GEOMETRIC REALIZATIONS OF LUSZTIG'S SYMMETRIES OF SYMMETRIZABLE QUANTUM GROUPS

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Abstract. Let $\mathbf{U}$ be the quantum group and $\mathbf{f}$ be the Lusztig's algebra associated with a symmetrizable generalized Cartan matrix. The algebra $\mathbf{f}$ can be viewed as the positive part of $\mathbf{U}$. Lusztig introduced some symmetries T_i on $\mathbf{U}$ for all $i \in I$. Since $T_i(\mathbf{f})$ is not contained in $\mathbf{f}$, Lusztig considered two subalgebras ${}_i\mathbf{f}$ and ${}^i\mathbf{f}$ of $\mathbf{f}$ for any $i \in I$, where ${}_i\mathbf{f} = \{x \in \mathbf{f} \mid T_i(x) \in \mathbf{f}\}$ and ${}^i\mathbf{f} = \{x \in \mathbf{f} \mid T_i^{-1}(x) \in \mathbf{f}\}$. The restriction of T_i on ${}_i\mathbf{f}$ is also denoted by $T_i : {}_i\mathbf{f} \to {}^i\mathbf{f}$. The geometric realization of $\mathbf{f}$ and its canonical basis are introduced by Lusztig via some semisimple complexes on the variety consisting of representations of the corresponding quiver. When the generalized Cartan matrix is symmetric, Xiao and Zhao gave geometric realizations of Lusztig's symmetries in the sense of Lusztig. In this paper, we shall generalize this result and give geometric realizations of ${}_i\mathbf{f}$, ${}^i\mathbf{f}$ and $T_i : {}_i\mathbf{f} \to {}^i\mathbf{f}$ by using the language 'quiver with automorphism' introduced by Lusztig.

Contents

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1. Introduction

1.1. Let $\mathbf{U}$ be the quantum group and $\mathbf{f}$ be the Lusztig's algebra associated with a symmetrizable generalized Cartan matrix. There are two well-defined $\mathbb{Q}(v)$-algebra embeddings $^+ : \mathbf{f} \to \mathbf{U}$ and $^- : \mathbf{f} \to \mathbf{U}$ with images $\mathbf{U}^+$ and $\mathbf{U}^-$, where $\mathbf{U}^+$ and $\mathbf{U}^-$ are the positive part and the negative part of $\mathbf{U}$ respectively.

When the generalized Cartan matrix is symmetric, Lusztig introduced the geometric realization of $\mathbf{f}$ and the canonical basis of it in [9, 11]. In [14], Lusztig generalized the geometric realization to $\mathbf{f}$ associated with a symmetrizable generalized Cartan matrix.

Let $\tilde{Q} = (Q, a)$ be a quiver with automorphism corresponding to $\mathbf{f}$, where $Q = (\mathbf{I}, H)$. Let $\mathbf{V}$ be an $\mathbf{I}$-graded vector space with an isomorphism $a : \mathbf{V} \to \mathbf{V}$ such that $\underline{\dim}\mathbf{V} = \nu \in \mathbb{N}\mathbf{I}^a$. Consider the variety $E_{\mathbf{V}}$ consisting of representations of Q with dimension vector ν and a category $\mathcal{Q}_{\mathbf{V}}$ of some semisimple complexes ([1, 2, 6]) on $E_{\mathbf{V}}$.

The isomorphism $a : \mathbf{V} \to \mathbf{V}$ induces a functor $a^* : \mathcal{Q}_{\mathbf{V}} \to \mathcal{Q}_{\mathbf{V}}$. Lusztig defined a new category $\tilde{\mathcal{Q}}_{\mathbf{V}}$ consisting of objects $(\mathcal{L}, \phi)$, where $\mathcal{L}$ is an object in $\mathcal{Q}_{\mathbf{V}}$ and $\phi : a^*\mathcal{L} \to \mathcal{L}$ is an isomorphism. Lusztig considered a submodule $\mathbf{k}_\nu$ of $K(\tilde{\mathcal{Q}}_{\mathbf{V}})$, whose definition is given by Lusztig and similar to that of a Grothendieck group ([14]). Considering all dimension vectors, he proved that $\mathbf{k} = \bigoplus_{\nu \in \mathbb{N}I} \mathbf{k}_\nu$ is isomorphic to $\mathbf{f}$.

Lusztig also introduced some symmetries T_i on $\mathbf{U}$ for all $i \in I$ in [8, 10]. Since $T_i(\mathbf{U}^+)$ is not contained in $\mathbf{U}^+$, Lusztig introduced two subalgebras ${}_i\mathbf{f}$ and ${}^i\mathbf{f}$ of $\mathbf{f}$ for any $i \in I$, where ${}_i\mathbf{f} = \{x \in \mathbf{f} \mid T_i(x^+) \in \mathbf{U}^+\}$ and ${}^i\mathbf{f} = \{x \in \mathbf{f} \mid T_i^{-1}(x^+) \in \mathbf{U}^+\}$. Let $T_i : {}_i\mathbf{f} \to {}^i\mathbf{f}$ be the unique map satisfying $T_i(x^+) = T_i(x)^+$. For any $i \in I$, ${}_i\mathbf{f}$ and ${}^i\mathbf{f}$ are the subalgebras of $\mathbf{f}$ generated by $f(i,j;m)$ and $f'(i,j;m)$ for all $i \neq j \in I$ and $-a_{ij} \geq m \in \mathbb{N}$ respectively. The definitions of $f(i,j;m)$ and $f'(i,j;m)$ will be given in Section 2.2. At the same time, Lusztig pointed that ${}_i\mathbf{f} = \{x \in \mathbf{f} \mid {}_ir(x) = 0\}$ and ${}^i\mathbf{f} = \{x \in \mathbf{f} \mid r_i(x) = 0\}$. The definition of ${}_ir$ will be given in Section 4.1 and the definition of r_i is similar to that of ${}_ir$. These descriptions of ${}_i\mathbf{f}$ and ${}^i\mathbf{f}$ are closely relevant to the geometric interpretation of them.

Associated to a finite dimensional hereditary algebra, Ringel introduced the Hall algebra and its composition subalgebra in [15], which gives a realization of $\mathbf{U}^+$. Via the Hall algebra approach, one can apply BGP-reflection functors to quantum groups to give precise constructions of Lusztig's symmetries ([16, 13, 17, 18, 3, 19]).

1.2. Assume that the Cartan matrix is symmetric and let $Q = (I, H)$ be a quiver corresponding to $\mathbf{f}$. Let $i \in I$ be a sink (resp. source) of Q. Similarly to the geometric realization of $\mathbf{f}$, consider a subvariety ${}_iE_{\mathbf{V}}$ (resp. ${}^iE_{\mathbf{V}}$) of $E_{\mathbf{V}}$ and a category ${}_i\mathcal{Q}_{\mathbf{V}}$ (resp. ${}^i\mathcal{Q}_{\mathbf{V}}$) of some semisimple complexes on ${}_iE_{\mathbf{V}}$ (resp. ${}^iE_{\mathbf{V}}$). In [20], it was showed that $\oplus_{\nu \in \mathbb{N}I} K({}_i\mathcal{Q}_{\mathbf{V}})$ (resp. $\oplus_{\nu \in \mathbb{N}I} K({}^i\mathcal{Q}_{\mathbf{V}})$) realizes ${}_i\mathbf{f}$ (resp. ${}^i\mathbf{f}$).

Let $i \in I$ be a sink of Q and $Q' = \sigma_i Q$ be the quiver by reversing the directions of all arrows in Q containing i. Hence, i is a source of Q'. Consider two I-graded vector spaces $\mathbf{V}$ and $\mathbf{V}'$ such that $\underline{\dim}\mathbf{V}' = s_i(\underline{\dim}\mathbf{V})$. In the case of finite type, Kato introduced an equivalence $\tilde{\omega}_i : {}_i\mathcal{Q}_{\mathbf{V},Q} \to {}^i\mathcal{Q}_{\mathbf{V}',Q'}$ and studied the properties of this equivalence in [5], building on the technical tools he established in [4]. In [20], his construction was generalized to all symmetric cases. It was proved that the map induced by $\tilde{\omega}_i$ realizes the Lusztig's symmetry $T_i : {}_i\mathbf{f} \to {}^i\mathbf{f}$ by using the relations between $\tilde{\omega}_i$ and the Hall algebra approach to T_i in [13].

In [12], Lusztig showed that Lusztig's symmetries and canonical bases are compatible. The main result in [20] gives a geometric interpretation of Lusztig's result in [12].

1.3. In this paper, we shall generalize the construction in [20] and give geometric realizations of Lusztig's symmetries of symmetrizable quantum groups.

Let $\tilde{Q} = (Q, a)$ be a quiver with automorphism. Fix $i \in I = \mathbf{I}^a$ and assume that $\mathbf{i}$ is a sink (resp. source) for any $\mathbf{i} \in i$. Similarly to the category $\tilde{\mathcal{Q}}_{\mathbf{V}}$, we can define ${}_i\tilde{\mathcal{Q}}_{\mathbf{V}}$ (resp. ${}^i\tilde{\mathcal{Q}}_{\mathbf{V}}$). Consider a submodule ${}_i\mathbf{k}_\nu$ (resp. ${}^i\mathbf{k}_\nu$) of $K({}_i\tilde{\mathcal{Q}}_{\mathbf{V}})$ (resp. $K({}^i\tilde{\mathcal{Q}}_{\mathbf{V}})$). We verify that $\oplus_{\nu \in \mathbb{N}I}{}_i\mathbf{k}_\nu$ (resp. $\oplus_{\nu\in\mathbb{N}I}{}^i\mathbf{k}_\nu$) realizes ${}_i\mathbf{f}$ (resp. ${}^i\mathbf{f}$) by using the result in [20] and the relation between $\tilde{\mathcal{Q}}_{\mathbf{V}}$ and $\mathcal{Q}_{\mathbf{V}}$.

Let $i \in I = \mathbf{I}^a$ such that $\mathbf{i}$ is a sink, for any $\mathbf{i} \in i$. Let $Q' = \sigma_i Q$ be the quiver by reversing the directions of all arrows in Q containing $\mathbf{i} \in i$. So for any $\mathbf{i} \in i$, $\mathbf{i}$ is a source of Q'.

