GEOMETRIC REALIZATIONS OF LUSZTIG'S SYMMETRIES

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Abstract. In this paper, we give geometric realizations of Lusztig's symmetries. We also give projective resolutions of a kind of standard modules. By using the geometric realizations and the projective resolutions, we obtain the categorification of the formulas of Lusztig's symmetries.

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1. Introduction

Let $\mathbf{U}$ be the quantum group and $\mathbf{f}$ be the Lusztig's algebra associated with a Cartan datum. Denote by $\mathbf{U}^+$ and $\mathbf{U}^-$ the positive part and the negative part of $\mathbf{U}$ respectively. There are two well-defined $\mathbb{Q}(v)$-algebra homomorphisms $^+ : \mathbf{f} \to \mathbf{U}$ and $^- : \mathbf{f} \to \mathbf{U}$ with images $\mathbf{U}^+$ and $\mathbf{U}^-$ respectively.

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Lusztig introduced the canonical basis $\mathbf{B}$ of $\mathbf{f}$ in [15, 17, 20]. Let $Q=(I,H)$ be a quiver corresponding to $\mathbf{f}$ and $\mathbf{V}$ be an I-graded vector space such that $\underline{\dim}\mathbf{V}=\nu\in\mathbb{N}I$. He studied the variety $E_{\mathbf{V}}$ consisting of representations of Q with dimension vector ν, and a category $\mathcal{Q}_{\mathbf{V}}$ of some semisimple complexes on $E_{\mathbf{V}}$. Let $K(\mathcal{Q}_{\mathbf{V}})$ be the Grothendieck group of $\mathcal{Q}_{\mathbf{V}}$. Considering all dimension vectors, he proved that $\bigoplus_{\nu\in\mathbb{N}I}K(\mathcal{Q}_{\mathbf{V}})$ realizes $\mathbf{f}$ and the set of isomorphism classes of simple objects realizes the canonical basis $\mathbf{B}$.

Lusztig also introduced some symmetries T_i on $\mathbf{U}$ for all $i\in I$ in [14, 16]. Note that $T_i(\mathbf{U}^+)$ is not contained in $\mathbf{U}^+$. Hence, Lusztig introduced two subalgebras ${}_i\mathbf{f}$ and ${}^i\mathbf{f}$ of $\mathbf{f}$ for any $i\in I$, where ${}_i\mathbf{f}=\{x\in\mathbf{f}\mid T_i(x^+)\in\mathbf{U}^+\}$ and ${}^i\mathbf{f}=\{x\in\mathbf{f}\mid T_i^{-1}(x^+)\in\mathbf{U}^+\}$. Let $T_i:{}_i\mathbf{f}\to{}^i\mathbf{f}$ be the unique map satisfying $T_i(x^+)=T_i(x)^+$. The algebra $\mathbf{f}$ has the following direct sum decompositions $\mathbf{f}={}_i\mathbf{f}\bigoplus\theta_i\mathbf{f}={}^i\mathbf{f}\bigoplus\mathbf{f}\theta_i$. Denote by ${}_i\pi:\mathbf{f}\to{}_i\mathbf{f}$ and ${}^i\pi:\mathbf{f}\to{}^i\mathbf{f}$ the natural projections.

Associated to a finite dimensional hereditary algebra Λ, Ringel introduced the Hall algebra and the composition subalgebra $\mathcal{F}$ in [22], which gives a realization of the positive part of the quantum group $\mathbf{U}$. If we use the notations of Lusztig in [19], we have the canonical isomorphism between the composition subalgebra $\mathcal{F}$ and the Lusztig's algebra $\mathbf{f}$. Via the Hall algebra approach, one can apply BGP-reflection functors to quantum groups to give precise constructions of Lusztig's symmetries ([23, 19, 26, 28, 6, 29]).

To a Lusztig's algebra $\mathbf{f}$, Khovanov, Lauda ([9]) and Rouquier ([24]) introduced a series of algebras $\mathbf{R}_\nu$ respectively. The category of finitely generated projective modules of $\mathbf{R}_\nu$ gives a categorification of $\mathbf{f}$ and $\mathbf{R}_\nu$ are called Khovanov-Lauda-Rouquier (KLR) algebras. Varagnolo, Vasserot ([27]) and Rouquier ([25]) realized the KLR algebra $\mathbf{R}_\nu$ as the extension algebra of semisimple complexes in $\mathcal{Q}_{\mathbf{V}}$ and proved that the set of indecomposable projective modules of $\mathbf{R}_\nu$ can categorify the canonical basis $\mathbf{B}$.

In [7, 8], Kato gave the categorification of the PBW-type bases of quantum groups of finite type. He constructed some modules (which are called standard modules) of the KLR algebras $\mathbf{R}_\nu$ and proved that there standard modules can categorify the PBW-type basis of $\mathbf{f}$ by using the geometric realizations of $\mathbf{R}_\nu$ given by Varagnolo, Vasserot and Rouquier. He proved that the length of the projective resolution of any standard module is finite, which is the categorification of the following fact: the transition matrix between the PBW-type basis of $\mathbf{f}$ and the canonical basis $\mathbf{B}$ is triangular with diagonal entries equal to 1. This result implies that the global dimensions of the KLR algebras $\mathbf{R}_\nu$ are also finite. In [21, 4, 3, 11], Brundan, Kleshchev and McNamara proved the same result by using an algebraic method.

Let $i\in I$ be a sink (resp. source) of Q. Similarly to the geometric realization of $\mathbf{f}$, consider a subvariety ${}_iE_{\mathbf{V}}$ (resp. ${}^iE_{\mathbf{V}}$) of $E_{\mathbf{V}}$ and a category ${}_i\mathcal{Q}_{\mathbf{V}}$ (resp. ${}^i\mathcal{Q}_{\mathbf{V}}$) of some semisimple complexes on ${}_iE_{\mathbf{V}}$ (resp. ${}^iE_{\mathbf{V}}$). In Section 3.2, we verify that $\bigoplus_{\nu\in\mathbb{N}I}K({}_i\mathcal{Q}_{\mathbf{V}})$ (resp. $\bigoplus_{\nu\in\mathbb{N}I}K({}^i\mathcal{Q}_{\mathbf{V}})$) realizes ${}_i\mathbf{f}$ (resp. ${}^i\mathbf{f}$).

Let $i\in I$ be a sink of Q. Let $Q'=\sigma_iQ$ be the quiver by reversing the directions of all arrows in Q containing i. Hence, i is a source of Q'. Consider two I-graded vector spaces $\mathbf{V}$ and $\mathbf{V}'$ such that $\underline{\dim}\mathbf{V}'=s_i(\underline{\dim}\mathbf{V})$. In the case of finite type, Kato introduced an equivalence $\tilde{\omega}_i:{}_i\mathcal{Q}_{\mathbf{V},Q}\to{}^i\mathcal{Q}_{\mathbf{V}',Q'}$ and studied the properties of this equivalence in [7, 8]. In this paper, we generalize his construction to all cases and prove that the map induced by $\tilde{\omega}_i$ realizes the Lusztig's symmetry $T_i:{}_i\mathbf{f}\to{}^i\mathbf{f}$.

For the proof of the result, we shall study the relations between the map induced by $\tilde{\omega}_i$ and the Hall algebra approach to T_i in [19].

In [18], Lusztig showed that Lusztig's symmetries and canonical bases are compatible. Let ${}_i\mathbf{B} = {}_i\pi(\mathbf{B})$, which is a $\mathbb{Q}(v)$-basis of ${}_i\mathbf{f}$. Similarly, ${}^i\mathbf{B} = {}^i\pi(\mathbf{B})$ is a $\mathbb{Q}(v)$-basis of ${}^i\mathbf{f}$. Lusztig proved that $T_i : {}_i\mathbf{f} \to {}^i\mathbf{f}$ maps any element of ${}_i\mathbf{B}$ to an element of ${}^i\mathbf{B}$.

For any simple perverse sheaf $\mathcal{L}$ in $\mathcal{Q}_{\mathbf{V},Q}$, the restriction ${}_i\mathcal{L} = j^*_{\mathbf{V}}(\mathcal{L})$ on ${}_iE_{\mathbf{V},Q}$ is also a simple perverse sheaf and belongs to ${}_i\mathcal{Q}_{\mathbf{V},Q}$, where $j_{\mathbf{V}} : {}_iE_{\mathbf{V},Q} \to E_{\mathbf{V},Q}$ is the canonical embedding. Let ${}^i\mathcal{L} = \tilde{\omega}_i({}_i\mathcal{L}) \in {}^i\mathcal{Q}_{\mathbf{V}',Q'}$. The simple perverse sheaf ${}^i\mathcal{L}$ can be wrote as ${}^i\mathcal{L} = j^*_{\mathbf{V}'}(\mathcal{L}')$, where $\mathcal{L}'$ is a simple perverse sheaf in $\mathcal{Q}_{\mathbf{V}',Q'}$ and $j_{\mathbf{V}'} : {}^iE_{\mathbf{V}',Q'} \to E_{\mathbf{V}',Q'}$ is the canonical embedding. Since the map induced by $\tilde{\omega}_i$ realizes $T_i : {}_i\mathbf{f} \to {}^i\mathbf{f}$, this result gives a geometric interpretation of Lusztig's result in [18].

For any $m \leq -a_{ij}$, let

$$f(i,j;m) = \sum_{r+s=m} (-1)^r v^{-r(-a_{ij}-m+1)} \theta_i^{(r)} \theta_j \theta_i^{(s)} \in \mathbf{f},$$

and

$$f'(i,j;m) = \sum_{r+s=m} (-1)^r v^{-r(-a_{ij}-m+1)} \theta_i^{(s)} \theta_j \theta_i^{(r)} \in \mathbf{f}.$$

In [20], Lusztig proved that $T_i(f(i,j;m)) = f'(i,j;m')$, where $m' = -a_{ij} - m$. The following formula

$$T_i(E_j) = \sum_{r+s=-a_{ij}} (-1)^r v^{-r} E_i^{(s)} E_j E_i^{(r)}$$

is a special case of $T_i(f(i,j;m)) = f'(i,j;m')$. In this paper, our main result is the categorification of these formulas. Consider an I-graded vector space $\mathbf{V}$ such that $\underline{\dim}\mathbf{V} = \nu = mi + j$. Let $\mathcal{D}_{G_{\mathbf{V}}}(E_{\mathbf{V}})$ be the bounded $G_{\mathbf{V}}$-equivariant derived category of complexes of l-adic sheaves on $E_{\mathbf{V}}$. We construct a series of distinguished triangles in $\mathcal{D}_{G_{\mathbf{V}}}(E_{\mathbf{V}})$, which represent the constant sheaf $\mathbf{1}_{{}_iE_{\mathbf{V}}}$ in terms of some semisimple complexes $I_p \in \mathcal{D}_{G_{\mathbf{V}}}(E_{\mathbf{V}})$ geometrically. Note that, $\mathbf{1}_{{}_iE_{\mathbf{V}}}$ corresponds to a standard module K_m of the KLR algebra $\mathbf{R}_\nu$ and I_p correspond to projective modules of $\mathbf{R}_\nu$. This result means that we find projective resolutions of the standard modules K_m. Consider two I-graded vector spaces $\mathbf{V}$ and $\mathbf{V}'$ such that $\underline{\dim}\mathbf{V} = mi + j$ and $\underline{\dim}\mathbf{V}' = s_i(\underline{\dim}\mathbf{V}) = m'i + j$. Applying to the Grothendieck group, $\mathbf{1}_{{}_iE_{\mathbf{V},Q}}$ (resp. $\mathbf{1}_{{}^iE_{\mathbf{V}',Q'}}$) corresponds to $f(i,j;m)$ (resp. $f'(i,j;m')$). The property of BGP-reflection functors implies $\tilde{\omega}_i(v^{-mN}\mathbf{1}_{{}_iE_{\mathbf{V},Q}}) = v^{-m'N}\mathbf{1}_{{}^iE_{\mathbf{V}',Q'}}$, therefore $T_i(f(i,j;m)) = f'(i,j;m')$.

In Example D of [8], Kato constructed a short exact sequence

$$0 \longrightarrow P_1 * P_2[2] \longrightarrow P_2 * P_1 \longrightarrow Q_{21} \longrightarrow 0,$$

which coincides with the projection resolution in our main result in the case of finite type. In Theorem 4.10 of [4], Brundan, Kleshchev and McNamara constructed a shout exact sequence of standard modules

$$0 \longrightarrow v^{-\beta\cdot\gamma}\Delta(\beta) \circ \Delta(\gamma) \longrightarrow \Delta(\gamma) \circ \Delta(\beta) \longrightarrow [p_{\beta,\gamma} + 1]\Delta(\alpha) \longrightarrow 0.$$

In the case of finite type, the projection resolution in our main result is a special case of the shout exact sequence above where $\alpha = \alpha_i + \alpha_j$.

2. Quantum groups and Lusztig's symmetries

2.1. **Quantum groups.** Let I be a finite index set with $|I| = n$ and $A = (a_{ij})_{i,j\in I}$ be a generalized Cartan matrix. Let $(A, \Pi, \Pi^\vee, P, P^\vee)$ be a Cartan datum associated with A, where

(1) $\Pi = \{\alpha_i \mid i \in I\}$ is the set of simple roots;
(2) $\Pi^\vee = \{h_i \mid i \in I\}$ is the set of simple coroots;
(3) P is the weight lattice;
(4) $P^\vee$ is the dual weight lattice.

In this paper, we always assume that the generalized Cartan matrix A is symmetric. Fix an indeterminate v. For any $n \in \mathbb{Z}$, set $[n]_v = \frac{v^n - v^{-n}}{v - v^{-1}} \in \mathbb{Q}(v)$. Let $[0]_v! = 1$ and $[n]_v! = [n]_v[n-1]_v \cdots [1]_v$ for any $n \in \mathbb{Z}_{>0}$.

