$
$$f(x) = x^2 \pm x - 3: \quad \tilde{\alpha} = 2, \ \tilde{\beta} = 1. \tag{2.4}$$

3 Proof of Theorem 1.1

For the digit set $\mathcal{D} = \{0, v, kAv\}$, denote by $\Delta\mathcal{D} = \{0, \pm v, \pm(kAv - v), \pm kAv\}$ the difference set. We separate the proof of Theorem 1.1 into three parts (A,B and C) according to the characteristic polynomials of A.

Part A: $f(x) = x^2 \pm 3$.

Since the case of $f(x) = x^2 - 3$ is more or less the same as that of $f(x) = x^2 + 3$, it suffices to show the last one. If $k = 1$, then $\Delta\mathcal{D} = \{0, \pm v, \pm(Av - v), \pm Av\}$. From $f(A) = A^2 + 3I = 0$, we have

$$I = -2A^{-2}(I + A^{-2})^{-1} = 2\sum_{n=1}^{\infty}(-1)^n A^{-2n} \tag{3.1}$$

and

$$v = 2\sum_{n=1}^{\infty}(-1)^n A^{-2n} v = \sum_{n=0}^{\infty} A^{-4n}\big(A^{-2}(-v) + A^{-3}(-Av) + A^{-4}v + A^{-5}(Av)\big). \tag{3.2}$$

Hence $v \in T - T$, or equivalently $T \cap (T + v) \neq \emptyset$. Moreover,

$$Av = \sum_{n=0}^{\infty} A^{-4n}\big(A^{-1}(-v) + A^{-2}(-Av) + A^{-3}v + A^{-4}(Av)\big) \tag{3.3}$$

which implies $T \cap (T + Av) \neq \emptyset$. Consequently, by Proposition 2.1, T is connected (see Fig. 1a).

If $k = -1$, then $\Delta\mathcal{D} = \{0, \pm v, \pm(Av + v), \pm Av\}$, and (3.1, 3.2, 3.3) still hold. Hence T is also connected.

If $|k| > 1$, let $k_i Av + l_i v \in \Delta\mathcal{D}$ for $i \geq 1$, then a point of $T - T$ can be written as

$$\begin{aligned}
\sum_{i=1}^{\infty} A^{-i}(k_i Av + l_i v) &= \sum_{i=1}^{\infty} A^{-2i}(k_{2i} Av + l_{2i} v) + \sum_{i=1}^{\infty} A^{-2i+1}(k_{2i-1} Av + l_{2i-1} v) \\
&= \sum_{i=1}^{\infty}(-\frac{1}{3})^i (k_{2i} Av + l_{2i} v) + \sum_{i=1}^{\infty}(-\frac{1}{3})^i(-3k_{2i-1} v + l_{2i-1} Av) \\
&= \left(k_1 + \sum_{i=1}^{\infty}(-\frac{1}{3})^i (l_{2i} + k_{2i+1})\right) v + \left(\sum_{i=1}^{\infty}(-\frac{1}{3})^i (l_{2i-1} + k_{2i})\right) Av \\
&:= Lv + KAv.
\end{aligned}$$

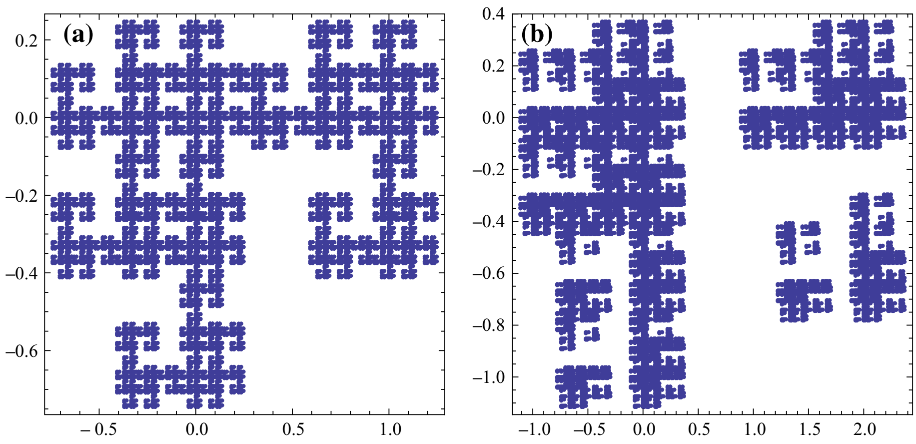

Fig. 1 Two cases for $f(x) = x^2 + 3$ and $v = (1, 0)^t$

As $|l_i + k_{i+1}| \leq 1 + |k|$, it follows that $|K| \leq (1 + |k|) \sum_{i=1}^{\infty} (\frac{1}{3})^i = (1 + |k|)/2 < |k|$. Hence $T \cap (T + kAv) = \emptyset$ and $(T + v) \cap (T + kAv) = \emptyset$, which imply that T is disconnected (see Fig. 1b).