Consider two $\mathbf{I}$-graded vector spaces $\mathbf{V}$ and $\mathbf{V}'$ with isomorphisms $a : \mathbf{V} \to \mathbf{V}$ and $a : \mathbf{V}' \to \mathbf{V}'$ such that $\underline{\dim}\mathbf{V}' = s_i(\underline{\dim}\mathbf{V})$. In this paper, it is proved that the equivalence $\tilde{\omega}_i : {}_i\mathcal{Q}_{\mathbf{V},Q} \to {}^i\mathcal{Q}_{\mathbf{V}',Q'}$ is compatible with a^*. Hence we get a functor $\tilde{\omega}_i : {}_i\tilde{\mathcal{Q}}_{\mathbf{V},Q} \to {}^i\tilde{\mathcal{Q}}_{\mathbf{V}',Q'}$ and a map $\tilde{\omega}_i : {}_i\mathbf{k} \to {}^i\mathbf{k}$. We also prove $\tilde{\omega}_i : {}_i\mathbf{k} \to {}^i\mathbf{k}$ is an isomorphism of algebras.

Assume that $\underline{\dim}\mathbf{V} = m\gamma_i + \gamma_j$, where $\gamma_i = \sum_{\mathbf{i}\in i}\mathbf{i}$ and $\gamma_j = \sum_{\mathbf{i}\in j}\mathbf{i}$. We construct a series of distinguished triangles in $\mathcal{D}_{G_{\mathbf{V},Q}}(E_{\mathbf{V},Q})$, which represent the constant sheaf $\mathbf{1}_{{}_iE_{\mathbf{V},Q}}$ in terms of some semisimple complexes $I_p \in \mathcal{D}_{G_{\mathbf{V},Q}}(E_{\mathbf{V},Q})$ geometrically. Applying to the Grothendieck group, $\mathbf{1}_{{}_iE_{\mathbf{V},Q}}$ corresponds to $f(i,j;m)$. Assume that $\underline{\dim}\mathbf{V}' = s_i(\underline{\dim}\mathbf{V}) = m'\gamma_i + \gamma_j$. Applying to the Grothendieck group, $\mathbf{1}_{{}^iE_{\mathbf{V}',Q'}}$ corresponds to $f'(i,j;m')$ similarly. The properties of BGP-reflection functors imply $\tilde{\omega}_i(v^{-mN}\mathbf{1}_{{}_iE_{\mathbf{V},Q}}) = v^{-m'N}\mathbf{1}_{{}^iE_{\mathbf{V}',Q'}}$. Since ${}_i\mathbf{f}$ (resp. ${}^i\mathbf{f}$) is generated by $f(i,j;m)$ (resp. $f'(i,j;m)$), we have the following commutative diagram

$$\begin{array}{ccc} {}_i\mathbf{k} & \xrightarrow{\tilde{\omega}_i} & {}^i\mathbf{k} \\ \downarrow & & \downarrow \\ {}_i\mathbf{f}_{\mathcal{A}} & \xrightarrow{T_i} & {}^i\mathbf{f}_{\mathcal{A}}. \end{array}$$

That is, $\tilde{\omega}_i$ gives a geometric realization of Lusztig's symmetry T_i for any $i \in I$.

2. Quantum groups and Lusztig's symmetries

2.1. **Quantum groups.** Fix a finite index set I with $|I| = n$. Let $A = (a_{ij})_{i,j\in I}$ be a symmetrizable generalized Cartan matrix and $D = \mathrm{diag}(\varepsilon_i \mid i \in I)$ be a diagonal matrix such that DA is symmetric. Let $(A, \Pi, \Pi^\vee, P, P^\vee)$ be a Cartan datum associated with A, where

(1) $\Pi = \{\alpha_i \mid i \in I\}$ is the set of simple roots;
(2) $\Pi^\vee = \{h_i \mid i \in I\}$ is the set of simple coroots;
(3) P is the weight lattice;
(4) $P^\vee$ is the dual weight lattice.

Let $\mathfrak{h}=\mathbb{Q}\otimes_{\mathbb{Z}}P^\vee$ and there exist a symmetric bilinear form $(-,-)$ on $\mathfrak{h}^*$ such that $(\alpha_i,\alpha_j)=\varepsilon_i a_{ij}$ for any $i,j\in I$ and $\lambda(h_i)=2\frac{(\alpha_i,\lambda)}{(\alpha_i,\alpha_i)}$ for any $\lambda\in\mathfrak{h}^*$ and $i\in I$.

Fix an indeterminate v. Let $v_i=v^{\varepsilon_i}$. For any $n\in\mathbb{Z}$, set

$$[n]_{v_i}=\frac{v_i^n-v_i^{-n}}{v_i-v_i^{-1}}\in\mathbb{Q}(v).$$

Let $[0]_{v_i}!=1$ and $[n]_{v_i}!=[n]_{v_i}[n-1]_{v_i}\cdots[1]_{v_i}$ for any $n\in\mathbb{Z}_{>0}$.

Let $\mathbf{U}$ be the quantum group corresponding to $(A,\Pi,\Pi^\vee,P,P^\vee)$ generated by the elements $E_i,F_i(i\in I)$ and $K_\mu(\mu\in P^\vee)$. Let $\mathbf{U}^+$ (resp. $\mathbf{U}^-$) be the positive (resp. negative) part of $\mathbf{U}$ generated by E_i (resp. F_i) for all $i\in I$, and $\mathbf{U}^0$ be the Cartan part of $\mathbf{U}$ generated by K_μ for all $\mu\in P^\vee$. The quantum group $\mathbf{U}$ has the following triangular decomposition

$$\mathbf{U}\cong\mathbf{U}^-\otimes\mathbf{U}^0\otimes\mathbf{U}^+.$$

Let $\mathbf{f}$ be the associative algebra defined by Lusztig in [14]. The algebra $\mathbf{f}$ is generated by $\theta_i(i\in I)$ subject to the quantum Serre relations. Let $\mathcal{A}=\mathbb{Z}[v,v^{-1}]$ and $\mathbf{f}_{\mathcal{A}}$ be the integral form of $\mathbf{f}$. There are two well-defined $\mathbb{Q}(v)$-algebra homomorphisms $^+:\mathbf{f}\to\mathbf{U}$ and $^-:\mathbf{f}\to\mathbf{U}$ satisfying $E_i=\theta_i^+$ and $F_i=\theta_i^-$ for all $i\in I$. The images of $^+$ and $^-$ are $\mathbf{U}^+$ and $\mathbf{U}^-$ respectively.

2.2. **Lusztig's symmetries.** Corresponding to $i\in I$, Lusztig introduced the Lusztig's symmetry $T_i:\mathbf{U}\to\mathbf{U}$ ([8, 10, 14]). The formulas of T_i on the generators are:

$$T_i(E_i)=-F_i\tilde{K}_i,\ \ T_i(F_i)=-\tilde{K}_{-i}E_i;$$
$$T_i(E_j)=\sum_{r+s=-a_{ij}}(-1)^r v_i^{-r}E_i^{(s)}E_jE_i^{(r)}\ \text{ for any } i\neq j\in I;$$
$$T_i(F_j)=\sum_{r+s=-a_{ij}}(-1)^r v_i^{r}F_i^{(r)}F_jF_i^{(s)}\ \text{ for any } i\neq j\in I;$$
$$T_i(K_\mu)=K_{\mu-\alpha_i(\mu)h_i}\ \text{ for any } \mu\in P^\vee,$$

where $E_i^{(n)}=E_i^n/[n]_{v_i}!$, $F_i^{(n)}=F_i^n/[n]_{v_i}!$ and $\tilde{K}_{\pm i}=K_{\pm\varepsilon_i h_i}$.

Let ${}_i\mathbf{f}=\{x\in\mathbf{f}\mid T_i(x^+)\in\mathbf{U}^+\}$ and ${}^i\mathbf{f}=\{x\in\mathbf{f}\mid T_i^{-1}(x^+)\in\mathbf{U}^+\}$. Lusztig symmetry T_i induces a unique map $T_i:{}_i\mathbf{f}\to{}^i\mathbf{f}$ such that $T_i(x^+)=T_i(x)^+$.

For any $i\neq j\in I$ and $m\in\mathbb{N}$, let

$$f(i,j;m)=\sum_{r+s=m}(-1)^r v_i^{-r(-a_{ij}-m+1)}\theta_i^{(r)}\theta_j\theta_i^{(s)}\in\mathbf{f},$$

and

$$f'(i,j;m)=\sum_{r+s=m}(-1)^r v_i^{-r(-a_{ij}-m+1)}\theta_i^{(s)}\theta_j\theta_i^{(r)}\in\mathbf{f},$$

where $\theta_i^{(n)}=\theta_i^n/[n]_{v_i}!$.

Proposition 2.1 (Proposition 38.1.6 in [14]). *For any $i\in I$,*
(1) ${}_i\mathbf{f}$ (resp. ${}^i\mathbf{f}$) is the subalgebra of $\mathbf{f}$ generated by $f(i,j;m)$ (resp. $f'(i,j;m)$) for all $i\neq j\in I$ and $-a_{ij}\geq m\in\mathbb{N}$;
(2) $T_i:{}_i\mathbf{f}\to{}^i\mathbf{f}$ is an isomorphism of algebras and

$$T_i(f(i,j;m))=f'(i,j;-a_{ij}-m)$$

for all $i\neq j\in I$ and $-a_{ij}\geq m\in\mathbb{N}$.

Lusztig also showed that $\mathbf{f}$ has the following direct sum decompositions

$$\mathbf{f} = {}_i\mathbf{f} \oplus \theta_i\mathbf{f} = {}^i\mathbf{f} \oplus \mathbf{f}\theta_i.$$

Denote by ${}_i\pi : \mathbf{f} \to {}_i\mathbf{f}$ and ${}^i\pi : \mathbf{f} \to {}^i\mathbf{f}$ the natural projections.

3. Geometric realization of $\mathbf{f}$

In this section, we shall review the geometric realization of $\mathbf{f}$ given by Lusztig in [9, 11, 14, 13, 7].

3.1. **Quivers with automorphisms.** Let $Q = (\mathbf{I}, H, s, t)$ be a quiver, where $\mathbf{I}$ is the set of vertices, H is the set of arrows, and $s, t : H \to \mathbf{I}$ are two maps such that an arrow $\rho \in H$ starts at $s(\rho)$ and terminates at $t(\rho)$. From now on, assume that $s(\rho) \neq t(\rho)$ for any $\rho \in H$.

An admissible automorphism a of Q consists of permutations $a : \mathbf{I} \to \mathbf{I}$ and $a : H \to H$ satisfying the following conditions:

(1) for any $h \in H$, $s(a(h)) = a(s(h))$ and $t(a(h)) = a(t(h))$;
(2) there are no arrows between two vertices in the same a-orbit.

From now on, $\tilde{Q} = (Q, a)$ is called a quiver with automorphism. Assume that $a^{\mathbf{n}} = \mathrm{id}$ for a given positive integer $\mathbf{n}$.