The quantum group $\mathbf{U}$ associated with a Cartan datum $(A, \Pi, \Pi^\vee, P, P^\vee)$ is an associative algebra over $\mathbb{Q}(v)$ with unit element $\mathbf{1}$, generated by the elements E_i, $F_i (i \in I)$ and $K_\mu (\mu \in P^\vee)$ subject to the following relations

$$K_0 = \mathbf{1}, \quad K_\mu K_{\mu'} = K_{\mu+\mu'} \text{ for all } \mu, \mu' \in P^\vee;$$

$$K_\mu E_i K_{-\mu} = v^{\alpha_i(\mu)} E_i \text{ for all } i \in I,\ \mu \in P^\vee;$$

$$K_\mu F_i K_{-\mu} = v^{-\alpha_i(\mu)} F_i \text{ for all } i \in I,\ \mu \in P^\vee;$$

$$E_i F_j - F_j E_i = \delta_{ij} \frac{K_i - K_{-i}}{v - v^{-1}} \text{ for all } i, j \in I;$$

$$\sum_{k=0}^{1-a_{ij}} (-1)^k E_i^{(k)} E_j E_i^{(1-a_{ij}-k)} = 0 \text{ for all } i \neq j \in I;$$

$$\sum_{k=0}^{1-a_{ij}} (-1)^k F_i^{(k)} F_j F_i^{(1-a_{ij}-k)} = 0 \text{ for all } i \neq j \in I.$$

Here, $K_i = K_{h_i}$ and $E_i^{(n)} = E_i^n/[n]_v!$, $F_i^{(n)} = F_i^n/[n]_v!$.

Let $\mathbf{U}^+$ (resp. $\mathbf{U}^-$) be the subalgebra of $\mathbf{U}$ generated by E_i (resp. F_i) for all $i \in I$, and $\mathbf{U}^0$ be the subalgebra of $\mathbf{U}$ generated by K_μ for all $\mu \in P^\vee$. The quantum group $\mathbf{U}$ has the following triangular decomposition

$$\mathbf{U} \cong \mathbf{U}^- \otimes \mathbf{U}^0 \otimes \mathbf{U}^+.$$

Let $\mathbf{f}$ be the associative algebra defined by Lusztig in [20]. The algebra $\mathbf{f}$ is generated by $\theta_i (i \in I)$ subject to the following relations

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \theta_i^{(k)} \theta_j \theta_i^{(1-a_{ij}-k)} = 0 \text{ for all } i \neq j \in I,$$

where $\theta_i^{(n)} = \theta_i^n/[n]_v!$.

There are two well-defined $\mathbb{Q}(v)$-algebra homomorphisms $^+ : \mathbf{f} \to \mathbf{U}$ and $^- : \mathbf{f} \to \mathbf{U}$ satisfying $E_i = \theta_i^+$ and $F_i = \theta_i^-$ for all $i \in I$. The images of $^+$ and $^-$ are $\mathbf{U}^+$ and $\mathbf{U}^-$ respectively.

2.2. **Lusztig's symmetries.** Corresponding to $i \in I$, Lusztig introduced the Lusztig's symmetry $T_i : \mathbf{U} \to \mathbf{U}$ ([14, 16, 20]). The formulas of T_i on the generators are:

$$T_i(E_i) = -F_iK_i, \;\; T_i(F_i) = -K_{-i}E_i;$$

$$T_i(E_j) = \sum_{r+s=-a_{ij}} (-1)^r v^{-r} E_i^{(s)} E_j E_i^{(r)} \text{ for } i \neq j \in I; \tag{1}$$

$$T_i(F_j) = \sum_{r+s=-a_{ij}} (-1)^r v^{r} F_i^{(r)} F_j F_i^{(s)} \text{ for } i \neq j \in I; \tag{2}$$

$$T_i(K_\mu) = K_{\mu - \alpha_i(\mu) h_i}.$$

Lusztig introduced two subalgebras ${}_i\mathbf{f}$ and ${}^i\mathbf{f}$ of $\mathbf{f}$. For any $j \in I$, $i \neq j$, $m \in \mathbb{N}$, define

$$f(i,j;m) = \sum_{r+s=m} (-1)^r v^{-r(-a_{ij}-m+1)} \theta_i^{(r)} \theta_j \theta_i^{(s)} \in \mathbf{f},$$

and

$$f'(i,j;m) = \sum_{r+s=m} (-1)^r v^{-r(-a_{ij}-m+1)} \theta_i^{(s)} \theta_j \theta_i^{(r)} \in \mathbf{f}.$$

The subalgebras ${}_i\mathbf{f}$ and ${}^i\mathbf{f}$ are generated by $f(i,j;m)$ and $f'(i,j;m)$ respectively.

Note that ${}_i\mathbf{f} = \{x \in \mathbf{f} \mid T_i(x^+) \in \mathbf{U}^+\}$ and ${}^i\mathbf{f} = \{x \in \mathbf{f} \mid T_i^{-1}(x^+) \in \mathbf{U}^+\}$ ([20]). Hence there exists a unique $T_i : {}_i\mathbf{f} \to {}^i\mathbf{f}$ such that $T_i(x^+) = T_i(x)^+$. Lusztig also showed that $\mathbf{f}$ has the following direct sum decompositions $\mathbf{f} = {}_i\mathbf{f} \bigoplus \theta_i\mathbf{f} = {}^i\mathbf{f} \bigoplus \mathbf{f}\theta_i$. Denote by ${}_i\pi : \mathbf{f} \to {}_i\mathbf{f}$ and ${}^i\pi : \mathbf{f} \to {}^i\mathbf{f}$ the natural projections.

Lusztig also proved the following formulas.

Proposition 2.1 ([20])**.** *For any $-a_{ij} \geq m \in \mathbb{N}$, $T_i(f(i,j;m)) = f'(i,j;-a_{ij}-m)$.*

The formulas (1) and (2) are two special cases of Proposition 2.1.

3. Geometric realizations

3.1. **Geometric realization and canonical basis of f.** In this subsection, we shall review the geometric realization of $\mathbf{f}$ introduced by Lusztig ([15, 17, 20]).

A quiver $Q = (I, H, s, t)$ consists of a vertex set I, an arrow set H, and two maps $s, t : H \to I$ such that an arrow $\rho \in H$ starts at $s(\rho)$ and terminates at $t(\rho)$. From now on, assume that $s(\rho) \neq t(\rho)$ for any $\rho \in H$.

For any $i, j \in I$, let

$$a_{ij} = \begin{cases} -\#\{i \to j\} - \#\{j \to i\}, & \text{if } i \neq j; \\ 2, & \text{if } i = j. \end{cases}$$

The matrix $A = (a_{ij})_{i,j \in I}$ is a symmetric generalized Cartan matrix. Let $\mathbf{f}$ be the Lusztig's algebra corresponding to A. Let p be a prime and q be a power of p. Denote by $\mathbb{F}_q$ the finite field with q elements and $\mathbb{K} = \overline{\mathbb{F}}_q$.

For a finite dimensional I-graded $\mathbb{K}$-vector space $\mathbf{V} = \bigoplus_{i \in I} V_i$, define

$$E_\mathbf{V} = \bigoplus_{\rho \in H} \mathrm{Hom}_\mathbb{K}(V_{s(\rho)}, V_{t(\rho)}).$$

The dimension vector of $\mathbf{V}$ is defined as $\underline{\dim}\mathbf{V} = \sum_{i \in I} (\dim_\mathbb{K} V_i) i \in \mathbb{N}I$. The algebraic group $G_\mathbf{V} = \prod_{i \in I} GL_\mathbb{K}(V_i)$ acts on $E_\mathbf{V}$ naturally.

Fix a nonzero element $\nu \in \mathbb{N}I$. Let

$$Y_\nu = \{\mathbf{y} = (\mathbf{i}, \mathbf{a}) \mid \sum_{l=1}^{k} a_l i_l = \nu\},$$

where $\mathbf{i} = (i_1, i_2, \ldots, i_k),\ i_l \in I$, $\mathbf{a} = (a_1, a_2, \ldots, a_k),\ a_l \in \mathbb{N}$, and

$$I^\nu = \{\mathbf{i} = (i_1, i_2, \ldots, i_k) \mid \sum_{l=1}^{k} i_l = \nu\}.$$

Fix a finite dimensional I-graded $\mathbb{K}$-vector space $\mathbf{V}$ such that $\underline{\dim}\mathbf{V} = \nu$. For any element $\mathbf{y} = (\mathbf{i}, \mathbf{a})$, a flag of type $\mathbf{y}$ in $\mathbf{V}$ is a sequence

$$\phi = (\mathbf{V} = \mathbf{V}^k \supset \mathbf{V}^{k-1} \supset \cdots \supset \mathbf{V}^0 = 0)$$

of I-graded $\mathbb{K}$-vector spaces such that $\underline{\dim}\mathbf{V}^l/\mathbf{V}^{l-1} = a_l i_l$. Let $F_\mathbf{y}$ be the variety of all flags of type $\mathbf{y}$ in $\mathbf{V}$. For any $x \in E_\mathbf{V}$, a flag ϕ is called x-stable if $x_\rho(V^l_{s(\rho)}) \subset V^l_{t(\rho)}$ for all l and all $\rho \in H$. Let

$$\tilde{F}_\mathbf{y} = \{(x, \phi) \in E_\mathbf{V} \times F_\mathbf{y} \mid \phi \text{ is } x\text{-stable}\}$$

and $\pi_\mathbf{y} : \tilde{F}_\mathbf{y} \to E_\mathbf{V}$ be the projection to $E_\mathbf{V}$.

Let $\bar{\mathbb{Q}}_l$ be the l-adic field and $\mathcal{D}_{G_\mathbf{V}}(E_\mathbf{V})$ be the bounded $G_\mathbf{V}$-equivariant derived category of complexes of l-adic sheaves on $E_\mathbf{V}$. For each $\mathbf{y} \in Y_\nu$, consider $\mathcal{L}_\mathbf{y} = (\pi_\mathbf{y})_!(\mathbf{1}_{\tilde{F}_\mathbf{y}})[d_\mathbf{y}] \in \mathcal{D}_{G_\mathbf{V}}(E_\mathbf{V})$, where $d_\mathbf{y} = \dim \tilde{F}_\mathbf{y}$. Since $F_\mathbf{y}$ is a smooth, irreducible, projective variety and the projection $\tilde{F}_\mathbf{y} \to F_\mathbf{y}$ is a vector bundle, $\tilde{F}_\mathbf{y}$ is a smooth, irreducible variety. At the same time, $\pi_\mathbf{y} : \tilde{F}_\mathbf{y} \to E_\mathbf{V}$ is a proper $G_\mathbf{V}$-equivariant morphism. Hence, by BBD decomposition theorem ([1, 5]), $\mathcal{L}_\mathbf{y}$ is a semisimple complex. Let $\mathcal{P}_\mathbf{V}$ be the set of isomorphism classes of simple perverse sheaves $\mathcal{L}$ on $E_\mathbf{V}$ such that $\mathcal{L}[r]$ appears as a direct summand of $\mathcal{L}_\mathbf{i}$ for some $\mathbf{i} \in I^\nu$ and $r \in \mathbb{Z}$. Let $\mathcal{Q}_\mathbf{V}$ be the full subcategory of $\mathcal{D}_{G_\mathbf{V}}(E_\mathbf{V})$ consisting of all complexes which are isomorphic to finite direct sums of complexes in the set $\{\mathcal{L}[r] \mid \mathcal{L} \in \mathcal{P}_\mathbf{V}, r \in \mathbb{Z}\}$.

Let $K(\mathcal{Q}_\mathbf{V})$ be the Grothendieck group of $\mathcal{Q}_\mathbf{V}$. Define

$$v^{\pm}[\mathcal{L}] = [\mathcal{L}[\pm 1](\pm\frac{1}{2})],$$

where $\mathcal{L}(d)$ is the Tate twist of $\mathcal{L}$. Then, $K(\mathcal{Q}_\mathbf{V})$ is a free $\mathcal{A}$-module, where $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$. Define

$$K(\mathcal{Q}) = \bigoplus_{\nu \in \mathbb{N}I} K(\mathcal{Q}_\mathbf{V}).$$

For $\nu, \nu', \nu'' \in \mathbb{N}I$ such that $\nu = \nu' + \nu''$ and three I-graded $\mathbb{K}$-vector spaces $\mathbf{V}$, $\mathbf{V}'$, $\mathbf{V}''$ such that $\underline{\dim}\mathbf{V} = \nu$, $\underline{\dim}\mathbf{V}' = \nu'$, $\underline{\dim}\mathbf{V}'' = \nu''$, Lusztig constructed a functor

$$* : \mathcal{Q}_{\mathbf{V}'} \times \mathcal{Q}_{\mathbf{V}''} \to \mathcal{Q}_\mathbf{V}.$$

This functor induces an associative $\mathcal{A}$-bilinear multiplication

$$\begin{array}{rcl} \circledast : K(\mathcal{Q}_{\mathbf{V}'}) \times K(\mathcal{Q}_{\mathbf{V}''}) & \to & K(\mathcal{Q}_\mathbf{V}) \\ ([\mathcal{L}'], [\mathcal{L}'']) & \mapsto & [\mathcal{L}'] \circledast [\mathcal{L}''] = [\mathcal{L}' \circledast \mathcal{L}''], \end{array}$$

where $\mathcal{L}' \circledast \mathcal{L}'' = (\mathcal{L}' * \mathcal{L}'')[m_{\nu'\nu''}](\frac{m_{\nu'\nu''}}{2})$ and $m_{\nu'\nu''} = \sum_{\rho \in H} \nu'_{s(\rho)}\nu''_{t(\rho)} - \sum_{i \in I} \nu'_i \nu''_i$. Then $K(\mathcal{Q})$ becomes an associative $\mathcal{A}$-algebra and the set $\{[\mathcal{L}] \mid \mathcal{L} \in \mathcal{P}_\mathbf{V}\}$ is a basis of $K(\mathcal{Q}_\mathbf{V})$.

Theorem 3.1 ([17])**.** *There is a unique $\mathcal{A}$-algebra isomorphism*

$$\lambda_{\mathcal{A}} : K(\mathcal{Q}) \to \mathbf{f}_{\mathcal{A}}$$

such that $\lambda_{\mathcal{A}}(\mathcal{L}_{\mathbf{y}}) = \theta_{\mathbf{y}}$ for all $\mathbf{y} \in Y_\nu$, where $\theta_{\mathbf{y}} = \theta_{i_1}^{(a_1)}\theta_{i_2}^{(a_2)}\cdots\theta_{i_k}^{(a_k)}$ and $\mathbf{f}_{\mathcal{A}}$ is the integral form of $\mathbf{f}$.