Part B: $f(x) = x^2 + px \pm 3$ where $p > 0$.

For the cases of $f(x) = x^2 + px + 3$ with $0 < p < 3$, by using $0 = f(A) = f(A)(A - I) = A^3 + (p-1)A^2 + (3-p)A - 3I$, we obtain

$$I = \sum_{i=1}^{\infty} A^{-3i}\big((1-p)A^2 - (3-p)A + 2I\big). \tag{3.4}$$

Case 1 $f(x) = x^2 + x + 3$: For $k = 1$, then $\Delta\mathcal{D} = \{0, \pm v, \pm(Av - v), \pm Av\}$. By (3.4), $I = \sum_{i=1}^{\infty} A^{-3i}\big(-2A + 2I\big)$ and

$$v = \sum_{i=1}^{\infty} A^{-3i}\big(-2Av + 2v\big) = \sum_{i=0}^{\infty} A^{-3i}\big(A^{-2}(-v) + A^{-3}(v - Av) + A^{-4}(Av)\big). \tag{3.5}$$

Hence $T \cap (T + v) \neq \emptyset$. Moreover,

$$Av = \sum_{i=0}^{\infty} A^{-3i}\big(A^{-1}(-v) + A^{-2}(v - Av) + A^{-3}(Av)\big) \tag{3.6}$$

which implies $T \cap (T + Av) \neq \emptyset$. Consequently, T is connected (see Fig. 2a).

For $k = -1$, then $\Delta\mathcal{D} = \{0, \pm v, \pm(Av + v), \pm Av\}$. From $f(A) = 0$, we deduce that $I = (-A - 2I)(A^2 + I)^{-1}$, which in turn gives

$$\begin{aligned} v &= -A^{-1}v - 2A^{-2}v + A^{-3}v + 2A^{-4}v - A^{-5}v - 2A^{-6}v + A^{-7}v + 2A^{-8}v - \cdots \\ &= A^{-2}(-Av - v) + A^{-3}(-Av) + A^{-4}(Av + v) + A^{-5}(Av) + A^{-6}(-Av - v) \\ &\quad + A^{-7}(-Av) + A^{-8}(Av + v) + A^{-9}(Av) + \cdots \\ &\in T - T. \end{aligned}$$