Let $I = \mathbf{I}^a$ be the set of a-orbits in $\mathbf{I}$. For any $i, j \in I$, let

$$a_{ij} = \begin{cases} -|\{\mathbf{i} \to \mathbf{j} \mid \mathbf{i} \in i, \mathbf{j} \in j\}| - |\{\mathbf{j} \to \mathbf{i} \mid \mathbf{i} \in i, \mathbf{j} \in j\}|, & \text{if } i \neq j; \\ 2|i|, & \text{if } i = j. \end{cases}$$

The matrix $A = (a_{ij})_{i,j \in I}$ is a symmetrizable generalized Cartan matrix.

Proposition 3.1 (Proposition 14.1.2 in [14]). *For any symmetrizable generalized Cartan matrix A, there exists a quiver with automorphism $\tilde{Q}$, such that the generalized Cartan matrix correspongding to $\tilde{Q}$ is A.*

3.2. **Geometric realization of Lusztig's algebra $\hat{\mathbf{f}}$ corresponding to Q.** Let p be a prime and $q = p^e$. Denote by $\mathbb{F}_q$ the finite field with q elements and $\mathbb{K} = \overline{\mathbb{F}}_q$.

Let $Q = (\mathbf{I}, H, s, t)$ be a quiver. Consider the category $\mathcal{C}'$, whose objects are finite dimensional $\mathbf{I}$-graded $\mathbb{K}$-vector spaces $\mathbf{V} = \bigoplus_{\mathbf{i} \in \mathbf{I}} V_{\mathbf{i}}$, and morphisms are graded linear maps. For any $\nu \in \mathbb{N}\mathbf{I}$, let $\mathcal{C}'_\nu$ be the subcategory of $\mathcal{C}'$ consisting of the objects $\mathbf{V} = \bigoplus_{\mathbf{i} \in \mathbf{I}} V_{\mathbf{i}}$ such that the dimension vector $\underline{\dim}\mathbf{V} = \sum_{\mathbf{i} \in \mathbf{I}} (\dim_{\mathbb{K}} V_{\mathbf{i}})\mathbf{i} = \nu$.

For any $\mathbf{V} \in \mathcal{C}'$, define

$$E_{\mathbf{V}} = \bigoplus_{\rho \in H} \mathrm{Hom}_{\mathbb{K}}(V_{s(\rho)}, V_{t(\rho)}).$$

The algebraic group $G_{\mathbf{V}} = \prod_{\mathbf{i} \in \mathbf{I}} GL_{\mathbb{K}}(V_{\mathbf{i}})$ acts on $E_{\mathbf{V}}$ naturally.

For any $\nu = \nu_{\mathbf{i}}\mathbf{i} \in \mathbb{N}\mathbf{I}$, ν is called discrete if there is no $h \in H$ such that $\{s(h), t(h)\} \in \{\mathbf{i} \in \mathbf{I} \mid \nu_{\mathbf{i}} \neq 0\}$. Fix a nonzero element $\nu \in \mathbb{N}\mathbf{I}$. Let

$$Y_\nu = \{\mathbf{y} = (\nu^1, \nu^2, \ldots, \nu^k) \mid \nu^l \in \mathbb{N}\mathbf{I} \text{ is discrete and } \sum_{l=1}^{k} \nu^l = \nu\}.$$

Fix $\mathbf{V} \in \mathcal{C}'_\nu$. For any element $\mathbf{y} \in Y_\nu$, a flag of type $\mathbf{y}$ in $\mathbf{V}$ is a sequence

$$\phi = (\mathbf{V} = \mathbf{V}^k \supset \mathbf{V}^{k-1} \supset \cdots \supset \mathbf{V}^0 = 0),$$

where $\mathbf{V}^l \in \mathcal{C}'$ such that $\underline{\dim}\mathbf{V}^l/\mathbf{V}^{l-1} = \nu^l$. Let $F_{\mathbf{y}}$ be the variety of all flags of type $\mathbf{y}$ in $\mathbf{V}$. For any $x \in E_{\mathbf{V}}$, a flag ϕ is called x-stable if $x_\rho(V^l_{s(\rho)}) \subset V^l_{t(\rho)}$ for all l and all $\rho \in H$. Let

$$\tilde{F}_{\mathbf{y}} = \{(x,\phi) \in E_{\mathbf{V}} \times F_{\mathbf{y}} \mid \phi \text{ is } x\text{-stable}\}$$

and $\pi_{\mathbf{y}} : \tilde{F}_{\mathbf{y}} \to E_{\mathbf{V}}$ be the projection to $E_{\mathbf{V}}$.

Let $\bar{\mathbb{Q}}_l$ be the l-adic field and $\mathcal{D}_{G_{\mathbf{V}}}(E_{\mathbf{V}})$ be the bounded $G_{\mathbf{V}}$-equivariant derived category of complexes of l-adic sheaves on $E_{\mathbf{V}}$. For each $\mathbf{y} \in Y_\nu$,

$$\mathcal{L}_{\mathbf{y}} = \pi_{\mathbf{y}!}\mathbf{1}_{\tilde{F}_{\mathbf{y}}}[d_{\mathbf{y}}](\frac{d_{\mathbf{y}}}{2}) \in \mathcal{D}_{G_{\mathbf{V}}}(E_{\mathbf{V}})$$

is a semisimple complex, where $d_{\mathbf{y}} = \dim \tilde{F}_{\mathbf{y}}$, $[-]$ is the shift functor and $(-)$ is the Tate twist. Let $\mathcal{P}_{\mathbf{V}}$ be the set of isomorphism classes of simple perverse sheaves $\mathcal{L}$ on $E_{\mathbf{V}}$ such that $\mathcal{L}[r](\frac{r}{2})$ appears as a direct summand of $\mathcal{L}_{\mathbf{y}}$ for some $\mathbf{y} \in Y_\nu$ and $r \in \mathbb{Z}$. Let $\mathcal{Q}_{\mathbf{V}}$ be the full subcategory of $\mathcal{D}_{G_{\mathbf{V}}}(E_{\mathbf{V}})$ consisting of all complexes which are isomorphic to finite direct sums of complexes in the set $\{\mathcal{L}[r](\frac{r}{2}) \mid \mathcal{L} \in \mathcal{P}_{\mathbf{V}}, r \in \mathbb{Z}\}$.

Let $K(\mathcal{Q}_{\mathbf{V}})$ be the Grothendieck group of $\mathcal{Q}_{\mathbf{V}}$. Define

$$v^{\pm}[\mathcal{L}] = [\mathcal{L}[\pm 1](\pm\frac{1}{2})].$$

Then, $K(\mathcal{Q}_{\mathbf{V}})$ is a free $\mathcal{A}$-module. Define

$$K(\mathcal{Q}) = \bigoplus_{\nu \in \mathbb{N}I} K(\mathcal{Q}_{\mathbf{V}}).$$

For any $\nu, \nu', \nu'' \in \mathbb{N}\mathbf{I}$ such that $\nu = \nu' + \nu''$, fix $\mathbf{V} \in \mathcal{C}'_\nu$, $\mathbf{V}' \in \mathcal{C}'_{\nu'}$, $\mathbf{V}'' \in \mathcal{C}'_{\nu''}$. Consider the following diagram

$$E_{\mathbf{V}'} \times E_{\mathbf{V}''} \xleftarrow{p_1} E' \xrightarrow{p_2} E'' \xrightarrow{p_3} E_{\mathbf{V}}\ ,$$

where

(1) $E'' = \{(x, \mathbf{W})\}$, where $x \in E_{\mathbf{V}}$ and $\mathbf{W} \in \mathcal{C}'_\nu$ is an x-stable subspace of $\mathbf{V}$;
(2) $E' = \{(x, \mathbf{W}, R'', R')\}$, where $(x, \mathbf{W}) \in E''$, $R'' : \mathbf{V}'' \simeq \mathbf{W}$ and $R' : \mathbf{V}' \simeq \mathbf{V}/\mathbf{W}$;
(3) $p_1(x, \mathbf{W}, R'', R') = (x', x'')$, where x' and x'' are induced through the following commutative diagrams

$$\begin{array}{ccc} \mathbf{V}'_{s(\rho)} & \xrightarrow{x'_\rho} & \mathbf{V}'_{t(\rho)} \\ \downarrow{\scriptstyle R'_{s(\rho)}} & & \downarrow{\scriptstyle R'_{t(\rho)}} \\ (\mathbf{V}/\mathbf{W})_{s(\rho)} & \xrightarrow{x_\rho} & (\mathbf{V}/\mathbf{W})_{t(\rho)} \end{array}$$

and

$$\begin{array}{ccc} \mathbf{V}''_{s(\rho)} & \xrightarrow{x''_\rho} & \mathbf{V}''_{t(\rho)} \\ \downarrow{\scriptstyle R''_{s(\rho)}} & & \downarrow{\scriptstyle R''_{t(\rho)}} \\ \mathbf{W}_{s(\rho)} & \xrightarrow{x_\rho} & \mathbf{W}_{t(\rho)}; \end{array}$$

(4) $p_2(x, \mathbf{W}, R'', R') = (x, \mathbf{W})$;
(5) $p_3(x, \mathbf{W}) = x$.

For any two complexes $\mathcal{L}' \in \mathcal{D}_{G_{\mathbf{V}'}}(E_{\mathbf{V}'})$ and $\mathcal{L}'' \in \mathcal{D}_{G_{\mathbf{V}''}}(E_{\mathbf{V}''})$, $\mathcal{L} = \mathcal{L}' * \mathcal{L}''$ is defined as follows.

Let $\mathcal{L}_1 = \mathcal{L}' \otimes \mathcal{L}''$ and $\mathcal{L}_2 = p_1^* \mathcal{L}_1$. Since p_1 is smooth with connected fibres and p_2 is a $G_{\mathbf{V}'} \times G_{\mathbf{V}''}$-principal bundle, there exists a complex $\mathcal{L}_3$ on E' such that $p_2^*(\mathcal{L}_3) = \mathcal{L}_2$. The complex $\mathcal{L}$ is defined as $(p_3)_! \mathcal{L}_3$.