Let $\mathbf{B}_\nu = \{\mathbf{b}_{\mathcal{L}} = \lambda_{\mathcal{A}}([\mathcal{L}]) \mid \mathcal{L} \in \mathcal{P}_{\mathbf{V}}\}$ and $\mathbf{B} = \sqcup_{\nu\in\mathbb{N}I}\mathbf{B}_\nu$. Then $\mathbf{B}$ is the canonical basis of $\mathbf{f}$ introduced by Lusztig in [15, 17].

3.2. **Geometric realizations of ${}_i\mathbf{f}$ and ${}^i\mathbf{f}$.** Assume that $i \in I$ is a sink. Let $\mathbf{V}$ be a finite dimensional I-graded $\mathbb{K}$-vector space such that $\underline{\dim}\mathbf{V} = \nu$. Consider a subvariety ${}_iE_{\mathbf{V}}$ of $E_{\mathbf{V}}$

$${}_iE_{\mathbf{V}} = \{x \in E_{\mathbf{V}} \mid \bigoplus_{h\in H, t(h)=i} x_h : \bigoplus_{h\in H, t(h)=i} V_{s(h)} \to V_i \text{ is surjective}\}.$$

Let $j_{\mathbf{V}} : {}_iE_{\mathbf{V}} \to E_{\mathbf{V}}$ be the canonical embedding. For any $\mathbf{y} = (\mathbf{i}, \mathbf{a}) \in Y_\nu$, let

$${}_i\tilde{F}_{\mathbf{y}} = \{(x, \phi) \in {}_iE_{\mathbf{V}} \times F_{\mathbf{y}} \mid \phi \text{ is } x\text{-stable}\}$$

and ${}_i\pi_{\mathbf{y}} : {}_i\tilde{F}_{\mathbf{y}} \to {}_iE_{\mathbf{V}}$ be the projection to ${}_iE_{\mathbf{V}}$.

For any $\mathbf{y} \in Y_\nu$, let ${}_i\mathcal{L}_{\mathbf{y}} = ({}_i\pi_{\mathbf{y}})_!(\mathbf{1}_{{}_i\tilde{F}_{\mathbf{y}}})[d_{\mathbf{y}}] \in \mathcal{D}_{G_{\mathbf{V}}}({}_iE_{\mathbf{V}})$. Since ${}_iE_{\mathbf{V}}$ is an open subvariety of $E_{\mathbf{V}}$, it is also a smooth variety. At the same time, ${}_i\pi_{\mathbf{y}} : {}_i\tilde{F}_{\mathbf{y}} \to {}_iE_{\mathbf{V}}$ is a projection. Hence, BBD decomposition theorem implies that ${}_i\mathcal{L}_{\mathbf{y}}$ is a semisimple complex. Let ${}_i\mathcal{P}_{\mathbf{V}}$ be the set of isomorphism classes of simple perverse sheaves $\mathcal{L}$ on ${}_iE_{\mathbf{V}}$ such that $\mathcal{L}[r]$ appears as a direct summand of ${}_i\mathcal{L}_{\mathbf{i}}$ for some $\mathbf{i} \in I^\nu$ and $r \in \mathbb{Z}$. Let ${}_i\mathcal{Q}_{\mathbf{V}}$ be the full subcategory of $\mathcal{D}_{G_{\mathbf{V}}}({}_iE_{\mathbf{V}})$ consisting of all complexes which are isomorphic to finite direct sums of complexes in the set $\{\mathcal{L}[r] \mid \mathcal{L} \in {}_i\mathcal{P}_{\mathbf{V}}, r \in \mathbb{Z}\}$.

Let $K({}_i\mathcal{Q}_{\mathbf{V}})$ be the Grothendieck group of ${}_i\mathcal{Q}_{\mathbf{V}}$ and

$$K({}_i\mathcal{Q}) = \bigoplus_{\nu\in\mathbb{N}I} K({}_i\mathcal{Q}_{\mathbf{V}}).$$

Naturally, we have two functors $j_{\mathbf{V}!} : \mathcal{D}_{G_{\mathbf{V}}}({}_iE_{\mathbf{V}}) \to \mathcal{D}_{G_{\mathbf{V}}}(E_{\mathbf{V}})$ and $j_{\mathbf{V}}^* : \mathcal{D}_{G_{\mathbf{V}}}(E_{\mathbf{V}}) \to \mathcal{D}_{G_{\mathbf{V}}}({}_iE_{\mathbf{V}})$.

For any $\mathbf{y} \in Y_\nu$, we have the following fiber product

$$\begin{array}{ccc} {}_i\tilde{F}_{\mathbf{y}} & \xrightarrow{\tilde{j}_{\mathbf{V}}} & \tilde{F}_{\mathbf{y}} \\ \downarrow {\scriptstyle {}_i\pi_y} & & \downarrow {\scriptstyle \pi_y} \\ {}_iE_{\mathbf{V}} & \xrightarrow{j_{\mathbf{V}}} & E_{\mathbf{V}}. \end{array}$$

By the smooth base change formula, we have

$$j_{\mathbf{V}}^*\mathcal{L}_{\mathbf{y}} = j_{\mathbf{V}}^*(\pi_y)_!(\mathbf{1}_{\tilde{F}_{\mathbf{y}}})[d_{\mathbf{y}}] = ({}_i\pi_y)_!\tilde{j}_{\mathbf{V}}^*(\mathbf{1}_{\tilde{F}_{\mathbf{y}}})[d_{\mathbf{y}}] = ({}_i\pi_y)_!(\mathbf{1}_{{}_i\tilde{F}_{\mathbf{y}}})[d_{\mathbf{y}}] = {}_i\mathcal{L}_{\mathbf{y}}. \tag{3}$$

That is $j_{\mathbf{V}}^*(\mathcal{Q}_{\mathbf{V}}) = {}_i\mathcal{Q}_{\mathbf{V}}$. Hence $j_{\mathbf{V}}^* : \mathcal{Q}_{\mathbf{V}} \to {}_i\mathcal{Q}_{\mathbf{V}}$ and $j^* : K(\mathcal{Q}) \to K({}_i\mathcal{Q})$ can be defined.

Consider the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \theta_i\mathbf{f}_{\mathcal{A}} & \xrightarrow{i} & \mathbf{f}_{\mathcal{A}} & \xrightarrow{{}_i\pi_{\mathcal{A}}} & {}_i\mathbf{f}_{\mathcal{A}} & \longrightarrow & 0 \\ & & & & \downarrow {\scriptstyle \lambda'_{\mathcal{A}}} & & \vdots & & \\ & & & & K(\mathcal{Q}) & \xrightarrow{j^*} & K({}_i\mathcal{Q}) & \longrightarrow & 0, \end{array} \tag{4}$$

where $\lambda'_{\mathcal{A}}$ is the inverse of $\lambda_{\mathcal{A}}$. Since $j^* \circ \lambda'_{\mathcal{A}} \circ i = 0$, there exists a map ${}_i\lambda'_{\mathcal{A}} : {}_i\mathbf{f}_{\mathcal{A}} \to K({}_i\mathcal{Q})$ such that the above diagram (4) commutes.

Proposition 3.2. *The map* ${}_i\lambda'_{\mathcal{A}} : {}_i\mathbf{f}_{\mathcal{A}} \to K({}_i\mathcal{Q})$ *is an isomorphism of* $\mathcal{A}$*-algebras.*

The proof of Proposition 3.2 will be given in Section 4.2.

Assume that $i \in I$ is a source. We can give a geometric realization of ${}^i\mathbf{f}$ similarly. Consider a subvariety ${}^iE_{\mathbf{V}}$ of $E_{\mathbf{V}}$

$$ {}^iE_{\mathbf{V}} = \{x \in E_{\mathbf{V}} \mid \bigoplus_{h\in H, s(h)=i} x_h : V_i \to \bigoplus_{h\in H, s(h)=i} V_{t(h)} \text{ is injective}\}. $$

Let $j_{\mathbf{V}} : {}^iE_{\mathbf{V}} \to E_{\mathbf{V}}$ be the canonical embedding. Since ${}^iE_{\mathbf{V}}$ is an open subvariety of $E_{\mathbf{V}}$, it is also a smooth variety. The definitions of ${}^i\mathcal{Q}_{\mathbf{V}}$, $K({}^i\mathcal{Q}_{\mathbf{V}})$ and $K({}^i\mathcal{Q})$ are similar to those of ${}_i\mathcal{Q}_{\mathbf{V}}$, $K({}_i\mathcal{Q}_{\mathbf{V}})$ and $K({}_i\mathcal{Q})$ respectively. We can also define $j^*_{\mathbf{V}} : \mathcal{Q}_{\mathbf{V}} \to {}^i\mathcal{Q}_{\mathbf{V}}$, $j^* : K(\mathcal{Q}) \to K({}^i\mathcal{Q})$ and ${}^i\lambda'_{\mathcal{A}} : {}^i\mathbf{f}_{\mathcal{A}} \to K({}^i\mathcal{Q})$.

Similarly to Proposition 3.2, we have the following proposition.

Proposition 3.3. *The map* ${}^i\lambda'_{\mathcal{A}} : {}^i\mathbf{f}_{\mathcal{A}} \to K({}^i\mathcal{Q})$ *is an isomorphism of* $\mathcal{A}$*-algebras.*

□

3.3. **Geometric realization of** $T_i : {}_i\mathbf{f} \to {}^i\mathbf{f}$**.** Assume that i is a sink of $Q = (I, H, s, t)$. So i is a source of $Q' = \sigma_i Q = (I, H', s, t)$, where $\sigma_i Q$ is the quiver by reversing the directions of all arrows in Q containing i. For any $\nu, \nu' \in \mathbb{N}I$ such that $\nu' = s_i\nu = \nu - \nu(h_i)i$ and I-graded $\mathbb{K}$-vector spaces $\mathbf{V}$, $\mathbf{V}'$ such that $\underline{\dim}\mathbf{V} = \nu$, $\underline{\dim}\mathbf{V}' = \nu'$, consider the following correspondence ([19, 8])

$$ {}_iE_{\mathbf{V},Q} \xleftarrow{\alpha} Z_{\mathbf{V}\mathbf{V}'} \xrightarrow{\beta} {}^iE_{\mathbf{V}',Q'} , \tag{5} $$

where

(1) $Z_{\mathbf{V}\mathbf{V}'}$ is the subset in $E_{\mathbf{V},Q} \times E_{\mathbf{V}',Q'}$ consisting of all (x, y) satisfying the following conditions
   (a) for any $h \in H$ such that $t(h) \neq i$ and $h \in H'$, $x_h = y_h$;
   (b) the following sequence is exact

$$ 0 \longrightarrow V'_i \xrightarrow{\bigoplus_{h\in H', s(h)=i} y_h} \bigoplus_{h\in H, t(h)=i} V_{s(h)} \xrightarrow{\bigoplus_{h\in H, t(h)=i} x_h} V_i \longrightarrow 0. $$

(2) $\alpha(x, y) = x$ and $\beta(x, y) = y$.

From now on, ${}_iE_{\mathbf{V},Q}$ is denoted by ${}_iE_{\mathbf{V}}$ and ${}^iE_{\mathbf{V}',Q'}$ is denoted by ${}^iE_{\mathbf{V}'}$. Let

$$ G_{\mathbf{V}\mathbf{V}'} = GL(V_i) \times GL(V'_i) \times \prod_{j\neq i} GL(V_j) \cong GL(V_i) \times GL(V'_i) \times \prod_{j\neq i} GL(V'_j), $$

which acts on $Z_{\mathbf{V}\mathbf{V}'}$ naturally.

By (5), we have

$$ \mathcal{D}_{G_{\mathbf{V}}}({}_iE_{\mathbf{V}}) \xrightarrow{\alpha^*} \mathcal{D}_{G_{\mathbf{V}\mathbf{V}'}}(Z_{\mathbf{V}\mathbf{V}'}) \xleftarrow{\beta^*} \mathcal{D}_{G_{\mathbf{V}'}}({}^iE_{\mathbf{V}'}) . \tag{6} $$

Since α and β are principal bundles with fibers $Aut(V'_i)$ and $Aut(V_i)$ respectively, α^* and β^* are equivalences of categories by Section 2.2.5 in [2]. Hence, for any

$\mathcal{L} \in \mathcal{D}_{G_{\mathbf{V}}}({}_iE_{\mathbf{V}})$ there exists a unique $\mathcal{L}' \in \mathcal{D}_{G_{\mathbf{V}'}}({}^iE_{\mathbf{V}'})$ such that $\alpha^*(\mathcal{L}) = \beta^*(\mathcal{L}')$. Define

$$\begin{aligned} \tilde{\omega}_i : \mathcal{D}_{G_{\mathbf{V}}}({}_iE_{\mathbf{V}}) &\to \mathcal{D}_{G_{\mathbf{V}'}}({}^iE_{\mathbf{V}'}) \\ \mathcal{L} &\mapsto \mathcal{L}'[-s(\mathbf{V})](-\frac{s(\mathbf{V})}{2}), \end{aligned}$$

where $s(\mathbf{V}) = \dim \mathrm{GL}(V_i) - \dim \mathrm{GL}(V_i')$. Since α^* and β^* are equivalences of categories, $\tilde{\omega}_i$ is also an equivalence of categories.

Proposition 3.4. *It holds that* $\tilde{\omega}_i({}_i\mathcal{Q}_{\mathbf{V}}) = {}^i\mathcal{Q}_{\mathbf{V}'}$.

The proof of Proposition 3.4 will be given in Section 4.3.

Hence, we can define $\tilde{\omega}_i : {}_i\mathcal{Q}_{\mathbf{V}} \to {}^i\mathcal{Q}_{\mathbf{V}'}$ and $\tilde{\omega}_i : K({}_i\mathcal{Q}) \to K({}^i\mathcal{Q})$. We have the following theorem.

Theorem 3.5. *We have the following commutative diagram*

$$\begin{array}{ccc} {}_i\mathbf{f}_{\mathcal{A}} & \xrightarrow{T_i} & {}^i\mathbf{f}_{\mathcal{A}} \\ \downarrow {}_i\lambda'_{\mathcal{A}} & & \downarrow {}^i\lambda'_{\mathcal{A}} \\ K({}_i\mathcal{Q}) & \xrightarrow{\tilde{\omega}_i} & K({}^i\mathcal{Q}). \end{array}$$

The proof of Theorem 3.5 will be given in Section 4.3.