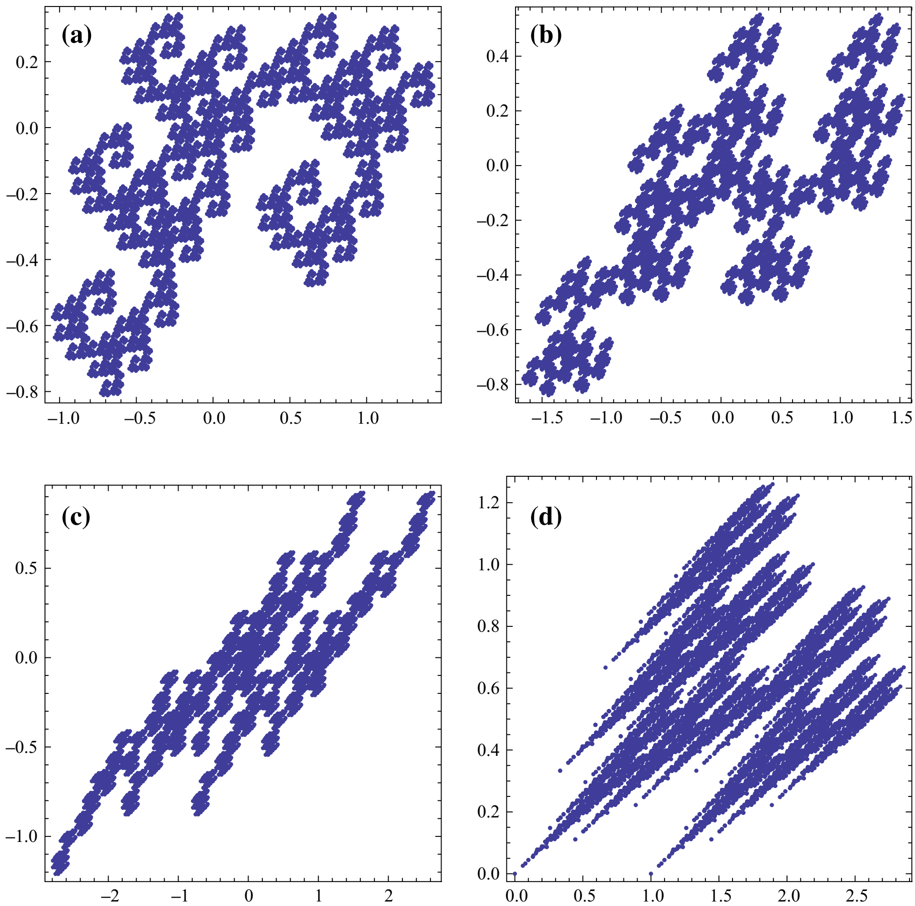

Fig. 2 Connected cases for $v = (1, 0)^t$ and $k = 1$

Hence $T \cap (T + v) \neq \emptyset$. Multiplying the above expression by A, we have

$$\begin{aligned} Av &= A^{-1}(-Av - v) + A^{-2}(-Av) + A^{-3}(Av + v) + A^{-4}(Av) \\ &\quad + A^{-5}(-Av - v) + A^{-6}(-Av) + A^{-7}(Av + v) + A^{-8}(Av) + \cdots \\ &\in T - T \end{aligned}$$

which implies $T \cap (T + Av) \neq \emptyset$. It follows that T is connected.

For $|k| > 1$. A point of $T - T$ can be written as

$$\sum_{i=1}^{\infty} A^{-i}(k_i Av + l_i v)$$

where $k_i Av + l_i v \in \Delta\mathcal{D}$ for $i \geq 1$. By using the relation $A^{-i}v = \alpha_i v + \beta_i Av$,

$$\sum_{i=1}^{\infty} A^{-i}(k_i Av + l_i v) = \sum_{i=1}^{\infty}(k_i A^{-i+1} v + l_i A^{-i} v)$$