Lemma 3.2 (Lemma 3.2 in [11], Lemma 9.2.3 in [14])**.** *For any $\mathcal{L}' \in \mathcal{Q}_{\mathbf{V}'}$ and $\mathcal{L}'' \in \mathcal{Q}_{\mathbf{V}''}$, $\mathcal{L}' * \mathcal{L}'' \in \mathcal{Q}_{\mathbf{V}}$.*

Hence, we get a functor

$$* : \mathcal{Q}_{\mathbf{V}'} \times \mathcal{Q}_{\mathbf{V}''} \to \mathcal{Q}_{\mathbf{V}}.$$

This functor induces an associative $\mathcal{A}$-bilinear multiplication

$$\begin{array}{rcl} \circledast : K(\mathcal{Q}_{\mathbf{V}'}) \times K(\mathcal{Q}_{\mathbf{V}''}) & \to & K(\mathcal{Q}_{\mathbf{V}}) \\ ([\mathcal{L}'], [\mathcal{L}'']) & \mapsto & [\mathcal{L}'] \circledast [\mathcal{L}''] = [\mathcal{L}' \circledast \mathcal{L}''], \end{array}$$

where $\mathcal{L}' \circledast \mathcal{L}'' = (\mathcal{L}' * \mathcal{L}'')[m_{\nu'\nu''}](\frac{m_{\nu'\nu''}}{2})$ and $m_{\nu'\nu''} = \sum_{\rho \in H} \nu'_{s(\rho)} \nu''_{t(\rho)} - \sum_{i \in I} \nu'_i \nu''_i$. Then $K(\mathcal{Q})$ becomes an associative $\mathcal{A}$-algebra and the set $\{[\mathcal{L}] \mid \mathcal{L} \in \mathcal{P}_{\mathbf{V}}\}$ is a basis of $K(\mathcal{Q}_{\mathbf{V}})$.

Let A be the generalized Cartan matrix corresponding to Q and $\hat{\mathbf{f}}$ be the Lusztig's algebra corresponding to A. For any

$$\mathbf{y} = (a_1 \mathbf{i}_1, a_2 \mathbf{i}_2, \ldots, a_k \mathbf{i}_k) \in Y_\nu,$$

let $\theta_{\mathbf{y}} = \theta_{\mathbf{i}_1}^{(a_1)} \theta_{\mathbf{i}_2}^{(a_2)} \cdots \theta_{\mathbf{i}_k}^{(a_k)}$.

Theorem 3.3 (Theorem 10.17 in [11], Theorem 13.2.11 in [14])**.** *There is a unique $\mathcal{A}$-algebra isomorphism*

$$\hat{\lambda}_{\mathcal{A}} : K(\mathcal{Q}) \to \hat{\mathbf{f}}_{\mathcal{A}}$$

such that $\hat{\lambda}_{\mathcal{A}}([\mathcal{L}_{\mathbf{y}}]) = \theta_{\mathbf{y}}$ for all $\mathbf{y} = (a_1 \mathbf{i}_1, a_2 \mathbf{i}_2, \ldots, a_k \mathbf{i}_k) \in Y_\nu$.

Let $\hat{\mathbf{B}}_\nu = \{[\mathcal{L}] \mid \mathcal{L} \in \mathcal{P}_{\mathbf{V}}\}$ and $\hat{\mathbf{B}} = \bigsqcup_{\nu \in \mathbb{N}\mathbf{I}} \hat{\mathbf{B}}_\nu$, which is an $\mathcal{A}$-basis of $K(\mathcal{Q})$ and is called the canonical basis by Lusztig.

3.3. **Geometric realization of f.**

3.3.1. Let $\tilde{Q} = (Q, a)$ be a quiver with automorphism, where $Q = (\mathbf{I}, H, s, t)$. Let $\tilde{\mathcal{C}}$ be the category of $\mathbf{V} = \bigoplus_{\mathbf{i} \in \mathbf{I}} V_{\mathbf{i}} \in \mathcal{C}'$ with a linear map $a : \mathbf{V} \to \mathbf{V}$ satisfying the following conditions:

(1) for any $\mathbf{i} \in \mathbf{I}$, $a(V_{\mathbf{i}}) = V_{a(\mathbf{i})}$;
(2) for any $\mathbf{i} \in \mathbf{I}$ and $k \in \mathbb{N}$ such that $a^k(\mathbf{i}) = \mathbf{i}$, $a^k|_{V_{\mathbf{i}}} = \mathrm{id}_{V_{\mathbf{i}}}$.

The morphisms in $\tilde{\mathcal{C}}$ are the graded linear maps $f = (f_{\mathbf{i}})_{\mathbf{i} \in \mathbf{I}}$ such that the following diagram commutes

$$\begin{array}{ccc} V_{\mathbf{i}} & \xrightarrow{a} & V_{a(\mathbf{i})} \\ \downarrow{\scriptstyle f_{\mathbf{i}}} & & \downarrow{\scriptstyle f_{a(\mathbf{i})}} \\ V_{\mathbf{i}} & \xrightarrow{a} & V_{a(\mathbf{i})}. \end{array}$$

Let $\mathbb{N}\mathbf{I}^a = \{\nu \in \mathbb{N}\mathbf{I} \mid \nu_{\mathbf{i}} = \nu_{a(\mathbf{i})}\}$. There is a bijection between $\mathbb{N}I$ and $\mathbb{N}\mathbf{I}^a$ sending i to $\gamma_i = \sum_{\mathbf{i} \in i} \mathbf{i}$. From now on, $\mathbb{N}\mathbf{I}^a$ is identified with $\mathbb{N}I$. For any $\nu \in \mathbb{N}\mathbf{I}^a$, let $\tilde{\mathcal{C}}_\nu$ be the subcategory of $\tilde{\mathcal{C}}$ consisting of the objects $\mathbf{V} = \bigoplus_{\mathbf{i} \in \mathbf{I}} V_{\mathbf{i}}$ such that the dimension vector $\underline{\dim} \mathbf{V} = \nu$.

For any $(\mathbf{V}, a) \in \tilde{\mathcal{C}}$, $E_{\mathbf{V}}$ and $G_{\mathbf{V}}$ are defined in Section 3.2. Let $a : G_{\mathbf{V}} \to G_{\mathbf{V}}$ be the automorphism defined by

$$a(g)(v) = a(g(a^{-1}(v)))$$

for any $g \in G_{\mathbf{V}}$ and $v \in \mathbf{V}$. Denote by $a : E_{\mathbf{V}} \to E_{\mathbf{V}}$ the automorphism such that the following diagram commutes for any $h \in H$

$$\begin{array}{ccc} \mathbf{V}_{s(h)} & \xrightarrow{x_h} & \mathbf{V}_{t(h)} \\ \downarrow a & & \downarrow a \\ \mathbf{V}_{a(s(h))} & \xrightarrow{a(x)_{a(h)}} & \mathbf{V}_{a(t(h))}. \end{array}$$

Since $a(gx) = a(g)a(x)$, we have a functor $a^* : \mathcal{D}_{G_{\mathbf{V}}}(E_{\mathbf{V}}) \to \mathcal{D}_{G_{\mathbf{V}}}(E_{\mathbf{V}})$.

Lemma 3.4 ([14])**.** *It holds that* $a^*(\mathcal{Q}_{\mathbf{V}}) = \mathcal{Q}_{\mathbf{V}}$ *and* $a^*(\mathcal{P}_{\mathbf{V}}) = \mathcal{P}_{\mathbf{V}}$.

Lusztig introduced the following categories $\tilde{\mathcal{Q}}_{\mathbf{V}}$ and $\tilde{\mathcal{P}}_{\mathbf{V}}$ in Section 11.1.2 of [14]. The objects in $\tilde{\mathcal{Q}}_{\mathbf{V}}$ are pairs $(\mathcal{L}, \phi)$, where $\mathcal{L} \in \mathcal{Q}_{\mathbf{V}}$ and $\phi : a^*\mathcal{L} \to \mathcal{L}$ is an isomorphism such that

$$\mathcal{L} = a^{*\mathbf{n}}\mathcal{L} \to a^{*(\mathbf{n}-1)}\mathcal{L} \to \ldots \to a^*\mathcal{L} \to \mathcal{L}$$

is the identity map of $\mathcal{L}$. A morphism in $\mathrm{Hom}_{\tilde{\mathcal{Q}}_{\mathbf{V}}}((\mathcal{L}, \phi), (\mathcal{L}', \phi'))$ is a morphism $f \in \mathrm{Hom}_{\mathcal{Q}_{\mathbf{V}}}(\mathcal{L}, \mathcal{L}')$ such that the following diagram commutes

$$\begin{array}{ccc} a^*\mathcal{L} & \xrightarrow{\phi} & \mathcal{L} \\ \downarrow a^*f & & \downarrow f \\ a^*\mathcal{L}' & \xrightarrow{\phi'} & \mathcal{L}'. \end{array}$$

Let $\mathcal{O}$ be the subring of $\bar{\mathbb{Q}}_l$ consisting of all $\mathbb{Z}$-linear combinations of $\mathbf{n}$-th roots of 1. In Section 11.1.5 of [14], Lusztig introduced two $\mathcal{O}$-modules $K(\tilde{\mathcal{Q}}_{\mathbf{V}})$ and $K(\tilde{\mathcal{P}}_{\mathbf{V}})$, whose definitions are similar to that of a Grothendieck group. In this paper, $K(\tilde{\mathcal{Q}}_{\mathbf{V}})$ and $K(\tilde{\mathcal{P}}_{\mathbf{V}})$ are called the "Grothendieck groups" of $\tilde{\mathcal{Q}}_{\mathbf{V}}$ and $\tilde{\mathcal{P}}_{\mathbf{V}}$ respectively. Let $\mathcal{O}' = \mathcal{O}[v, v^{-1}]$. Since a^* commutes with the shift functor and the Tate twist, we can define

$$v^{\pm}[\mathcal{L}, \phi] = [\mathcal{L}[\pm 1](\pm\frac{1}{2}), \phi[\pm 1](\pm\frac{1}{2})].$$

Then $K(\tilde{\mathcal{Q}}_{\mathbf{V}})$ has a natural $\mathcal{O}'$-module structure. Note that $K(\tilde{\mathcal{Q}}_{\mathbf{V}}) = \mathcal{O}' \otimes_{\mathcal{O}} K(\tilde{\mathcal{P}}_{\mathbf{V}})$. Define

$$K(\tilde{\mathcal{Q}}) = \bigoplus_{\nu \in \mathbb{N}I} K(\tilde{\mathcal{Q}}_{\mathbf{V}}).$$

For $\nu, \nu', \nu'' \in \mathbb{N}I = \mathbb{N}\mathbf{I}^a$ such that $\nu = \nu' + \nu''$, fix $\mathbf{V} \in \tilde{\mathcal{C}}_{\nu}$, $\mathbf{V}' \in \tilde{\mathcal{C}}_{\nu'}$, $\mathbf{V}'' \in \tilde{\mathcal{C}}_{\nu''}$. Lusztig proved the following lemma in Section 12.1.5 of [14].