3.4. $T_i : {}_i\mathbf{f} \to {}^i\mathbf{f}$ **and canonical bases.** In [18], Lusztig showed that Lusztig's symmetries and canonical bases are compatible. In this section, we shall give a geometric interpretation of this result by using the geometric realization of T_i.

Let $\mathbf{B}$ be the canonical basis of $\mathbf{f}$. Since $\theta_i\mathbf{f}$ is the kernel of ${}_i\pi : \mathbf{f} \to {}_i\mathbf{f}$ and $\mathbf{B}\cap\theta_i\mathbf{f}$ is a $\mathbb{Q}(v)$-basis of $\theta_i\mathbf{f}$, ${}_i\mathbf{B} = {}_i\pi(\mathbf{B})$ is a $\mathbb{Q}(v)$-basis of ${}_i\mathbf{f}$. Similarly, ${}^i\mathbf{B} = {}^i\pi(\mathbf{B})$ is a $\mathbb{Q}(v)$-basis of ${}^i\mathbf{f}$.

Lusztig proved the following theorem.

Theorem 3.6 ([18])**.** *Lusztig's symmetry* $T_i : {}_i\mathbf{f} \to {}^i\mathbf{f}$ *maps any element of* ${}_i\mathbf{B}$ *to an element of* ${}^i\mathbf{B}$*. Thus, there exists a unique bijection* $\kappa_i : \mathbf{B} - \mathbf{B}\cap\theta_i\mathbf{f} \to \mathbf{B} - \mathbf{B}\cap\mathbf{f}\theta_i$ *such that* $T_i({}_i\pi(b)) = {}^i\pi(\kappa_i(b))$*.*

Let i be a sink of a quiver Q. So i is a source of $Q' = \sigma_iQ$. By Theorem 3.1, Proposition 3.3, the formula (3) and the commutative diagram (4), we have

$$(7) \qquad {}_i\mathbf{B} = \sqcup_{\nu\in\mathbb{N}I}\{\mathbf{b}_{\mathcal{L}} = {}_i\lambda_{\mathcal{A}}([\mathcal{L}]) \mid \mathcal{L} \in {}_i\mathcal{P}_{\mathbf{V}},\ \underline{\dim}\mathbf{V} = \nu\}.$$

Similarly, we have

$$(8) \qquad {}^i\mathbf{B} = \sqcup_{\nu'\in\mathbb{N}I}\{\mathbf{b}_{\mathcal{L}} = {}^i\lambda_{\mathcal{A}}([\mathcal{L}]) \mid \mathcal{L} \in {}^i\mathcal{P}_{\mathbf{V}'},\ \underline{\dim}\mathbf{V}' = \nu'\}.$$

Fix any $\nu, \nu' \in \mathbb{N}I$ such that $\nu' = s_i\nu$ and I-graded $\mathbb{K}$-vector spaces $\mathbf{V}$, $\mathbf{V}'$ such that $\underline{\dim}\mathbf{V} = \nu$, $\underline{\dim}\mathbf{V}' = \nu'$.

In (6), the functors α^* and β^* are equivalences of categories. Hence the functor

$$\tilde{\omega}_i : {}_i\mathcal{Q}_{\mathbf{V}} \to {}^i\mathcal{Q}_{\mathbf{V}'}$$

maps any simple perverse sheaf in ${}_i\mathcal{Q}_{\mathbf{V}}$ to a simple perverse sheaf in ${}^i\mathcal{Q}_{\mathbf{V}'}$. That is, $\tilde{\omega}_i({}_i\mathcal{P}_{\mathbf{V}}) = {}^i\mathcal{P}_{\mathbf{V}'}$. So the map

$$\tilde{\omega}_i : K({}_i\mathcal{Q}) \to K({}^i\mathcal{Q})$$

satisfies

$$\tilde{\omega}_i(\{[\mathcal{L}] \mid \mathcal{L} \in {}_i\mathcal{P}_{\mathbf{V}}\}) = \{[\mathcal{L}] \mid \mathcal{L} \in {}^i\mathcal{P}_{\mathbf{V}'}\}.$$

By Theorem 3.5, (7) and (8), it holds that $T_i({}_i\mathbf{B}) = {}^i\mathbf{B}$ and we get a geometric interpretation of Theorem 3.6.

4. Hall algebra approaches

4.1. **Hall algebra approach to f.** In this subsection, we shall review the Hall algebra approach to $\mathbf{f}$ ([22, 19, 12, 13]).

Let $Q = (I, H, s, t)$ be a quiver. In Section 3.1, $E_{\mathbf{V}}$ and $G_{\mathbf{V}}$ are defined for any I-graded $\mathbb{K}$-vector space $\mathbf{V}$. Let F^n be the Frobenius morphism. The sets $E_{\mathbf{V}}^{F^n}$ and $G_{\mathbf{V}}^{F^n}$ consist of the F^n-fixed points in $E_{\mathbf{V}}$ and $G_{\mathbf{V}}$ respectively.

Lusztig defined $\underline{\mathcal{F}}_{\mathbf{V}}^n$ as the set of all $G_{\mathbf{V}}^{F^n}$-invariant $\bar{\mathbb{Q}}_l$-functions on $E_{\mathbf{V}}^{F^n}$ and we can give a multiplication on $\underline{\mathcal{F}}^n = \bigoplus_{\nu\in\mathbb{N}I} \underline{\mathcal{F}}_{\mathbf{V}}^n$ to obtain the Hall algebra. For any $i \in I$, let $\mathbf{V}_i$ be the I-graded $\mathbb{K}$-vector space with dimension vector i and f_i be the constant function on $E_{\mathbf{V}_i}^{F^n}$ with value 1. Denote by $\mathcal{F}^n$ the composition subalgebra of $\underline{\mathcal{F}}^n$ generated by f_i and $\mathcal{F}_{\mathbf{V}}^n = \underline{\mathcal{F}}_{\mathbf{V}}^n \cap \mathcal{F}^n$. Let $\mathcal{F} = \bigoplus_{\nu\in\mathbb{N}I} \mathcal{F}_{\mathbf{V}}$ be the generic form of $\mathcal{F}^n$ and $\mathcal{F}_{\mathcal{A}}$ be the integral form of $\mathcal{F}$ ([19]).

Theorem 4.1 ([22, 19])**.** *There exists an isomorphism of $\mathcal{A}$-algebras*

$$\varpi_{\mathcal{A}} : \mathbf{f}_{\mathcal{A}} \to \mathcal{F}_{\mathcal{A}}$$

such that $\varpi_{\mathcal{A}}(\theta_i) = f_i$.

For any $\mathcal{L} \in \mathcal{D}_{G_{\mathbf{V}}}(E_{\mathbf{V}})$, there is a function $\chi_{\mathcal{L}}^n : E_{\mathbf{V}}^{F^n} \to \bar{\mathbb{Q}}_l$ (Section I.2.12 in [10]). Hence, we have the following map

$$\begin{array}{rcl} \chi^n : \mathcal{D}_{G_{\mathbf{V}}}(E_{\mathbf{V}}) & \to & \underline{\mathcal{F}}_{\mathbf{V}}^n \\ \mathcal{L} & \mapsto & \chi_{\mathcal{L}}^n. \end{array}$$

The restriction of this map on the subcategory $\mathcal{Q}_{\mathbf{V}}$ is also denoted by

$$\chi^n : \mathcal{Q}_{\mathbf{V}} \to \underline{\mathcal{F}}_{\mathbf{V}}^n.$$

Lusztig proved that $\chi^n(\mathcal{Q}_{\mathbf{V}}) \subset \mathcal{F}_{\mathbf{V}}^n$ in [19]. Hence, we can define $\chi^n : \mathcal{Q}_{\mathbf{V}} \to \mathcal{F}_{\mathbf{V}}^n$, which induces $\chi : \mathcal{Q}_{\mathbf{V}} \to \mathcal{F}_{\mathbf{V}}$ naturally. Hence, we get a map $\chi_{\mathcal{A}} : K(\mathcal{Q}) \to \mathcal{F}_{\mathcal{A}}$.

Lusztig proved the following proposition.

Proposition 4.2 ([19])**.** *$\chi_{\mathcal{A}} : K(\mathcal{Q}) \to \mathcal{F}_{\mathcal{A}}$ is an isomorphism of $\mathcal{A}$-algebras such that $\chi_{\mathcal{A}}([\mathcal{L}_i]) = f_i$ and the following diagram is commutative*

$$\begin{array}{ccc} K(\mathcal{Q}) & \xrightarrow{\lambda_{\mathcal{A}}} & \mathbf{f}_{\mathcal{A}}. \\ \downarrow{\scriptstyle \chi_{\mathcal{A}}} & \swarrow{\scriptstyle \varpi_{\mathcal{A}}} & \\ \mathcal{F}_{\mathcal{A}} & & \end{array}$$

4.2. **Hall algebra approaches to ${}_i\mathbf{f}$ and ${}^i\mathbf{f}$.** Let i be a sink of Q. In Section 3.2, ${}_iE_{\mathbf{V}}$ and $j_{\mathbf{V}} : {}_iE_{\mathbf{V}} \to E_{\mathbf{V}}$ are defined for any I-graded $\mathbb{K}$-vector space $\mathbf{V}$. Similarly, ${}_iE_{\mathbf{V}}^{F^n}$ is defined as the set of F_n-fixed points in ${}_iE_{\mathbf{V}}$ and we have $j_{\mathbf{V}} : {}_iE_{\mathbf{V}}^{F^n} \to E_{\mathbf{V}}^{F^n}$.

Lusztig also defined ${}_i\underline{\mathcal{F}}_{\mathbf{V}}^n$ as the set of all $G_{\mathbf{V}}^{F^n}$-invariant $\bar{\mathbb{Q}}_l$-functions on ${}_iE_{\mathbf{V}}^{F^n}$. Similarly to the case in Section 4.1, the Hall algebra is denoted by ${}_i\underline{\mathcal{F}}^n = \bigoplus_{\nu\in\mathbb{N}I} {}_i\underline{\mathcal{F}}_{\mathbf{V}}^n$, the composition subalgebra is denoted by ${}_i\mathcal{F}^n = \bigoplus_{\nu\in\mathbb{N}I} {}_i\mathcal{F}_{\mathbf{V}}^n$ and the generic form is denoted by ${}_i\mathcal{F} := \bigoplus_{\nu\in\mathbb{N}I} {}_i\mathcal{F}_{\mathbf{V}}$.

Naturally, we have two maps $j_{\mathbf{V}}^* : \mathcal{F}_{\mathbf{V}} \to {}_i\mathcal{F}_{\mathbf{V}}$ and $j_{\mathbf{V}!} : {}_i\mathcal{F}_{\mathbf{V}} \to \mathcal{F}_{\mathbf{V}}$. Considering all dimension vectors, we have $j_! : {}_i\mathcal{F} \to \mathcal{F}$ and $j^* : \mathcal{F} \to {}_i\mathcal{F}$.

Proposition 4.3 ([19])**.** *We have the following commutative diagram*

$$\begin{array}{ccccc} {}_i\mathbf{f} & \longrightarrow & \mathbf{f} & \xrightarrow{{}_i\pi} & {}_i\mathbf{f} \\ \cong\big\downarrow {}_i\varpi & & \cong\big\downarrow \varpi & & \cong\big\downarrow {}_i\varpi \\ {}_i\mathcal{F} & \xrightarrow{j_!} & \mathcal{F} & \xrightarrow{j^*} & {}_i\mathcal{F}, \end{array}$$

where ${}_i\varpi$ is the isomorphism induced by ϖ.

Next, we shall prove Proposition 3.2.

For any $\mathcal{L} \in \mathcal{D}_{G_{\mathbf{V}}}({}_iE_{\mathbf{V}})$, there is also a function $\chi_{\mathcal{L}}^n : {}_iE_{\mathbf{V}}^{F^n} \to \bar{\mathbb{Q}}_l$. Hence, we have the following map

$$\begin{array}{rcl} {}_i\chi^n : \mathcal{D}_{G_{\mathbf{V}}}({}_iE_{\mathbf{V}}) & \to & {}_i\underline{\mathcal{F}}_{\mathbf{V}}^n \\ \mathcal{L} & \mapsto & \chi_{\mathcal{L}}^n. \end{array}$$

The restriction of this map on the subcategory ${}_i\mathcal{Q}_{\mathbf{V}}$ is also denoted by

$${}_i\chi^n : {}_i\mathcal{Q}_{\mathbf{V}} \to {}_i\underline{\mathcal{F}}_{\mathbf{V}}^n.$$

Proposition 4.4. *It holds that ${}_i\chi^n({}_i\mathcal{Q}_{\mathbf{V}}) \subset {}_i\mathcal{F}_{\mathbf{V}}^n$.*

Proof. By the properties of χ and ${}_i\chi$ (Theorem III.12.1(5) in [10]), we have the following commutative diagram

$$\begin{array}{ccc} \mathcal{Q}_{\mathbf{V}} & \xrightarrow{j_{\mathbf{V}}^*} & {}_i\mathcal{Q}_{\mathbf{V}} \\ \big\downarrow \chi^n & & \big\downarrow {}_i\chi^n \\ \mathcal{F}_{\mathbf{V}}^n & \xrightarrow{j_{\mathbf{V}}^*} & {}_i\underline{\mathcal{F}}_{\mathbf{V}}^n. \end{array} \tag{9}$$

By the commutative diagram (9), $j_{\mathbf{V}}^*(\mathcal{F}_{\mathbf{V}}^n) \subset {}_i\mathcal{F}_{\mathbf{V}}^n$ and $j_{\mathbf{V}}^*(\mathcal{Q}_{\mathbf{V}}) = {}_i\mathcal{Q}_{\mathbf{V}}$, we have ${}_i\chi^n({}_i\mathcal{Q}_{\mathbf{V}}) \subset {}_i\mathcal{F}_{\mathbf{V}}^n$.