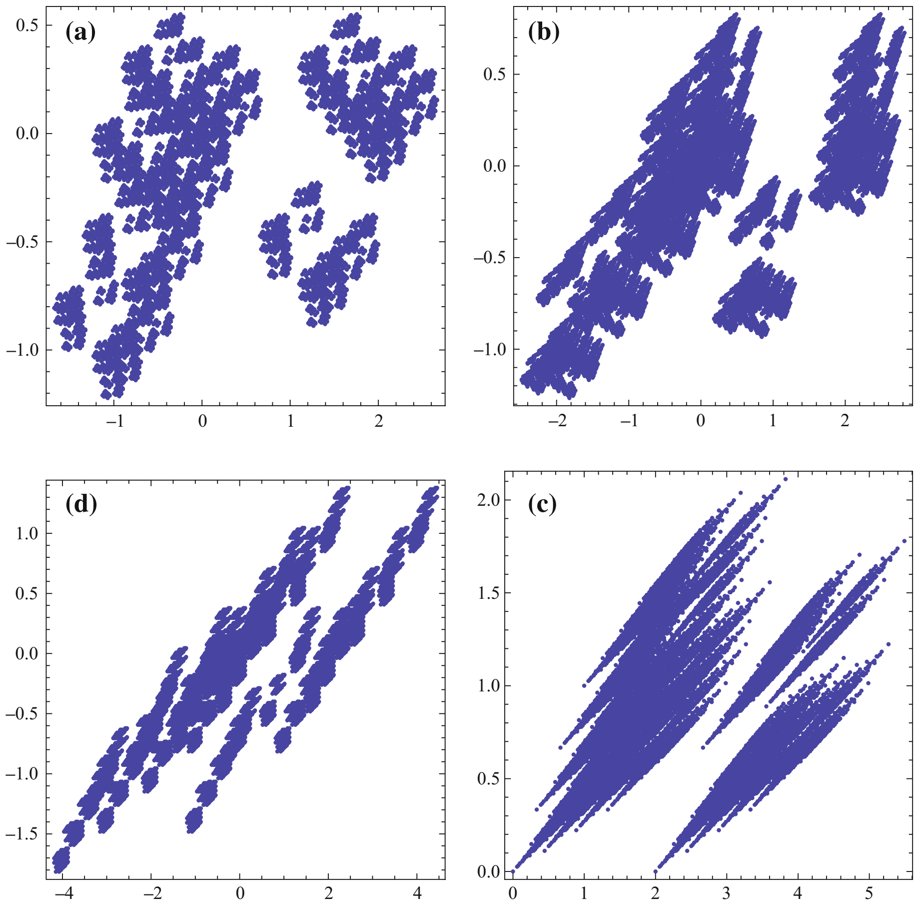

Fig. 3 Disconnected cases for $v = (1, 0)^t$ and $k = 2$

$$
\begin{aligned}
&= \sum_{i=1}^{\infty} k_i(\alpha_{i-1}v + \beta_{i-1}Av) + \sum_{i=1}^{\infty} l_i(\alpha_i v + \beta_i Av) \\
&= \left(k_1 + \sum_{i=1}^{\infty}(k_{i+1} + l_i)\alpha_i\right)v + \left(\sum_{i=1}^{\infty}(k_{i+1} + l_i)\beta_i\right)Av \\
&:= Lv + KAv. \qquad (3.7)
\end{aligned}
$$

As $|l_i + k_{i+1}| \le 1 + |k|$ and $\tilde{\beta} < 0.63$ (2.1), we conclude $|K| \le 0.63(1 + |k|) < |k|$, which yields $T \cap (T + kAv) = \emptyset$ and $(T + v) \cap (T + kAv) = \emptyset$. Hence T is disconnected (see Fig. 3(a)).

Case 2 $f(x) = x^2 + 2x + 3$: For $k = 1$. By (3.4), $I = \sum_{i=1}^{\infty} A^{-3i}\big(-A^2 - A + 2I\big)$ and

$$
v = \sum_{i=1}^{\infty} A^{-3i}\big(-A^2 v - Av + 2v\big) = A^{-1}(-v) + \sum_{i=0}^{\infty} A^{-3i}\big(A^{-2}(-v) + A^{-3}v + A^{-4}(Av - v)\big).
$$

Hence $T \cap (T+v) \neq \emptyset$. Moreover,

$$Av = A^{-1}(-Av) + \sum_{i=0}^{\infty} A^{-3i}\big(A^{-1}(-v) + A^{-2}v + A^{-3}(Av - v)\big)$$

which implies $T \cap (T + Av) \neq \emptyset$. Consequently, T is connected (see Fig. 2b).

For $k = -1$. We obtain from (3.4) that

$$\begin{aligned} v &= -A^{-1}v - A^{-2}v + 2A^{-3}v - A^{-4}v - A^{-5}v \\ &\quad +2A^{-6}v - A^{-7}v - A^{-8}v + 2A^{-9}v + \cdots \\ &= A^{-1}(-v) + A^{-2}(-v) + A^{-3}v + A^{-4}(Av) + A^{-5}(-Av - v) \\ &\quad +A^{-6}v + A^{-7}(Av) + A^{-8}(-Av - v) + \cdots \\ &\in T - T, \end{aligned}$$

implying $T \cap (T+v) \neq \emptyset$. Multiplying the above expression by A, we have

$$\begin{aligned} Av + v &= A^{-1}(-v) + A^{-2}v + A^{-3}(Av) + A^{-4}(-Av - v) \\ &\quad +A^{-5}v + A^{-6}(Av) + A^{-7}(-Av - v) + \cdots \\ &\in T - T, \end{aligned}$$

implying $(T+v) \cap (T - Av) \neq \emptyset$. Hence T is connected.

For $|k| > 1$. By (3.7) and $\tilde{\beta} < 0.73$ (2.2), we have $|K| \leq (1+|k|)\tilde{\beta} < 0.73(1+|k|)$. When $|k| \geq 3$, $|K| < 0.73(1+|k|) < |k|$, which yields $T \cap (T + kAv) = \emptyset$ and $(T+v) \cap (T + kAv) = \emptyset$. Hence T is disconnected. When $k = 2$, suppose $(T + 2Av) \cap T \neq \emptyset$ or $(T + 2Av) \cap (T + v) \neq \emptyset$, i.e., $2Av + lv \in T - T$ for $l = 0$ or -1. By (3.7), we obtain