Lemma 3.5 ([14])**.** *The induction functor*

$$* : \mathcal{Q}_{\mathbf{V}'} \times \mathcal{Q}_{\mathbf{V}''} \to \mathcal{Q}_{\mathbf{V}}$$

satisfies that the following diagram commutes

$$\begin{array}{ccc} \mathcal{Q}_{\mathbf{V}'} \times \mathcal{Q}_{\mathbf{V}''} & \xrightarrow{*} & \mathcal{Q}_{\mathbf{V}} \\ \downarrow{\scriptstyle a^*\times a^*} & & \downarrow{\scriptstyle a^*} \\ \mathcal{Q}_{\mathbf{V}'} \times \mathcal{Q}_{\mathbf{V}''} & \xrightarrow{*} & \mathcal{Q}_{\mathbf{V}}. \end{array}$$

For any $(\mathcal{L}', \phi') \in \tilde{\mathcal{Q}}_{\mathbf{V}'}$ and $(\mathcal{L}'', \phi'') \in \tilde{\mathcal{Q}}_{\mathbf{V}''}$, Lemma 3.5 implies that

$$a^*(\mathcal{L}' * \mathcal{L}'') = a^*\mathcal{L}' * a^*\mathcal{L}''.$$

Hence there is a functor

$$\begin{array}{rcl} * : \tilde{\mathcal{Q}}_{\mathbf{V}'} \times \tilde{\mathcal{Q}}_{\mathbf{V}''} & \to & \tilde{\mathcal{Q}}_{\mathbf{V}} \\ ((\mathcal{L}', \phi'), (\mathcal{L}'', \phi'')) & \mapsto & (\mathcal{L}' * \mathcal{L}'', \phi), \end{array}$$

where

$$\phi = \phi' * \phi'' : \mathcal{L}' * \mathcal{L}'' \to a^*\mathcal{L}' * a^*\mathcal{L}'' = a^*(\mathcal{L}' * \mathcal{L}'').$$

This functor induces an associative $\mathcal{O}'$-bilinear multiplication

$$\begin{array}{rcl} \circledast : K(\tilde{\mathcal{Q}}_{\mathbf{V}'}) \times K(\tilde{\mathcal{Q}}_{\mathbf{V}''}) & \to & K(\tilde{\mathcal{Q}}_{\mathbf{V}}) \\ ([\mathcal{L}', \phi'] , [\mathcal{L}'', \phi'']) & \mapsto & [\mathcal{L}, \phi], \end{array}$$

where $(\mathcal{L}, \phi) = (\mathcal{L}', \phi') \circledast (\mathcal{L}'', \phi'') = ((\mathcal{L}', \phi') * (\mathcal{L}'', \phi''))[m_{\nu'\nu''}](\frac{m_{\nu'\nu''}}{2})$. Then $K(\tilde{\mathcal{Q}})$ becomes an associative $\mathcal{O}'$-algebra.

3.3.2. Fix a nonzero element $\nu \in \mathbb{N}\mathbf{I}^a$. Let

$$Y_\nu^a = \{\mathbf{y} = (\nu^1, \nu^2, \ldots, \nu^k) \in Y_\nu \mid \nu^l \in \mathbb{N}\mathbf{I}^a\}.$$

Fix $\mathbf{V} \in \tilde{\mathcal{C}}_\nu$. For any element $\mathbf{y} \in Y_\nu^a$, the automorphism $a : F_{\mathbf{y}} \to F_{\mathbf{y}}$ is defined as

$$a(\phi) = (\mathbf{V} = a(\mathbf{V}^k) \supset a(\mathbf{V}^{k-1}) \supset \cdots \supset a(\mathbf{V}^0) = 0)$$

for any

$$\phi = (\mathbf{V} = \mathbf{V}^k \supset \mathbf{V}^{k-1} \supset \cdots \supset \mathbf{V}^0 = 0) \in F_{\mathbf{y}}.$$

There also exists an automorphism $a : \tilde{F}_{\mathbf{y}} \to \tilde{F}_{\mathbf{y}}$, defined as $a((x, \phi)) = (a(x), a(\phi))$ for any $(x, \phi) \in \tilde{F}_{\mathbf{y}}$.

The automorphism $a : \tilde{F}_{\mathbf{y}} \to \tilde{F}_{\mathbf{y}}$ induces a natural isomorphism $a^*\mathbf{1}_{\tilde{F}_{\mathbf{y}}} \cong \mathbf{1}_{\tilde{F}_{\mathbf{y}}}$. Hence, there exists an isomorphism

$$\phi_0 : a^*\mathcal{L}_{\mathbf{y}} = a^*\pi_{\mathbf{y}!}\mathbf{1}_{\tilde{F}_{\mathbf{y}}}[d_{\mathbf{y}}](\frac{d_{\mathbf{y}}}{2}) = \pi_{\mathbf{y}!}a^*\mathbf{1}_{\tilde{F}_{\mathbf{y}}}[d_{\mathbf{y}}](\frac{d_{\mathbf{y}}}{2}) \cong \pi_{\mathbf{y}!}\mathbf{1}_{\tilde{F}_{\mathbf{y}}}[d_{\mathbf{y}}](\frac{d_{\mathbf{y}}}{2}) = \mathcal{L}_{\mathbf{y}}.$$

That is, $(\mathcal{L}_{\mathbf{y}}, \phi_0)$ is an object in $\tilde{\mathcal{Q}}_{\mathbf{V}}$.

Let $\mathbf{k}_\nu$ be the $\mathcal{A}$-submodule of $K(\tilde{\mathcal{Q}}_{\mathbf{V}})$ spanned by $(\mathcal{L}_{\mathbf{y}}, \phi_0)$ for all $\mathbf{y} \in Y_\nu^a$. Let $\mathbf{k} = \bigoplus_{\nu \in \mathbb{N}I} \mathbf{k}_\nu$. Lusztig proved that $\mathbf{k}$ is also a subalgebra of $K(\tilde{\mathcal{Q}})$ (Section 13.2 in [14]).

Let $i \in I$ and $\gamma_i = \sum_{\mathbf{i} \in i} \mathbf{i}$. Define $\mathbf{1}_i = [\mathbf{1}, \mathrm{id}] \in K(\tilde{\mathcal{Q}}_{\mathbf{V}})$, where $\mathbf{V} \in \tilde{C}_{\gamma_i}$. Let A be the generalized Cartan matrix corresponding to (Q, a) and $\mathbf{f}$ be the Lusztig's algebra corresponding to A.

Theorem 3.6 (Theorem 13.2.11 in [14])**.** *There is a unique $\mathcal{A}$-algebra isomorphism*

$$\lambda_{\mathcal{A}} : \mathbf{k} \to \mathbf{f}_{\mathcal{A}}$$

such that $\lambda_{\mathcal{A}}(\mathbf{1}_i) = \theta_i$ for all $i \in I$.

On the canonical basis of $\mathbf{f}$, Lusztig gave the following theorem.

Theorem 3.7 (Proposition 12.5.2 and 12.6.3 in [14]). *(1) For any $\mathcal{L} \in \mathcal{P}_{\mathbf{V}}$ such that $a^*\mathcal{L} \cong \mathcal{L}$, there exists an isomorphism $\phi : a^*\mathcal{L} \cong \mathcal{L}$ such that $(\mathcal{L}, \phi) \in \tilde{\mathcal{P}}_{\mathbf{V}}$ and $(D(\mathcal{L}), D(\phi)^{-1})$ is isomorphic to $(\mathcal{L}, \phi)$ as objects of $\tilde{P}_{\mathbf{V}}$, where D is the Verdier duality. Moreover, ϕ is unique, if $\mathbf{n}$ is odd, and unique up to multiplication by ± 1, if $\mathbf{n}$ is even.*
(2) $\mathbf{k}_\nu$ is generated by $[\mathcal{L}, \phi]$ in (1) as an $\mathcal{A}$-submodule of $K(\tilde{\mathcal{Q}}_{\mathbf{V}})$.

If $\mathbf{n}$ is odd, let $\mathcal{B}_\nu$ be the subset of $\mathbf{k}_\nu$ consisting of all elements $[\mathcal{L}, \phi]$ in Theorem 3.7. If $\mathbf{n}$ is even, let $\mathcal{B}_\nu$ be the subset of $\mathbf{k}_\nu$ consisting of all elements $\pm[\mathcal{L}, \phi]$ in Theorem 3.7. The set $\mathcal{B}_\nu$ is called a signed basis of $\mathbf{k}_\nu$ by Lusztig. Lusztig gave a non-geometric way to choose a subset $\mathbf{B}_\nu$ of $\mathcal{B}_\nu$ such that $\mathcal{B}_\nu = \mathbf{B}_\nu \cup -\mathbf{B}_\nu$ and $\mathbf{B}_\nu$ is an $\mathcal{A}$-basis of $\mathbf{k}_\nu$ (Section 14.4.2 in [14]). The set $\mathbf{B} = \sqcup_{\nu \in \mathbb{N}I} \mathbf{B}_\nu$ is called the canonical basis of $\mathbf{k}$.

At last, let us recall the relation between $\hat{\mathbf{f}}$ and $\mathbf{f}$. Define $\tilde{\delta} : \mathbf{k} \to K(\mathcal{Q})$ by $\tilde{\delta}([\mathcal{L}, \phi]) = [\mathcal{L}]$ for any $[\mathcal{L}, \phi] \in \mathbf{B}$. It is clear that this is an injection and induces an embedding $\delta : \mathbf{f} \to \hat{\mathbf{f}}$. Note that $\delta(\mathbf{B}) = \hat{\mathbf{B}}^a$.

4. Geometric realizations of subalgebras ${}_i\mathbf{f}$ and ${}^i\mathbf{f}$

4.1. **The algebra ${}_{\mathfrak{i}}\hat{\mathbf{f}}$.** Let $Q = (\mathbf{I}, H, s, t)$ be a quiver and $\hat{\mathbf{f}}$ be the corresponding Lusztig's algebra. Let ${}_{\mathbf{i}}\hat{\mathbf{f}}$ be the subalgebra of $\hat{\mathbf{f}}$ generated by $f(\mathbf{i}, \mathbf{j}; m)$ for all $\mathbf{i} \neq \mathbf{j} \in \mathbf{I}$ and integer m. Let $\mathfrak{i}$ be a subset of $\mathbf{I}$ and define

$$
{}_{\mathfrak{i}}\hat{\mathbf{f}} = \bigcap_{\mathbf{i} \in \mathfrak{i}} {}_{\mathbf{i}}\hat{\mathbf{f}},
$$

which is also a subalgebra of $\hat{\mathbf{f}}$.