□

Hence, we can define ${}_i\chi^n : {}_i\mathcal{Q}_{\mathbf{V}} \to {}_i\mathcal{F}_{\mathbf{V}}^n$, which induces ${}_i\chi : {}_i\mathcal{Q}_{\mathbf{V}} \to {}_i\mathcal{F}_{\mathbf{V}}$ and ${}_i\chi_{\mathcal{A}} : K({}_i\mathcal{Q}) \to {}_i\mathcal{F}_{\mathcal{A}}$.

The commutative diagram (9) implies the following proposition.

Proposition 4.5. *We have the following commutative diagram*

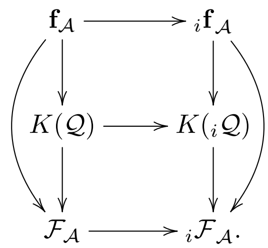
$$\begin{array}{ccc} K(\mathcal{Q}) & \xrightarrow{j^*} & K({}_i\mathcal{Q}) \\ \big\downarrow \chi_{\mathcal{A}} & & \big\downarrow {}_i\chi_{\mathcal{A}} \\ \mathcal{F}_{\mathcal{A}} & \xrightarrow{j^*} & {}_i\mathcal{F}_{\mathcal{A}}. \end{array}$$

□

Proof of Proposition 3.2. First, we shall prove the following commutative diagram

$$\begin{array}{ccc} {}_i\mathbf{f}_{\mathcal{A}} & \xrightarrow{{}_i\varpi_{\mathcal{A}}} & {}_i\mathcal{F}_{\mathcal{A}}. \\ \big\downarrow {}_i\lambda'_{\mathcal{A}} & \nearrow {}_i\chi_{\mathcal{A}} & \\ K({}_i\mathcal{Q}) & & \end{array}$$

Consider the following diagram



Since three squares and the triangle in the left are commutative, the triangle in the right is also commutative.

Proposition 4.3 implies that ${}_i\varpi_{\mathcal{A}} : {}_i\mathbf{f}_{\mathcal{A}} \to {}_i\mathcal{F}_{\mathcal{A}}$ is isomorphic. Hence ${}_i\lambda'_{\mathcal{A}} : {}_i\mathbf{f}_{\mathcal{A}} \to K({}_i\mathcal{Q})$ is injective. The commutative diagram (4) in the definition of ${}_i\lambda'_{\mathcal{A}}$ implies that ${}_i\lambda'_{\mathcal{A}} : {}_i\mathbf{f}_{\mathcal{A}} \to K({}_i\mathcal{Q})$ is surjection. Hence, ${}_i\lambda'_{\mathcal{A}} : {}_i\mathbf{f}_{\mathcal{A}} \to K({}_i\mathcal{Q})$ is isomorphic.

□

In the proof, we get the following proposition.

Proposition 4.6. *We have the following commutative diagram*

$$\begin{array}{ccc} K({}_i\mathcal{Q}) & \xrightarrow{{}_i\lambda_{\mathcal{A}}} & {}_i\mathbf{f}_{\mathcal{A}}, \\ \downarrow {\scriptstyle {}_i\chi_{\mathcal{A}}} & \swarrow {\scriptstyle {}_i\varpi_{\mathcal{A}}} & \\ {}_i\mathcal{F}_{\mathcal{A}} & & \end{array}$$

where all maps are isomorphisms of $\mathcal{A}$-algebras and ${}_i\lambda_{\mathcal{A}}$ is the inverse of ${}_i\lambda'_{\mathcal{A}}$.

□

Assume that i is a source of Q. The notations and results in this case are completely similar to the case that i is a sink. We can define ${}^i\underline{\mathcal{F}}^n_{\mathbf{V}}$, ${}^i\mathcal{F}^n = \bigoplus_{\nu\in\mathbb{N}I} {}^i\mathcal{F}^n_{\mathbf{V}}$ and ${}^i\mathcal{F} = \bigoplus_{\nu\in\mathbb{N}I} {}^i\mathcal{F}_{\mathbf{V}}$. We also have two maps $j^*_{\mathbf{V}} : \mathcal{F}_{\mathbf{V}} \to {}^i\mathcal{F}_{\mathbf{V}}$ and $j_{\mathbf{V}!} : {}^i\mathcal{F}_{\mathbf{V}} \to \mathcal{F}_{\mathbf{V}}$. Considering all dimension vectors, we have $j_! : {}^i\mathcal{F} \to \mathcal{F}$ and $j^* : \mathcal{F} \to {}^i\mathcal{F}$.

Proposition 4.7 ([19])**.** *We have the following commutative diagram*

$$\begin{array}{ccccc} {}^i\mathbf{f} & \longrightarrow & \mathbf{f} & \xrightarrow{{}^i\pi} & {}^i\mathbf{f} \\ \cong \downarrow {\scriptstyle {}^i\varpi} & & \cong \downarrow {\scriptstyle \varpi} & & \cong \downarrow {\scriptstyle {}^i\varpi} \\ {}^i\mathcal{F} & \xrightarrow{j_!} & \mathcal{F} & \xrightarrow{j^*} & {}^i\mathcal{F}, \end{array}$$

where ${}^i\varpi$ is the isomorphism induced by ϖ.

We can also define ${}^i\chi : {}^i\mathcal{Q}_{\mathbf{V}} \to {}^i\mathcal{F}_{\mathbf{V}}$ and ${}^i\chi_{\mathcal{A}} : K({}^i\mathcal{Q}) \to {}^i\mathcal{F}_{\mathcal{A}}$.

Proposition 4.8. *We have the following commutative diagram*

$$\begin{array}{ccc} K(\mathcal{Q}) & \xrightarrow{j^*} & K({}^i\mathcal{Q}) \\ \downarrow {\scriptstyle \chi_{\mathcal{A}}} & & \downarrow {\scriptstyle {}^i\chi_{\mathcal{A}}} \\ \mathcal{F}_{\mathcal{A}} & \xrightarrow{j^*} & {}^i\mathcal{F}_{\mathcal{A}}. \end{array}$$

□

Proposition 4.9. *We have the following commutative diagram*

$$\begin{array}{ccc} K({}^i\mathcal{Q}) & \xrightarrow{{}^i\lambda_{\mathcal{A}}} & {}^i\mathbf{f}_{\mathcal{A}}, \\ \downarrow {}^i\chi_{\mathcal{A}} & \swarrow {}^i\varpi_{\mathcal{A}} & \\ {}^i\mathcal{F}_{\mathcal{A}} & & \end{array}$$

where all maps are isomorphisms of $\mathcal{A}$-algebras and ${}^i\lambda_{\mathcal{A}}$ is the inverse of ${}^i\lambda'_{\mathcal{A}}$.

□

4.3. **Hall algebra approach to $T_i : {}_i\mathbf{f} \to {}^i\mathbf{f}$ and the proof of Theorem 3.5.** Let i be a sink of a quiver $Q = (I, H, s, t)$. So i is a source of $Q' = \sigma_i Q = (I, H', s, t)$. For any ν and $\nu' \in \mathbb{N}I$ such that $\nu' = s_i\nu$, and two I-graded $\mathbb{K}$-vector spaces $\mathbf{V}$ and $\mathbf{V}'$ such that $\underline{\dim}\mathbf{V} = \nu$ and $\underline{\dim}\mathbf{V}' = \nu'$, the following correspondence is considered in Section 3.3

$${}_iE_{\mathbf{V},Q} \xleftarrow{\alpha} Z_{\mathbf{V}\mathbf{V}'} \xrightarrow{\beta} {}^iE_{\mathbf{V}',Q'} .$$

Similarly, $Z^{F^n}_{\mathbf{V}\mathbf{V}'}$ is defined as the F_n-fixed points set in $Z_{\mathbf{V}\mathbf{V}'}$ and we have

$${}_iE^{F^n}_{\mathbf{V},Q} \xleftarrow{\alpha} Z^{F^n}_{\mathbf{V}\mathbf{V}'} \xrightarrow{\beta} {}^iE^{F^n}_{\mathbf{V}',Q'} .$$

Note that α and β are principal bundles with fibers $Aut(V'_i)$ and $Aut(V_i)$ respectively. Hence, for any $f \in {}_i\underline{\mathcal{F}}^n_{\mathbf{V}}$, there exists a unique $g \in {}^i\underline{\mathcal{F}}^n_{\mathbf{V}'}$ such that $\alpha^*(f) = \beta^*(g)$. Define

$$\begin{aligned} \omega_i : {}_i\underline{\mathcal{F}}^n_{\mathbf{V}} &\to {}^i\underline{\mathcal{F}}^n_{\mathbf{V}'} \\ f &\mapsto (p^n)^{-\frac{s(\mathbf{V})}{2}} g. \end{aligned}$$

Lusztig proved that $\omega_i({}_i\mathcal{F}^n_{\mathbf{V}}) \subset {}^i\mathcal{F}^n_{\mathbf{V}'}$. Hence, we have $\omega_i : {}_i\mathcal{F}^n_{\mathbf{V}} \to {}^i\mathcal{F}^n_{\mathbf{V}'}$ and $\omega_i : {}_i\mathcal{F}_{\mathbf{V}} \to {}^i\mathcal{F}_{\mathbf{V}'}$. Considering all dimension vectors, we have $\omega_i : {}_i\mathcal{F} \to {}^i\mathcal{F}$.

Lusztig proved the following theorem.

Theorem 4.10 ([19])**.** *We have the following commutative diagram*

$$\begin{array}{ccc} {}_i\mathbf{f} & \xrightarrow{T_i} & {}^i\mathbf{f} \\ \downarrow {}_i\varpi & & \downarrow {}^i\varpi \\ {}_i\mathcal{F} & \xrightarrow{\omega_i} & {}^i\mathcal{F}. \end{array}$$

Proof of Proposition 3.4. By the properties of ${}_i\chi^n$ and ${}^i\chi^n$ (Theorem III.12.1(4,5) in [10]), we have the following commutative diagram

$$\begin{array}{ccc} \mathcal{D}_{G_{\mathbf{V}}}({}_iE_{\mathbf{V}}) & \xrightarrow{\tilde{\omega}_i} & \mathcal{D}_{G_{\mathbf{V}'}}({}^iE_{\mathbf{V}'}) \\ \downarrow {}_i\chi^n & & \downarrow {}^i\chi^n \\ {}_i\underline{\mathcal{F}}^n_{\mathbf{V}} & \xrightarrow{\omega^n_i} & {}^i\underline{\mathcal{F}}^n_{\mathbf{V}'}. \end{array}$$

Hence, we have

$$\begin{CD}
\mathcal{D}_{G_{\mathbf{V}}}({}_iE_{\mathbf{V}}) @>{\tilde{\omega}_i}>> \mathcal{D}_{G_{\mathbf{V}'}}({}^iE_{\mathbf{V}'})\\
@V{\prod_{n\in\mathbb{Z}_{\geq 1}} {}_i\chi^n}VV @VV{\prod_{n\in\mathbb{Z}_{\geq 1}} {}^i\chi^n}V\\
\prod_{n\in\mathbb{Z}_{\geq 1}} {}_i\underline{\mathcal{F}}^n_{\mathbf{V}} @>{\prod_{n\in\mathbb{Z}_{\geq 1}} \omega_i^n}>> \prod^i_{n\in\mathbb{Z}_{\geq 1}} \underline{\mathcal{F}}^n_{\mathbf{V}'}.
\end{CD}$$

By Proposition 4.4, ${}_i\chi^n({}_i\mathcal{Q}_{\mathbf{V}}) \subset {}_i\mathcal{F}^n_{\mathbf{V}}$. Hence, we have

$$\begin{CD}
{}_i\mathcal{Q}_{\mathbf{V}} @>{\tilde{\omega}_i}>> \mathcal{D}_{G_{\mathbf{V}'}}({}^iE_{\mathbf{V}'})\\
@V{{}_i\chi}VV @VV{\prod_{n\in\mathbb{Z}_{\geq 1}} {}^i\chi^n}V\\
{}_i\mathcal{F}_{\mathbf{V}} @>{\omega_i}>> \prod_{n\in\mathbb{Z}_{\geq 1}} {}^i\underline{\mathcal{F}}^n_{\mathbf{V}'}.
\end{CD}$$

Hence,

$$(\prod_{n\in\mathbb{Z}_{\geq 1}} {}^i\chi^n) \circ \tilde{\omega}_i({}_i\mathcal{Q}_{\mathbf{V}}) \subset \omega_i \circ {}_i\chi({}_i\mathcal{Q}_{\mathbf{V}}).$$

Since $\omega_i({}_i\mathcal{F}_{\mathbf{V}}) \subset {}^i\mathcal{F}_{\mathbf{V}'}$,

$$(\prod_{n\in\mathbb{Z}_{\geq 1}} {}^i\chi^n) \circ \tilde{\omega}_i({}_i\mathcal{Q}_{\mathbf{V}}) \subset {}^i\mathcal{F}_{\mathbf{V}'}.$$

For any two semisimple complexes $\mathcal{L}$ and $\mathcal{L}'$ in $\mathcal{D}_{G_{\mathbf{V}'}}({}^iE_{\mathbf{V}'})$ such that

$$(\prod_{n\in\mathbb{Z}_{\geq 1}} {}^i\chi^n)(\mathcal{L}) = (\prod_{n\in\mathbb{Z}_{\geq 1}} {}^i\chi^n)(\mathcal{L}'),$$

$\mathcal{L}$ is isomorphic to $\mathcal{L}'$ by Theorem III.12.1(3) in [10]. Since $(\prod_{n\in\mathbb{Z}_{\geq 1}} {}^i\chi^n)({}^i\mathcal{Q}_{\mathbf{V}'}) = {}^i\mathcal{F}_{\mathbf{V}'}$ and the objects in $\tilde{\omega}_i({}_i\mathcal{Q}_{\mathbf{V}})$ are semisimple, $\tilde{\omega}_i({}_i\mathcal{Q}_{\mathbf{V}}) \subset {}^i\mathcal{Q}_{\mathbf{V}'}$.