$$(k_2 + l_1)\beta_1 = 2 - \sum_{i=2}^{\infty}(k_{i+1} + l_i)\beta_i \geq 2 - (1+2)(\tilde{\beta} - |\beta_1|) > 0.8 \tag{3.8}$$

where $\beta_1 = -1/3$. It follows that $l_1 = -1,\ k_2 = -2$. Then using $f(A) = 0$, we have

$$A(2Av + lv) - k_1 Av - l_1 v = (-4 + l - k_1)Av - (6 + l_1)v \in T - T.$$

It follows from $l_1 = -1$ that $k_1 = 0$ or 2. Hence $-4 + l - k_1 \leq -4 + 0 + 0 = -4$, which contradicts the inequality $|K| \leq (1+|k|)\tilde{\beta} < 3 \times 0.73 = 2.19$. So T is disconnected for $k = 2$ (see Fig. 3b).

When $k = -2$, suppose $2Av + lv \in T - T$ where $l = 0$ or 1. Similarly, it yields from (3.8) that $l_1 = -1,\ k_2 = -2$ and $k_1 = -2$ or 0. If $k_1 = -2$, then $(l-2)Av - 5v \in T - T$. Multiplying the expression by A and using $f(A) = 0$, we obtain

$$(l-2)A^2 v - 5Av - (k_2 Av + l_2 v) = (1 - 2l)Av + (6 - 3l - l_2)v \in T - T$$

and $6 - 3l - l_2 \geq 6 - 3 - 1 = 2$. On the other hand, by (3.7) and (2.2), $-5.81 = -2 - 3\tilde{\alpha} \leq L \leq -2 + 3\tilde{\alpha} < 1.81$. This is ridiculous. If $k_1 = 0$, then $(l-4)Av - 5v \in T - T$ and $|l - 4| \geq 3$ contracts $|l - 4| = |K| < 2.19$. Therefore T is disconnected for $k = -2$.

Case 3 $f(x) = x^2 + 3x + 3$: For $k = 1$. From $(A - I)f(A) = 0$, we get $A^2 + A - I = 2(A+I)^{-1}$, which yields $I = -A^{-1} + A^{-2} + 2\sum_{i=3}^{\infty}(-1)^{i+1}A^{-i}$ and

$$\begin{aligned} v &= -A^{-1}v + A^{-2}v + 2\sum_{i=3}^{\infty}(-1)^{i+1}A^{-i}v \\ &= A^{-2}(v - Av) + A^{-3}v + \sum_{i=0}^{\infty} A^{-2i}\big(A^{-4}(Av - v) + A^{-5}(v - Av)\big). \end{aligned}$$

Hence $T \cap (T + v) \neq \emptyset$. Moreover,

$$Av = A^{-1}(v - Av) + A^{-2}v + \sum_{i=0}^{\infty} A^{-2i}\big(A^{-3}(Av - v) + A^{-4}(v - Av)\big)$$

which implies $T \cap (T + Av) \neq \emptyset$. Consequently, T is connected (see Fig. 2c).

For $k = -1$. From $f(A) = 0$ we get $A + 2I = -(A + I)^{-1}$. It follows that $I = -2A^{-1} - A^{-2} + A^{-3} - A^{-4} + A^{-5} + \cdots$ and

$$\begin{aligned} v &= -2A^{-1}v - A^{-2}v + A^{-3}v - A^{-4}v + A^{-5}v + \cdots \\ &= A^{-1}(-v) + A^{-2}(-Av - v) + A^{-3}v + A^{-4}(-v) + A^{-5}v + A^{-6}(-v) + \cdots \\ &\in T - T, \end{aligned}$$

then $T \cap (T + v) \neq \emptyset$. Also we can deduce immediately that

$$\begin{aligned} Av + v &= A^{-1}(-Av - v) + A^{-2}v + A^{-3}(-v) + A^{-4}v + A^{-5}(-v) + \cdots \\ &\in T - T, \end{aligned}$$

which yields $(T + v) \cap (T - Av) \neq \emptyset$. As a result, T is connected.