For any $\mathbf{i} \in \mathbf{I}$, there exists a unique linear map ${}_{\mathbf{i}}r : \hat{\mathbf{f}} \to \hat{\mathbf{f}}$ such that ${}_{\mathbf{i}}r(\mathbf{1}) = 0$, ${}_{\mathbf{i}}r(\theta_{\mathbf{j}}) = \delta_{\mathbf{ij}}$ for all $\mathbf{j} \in \mathbf{I}$ and ${}_{\mathbf{i}}r(xy) = {}_{\mathbf{i}}r(x)y + v^{(\nu, \alpha_{\mathbf{i}})} x_{\mathbf{i}}r(y)$ for all homogeneous $x \in \hat{\mathbf{f}}_\nu$ and y. Denote by $(-,-)$ the non-degenerate symmetric bilinear form on $\hat{\mathbf{f}}$ introduced by Lusztig.

Proposition 4.1 (Proposition 38.1.6 in [14]). *It holds that ${}_{\mathbf{i}}\hat{\mathbf{f}} = \{x \in \hat{\mathbf{f}} \mid {}_{\mathbf{i}}r(x) = 0\}$.*

As a corollary of Proposition 4.1, we have

Corollary 4.2. *It holds that ${}_{\mathfrak{i}}\hat{\mathbf{f}} = \{x \in \hat{\mathbf{f}} \mid {}_{\mathbf{i}}r(x) = 0$ for any $\mathbf{i} \in \mathfrak{i}\}$.*

□

Proposition 4.3. *The algebra $\hat{\mathbf{f}}$ has the following decomposition*

$$
\hat{\mathbf{f}} = {}_{\mathfrak{i}}\hat{\mathbf{f}} \oplus \sum_{\mathbf{i} \in \mathfrak{i}} \theta_{\mathbf{i}} \hat{\mathbf{f}}.
$$

Proof. In the algebra $\hat{\mathbf{f}}$, $(\theta_{\mathbf{i}} y, x) = (\theta_{\mathbf{i}}, \theta_{\mathbf{i}})(y, {}_{\mathbf{i}}r(x))$. Hence the decomposition $\hat{\mathbf{f}}_\nu = {}_{\mathbf{i}}\hat{\mathbf{f}}_\nu \oplus \theta_{\mathbf{i}} \hat{\mathbf{f}}_{\nu - \mathbf{i}}$ is an orthogonal decomposition and $\theta_{\mathbf{i}} \hat{\mathbf{f}}_{\nu - \mathbf{i}} = {}_{\mathbf{i}}\hat{\mathbf{f}}_\nu^\perp$.

For the proof of this proposition, it is sufficient to show that ${}_{\mathfrak{i}}\hat{\mathbf{f}}_\nu^\perp = \sum_{\mathbf{i} \in \mathfrak{i}} {}_{\mathbf{i}}\hat{\mathbf{f}}_\nu^\perp$. That is $(\bigcap_{\mathbf{i} \in \mathfrak{i}} {}_{\mathbf{i}}\hat{\mathbf{f}}_\nu)^\perp = \sum_{\mathbf{i} \in \mathfrak{i}} {}_{\mathbf{i}}\hat{\mathbf{f}}_\nu^\perp$. It is clear that $(\bigcap_{\mathbf{i} \in \mathfrak{i}} {}_{\mathbf{i}}\hat{\mathbf{f}}_\nu)^\perp \supset \sum_{\mathbf{i} \in \mathfrak{i}} {}_{\mathbf{i}}\hat{\mathbf{f}}_\nu^\perp$. On the other hand, $x \in (\sum_{\mathbf{i}\in\mathfrak{i}} {}_{\mathbf{i}}\hat{\mathbf{f}}_\nu^\perp)^\perp$ implies $x \in \bigcap_{\mathbf{i}\in\mathfrak{i}} {}_{\mathbf{i}}\hat{\mathbf{f}}_\nu$. Hence $\bigcap_{\mathbf{i}\in\mathfrak{i}} {}_{\mathbf{i}}\hat{\mathbf{f}}_\nu \supset (\sum_{\mathbf{i}\in\mathfrak{i}} {}_{\mathbf{i}}\hat{\mathbf{f}}_\nu^\perp)^\perp$. That is $(\bigcap_{\mathbf{i}\in\mathfrak{i}} {}_{\mathbf{i}}\hat{\mathbf{f}}_\nu)^\perp \subset \sum_{\mathbf{i}\in\mathfrak{i}} {}_{\mathbf{i}}\hat{\mathbf{f}}_\nu^\perp$.

□

□

Denoted by ${}_{\mathfrak{i}}\pi : \hat{\mathbf{f}} \to {}_{\mathfrak{i}}\hat{\mathbf{f}}$ the canonical projection.

4.2. Geometric realization of ${}_{\mathfrak{i}}\hat{\mathbf{f}}$.

4.2.1. Let $\mathfrak{i}$ be a subset of $\mathbf{I}$ satisfying the following conditions: (1) for any $\mathbf{i} \in \mathfrak{i}$, $\mathbf{i}$ is a sink; (2) for any $\mathbf{i}, \mathbf{j} \in \mathfrak{i}$, there are no arrows between them.

For any $\mathbf{V} \in \mathcal{C}'_\nu$, consider a subvariety ${}_{\mathfrak{i}}E_{\mathbf{V}}$ of $E_{\mathbf{V}}$

$${}_{\mathfrak{i}}E_{\mathbf{V}} = \{x \in E_{\mathbf{V}} \mid \bigoplus_{h\in H, t(h)=\mathbf{i}} x_h : \bigoplus_{h\in H, t(h)=\mathbf{i}} V_{s(h)} \to V_{\mathbf{i}} \text{ is surjective for any } \mathbf{i} \in \mathfrak{i}\}.$$

Denote by ${}_{\mathfrak{i}}j_{\mathbf{V}} : {}_{\mathfrak{i}}E_{\mathbf{V}} \to E_{\mathbf{V}}$ the canonical embedding.

For any $\mathbf{y} \in Y_\nu$, let

$${}_{\mathfrak{i}}\tilde{F}_{\mathbf{y}} = \{(x, \phi) \in {}_{\mathfrak{i}}E_{\mathbf{V}} \times F_{\mathbf{y}} \mid \phi \text{ is } x\text{-stable}\}$$

and ${}_{\mathfrak{i}}\pi_{\mathbf{y}} : {}_{\mathfrak{i}}\tilde{F}_{\mathbf{y}} \to {}_{\mathfrak{i}}E_{\mathbf{V}}$ be the projection to ${}_{\mathfrak{i}}E_{\mathbf{V}}$.

For any $\mathbf{y} \in Y_\nu$, ${}_{\mathfrak{i}}\mathcal{L}_{\mathbf{y}} = {}_{\mathfrak{i}}\pi_{\mathbf{y}!}\mathbf{1}_{{}_{\mathfrak{i}}\tilde{F}_{\mathbf{y}}}[d_{\mathbf{y}}](\frac{d_{\mathbf{y}}}{2}) \in \mathcal{D}_{G_{\mathbf{V}}}({}_{\mathfrak{i}}E_{\mathbf{V}})$ is a semisimple complex. Let ${}_{\mathfrak{i}}\mathcal{P}_{\mathbf{V}}$ be the set of isomorphism classes of simple perverse sheaves $\mathcal{L}$ on ${}_{\mathfrak{i}}E_{\mathbf{V}}$ such that $\mathcal{L}[r](\frac{r}{2})$ appears as a direct summand of ${}_{\mathfrak{i}}\mathcal{L}_{\mathbf{y}}$ for some $\mathbf{y} \in Y_\nu$ and $r \in \mathbb{Z}$. Let ${}_{\mathfrak{i}}\mathcal{Q}_{\mathbf{V}}$ be the full subcategory of $\mathcal{D}_{G_{\mathbf{V}}}({}_{\mathfrak{i}}E_{\mathbf{V}})$ consisting of all complexes which are isomorphic to finite direct sums of complexes in the set $\{\mathcal{L}[r](\frac{r}{2}) \mid \mathcal{L} \in {}_{\mathfrak{i}}\mathcal{P}_{\mathbf{V}}, r \in \mathbb{Z}\}$.

Let $K({}_{\mathfrak{i}}\mathcal{Q}_{\mathbf{V}})$ be the Grothendieck group of ${}_{\mathfrak{i}}\mathcal{Q}_{\mathbf{V}}$. Define

$$v^{\pm}[\mathcal{L}] = [\mathcal{L}[\pm 1](\pm\frac{1}{2})].$$

Then, $K({}_{\mathfrak{i}}\mathcal{Q}_{\mathbf{V}})$ is a free $\mathcal{A}$-module. Define

$$K({}_{\mathfrak{i}}\mathcal{Q}) = \bigoplus_{\nu\in\mathbb{N}I} K({}_{\mathfrak{i}}\mathcal{Q}_{\mathbf{V}}).$$

The canonical embedding ${}_{\mathfrak{i}}j_{\mathbf{V}} : {}_{\mathfrak{i}}E_{\mathbf{V}} \to E_{\mathbf{V}}$ induces a functor

$${}_{\mathfrak{i}}j^*_{\mathbf{V}} : \mathcal{D}_{G_{\mathbf{V}}}(E_{\mathbf{V}}) \to \mathcal{D}_{G_{\mathbf{V}}}({}_{\mathfrak{i}}E_{\mathbf{V}}).$$

Lemma 4.4. *It holds that* ${}_{\mathfrak{i}}j^*_{\mathbf{V}}(\mathcal{Q}_{\mathbf{V}}) = {}_{\mathfrak{i}}\mathcal{Q}_{\mathbf{V}}$.

Proof. For any $\mathbf{y} \in Y_\nu$, we have the following fiber product

$$\begin{array}{ccc} {}_{\mathfrak{i}}\tilde{F}_{\mathbf{y}} & \xrightarrow{{}_{\mathfrak{i}}\tilde{j}_{\mathbf{V}}} & \tilde{F}_{\mathbf{y}} \\ \downarrow {}_{\mathfrak{i}}\pi_{\mathbf{y}} & & \downarrow \pi_{\mathbf{y}} \\ {}_{\mathfrak{i}}E_{\mathbf{V}} & \xrightarrow{{}_{\mathfrak{i}}j_{\mathbf{V}}} & E_{\mathbf{V}}. \end{array}$$

Hence we have

$$\begin{aligned} {}_{\mathfrak{i}}j^*_{\mathbf{V}}\mathcal{L}_{\mathbf{y}} &= {}_{\mathfrak{i}}j^*_{\mathbf{V}}\pi_{y!}\mathbf{1}_{\tilde{F}_{\mathbf{y}}}[d_{\mathbf{y}}](\frac{d_{\mathbf{y}}}{2}) \\ &= {}_{\mathfrak{i}}\pi_{y!}{}_{\mathfrak{i}}\tilde{j}^*_{\mathbf{V}}\mathbf{1}_{\tilde{F}_{\mathbf{y}}}[d_{\mathbf{y}}](\frac{d_{\mathbf{y}}}{2}) \\ &= {}_{\mathfrak{i}}\pi_{y!}\mathbf{1}_{{}_{\mathfrak{i}}\tilde{F}_{\mathbf{y}}}[d_{\mathbf{y}}](\frac{d_{\mathbf{y}}}{2}) = {}_{\mathfrak{i}}\mathcal{L}_{\mathbf{y}}. \end{aligned}$$

That is ${}_{\mathfrak{i}}j^*_{\mathbf{V}}(\mathcal{Q}_{\mathbf{V}}) = {}_{\mathfrak{i}}\mathcal{Q}_{\mathbf{V}}$.