□

Proposition 4.11. *We have the following commutative diagram*

$$\begin{CD}
K({}_i\mathcal{Q}) @>{\tilde{\omega}_i}>> K({}^i\mathcal{Q})\\
@V{{}_i\chi}VV @VV{{}^i\chi}V\\
{}_i\mathcal{F}_{\mathcal{A}} @>{\omega_i}>> {}^i\mathcal{F}_{\mathcal{A}}.
\end{CD}$$

Proof. By the properties of ${}_i\chi^n$ and ${}^i\chi^n$, we have the following commutative diagram

$$\begin{CD}
{}_i\mathcal{Q}_{\mathbf{V}} @>{\tilde{\omega}_i}>> {}^i\mathcal{Q}_{\mathbf{V}'}\\
@V{{}_i\chi^n}VV @VV{{}^i\chi^n}V\\
{}_i\mathcal{F}^n_{\mathbf{V}} @>{\omega_i^n}>> {}^i\mathcal{F}^n_{\mathbf{V}'}.
\end{CD}$$

Hence, we get the commutative diagram in this proposition.

□

At last, Theorem 4.10 and Proposition 4.11 imply Theorem 3.5.

5. Projective resolutions of a kind of standard modules

5.1. **KLR algebras.** First let us review the definitions of KLR algebras ([9, 27]).

Let $Q = (I, H, s, t)$ be a quiver corresponding to the Lusztig's algebra $\mathbf{f}$. Let $\mathbb{K}$ be an algebraic closed field. Fix an I-graded $\mathbb{K}$-vector space $\mathbf{V}$ such that $\underline{\dim}\mathbf{V} = \nu \in \mathbb{N}I$. In Section 3.1, the semisimple complexes $\mathcal{L}_{\mathbf{i}} \in \mathcal{D}_{G_{\mathbf{V}}}(E_{\mathbf{V}})$ are defined for all $\mathbf{i} \in I^{\nu}$. Let

$$\mathcal{L}_{\nu} = \bigoplus_{\mathbf{i} \in I^{\nu}} \mathcal{L}_{\mathbf{i}}.$$

The KLR algebra $\mathbf{R}_{\nu}$ is defined as

$$\mathbf{R}_{\nu} = \bigoplus_{k \in \mathbb{Z}} \mathrm{Ext}^k_{\mathcal{D}_{G_{\mathbf{V}}}(E_{\mathbf{V}})}(\mathcal{L}_{\nu}, \mathcal{L}_{\nu}).$$

$\mathbf{R}_{\nu}$ is a graded algebra and the degree of any element in $\mathrm{Ext}^k_{\mathcal{D}_{G_{\mathbf{V}}}(E_{\mathbf{V}})}(\mathcal{L}_{\nu}, \mathcal{L}_{\nu})$ is k.

Let $\mathbf{R}_{\nu}$-gmod be the category of graded $\mathbf{R}_{\nu}$-modules and $\mathbf{R}_{\nu}$-proj be the category of finitely generated graded projective $\mathbf{R}_{\nu}$-modules. Let $K(\mathbf{R}_{\nu}\text{-proj})$ be the Grothendieck group of $\mathbf{R}_{\nu}$-proj.

Define $v^{\pm}[P] = [P[\pm 1]]$. So $K(\mathbf{R}_{\nu}\text{-proj})$ is a free $\mathcal{A}$-module. Define

$$K(\mathbf{R}\text{-proj}) = \bigoplus_{\nu \in \mathbb{N}I} K(\mathbf{R}_{\nu}\text{-proj}).$$

For $\nu, \nu', \nu'' \in \mathbb{N}I$ such that $\nu = \nu' + \nu''$ and three I-graded $\mathbb{K}$-vector spaces $\mathbf{V}$, $\mathbf{V}'$, $\mathbf{V}''$ such that $\underline{\dim}\mathbf{V} = \nu$, $\underline{\dim}\mathbf{V}' = \nu'$, $\underline{\dim}\mathbf{V}'' = \nu''$, Khovanov and Lauda ([9]) defined a functor

$$\mathrm{Ind}_{\nu', \nu''} : \mathbf{R}_{\nu'}\text{-proj} \times \mathbf{R}_{\nu''}\text{-proj} \to \mathbf{R}_{\nu}\text{-proj},$$

which induces an $\mathcal{A}$-bilinear multiplication

$$[\mathrm{Ind}_{\nu', \nu''}] : K(\mathbf{R}_{\nu'}\text{-proj}) \otimes_{\mathcal{A}} K(\mathbf{R}_{\nu''}\text{-proj}) \to K(\mathbf{R}_{\nu}\text{-proj}).$$

Khovanov and Lauda ([9]) proved that $K(\mathbf{R}\text{-proj})$ becomes an associative $\mathcal{A}$-algebra.

For any $\mathbf{y} \in Y_{\nu}$, let

$$P_{\mathbf{y}} = \bigoplus_{k \in \mathbb{Z}} \mathrm{Ext}^k_{\mathcal{D}_{G_{\mathbf{V}}}(E_{\mathbf{V}})}(\mathcal{L}_{\mathbf{y}}, \mathcal{L}_{\nu}).$$

Theorem 5.1 ([9, 24])**.** *There is a unique isomorphism of $\mathcal{A}$-algebras*

$$\gamma_{\mathcal{A}} : \mathbf{f}_{\mathcal{A}} \to K(\mathbf{R}\text{-}proj)$$

such that $\gamma_{\mathcal{A}}(\theta_{\mathbf{y}}) = P_{\mathbf{y}}$ for all $\mathbf{y} \in Y_{\nu}$.

Let $\mathbf{B}_{\mathbb{Z}} = \{v^s b \mid b \in \mathbf{B}, s \in \mathbb{Z}\}$, which is a $\mathbb{Z}$-basis of $\mathbf{f}_{\mathcal{A}}$. Varagnolo, Vasserot and Rouquier proved the following theorem.

Theorem 5.2 ([27, 25])**.** *The map $\gamma_{\mathcal{A}}$ takes $\mathbf{B}_{\mathbb{Z}}$ to the $\mathbb{Z}$-basis of $K(\mathbf{R}\text{-}proj)$ consisting of all indecomposable projective modules.*

5.2. **Projective resolutions.** Let i and j be two vertices of the quiver Q such that there are no arrows from i to j. Let $N = \#\{j \to i\}$ and m be a non-negative integer such that $m \leq N$. Let $\nu^{(m)} = mi + j \in \mathbb{N}I$. Fix an I-graded $\mathbb{K}$-vector space $\mathbf{V}^{(m)}$ such that $\underline{\dim}\mathbf{V}^{(m)} = \nu^{(m)}$.

Denote by $\mathbf{1}_{{}_iE_{\mathbf{V}^{(m)}}} \in \mathcal{D}_{G_{\mathbf{V}^{(m)}}}({}_iE_{\mathbf{V}^{(m)}})$ the constant sheaf on ${}_iE_{\mathbf{V}^{(m)}}$. The following functor is defined in Section 3.2:

$$j_{\mathbf{V}^{(m)}!} : \mathcal{D}_{G_{\mathbf{V}^{(m)}}}({}_iE_{\mathbf{V}^{(m)}}) \to \mathcal{D}_{G_{\mathbf{V}^{(m)}}}(E_{\mathbf{V}^{(m)}}).$$

Define

$$\mathcal{E}^{(m)}=j_{\mathbf{V}^{(m)}!}(v^{-mN}\mathbf{1}_{{}_iE_{\mathbf{V}^{(m)}}})\in\mathcal{D}_{G_{\mathbf{V}^{(m)}}}(E_{\mathbf{V}^{(m)}})$$

and

$$K_m=\bigoplus_{k\in\mathbb{Z}}\mathrm{Ext}^k_{\mathcal{D}_{G_{\mathbf{V}^{(m)}}}(E_{\mathbf{V}^{(m)}})}(\mathcal{E}^{(m)},\mathcal{L}_{\nu^{(m)}}).$$

K_m is an object in $\mathbf{R}_{\nu^{(m)}}$-gmod for any m. Note that K_m is a standard module in the sense of Kato ([8]). We shall give projective resolutions of these standard modules. For convenience, the complex $j_{\mathbf{V}^{(m)}!}(\mathbf{1}_{{}_iE_{\mathbf{V}^{(m)}}})\in\mathcal{D}_{G_{\mathbf{V}^{(m)}}}(E_{\mathbf{V}^{(m)}})$ is also denoted by $\mathbf{1}_{{}_iE_{\mathbf{V}^{(m)}}}$.

For each $m\geq p\in\mathbb{N}$, consider the following variety

$$\tilde{S}^{(m)}_p=\{(x,W)\mid x\in E_{\mathbf{V}^{(m)}},\, W\subset V_i,\, \dim(W)=p,\, \mathrm{Im}\bigoplus_{h\in H,t(h)=i}x_h\subset W\}.$$

Let $\pi_p:\tilde{S}^{(m)}_p\to E_{\mathbf{V}^{(m)}}$ be the projection taking (x,W) to x and $S^{(m)}_p=\mathrm{Im}\pi_p$.

By the definitions of $S^{(m)}_p$, we have

$$E_{\mathbf{V}^{(m)}}=S^{(m)}_m\supset S^{(m)}_{m-1}\supset S^{(m)}_{m-2}\supset\cdots\supset S^{(m)}_0.$$

For each $1\leq p\leq m$, let

$$\mathcal{N}^{(m)}_p=S^{(m)}_p\backslash S^{(m)}_{p-1}.$$

Denote by $i^{(m)}_p:S^{(m)}_{p-1}\to S^{(m)}_p$ the close embedding and $j^{(m)}_p:\mathcal{N}^{(m)}_p\to S^{(m)}_p$ the open embedding.

Define

$$I^{(m)}_p=(\pi_p)_!(\mathbf{1}_{\tilde{S}^{(m)}_p})[\dim\tilde{S}^{(m)}_p].$$

In [17], Lusztig proved that $I^{(m)}_p$ are semisimple complexes in $\mathcal{D}_{G_{\mathbf{V}^{(m)}}}(E_{\mathbf{V}^{(m)}})$. Hence $I^{(m)}_p$ correspond to projective modules in $\mathbf{R}_\nu$-proj.

The following theorem is the main result in this section.

Theorem 5.3. *For $\mathcal{E}^{(m)}$, there exists $s_m\in\mathbb{N}$. For each $s_m\geq p\in\mathbb{N}$, there exists $\mathcal{E}^{(m)}_p\in\mathcal{D}_{G_{\mathbf{V}^{(m)}}}(E_{\mathbf{V}^{(m)}})$ such that*

(1) *$\mathcal{E}^{(m)}_{s_m}=\mathcal{E}^{(m)}$ and $\mathcal{E}^{(m)}_0$ is the direct sum of some semisimple complexes of the form $I^{(m)}_{p'}[l]$;*

(2) *for each $p\geq 1$, there exists a distinguished triangle*

$$\mathcal{E}^{(m)}_p\longrightarrow\mathcal{G}^{(m)}_p\longrightarrow\mathcal{E}^{(m)}_{p-1}\longrightarrow,$$

where $\mathcal{G}^{(m)}_p$ is the direct sum of some semisimple complexes of the form $I^{(m)}_{p'}[l]$.

The proof of Theorem 5.3 will be given in Section 5.3.

Let

$$P^{(m)}_0=\bigoplus_{k\in\mathbb{Z}}\mathrm{Ext}^k_{\mathcal{D}_{G_{\mathbf{V}^{(m)}}}(E_{\mathbf{V}^{(m)}})}(\mathcal{E}^{(m)}_0,\mathcal{L}_{\nu^{(m)}})$$

and

$$P^{(m)}_s=\bigoplus_{k\in\mathbb{Z}}\mathrm{Ext}^k_{\mathcal{D}_{G_{\mathbf{V}^{(m)}}}(E_{\mathbf{V}^{(m)}})}(\mathcal{G}^{(m)}_p,\mathcal{L}_{\nu^{(m)}})\quad(1\leq s\leq m),$$

which are projective modules in $\mathbf{R}_{\nu^{(m)}}$-proj.

As a corollary of Theorem 5.3, we have the following theorem.

Theorem 5.4. *For any $N \geq m \in \mathbb{N}$, there exists a finite length projective resolution of K_m:*

$$0 \longrightarrow P_0^{(m)} \longrightarrow P_1^{(m)} \longrightarrow \cdots \longrightarrow P_{s_m-1}^{(m)} \longrightarrow P_{s_m}^{(m)} \longrightarrow K_m \longrightarrow 0.$$

□

In the case of finite type, Kato proved that the projective dimension of any standard module is finite ([7, 8]). Theorem 5.4 show that the projective dimensions of a kind of standard modules are also finite in the general case.

5.3. **The proof of Theorem 5.3.** For convenience, a sheaf $\mathcal{A} \in \mathcal{D}_{G_{\mathbf{V}^{(m)}}}(E_{\mathbf{V}^{(m)}})$ is called with Property $\mathbf{A}$(m), if $\mathcal{A}$ satisfies the following conditions. There exists $s_{\mathcal{A}} \in \mathbb{N}$. For each $s_{\mathcal{A}} \geq p \in \mathbb{N}$, there exists $\mathcal{A}_p \in \mathcal{D}_{G_{\mathbf{V}^{(m)}}}(E_{\mathbf{V}^{(m)}})$ such that

(1) $\mathcal{A}_{s_{\mathcal{A}}} = \mathcal{A}$ and $\mathcal{A}_0$ is the direct sum of some semisimple complexes of the form $I_{p'}^{(m)}[l]$;
(2) for each $p \geq 1$, there exists a distinguished triangle

$$\mathcal{A}_p \longrightarrow \mathcal{G}_p^{\mathcal{A}} \longrightarrow \mathcal{A}_{p-1} \longrightarrow,$$

where $\mathcal{G}_p^{\mathcal{A}}$ is the direct sum of some semisimple complexes of the form $I_{p'}^{(m)}[l]$.

Theorem 5.3 means that $\mathcal{E}^{(m)}$ is with Property $\mathbf{A}$(m).

For the proof of Theorem 5.3, we need the following lemma.

Lemma 5.5. *Fix any distinguished triangle*

$$\mathcal{A} \longrightarrow \mathcal{A}' \longrightarrow \mathcal{A}'' \longrightarrow,$$

where $\mathcal{A}, \mathcal{A}', \mathcal{A}'' \in \mathcal{D}_{G_{\mathbf{V}^{(m)}}}(E_{\mathbf{V}^{(m)}})$. If $\mathcal{A}$ and $\mathcal{A}''$ are with Property $\boldsymbol{A}$(m), $\mathcal{A}'$ is with Property $\boldsymbol{A}$(m) and $s_{\mathcal{A}'} = s_{\mathcal{A}} + s_{\mathcal{A}''} + 1$.