For $k > 1$. By (3.7) and (2.3), we have $|L| \leq k + (1 + k)\tilde{\alpha} < 2.24 + 3.24k$. Suppose $kAv + lv \in T - T$ for $l = 0$ or -1. Multiplying (3.7) by A and then subtracting $k_1Av + l_1v$ from both sides, we see that

$$(-3k + l - k_1)Av - (3k + l_1)v \in T - T. \tag{3.9}$$

Repeating the process, we obtain

$$(6k - 3l + 3k_1 - l_1 - k_2)Av + (9k - 3l + 3k_1 - l_2)v \in T - T. \tag{3.10}$$

Since $9k - 3l + 3k_1 - l_2 \geq 9k - 0 - 3k - 1 = 6k - 1 > 2.24 + 3.24k$, which exceeds the upper bound of $|L|$. It concludes that $T \cap (T + kAv) = \emptyset$ and $(T + v) \cap (T + kAv) = \emptyset$, that is, T is disconnected (see Fig. 3c).

For $k \leq -3$, then $|L| \leq -k + (1 - k)\tilde{\alpha} < 2.24 - 3.24k$. It follows from (3.10) that $|9k - 3l + 3k_1 - l_2| \geq -9k - 3 + 3k - 1 = -6k - 4 > 2.24 - 3.24k$, which also exceeds the upper bound of $|L|$.

For $k = -2$, then $|K| < 3.24$ and $|L| < 8.72$. From (3.9), we have $6 + l - k_1 < 3.24$, hence $l = -1$ and $k_1 = 2$. From (3.10), we have $l_2 = -1$ and $3 + l_1 + k_2 < 3.24$, it follows that $l_1 = 0, k_2 = -2$, or $l_1 = 0, k_2 = 0$, or $l_1 = 1, k_2 = -2$.

When $l_1 = 0, k_2 = -2$. Multiplying (3.10) by A and then subtracting $k_3Av + l_3v$, we get

$$(-5 - k_3)Av + (3 - l_3)v \in T - T.$$

By $|5 + k_3| \leq 3.24$, it yields $k_3 = -2$. Repeating this process, we obtain

$$(12 - l_3 - k_4)Av + (9 - l_4)v \in T - T.$$

Hence we get a contradiction $|12 - l_3 - k_4| \geq 9 > 3.24$.

When $l_1 = 0, k_2 = 0$. Multiplying (3.10) by A and then subtracting $k_3 Av + l_3 v$, we get

$$(1 - k_3)Av + (9 - l_3)v \in T - T.$$

It yields $l_3 = 1$ and $k_3 = 0$ or 2. Repeating this process, we obtain

$$(3k_3 + 5 - k_4)Av + (3k_3 - 3 - l_{4)}v \in T - T.$$

If $k_3 = 0$, then $(5 - k_4)Av + (-3 - l_4)v \in T - T$, and $k_4 = 2$. Finally we get $(-12 - l_4 - k_5)Av + (-9 - l_5)v \in T - T$ and a contradiction $|12 + l_{4+}k_5| \geq 9 > 3.24$. If $k_3 = 2$, then $(11 - k_4)Av + (3 - l_4)v \in T - T$, and also $|11 - k_4| \geq 9 > 3.24$.

When $l_1 = 1, k_2 = -2$. By the same argument as above, we first get

$$(-2 - k_3)Av + (6 - l_3)v \in T - T.$$

It yields $k_3 = 0$. Repeating this process, we obtain $(12 - l_3 - k_4)Av + (6 - l_4)v \in T - T$ and $|12 - l_{3-}k_4| \geq 9 > 3.24$ follows. Therefore T is disconnected for $|k| > 1$.

Case 4 $f(x) = x^2 + x - 3$: For $k = 1$. We deduce from $f(A) = 0$ that $I = (A^2 - I)^{-1}(-A + 2I)$, which yields $I = \sum_{i=0}^{\infty} A^{-2i}(-A^{-1} + 2A^{-2})$ and

$$\begin{aligned} v &= \sum_{i=0}^{\infty} A^{-2i}(-A^{-1} + 2A^{-2})v \\ &= A^{-2}(v - Av) + \sum_{i=1}^{\infty} A^{-2i}\left(A^{-1}(Av - v) + A^{-2}v\right). \end{aligned}$$

Hence $T \cap (T + v) \neq \emptyset$. Moreover,

$$Av = A^{-1}(v - Av) + \sum_{i=1}^{\infty} A^{-2i}\left((Av - v) + A^{-1}v\right)$$

which implies $T \cap (T + Av) \neq \emptyset$. Consequently, T is connected (see Fig. 2d).