$\square$

The restriction of ${}_{\mathfrak{i}}j^*_{\mathbf{V}} : \mathcal{D}_{G_{\mathbf{V}}}(E_{\mathbf{V}}) \to \mathcal{D}_{G_{\mathbf{V}}}({}_{\mathfrak{i}}E_{\mathbf{V}})$ on $\mathcal{Q}_{\mathbf{V}}$ is also denoted by ${}_{\mathfrak{i}}j^*_{\mathbf{V}} : \mathcal{Q}_{\mathbf{V}} \to {}_{\mathfrak{i}}\mathcal{Q}_{\mathbf{V}}$. Considering all dimension vectors, we have ${}_{\mathfrak{i}}j^* : K(\mathcal{Q}) \to K({}_{\mathfrak{i}}\mathcal{Q})$.

Proposition 4.5. *There exists an isomorphism of vector spaces* ${}_{\mathfrak{i}}\hat{\lambda}_{\mathcal{A}} : K({}_{\mathfrak{i}}\mathcal{Q}) \to {}_{\mathfrak{i}}\hat{\mathbf{f}}_{\mathcal{A}}$ *such that the following diagram commutes*

$$\begin{array}{ccc} K(\mathcal{Q}) & \xrightarrow{{}_{\mathfrak{i}}j^*} & K({}_{\mathfrak{i}}\mathcal{Q}) \\ \downarrow{\scriptstyle \hat{\lambda}_{\mathcal{A}}} & & \downarrow{\scriptstyle {}_{\mathfrak{i}}\hat{\lambda}_{\mathcal{A}}} \\ \hat{\mathbf{f}}_{\mathcal{A}} & \xrightarrow{{}_{\mathfrak{i}}\pi_{\mathcal{A}}} & {}_{\mathfrak{i}}\hat{\mathbf{f}}_{\mathcal{A}}. \end{array}$$

Proof. Consider the following set of surjections:

$$\{{}_{\mathbf{i}}\pi_{\mathcal{A}} : \hat{\mathbf{f}}_{\mathcal{A}} \to {}_{\mathbf{i}}\hat{\mathbf{f}}_{\mathcal{A}} \mid \mathbf{i} \in \mathfrak{i}\}.$$

It is clear that the push out of this set is ${}_{\mathfrak{i}}\pi_{\mathcal{A}} : \hat{\mathbf{f}}_{\mathcal{A}} \to {}_{\mathfrak{i}}\hat{\mathbf{f}}_{\mathcal{A}}$.

For any $\mathbf{i} \in \mathfrak{i}$, there exists a canonical open embedding

$${}_{\mathbf{i}}j_{\mathbf{V}} : {}_{\mathbf{i}}E_{\mathbf{V}} \to E_{\mathbf{V}}.$$

Since ${}_{\mathfrak{i}}E_{\mathbf{V}} = \bigcap_{\mathbf{i}\in\mathfrak{i}} {}_{\mathbf{i}}E_{\mathbf{V}}$, we have the following commutative diagram

$$\begin{array}{ccc} {}_{\mathfrak{i}}E_{\mathbf{V}} & \longrightarrow & {}_{\mathbf{i}}E_{\mathbf{V}} \\ \downarrow & & \downarrow{\scriptstyle {}_{\mathbf{i}}j_{\mathbf{V}}} \\ {}_{\mathbf{j}}E_{\mathbf{V}} & \xrightarrow{{}_{\mathbf{j}}j_{\mathbf{V}}} & E_{\mathbf{V}} \end{array}$$

for any $\mathbf{i}, \mathbf{j} \in \mathfrak{i}$. Hence we have the following commutative diagram

$$\begin{array}{ccc} K(\mathcal{Q}_{\mathbf{V}}) & \xrightarrow{{}_{\mathbf{i}}j^*_{\mathbf{V}}} & K({}_{\mathbf{i}}\mathcal{Q}_{\mathbf{V}}) \\ \downarrow{\scriptstyle {}_{\mathbf{j}}j^*_{\mathbf{V}}} & & \downarrow \\ K({}_{\mathbf{j}}\mathcal{Q}_{\mathbf{V}}) & \longrightarrow & K({}_{\mathfrak{i}}\mathcal{Q}_{\mathbf{V}}). \end{array}$$

For any simple perverse sheaf $\mathcal{L}$ such that ${}_{\mathfrak{i}}j^*_{\mathbf{V}}\mathcal{L} = 0$, we have

$$\mathrm{supp}(\mathcal{L}) \subset E_{\mathbf{V}} - {}_{\mathfrak{i}}E_{\mathbf{V}} = E_{\mathbf{V}} - \bigcap_{\mathbf{i}\in\mathfrak{i}} {}_{\mathbf{i}}E_{\mathbf{V}} = \bigcup_{\mathbf{i}\in\mathfrak{i}} (E_{\mathbf{V}} - {}_{\mathbf{i}}E_{\mathbf{V}}).$$

Hence $\mathrm{supp}(\mathcal{L}) \subset E_{\mathbf{V}} - {}_{\mathbf{i}}E_{\mathbf{V}}$ for some $\mathbf{i} \in \mathfrak{i}$ (Section 9.3.4 in [14]). So ${}_{\mathbf{i}}j^*_{\mathbf{V}}\mathcal{L} = 0$. By the definition of push out, the push out of the following set

$$\{{}_{\mathbf{i}}j^* : K(\mathcal{Q}) \to K({}_{\mathbf{i}}\mathcal{Q}) \mid \mathbf{i} \in \mathfrak{i}\}$$

is ${}_{\mathfrak{i}}j^* : K(\mathcal{Q}) \to K({}_{\mathfrak{i}}\mathcal{Q})$.

In [20], it was proved that the following diagram commutes

$$\begin{array}{ccc} K(\mathcal{Q}) & \xrightarrow{{}_{\mathbf{i}}j^*} & K({}_{\mathbf{i}}\mathcal{Q}) \\ \downarrow{\scriptstyle \hat{\lambda}_{\mathcal{A}}} & & \downarrow{\scriptstyle {}_{\mathbf{i}}\hat{\lambda}_{\mathcal{A}}} \\ \hat{\mathbf{f}}_{\mathcal{A}} & \xrightarrow{{}_{\mathbf{i}}\pi_{\mathcal{A}}} & {}_{\mathbf{i}}\hat{\mathbf{f}}_{\mathcal{A}} \end{array}$$

for any $\mathbf{i} \in \mathfrak{i}$. Hence there exists an isomorphism ${}_{\imath}\hat{\lambda}_{\mathcal{A}} : K({}_{\imath}\mathcal{Q}) \to {}_{\imath}\hat{\mathbf{f}}_{\mathcal{A}}$ such that the following diagram commutes

$$\begin{CD}
K(\mathcal{Q}) @>{{}_{\imath}j^*}>> K({}_{\imath}\mathcal{Q}) \\
@V{\hat{\lambda}_{\mathcal{A}}}VV @VV{{}_{\imath}\hat{\lambda}_{\mathcal{A}}}V \\
\hat{\mathbf{f}}_{\mathcal{A}} @>{{}_{\imath}\pi_{\mathcal{A}}}>> {}_{\imath}\hat{\mathbf{f}}_{\mathcal{A}}.
\end{CD}$$

□

4.2.2. For any $\nu, \nu', \nu'' \in \mathbb{N}\mathbf{I}$ such that $\nu = \nu' + \nu''$, fix $\mathbf{V} \in \mathcal{C}'_{\nu}$, $\mathbf{V}' \in \mathcal{C}'_{\nu'}$, $\mathbf{V}'' \in \mathcal{C}'_{\nu''}$. Consider the following diagram

(1)
$$\begin{CD}
{}_{\imath}E_{\mathbf{V}'} \times {}_{\imath}E_{\mathbf{V}''} @<{p_1}<< {}_{\imath}E' @>{p_2}>> {}_{\imath}E'' @>{p_3}>> {}_{\imath}E_{\mathbf{V}} \\
@V{{}_{\imath}j_{\mathbf{V}'} \times {}_{\imath}j_{\mathbf{V}''}}VV @V{j_1}VV @V{j_2}VV @V{{}_{\imath}j_{\mathbf{V}}}VV \\
E_{\mathbf{V}'} \times E_{\mathbf{V}''} @<{p_1}<< E' @>{p_2}>> E'' @>{p_3}>> E_{\mathbf{V}},
\end{CD}$$

where

(1) ${}_{\imath}E' = p_1^{-1}({}_{\imath}E_{\mathbf{V}'} \times {}_{\imath}E_{\mathbf{V}''})$;
(2) ${}_{\imath}E'' = p_2({}_{\imath}E')$;
(3) the restrictions of p_1, p_2 and p_3 are also denoted by p_1, p_2 and p_3 respectively.

For any two complexes $\mathcal{L}' \in \mathcal{D}_{G_{\mathbf{V}'}}({}_{\imath}E_{\mathbf{V}'})$ and $\mathcal{L}'' \in \mathcal{D}_{G_{\mathbf{V}''}}({}_{\imath}E_{\mathbf{V}''})$, $\mathcal{L} = \mathcal{L}' * \mathcal{L}''$ is defined as follows.

Let $\mathcal{L}_1 = \mathcal{L}' \otimes \mathcal{L}''$ and $\mathcal{L}_2 = p_1^*\mathcal{L}_1$. Since p_1 is smooth with connected fibres and p_2 is a $G_{\mathbf{V}'} \times G_{\mathbf{V}''}$-principal bundle, there exists a complex $\mathcal{L}_3$ on ${}_{\imath}E'$ such that $p_2^*(\mathcal{L}_3) = \mathcal{L}_2$. $\mathcal{L}$ is defined as $(p_3)_!\mathcal{L}_3$. Note that, $\mathcal{L}$ is not a semisimple complex in general.