Proof. We shall prove this lemma by induction on $s_{\mathcal{A}''}$.

(1) For $s_{\mathcal{A}''} = 0$, $\mathcal{A}''$ is the direct sum of some semisimple complexes of the form $I_{p'}^{(m)}[l]$. Let $\mathcal{A}'_{s_{\mathcal{A}'}} = \mathcal{A}'$ and $\mathcal{A}'_p = \mathcal{A}_p[1]$ for any $0 \leq p \leq s_{\mathcal{A}} = s_{\mathcal{A}'} - 1$. Let $\mathcal{G}^{\mathcal{A}'}_{s_{\mathcal{A}'}} = \mathcal{A}''$ and $\mathcal{G}^{\mathcal{A}'}_p = \mathcal{G}^{\mathcal{A}}_p[1]$ for any $1 \leq p \leq s_{\mathcal{A}} = s_{\mathcal{A}'} - 1$. The distinguished triangle

$$\mathcal{A} \longrightarrow \mathcal{A}' \longrightarrow \mathcal{A}'' \longrightarrow$$

implies

$$\mathcal{A}'_{s_{\mathcal{A}'}} \longrightarrow \mathcal{G}^{\mathcal{A}'}_{s_{\mathcal{A}'}} \longrightarrow \mathcal{A}'_{s_{\mathcal{A}'}-1} \longrightarrow$$

and the distinguished triangles

$$\mathcal{A}_p \longrightarrow \mathcal{G}_p^{\mathcal{A}} \longrightarrow \mathcal{A}_{p-1} \longrightarrow$$

imply

$$\mathcal{A}'_p \longrightarrow \mathcal{G}_p^{\mathcal{A}} \longrightarrow \mathcal{A}'_{p-1} \longrightarrow$$

for $1 \leq p \leq s_{\mathcal{A}'} - 1$. Hence, $\mathcal{A}'$ is with Property $\mathbf{A}$(m).

(2) Assuming that the lemma is true for $s_{\mathcal{A}''} < k$, we shall prove the lemma for $s_{\mathcal{A}''} = k$.

Now we have the following two distinguished triangles

$$\mathcal{A}' \xrightarrow{u} \mathcal{A}'' \longrightarrow \mathcal{A}[1] \longrightarrow$$

and

$$\mathcal{A}'' \xrightarrow{v} \mathcal{G}_k^{\mathcal{A}''} \longrightarrow \mathcal{A}''_{k-1} \longrightarrow .$$

Then we can construct the following distinguished triangle

$$\mathcal{A}' \xrightarrow{vu} \mathcal{G}_k^{\mathcal{A}''} \longrightarrow \mathcal{B} \longrightarrow .$$

By the octahedral axiom, there exist two maps $f : \mathcal{A}[1] \to \mathcal{B}$ and $g : \mathcal{B} \to \mathcal{A}''_{k-1}$ such that the following diagram commutes and the third row is a distinguished triangle

$$\begin{array}{ccccccc}
\mathcal{A}' & \xrightarrow{\mathrm{id}} & \mathcal{A}' & & & & \\
\downarrow{\scriptstyle u} & & \downarrow{\scriptstyle vu} & & & & \\
\mathcal{A}'' & \xrightarrow{v} & \mathcal{G}_k^{\mathcal{A}''} & \longrightarrow & \mathcal{A}''_{k-1} & \longrightarrow & \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}} & & \\
\mathcal{A}[1] & \xrightarrow{f} & \mathcal{B} & \xrightarrow{g} & \mathcal{A}''_{k-1} & \longrightarrow & . \\
\downarrow & & \downarrow & & & &
\end{array}$$

Consider the following distinguished triangle

$$\mathcal{A}[1] \xrightarrow{f} \mathcal{B} \xrightarrow{g} \mathcal{A}''_{k-1} \longrightarrow .$$

Since $\mathcal{A}''_k = \mathcal{A}''$ is with Property $\mathbf{A}$(m), $\mathcal{A}''_{k-1}$ is also with Property $\mathbf{A}$(m) and $s_{\mathcal{A}''_{k-1}} = k-1$. By the induction hypothesis, $\mathcal{B}$ is with Property $\mathbf{A}$(m) and $s_{\mathcal{B}} = s_{\mathcal{A}} + k$. Hence, for each $s_{\mathcal{B}} \geq p \in \mathbb{N}$, there exists $\mathcal{B}_p \in \mathcal{D}_{G_{\mathbf{V}^{(m)}}}(E_{\mathbf{V}^{(m)}})$ such that

1) $\mathcal{B}_{s_{\mathcal{B}}} = \mathcal{B}$ and $\mathcal{B}_0$ is the direct sum of some semisimple complexes of the form $I_{p'}^{(m)}[l]$;
2) for each $p \geq 1$, there exists a distinguished triangle

$$\mathcal{B}_p \longrightarrow \mathcal{G}_p^{\mathcal{B}} \longrightarrow \mathcal{B}_{p-1} \longrightarrow ,$$

where $\mathcal{G}_p^{\mathcal{B}}$ is the direct sum of some semisimple complexes of the form $I_{p'}^{(m)}[l]$.

Note that $s_{\mathcal{A}'} = s_{\mathcal{B}} + 1$. Let $\mathcal{A}'_{s_{\mathcal{A}'}} = \mathcal{A}'$ and $\mathcal{A}'_p = \mathcal{B}_p$ for any $0 \leq p \leq s_{\mathcal{B}} = s_{\mathcal{A}'} - 1$. Let $\mathcal{G}_{s_{\mathcal{A}'}}^{\mathcal{A}'} = \mathcal{G}_k^{\mathcal{A}''}$ and $\mathcal{G}_p^{\mathcal{A}'} = \mathcal{G}_p^{\mathcal{B}}$ for any $1 \leq p \leq s_{\mathcal{B}} = s_{\mathcal{A}'} - 1$. The distinguished triangle

$$\mathcal{A}' \xrightarrow{vu} \mathcal{G}_k^{\mathcal{A}''} \longrightarrow \mathcal{B} \longrightarrow$$

implies

$$\mathcal{A}'_{s_{\mathcal{A}'}} \xrightarrow{vu} \mathcal{G}_{s_{\mathcal{A}'}}^{\mathcal{A}'} \longrightarrow \mathcal{A}'_{s_{\mathcal{A}'}-1} \longrightarrow$$

and the distinguished triangles

$$\mathcal{B}_p \longrightarrow \mathcal{G}_p^{\mathcal{B}} \longrightarrow \mathcal{B}_{p-1} \longrightarrow$$

imply

$$\mathcal{A}'_p \longrightarrow \mathcal{G}_p^{\mathcal{A}} \longrightarrow \mathcal{A}'_{p-1} \longrightarrow$$

for $1 \leq p \leq s_{\mathcal{A}'} - 1$. Hence, $\mathcal{A}'$ is with Property $\mathbf{A}$(m).

By induction, the proof is finished.

□

For the proof of Theorem 5.3, we also need the following proposition.

Proposition 5.6. *For each $m \geq p \in \mathbb{N}$, there exists $\mathcal{C}_p^{(m)} \in \mathcal{D}_{G_{\mathbf{V}^{(m)}}}(E_{\mathbf{V}^{(m)}})$ such that*

(1) $\mathcal{C}_m^{(m)} = I_m^{(m)}$ *and* $\mathcal{C}_0^{(m)} = v^{-mN}(\mathcal{L}_{mi} \circledast \mathbf{1}_{{}_iE_{\mathbf{V}^{(0)}}})$*;*

(2) *for each $p \geq 1$, there exists a distinguished triangle*

$$v^{a_p^{(m)}}(\mathcal{L}_{(m-p)i} \circledast \mathbf{1}_{{}_iE_{\mathbf{V}^{(p)}}}) \longrightarrow \mathcal{C}_p^{(m)} \longrightarrow \mathcal{C}_{p-1}^{(m)} \longrightarrow,$$

where $a_p^{(m)} = p(m-p) - mN$.

Proof. For each $m \geq p \in \mathbb{N}$, let $\mathcal{C}_p^{(m)} = v^{-mN}\mathbf{1}_{S_p^{(m)}}$. We shall prove that $\mathcal{C}_p^{(m)}$ satisfy the desired conditions for each p.

For $p = m$, $\mathcal{C}_m^{(m)} = v^{-mN}\mathbf{1}_{S_m^{(m)}} = v^{-mN}\mathbf{1}_{E_{\mathbf{V}^{(m)}}}$. Since $I_m^{(m)} \simeq v^{-mN}\mathbf{1}_{E_{\mathbf{V}^{(m)}}}$, we have $\mathcal{C}_m^{(m)} \simeq I_m^{(m)}$.

For each $p \geq 1$, we have the following distinguished triangle

$$(j_p^{(m)})_!(j_p^{(m)})^!(\mathcal{C}_p^{(m)}) \longrightarrow \mathcal{C}_p^{(m)} \longrightarrow (i_p^{(m)})_*(i_p^{(m)})^*(\mathcal{C}_p^{(m)}) \longrightarrow.$$

Since $\mathcal{C}_p^{(m)} \simeq v^{-mN}\mathbf{1}_{S_p^{(m)}}$,

$$(i_p^{(m)})_*(i_p^{(m)})^*(\mathcal{C}_p^{(m)}) \simeq v^{-mN}\mathbf{1}_{S_{p-1}^{(m)}} = \mathcal{C}_{p-1}^{(m)},$$

and

$$(j_p^{(m)})_!(j_p^{(m)})^!(\mathcal{C}_p^{(m)}) \simeq v^{-mN}\mathbf{1}_{\mathcal{N}_p^{(m)}}.$$

By the definition of $\mathcal{N}_p^{(m)}$,

$$\mathbf{1}_{\mathcal{N}_p^{(m)}} = \mathcal{L}_{(m-p)i} * \mathbf{1}_{{}_iE_{\mathbf{V}^{(p)}}}.$$

Hence

$$v^{-mN}\mathbf{1}_{\mathcal{N}_p^{(m)}} = v^{-mN}v^{p(m-p)}(\mathcal{L}_{(m-p)i} \circledast \mathbf{1}_{{}_iE_{\mathbf{V}^{(p)}}}) = v^{a_p^{(m)}}(\mathcal{L}_{(m-p)i} \circledast \mathbf{1}_{{}_iE_{\mathbf{V}^{(p)}}}),$$

that is

$$(j_p^{(m)})_!(j_p^{(m)})^!(\mathcal{C}_p^{(m)}) \simeq v^{a_p^{(m)}}(\mathcal{L}_{(m-p)i} \circledast \mathbf{1}_{{}_iE_{\mathbf{V}^{(p)}}}).$$

So, for each $p \geq 1$, there exists a distinguished triangle

$$v^{a_p^{(m)}}(\mathcal{L}_{(m-p)i} \circledast \mathbf{1}_{{}_iE_{\mathbf{V}^{(p)}}}) \longrightarrow \mathcal{C}_p^{(m)} \longrightarrow \mathcal{C}_{p-1}^{(m)} \longrightarrow,$$

and $\mathcal{C}_0^{(m)} = v^{-mN}(\mathcal{L}_{mi} \circledast \mathbf{1}_{{}_iE_{\mathbf{V}^{(0)}}})$.

□

In Section 4.1, we have $\chi : K(\mathcal{Q}) \to \mathcal{F}$. In this section, we identify the Lusztig's algebra $\mathbf{f}$ with the corresponding composition subalgebra $\mathcal{F}$.

Lusztig proved the following theorem.

Theorem 5.7 ([17])**.** *$\chi(I_p^{(m)}) = \theta_i^{(m-p)}\theta_j\theta_i^{(p)}$ for each $m \geq p \in \mathbb{N}$.*

By Proposition 5.6 and Theorem 5.7, we have the following corollary.

Corollary 5.8. *We have the following formula in* $\mathbf{f}$

$$\theta_j\theta_i^{(m)} = \sum_{p=0}^{m} v^{b_p^{(m)}}\theta_i^{(m-p)}\chi(\mathcal{E}^{(p)}),$$

where $b_p^{(m)} = (p-N)(m-p)$.

Proof. By Proposition 5.6 and Theorem 5.7, we have

$$\theta_j\theta_i^{(m)} = \sum_{p=0}^{m} v^{a_p^{(m)}}\theta_i^{(m-p)}\chi(\mathbf{1}_{iE_{\mathbf{V}^{(p)}}}) = \sum_{p=0}^{m} v^{a_p^{(m)}}v^{pN}\theta_i^{(m-p)}\chi(\mathcal{E}^{(p)}).$$

Since $a_p^{(m)} + pN = b_p^{(m)}$, we have

$$\theta_j\theta_i^{(m)} = \sum_{p=0}^{m} v^{b_p^{(m)}}\theta_i^{(m-p)}\chi(\mathcal{E}^{(p)}).$$

□

We shall use Lemma 5.5 and Proposition 5.6 to prove Theorem 5.3 by induction.

Proof of Theorem 5.3. We shall prove this result by induction on m.

(1) For $m = 0$, $\mathcal{E}^{(0)} = I_0^{(0)}$. It is clear that $\mathcal{E}^{(0)}$ is with Property $\mathbf{A}(0)$.

(2) For $m = 1$, by Proposition 5.6, there exists a distinguished triangle

$$v^{-N}\mathbf{1}_{iE_{\mathbf{V}^{(1)}}} \longrightarrow \mathcal{C}_1^{(1)} \longrightarrow \mathcal{C}_0^{(1)} \longrightarrow,$$

where $\mathcal{C}_1^{(1)} = I_1^{(1)}$ and $\mathcal{C}_0^{(1)} = v^{-N}(\mathcal{L}_i \circledast \mathbf{1}_{iE_{\mathbf{V}^{(0)}}})$. Since $\mathcal{E}^{(0)} = I_0^{(0)}$,

$$\mathcal{C}_0^{(1)} = v^{-N}(\mathcal{L}_i \circledast \mathbf{1}_{iE_{\mathbf{V}^{(0)}}}) = \mathcal{L}_i \circledast \mathcal{E}^{(0)}$$

is the direct sum of some semisimple complexes of the form $I_{p'}^{(1)}[l]$. Hence, $\mathcal{E}^{(1)} = v^{-N}\mathbf{1}_{iE_{\mathbf{V}^{(1)}}}$ is with Property $\mathbf{A}(1)$.