For $k = -1$. It follows from $v = A^{-1}(Av) \in T - T$ that $T \cap (T + v) \neq \emptyset$. Moreover, we can get $A + I = -I + (A - I)^{-1}$ from $f(A) = 0$. This implies

$$\begin{aligned} Av + v &= A^{-1}(-Av) + A^{-2}(Av) + A^{-3}(Av) + A^{-4}(Av) + \cdots \\ &\in T - T, \end{aligned}$$

that is, $(T + v) \cap (T - Av) \neq \emptyset$. Hence T is connected.

For $k > 1$. By (3.7) and (2.4), we have $|K| \leq (1+k)\tilde{\beta} = 1 + k$ and $|L| \leq k + (1+k)\tilde{\alpha} = 2 + 3k$. Suppose $kAv + lv \in T - T$ for $l = 0$ or -1. Multiplying (3.7) by A and then subtracting $k_1 Av + l_1 v$ from both sides, we have

$$(-k + l - k_1)Av + (3k - l_1)v \in T - T.$$

Repeating the process, we obtain

$$(4k - l + k_1 - l_1 - k_2)Av + (-3k + 3l - 3k_1 - l_2)v \in T - T. \tag{3.11}$$

Note $4k - l + k_1 - l_1 - k_2 \geq 2k - 1$. When $k \geq 3$, $2k - 1 > k + 1$ which contradicts the upper bound of $|K|$, hence T is disconnected; when $k = 2$, it forces $l = 0,\ k_1 = -2,\ l_1 = 1,\ k_2 = 2$ and $3Av - l_2 v \in T - T$, similarly which implies

$$(-3 - l_2 - k_3)Av + (9 - l_3)v \in T - T.$$

It is required that $|9-l_3| \leq 8$, hence $l_3 = 1$. Furthermore, from $(-3-l_2-k_3)Av+8v \in T-T$, we can deduce that

$$(11 + l_2 + k_3 - k_4)Av + (-9 - 3l_2 - 3k_2 - l_4)v \in T - T.$$

Since $11 + l_2 + k_3 - k_4 \geq 6 > 3$, we also get a contradiction, and T is disconnected. Consequently, T is disconnected for all $k > 1$ (see Fig. 3d).

For $k < -1$, then $|K| \leq 1-k$. From (3.11), it follows that $|K| \geq -4k+l-k_1+l_1+k_2 \geq -4k - 1 + k + 1 + k = -2k > 1 - k$, which is impossible. Therefore T is disconnected for $|k| > 1$.

Part C: $f(x) = x^2 - px \pm 3$ where $p > 0$.

The characteristic polynomial of A is $f(x) = x^2 - px \pm 3$ if and only if that of $B := -A$ is $g(x) = x^2 + px \pm 3$ which has been considered in Part B. The proof for the disconnectedness of $T(A, \mathcal{D})$ when $|k| > 1$ can be adapted easily from Part B by applying Corollary 2.3. Let

$$\mathcal{D}_1 = \{0, v, Bv\} = \{0, v, -Av\} \quad \text{and} \quad \mathcal{D}_2 = \{0, v, -Bv\} = \{0, v, Av\}.$$

For $|k| = 1$, we deduce the connectedness of $T_1 := T(A, \mathcal{D}_1)$ (respectively, $T_1' = T(A, \mathcal{D}_2)$) from that of $T(B, \mathcal{D}_1)$ (respectively, $T(B, \mathcal{D}_2)$).

We only show the case of $f(x) = x^2 - x + 3$. In Case 1 of Part B, from (3.5) we have

$$v = \sum_{i=0}^{\infty} (-A)^{-3i} \big(A^{-2}(-v) - A^{-3}(v + Av) + A^{-4}(-Av)\big) \in T_1 - T_1$$

and from (3.6) we have

$$Av = \sum_{i=0}^{\infty} (-A)^{-3i} \big(A^{-1}(-v) - A^{-2}(v + Av) + A^{-3}(-Av)\big) \in T_1 - T_1.$$

Hence T_1 is connected. Similarly, it can be verified that $v, Av \in T_1' - T_1'$ and T_1' is also connected.

Consequently, we finish the proof of Theorem 1.1.

4 Remarks

The proof of Part C in the last section is indeed an application of the following more general result.

Theorem 4.1 *Let $A \in M_n(\mathbb{Z})$ be an expanding matrix, and $\mathcal{D} \subset \mathbb{R}^n$ be a digit set. If a matrix B is similar to A, then there exists a digit set $\mathcal{D}'$ such that $T(A, \mathcal{D})$ is connected if and only if $T(B, \mathcal{D}')$ is connected.*

Proof Since B is similar to A, there exists an invertible matrix P such that $B = PAP^{-1}$. By letting $\mathcal{D}' = P\mathcal{D}$, then

$$\begin{aligned} T(B, \mathcal{D}') &= \left\{ \sum_{i=1}^{\infty} B^{-i} d'_{j_i} : d'_{j_i} \in \mathcal{D}' \right\} \\ &= \left\{ \sum_{i=1}^{\infty} PA^{-i}P^{-1} d'_{j_i} : d'_{j_i} \in \mathcal{D}' \right\} \end{aligned}$$

$$= P\left\{\sum_{i=1}^{\infty} A^{-i} d_{j_i} : d_{j_i} \in \mathcal{D}\right\}$$
$$= PT(A, \mathcal{D}).$$

Thus $T(A, \mathcal{D})$ and $T(B, \mathcal{D}')$ have the same connected property. □

Corollary 4.2 *Let $A, B \in M_2(\mathbb{Z})$ be two expanding matrices with characteristic polynomials $f(x) = x^2 + px + q$ and $g(x) = x^2 - px + q$ respectively. Let v be a vector such that $\{v, Av\}$ is linearly independent, and $L = \{aAv + bv : a, b \in \mathbb{R}\}$. If $\mathcal{D} \subset L$ is a digit set, then there exists a digit set $\mathcal{D}'$ such that $T_1 := T(A, \mathcal{D})$ is connected if and only if $T_2 := T(B, \mathcal{D}')$ is connected.*

Proof If $B = -A$, we let $\mathcal{D}' = \mathcal{D}$. For $aAv + bv = -aBv + bv \in \Delta\mathcal{D} = \Delta\mathcal{D}'$, we claim that $aAv + bv \in T_1 - T_1$ if and only if $-aBv + bv \in T_2 - T_2$. Then the desired result follows by Proposition 2.1. We now prove the claim: if $aAv + bv \in T_1 - T_1$ then

$$aAv + bv = \sum_{i=1}^{\infty} A^{-i}(c_i Av + d_i v)$$

where $c_i Av + d_i v \in \Delta\mathcal{D}$. Using the relation $A = -B$ and the symmetry of $\Delta\mathcal{D}$, it follows that

$$-aBv + bv = \sum_{i=1}^{\infty} B^{-i}\left((-1)^i(-c_i Bv + d_i v)\right) \in T_2 - T_2.$$

If otherwise, $B \neq -A$. Since v, Av are linearly independent, the minimal polynomial of A coincides with $f(x)$. Hence A is similar to the companion matrix of $f(x)$. So is B. By Theorem 4.1, we can assume that A, B are the companion matrices of $f(x)$, $g(x)$ respectively. Then $B = P(-A)P^{-1}$ where $P = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$. Combining the above argument and Theorem 4.1 yields the corollary. □

From the proof of Theorem 1.1 in the previous section, we can also deduce that

Corollary 4.3 *Let A be a 2×2 integral expanding matrix with $|\det A| = 3$. Let $v \in \mathbb{R}^2$ such that $\{v, Av\}$ is linearly independent. Then the self-affine set $T(A, \mathcal{D})$ is connected for $\mathcal{D} = \{0, v, Av + v\}$ or $\{0, v, -Av + v\}$.*

Proof Notice that the difference set $\Delta\{0, v, Av + v\} = \{0, \pm v, \pm Av, \pm(Av + v)\} = \Delta\{0, v, -Av\}$ and $\Delta\{0, v, -Av + v\} = \{0, \pm v, \pm Av, \pm(Av - v)\} = \Delta\{0, v, Av\}$. □

The connectedness of self-affine sets/tiles is far from known extensively. Even for the planar case, there are still a lot of unsolved questions. The following may be some interesting topics related to the paper.

Q1. Can we characterize the connectedness of $T(A, \mathcal{D})$ with $|\det A| = 3$ and $\mathcal{D} = \{0, v, kAv + lv\}$?

Q2. For a two dimensional digit set $\mathcal{D} = \{0, v, 2v, \ldots, (l-1)v, Av, 2Av, \ldots, kAv\}$ with $l + k = |\det A| > 3$, can we apply the same method to study the connectedness of $T(A, \mathcal{D})$?

Recently, Fu and Gabardo [5] devised an algorithm to find the integral points in $T - T$. Their method seems to be a different approach to the problems.

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