(3) Assume that the $\mathcal{E}^{(k)}$ is with Property $\mathbf{A}$(k) for all $k < m$. Let us prove that $\mathcal{E}^{(m)}$ is with Property $\mathbf{A}$(m).

For any $k < m$, there exists $s_k \in \mathbb{N}$. For each $s_k \geq p \in \mathbb{N}$, there exists $\mathcal{E}_p^{(k)} \in \mathcal{D}_{G_{\mathbf{V}^{(k}}}(E_{\mathbf{V}^{(k)}})$ such that

1) $\mathcal{E}_{s_k}^{(k)} = \mathcal{E}^{(k)}$ and $\mathcal{E}_0^{(k)}$ is the direct sum of some semisimple complexes of the form $I_{p'}^{(k)}[l]$;
2) for each $p \geq 1$, there exists a distinguished triangle

$$\mathcal{E}_p^{(k)} \longrightarrow \mathcal{G}_p^{(k)} \longrightarrow \mathcal{E}_{p-1}^{(k)} \longrightarrow,$$

where $\mathcal{G}_p^{(k)}$ is the direct sum of some semisimple complexes of the form $I_{p'}^{(k)}[l]$.

Hence, we have the following distinguished triangle for each $p \geq 1$

$$\mathcal{L}_{(m-k)i} \circledast \mathcal{E}_p^{(k)} \longrightarrow \mathcal{L}_{(m-k)i} \circledast \mathcal{G}_p^{(k)} \longrightarrow \mathcal{L}_{(m-k)i} \circledast \mathcal{E}_{p-1}^{(k)} \longrightarrow .$$

Denote $\tilde{\mathcal{E}}_p^{(k)} = \mathcal{L}_{(m-k)i} \circledast \mathcal{E}_p^{(k)}$ and $\tilde{\mathcal{G}}_p^{(k)} = \mathcal{L}_{(m-k)i} \circledast \mathcal{G}_p^{(k)}$. Then, we have

$$\tilde{\mathcal{E}}_p^{(k)} \longrightarrow \tilde{\mathcal{G}}_p^{(k)} \longrightarrow \tilde{\mathcal{E}}_{p-1}^{(k)} \longrightarrow .$$

Because $\tilde{\mathcal{E}}_0^{(k)}$ and $\tilde{\mathcal{G}}_p^{(k)}$ are the direct sums of some semisimple complexes of the form $I_{p'}^{(m)}[l]$, $\tilde{\mathcal{E}}_k^{(k)}$ is with Property $\mathbf{A}$(m). Since

$$\tilde{\mathcal{E}}_k^{(k)} = \mathcal{L}_{(m-k)i} \circledast \mathcal{E}_k^{(k)} = v^{-kN}(\mathcal{L}_{(m-k)i} \circledast \mathbf{1}_{{}_iE_{\mathbf{V}^{(k)}}}),$$

$\mathcal{L}_{(m-k)i} \circledast \mathbf{1}_{{}_iE_{\mathbf{V}^{(k)}}}$ is with Property $\mathbf{A}$(m).

By Proposition 5.6, for each $m \geq k \in \mathbb{N}$, there exists $\mathcal{C}_k^{(m)} \in \mathcal{D}_{G_{\mathbf{V}^{(m)}}}(E_{\mathbf{V}^{(m)}})$ such that

1) $\mathcal{C}_m^{(m)} = I_m^{(m)}$ and $\mathcal{C}_0^{(m)} = v^{-mN}(\mathcal{L}_{mi} \circledast \mathbf{1}_{{}_iE_{\mathbf{V}^{(0)}}})$;
2) for each $k \geq 1$, there exists a distinguished triangle

$$v^{a_k^{(m)}}(\mathcal{L}_{(m-k)i} \circledast \mathbf{1}_{{}_iE_{\mathbf{V}^{(k)}}}) \longrightarrow \mathcal{C}_k^{(m)} \longrightarrow \mathcal{C}_{k-1}^{(m)} \longrightarrow .$$

We have proved that $\mathcal{C}_0^{(m)}$ and $\mathcal{L}_{(m-k)i} \circledast \mathbf{1}_{{}_iE_{\mathbf{V}^{(k)}}}$ $(1 \leq k \leq m-1)$ are with Property $\mathbf{A}$(m). Hence, by Lemma 5.5, $\mathcal{C}_{m-1}^{(m)}$ is with Property $\mathbf{A}$(m). At last, by Lemma 5.5 and the distinguished triangle

$$\mathcal{C}_{m-1}^{(m)}[-1] \longrightarrow v^{-mN}\mathbf{1}_{{}_iE_{\mathbf{V}^{(p)}}} \longrightarrow I_m^{(m)} \longrightarrow ,$$

$\mathcal{E}^{(m)} = v^{-mN}\mathbf{1}_{{}_iE_{\mathbf{V}^{(p)}}}$ is with Property $\mathbf{A}$(m).

By induction, the proof is finished. □

As a corollary of Theorem 5.3, we have

Corollary 5.9. *For each $N \geq m \in \mathbb{N}$, we have the following formula*

$$\chi(\mathcal{E}^{(m)}) = \sum_{p=0}^{m} (-1)^p v^{-p(1+N-m)} \theta_i^{(p)} \theta_j \theta_i^{(m-p)} = f(i,j;m).$$

Proof. By Theorem 5.3, we have

$$\chi(\mathcal{E}^{(m)}) = \sum_{p=0}^{m} c_p^{(m)} \theta_i^{(p)} \theta_j \theta_i^{(m-p)}.$$

We shall prove that $c_p^{(m)} = (-1)^p v^{-p(1+N-m)}$ $(0 \leq p \leq m)$ by induction on m.

(1) For $m = 0$, by Corollary 5.8,

$$\theta_j = \chi(\mathcal{E}^{(0)}).$$

That is $c_0^{(0)} = 1$. Hence, the corollary is true in this case.

(2) Assume that $c_p^{(k)} = (-1)^p v^{-p(1+N-k)}$ $(0 \le p \le k)$ for any $k < m$. We shall prove that $c_q^{(m)} = (-1)^q v^{-q(1+N-m)}$ $(0 \le q \le m)$.

By Corollary 5.8,

$$\begin{aligned}
\theta_j\theta_i^{(m)} &= \sum_{k=0}^{m} v^{b_k^{(m)}}\theta_i^{(m-k)}\chi(\mathcal{E}^{(k)}) \\
&= \sum_{k=0}^{m-1} v^{b_k^{(m)}}\theta_i^{(m-k)}\chi(\mathcal{E}^{(k)}) + v^{b_m^{(m)}}\chi(\mathcal{E}^{(m)}) \\
&= \sum_{k=0}^{m-1} v^{b_k^{(m)}}\theta_i^{(m-k)}\chi(\mathcal{E}^{(k)}) + \chi(\mathcal{E}^{(m)}).
\end{aligned}$$

Hence,

$$\begin{aligned}
\chi(\mathcal{E}^{(m)}) &= \theta_j\theta_i^{(m)} - \sum_{k=0}^{m-1} v^{b_k^{(m)}}\theta_i^{(m-k)}\chi(\mathcal{E}^{(k)}) \\
&= \theta_j\theta_i^{(m)} - \sum_{k=0}^{m-1} v^{b_k^{(m)}}\theta_i^{(m-k)}\sum_{p=0}^{k} c_p^{(k)}\theta_i^{(p)}\theta_j\theta_i^{(k-p)} \\
&= \theta_j\theta_i^{(m)} - \sum_{k=0}^{m-1}\sum_{p=0}^{k} v^{b_k^{(m)}} c_p^{(k)}\frac{[m-k+p]_v!}{[m-k]_v![p]_v!}\theta_i^{(m-k+p)}\theta_j\theta_i^{(k-p)}.
\end{aligned}$$

For any $q \ge 1$,

$$\begin{aligned}
c_q^{(m)} &= -\sum_{k=m-q}^{m-1} v^{b_k^{(m)}} c_{q+k-m}^{(k)}\frac{[q]_v!}{[m-k]_v![q+k-m]_v!} \\
&= -\sum_{k=0}^{q-1} v^{b_{k+m-q}^{(m)}} c_k^{(k+m-q)}\frac{[q]_v!}{[q-k]_v![k]_v!}.
\end{aligned}$$

By the induction hypothesis,

$$\begin{aligned}
c_q^{(m)} &= -\sum_{k=0}^{q-1} v^{b_{k+m-q}^{(m)}} c_k^{(k+m-q)}\frac{[q]_v!}{[q-k]_v![k]_v!} \\
&= -\sum_{k=0}^{q-1}(-1)^k v^{(k+m-q-N)(q-k)}v^{-k(1+N-k-m+q)}\frac{[q]_v!}{[q-k]_v![k]_v!} \\
&= -v^{q(m-q-N)}\sum_{k=0}^{q-1}(-1)^k v^{k(q-1)}\frac{[q]_v!}{[q-k]_v![k]_v!} \\
&= -v^{q(m-q-N)}\sum_{k=0}^{q}(-1)^k v^{k(q-1)}\frac{[q]_v!}{[q-k]_v![k]_v!} + v^{q(m-q-N)}(-1)^q v^{q(q-1)} \\
&= v^{q(m-q-N)}(-1)^q v^{q(q-1)} = (-1)^q v^{-q(1+N-m)}.
\end{aligned}$$

Note that

$$c_0^{(m)} = 1 = (-1)^0 v^{-0(1+N-m)}.$$

Hence, $c_q^{(m)} = (-1)^q v^{-q(1+N-m)}$ for any $0 \le q \le m$.

By induction, for each $N \geq m \in \mathbb{N}$, $c_p^{(m)} = (-1)^p v^{-p(1+N-m)}$ $(0 \leq p \leq m)$ and

$$\chi(\mathcal{E}^{(m)}) = \sum_{p=0}^{m} (-1)^p v^{-p(1+N-m)} \theta_i^{(p)} \theta_j \theta_i^{(m-p)}.$$

□

5.4. **The formulas of Lusztig's symmetries.** In this section, we shall give a new proof of Proposition 2.1.

Consider the following quiver

$$Q:\ i \overset{\vdots}{\leftleftarrows} j$$

with vertex set $I = \{i, j\}$ and N arrows from j to i. Let $Q' = \sigma_i Q$ be the quiver by reversing the directions of all arrows

$$Q':\ i \overset{\vdots}{\rightrightarrows} j\ .$$

Let m be a non-negative integer such that $m \leq N$ and $m' = N - m$. Let $\nu = mi + j \in \mathbb{N}I$ and $\nu' = s_i \nu = m'i + j \in \mathbb{N}I$. Fix two I-graded $\mathbb{K}$-vector spaces $\mathbf{V}$ and $\mathbf{V}'$ such that $\underline{\dim}\mathbf{V} = \nu$ and $\underline{\dim}\mathbf{V}' = \nu'$.

Denote by $\mathbf{1}_{{}_iE_{\mathbf{V},Q}} \in \mathcal{D}_{G_\mathbf{V}}({}_iE_{\mathbf{V},Q})$ the constant sheaf on ${}_iE_{\mathbf{V},Q}$ and $\mathbf{1}_{{}^iE_{\mathbf{V}',Q'}} \in \mathcal{D}_{G_{\mathbf{V}'}}({}^iE_{\mathbf{V}',Q'})$ the constant sheaf on ${}^iE_{\mathbf{V}',Q'}$. For convenience, denote ${}_iE_{\mathbf{V},Q}$ (resp. ${}^iE_{\mathbf{V}',Q'}$) by ${}_iE_\mathbf{V}$ (resp. ${}^iE_{\mathbf{V}'}$) and $\mathbf{1}_{{}_iE_{\mathbf{V},Q}}$ (resp. $\mathbf{1}_{{}^iE_{\mathbf{V}',Q'}}$) by $\mathbf{1}_{{}_iE_\mathbf{V}}$ (resp. $\mathbf{1}_{{}^iE_{\mathbf{V}'}}$).

Denote

$$\mathcal{E}^{(m)} = j_{\mathbf{V}!}(v^{-mN} \mathbf{1}_{{}_iE_\mathbf{V}}) \in \mathcal{D}_{G_\mathbf{V}}(E_\mathbf{V})$$

and

$$\mathcal{E}'^{(m')} = j_{\mathbf{V}'!}(v^{-m'N} \mathbf{1}_{{}^iE_{\mathbf{V}'}}) \in \mathcal{D}_{G_{\mathbf{V}'}}(E_{\mathbf{V}'}).$$

In Section 3.3, we give the following geometric realization of the Lusztig's symmetry T_i:

$$\tilde{\omega}_i : \mathcal{D}_{G_\mathbf{V}}({}_iE_\mathbf{V}) \to \mathcal{D}_{G_{\mathbf{V}'}}({}^iE_{\mathbf{V}'}).$$

Proposition 5.10. *For any $N \geq m \in \mathbb{N}$, $\tilde{\omega}_i(v^{-mN} \mathbf{1}_{{}_iE_\mathbf{V}}) = v^{-m'N} \mathbf{1}_{{}^iE_{\mathbf{V}'}}$.*

Proof. By the definitions of α and β in the diagram (5) of Section 3.3,

$$\alpha^*(\mathbf{1}_{{}_iE_\mathbf{V}}) = \mathbf{1}_{Z_{\mathbf{V}\mathbf{V}'}} = \beta^*(\mathbf{1}_{{}^iE_{\mathbf{V}'}}).$$

Hence

$$\tilde{\omega}_i(\mathbf{1}_{{}_iE_\mathbf{V}}) = v^{(m-m')N} \mathbf{1}_{{}^iE_{\mathbf{V}'}}.$$

That is

$$\tilde{\omega}_i(v^{-mN} \mathbf{1}_{{}_iE_\mathbf{V}}) = v^{-m'N} \mathbf{1}_{{}^iE_{\mathbf{V}'}}.$$

□

Corollary 5.9 implies $\chi(\mathcal{E}^{(m)}) = f(i, j; m)$. Similarly, we have $\chi(\mathcal{E}'^{(m')}) = f'(i, j; m')$. Hence, Proposition 5.10 implies Proposition 2.1.

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