The canonical embedding ${}_{\imath}j_{\mathbf{V}} : {}_{\imath}E_{\mathbf{V}} \to E_{\mathbf{V}}$ also induces a functor

$${}_{\imath}j_{\mathbf{V}!} : \mathcal{D}_{G_{\mathbf{V}}}({}_{\imath}E_{\mathbf{V}}) \to \mathcal{D}_{G_{\mathbf{V}}}(E_{\mathbf{V}}).$$

Lemma 4.6. *It holds that* ${}_{\imath}j_{\mathbf{V}!}(\mathcal{L}' * \mathcal{L}'') = {}_{\imath}j_{\mathbf{V}'!}(\mathcal{L}') * {}_{\imath}j_{\mathbf{V}''!}(\mathcal{L}'')$.

Proof. Let $\hat{\mathcal{L}}' = {}_{\imath}j_{\mathbf{V}'!}\mathcal{L}'$ and $\hat{\mathcal{L}}'' = {}_{\imath}j_{\mathbf{V}''!}\mathcal{L}''$. Since

$$\begin{CD}
{}_{\imath}E_{\mathbf{V}'} \times {}_{\imath}E_{\mathbf{V}''} @<{p_1}<< {}_{\imath}E' \\
@V{{}_{\imath}j_{\mathbf{V}'} \times {}_{\imath}j_{\mathbf{V}''}}VV @VV{j_1}V \\
E_{\mathbf{V}'} \times E_{\mathbf{V}''} @<{p_1}<< E'
\end{CD}$$

is a fiber product, we have $\hat{\mathcal{L}}_2 := p_1^*(\hat{\mathcal{L}}' \otimes \hat{\mathcal{L}}'') = j_{1!}p_1^*(\mathcal{L}' \otimes \mathcal{L}'') = j_{1!}\mathcal{L}_2$.

There exists a complex $\hat{\mathcal{L}}_3$ on E' such that $p_2^*(\hat{\mathcal{L}}_3) = \hat{\mathcal{L}}_2$. Since p_2^* are equivalences of categories, $\hat{\mathcal{L}}_3 = j_{2!}\mathcal{L}_3$.

At last, ${}_{\imath}j_{\mathbf{V}'!}(\mathcal{L}') * {}_{\imath}j_{\mathbf{V}''!}(\mathcal{L}'') = \hat{\mathcal{L}}' * \hat{\mathcal{L}}'' = p_{3!}\hat{\mathcal{L}}_3 = p_{3!}j_{2!}\mathcal{L}_3 = {}_{\imath}j_{\mathbf{V}!}p_{3!}\mathcal{L}_3 = {}_{\imath}j_{\mathbf{V}!}(\mathcal{L}' * \mathcal{L}'')$.

□

Let $K(\mathcal{D}_{G_{\mathbf{V}}}({}_{\mathfrak{i}}E_{\mathbf{V}}))$ be the Grothendieck group of $\mathcal{D}_{G_{\mathbf{V}}}({}_{\mathfrak{i}}E_{\mathbf{V}})$ and $K(\mathcal{D}_{G_{\mathbf{V}}}(E_{\mathbf{V}}))$ be the Grothendieck group of $\mathcal{D}_{G_{\mathbf{V}}}(E_{\mathbf{V}})$. Since ${}_{\mathfrak{i}}j_{\mathbf{V}} : {}_{\mathfrak{i}}E_{\mathbf{V}} \to E_{\mathbf{V}}$ is an open embedding, the functor

$$ {}_{\mathfrak{i}}j_{\mathbf{V}!} : \mathcal{D}_{G_{\mathbf{V}}}({}_{\mathfrak{i}}E_{\mathbf{V}}) \to \mathcal{D}_{G_{\mathbf{V}}}(E_{\mathbf{V}}) $$

induces a map

$$ {}_{\mathfrak{i}}j_{\mathbf{V}!} : K(\mathcal{D}_{G_{\mathbf{V}}}({}_{\mathfrak{i}}E_{\mathbf{V}})) \to K(\mathcal{D}_{G_{\mathbf{V}}}(E_{\mathbf{V}})). $$

Lemma 4.7. *It holds that* ${}_{\mathfrak{i}}j_{\mathbf{V}!}(K({}_{\mathfrak{i}}\mathcal{Q}_{\mathbf{V}})) \in K(\mathcal{Q}_{\mathbf{V}})$.

Proof. Consider the following diagram

$$ \begin{array}{ccccc} K({}_{\mathfrak{i}}\mathcal{Q}) & \xrightarrow{{}_{\mathfrak{i}}j_!} & \bigoplus_{\nu} K(\mathcal{D}_{G_{\mathbf{V}}}(E_{\mathbf{V}})) & \xrightarrow{{}_{\mathfrak{i}}j^*} & \bigoplus_{\nu} K(\mathcal{D}_{G_{\mathbf{V}}}({}_{\mathfrak{i}}E_{\mathbf{V}})) \\ \uparrow & & \uparrow & & \uparrow \\ \uparrow {\scriptstyle {}_{\mathfrak{i}}\hat{\lambda}_{\mathcal{A}}^{-1}} & & K(\mathcal{Q}) & \xrightarrow{{}_{\mathfrak{i}}j^*} & K({}_{\mathfrak{i}}\mathcal{Q}) \\ \uparrow & & \uparrow {\scriptstyle \hat{\lambda}_{\mathcal{A}}^{-1}} & & \uparrow {\scriptstyle {}_{\mathfrak{i}}\hat{\lambda}_{\mathcal{A}}^{-1}} \\ {}_{\mathfrak{i}}\hat{\mathbf{f}}_{\mathcal{A}} & \longrightarrow & \hat{\mathbf{f}}_{\mathcal{A}} & \xrightarrow{{}_{\mathfrak{i}}\pi_{\mathcal{A}}} & {}_{\mathfrak{i}}\hat{\mathbf{f}}_{\mathcal{A}}. \end{array} $$

Since the compositions

$$ K({}_{\mathfrak{i}}\mathcal{Q}) \xrightarrow{{}_{\mathfrak{i}}j_!} \bigoplus_{\nu} K(\mathcal{D}_{G_{\mathbf{V}}}(E_{\mathbf{V}})) \xrightarrow{{}_{\mathfrak{i}}j^*} \bigoplus_{\nu} K(\mathcal{D}_{G_{\mathbf{V}}}({}_{\mathfrak{i}}E_{\mathbf{V}})) $$

and

$$ {}_{\mathfrak{i}}\hat{\mathbf{f}}_{\mathcal{A}} \longrightarrow \hat{\mathbf{f}}_{\mathcal{A}} \xrightarrow{{}_{\mathfrak{i}}\pi_{\mathcal{A}}} {}_{\mathfrak{i}}\hat{\mathbf{f}}_{\mathcal{A}} $$

are identifies, the following diagram commutes

(2)
$$ \begin{array}{ccccc} K({}_{\mathfrak{i}}\mathcal{Q}) & \xrightarrow{{}_{\mathfrak{i}}j_!} & \bigoplus_{\nu} K(\mathcal{D}_{G_{\mathbf{V}}}(E_{\mathbf{V}})) & \xrightarrow{{}_{\mathfrak{i}}j^*} & \bigoplus_{\nu} K(\mathcal{D}_{G_{\mathbf{V}}}({}_{\mathfrak{i}}E_{\mathbf{V}})) \\ \uparrow & & & & \uparrow \\ \uparrow {\scriptstyle {}_{\mathfrak{i}}\hat{\lambda}_{\mathcal{A}}^{-1}} & & & & K({}_{\mathfrak{i}}\mathcal{Q}) \\ \uparrow & & & & \uparrow {\scriptstyle {}_{\mathfrak{i}}\hat{\lambda}_{\mathcal{A}}^{-1}} \\ {}_{\mathfrak{i}}\hat{\mathbf{f}}_{\mathcal{A}} & \longrightarrow & \hat{\mathbf{f}}_{\mathcal{A}} & \xrightarrow{{}_{\mathfrak{i}}\pi_{\mathcal{A}}} & {}_{\mathfrak{i}}\hat{\mathbf{f}}_{\mathcal{A}}. \end{array} $$

For any homogeneous $x \in {}_{\mathfrak{i}}\hat{\mathbf{f}}_{\mathcal{A}}$, choose $\mathcal{L} \in {}_{\mathfrak{i}}\mathcal{Q}_{\mathbf{V}}$ such that $[\mathcal{L}] = {}_{\mathfrak{i}}\hat{\lambda}_{\mathcal{A}}^{-1}(x)$. Let $\mathcal{L}_1 = {}_{\mathfrak{i}}j_!\mathcal{L}$. It is clear that $\mathrm{supp}(\mathcal{L}_1) \in {}_{\mathfrak{i}}E_{\mathbf{V}}$.

The subalgebra ${}_{\mathfrak{i}}\hat{\mathbf{f}}$ of $\hat{\mathbf{f}}$ is generated by $f(\mathbf{i},\mathbf{j};m)$ for all $\mathbf{i} \in \mathfrak{i}$, $\mathbf{j} \notin \mathfrak{i}$ and $m \leq -a_{\mathbf{ij}}$. Let $\nu^{(m)} = m\mathbf{i} + \mathbf{j} \in \mathbb{N}\mathbf{I}$. Fix an object $\mathbf{V}^{(m)} \in \mathcal{C}'$ such that $\underline{\dim}\mathbf{V}^{(m)} = \nu^{(m)}$. Denote by $\mathbf{1}_{{}_{\mathfrak{i}}E_{\mathbf{V}^{(m)}}} \in \mathcal{D}_{G_{\mathbf{V}^{(m)}}}({}_{\mathfrak{i}}E_{\mathbf{V}^{(m)}})$ the constant sheaf on ${}_{\mathfrak{i}}E_{\mathbf{V}^{(m)}}$. Define

$$ \mathcal{E}^{(m)} = j_{\mathbf{V}^{(m)}!}(v^{-mN}\mathbf{1}_{{}_{\mathfrak{i}}E_{\mathbf{V}^{(m)}}}) \in \mathcal{D}_{G_{\mathbf{V}^{(m)}}}(E_{\mathbf{V}^{(m)}}). $$

In [20], it was proved that $[\mathcal{E}^{(m)}] = \hat{\lambda}_{\mathcal{A}}^{-1}(f(\mathbf{i},\mathbf{j};m))$. Since $\mathrm{supp}(\mathcal{E}^{(m)}) \in {}_{\mathfrak{i}}E_{\mathbf{V}^{(m)}}$, there exists $\mathcal{L}_2$ such that $[\mathcal{L}_2] = \hat{\lambda}_{\mathcal{A}}^{-1}(x)$ and $\mathrm{supp}(\mathcal{L}_2) \in {}_{\mathfrak{i}}E_{\mathbf{V}}$.

The Diagram (2) implies $[{}_{\mathfrak{i}}j^*\mathcal{L}_1] = [{}_{\mathfrak{i}}j^*\mathcal{L}_2]$. Hence $[\mathcal{L}_1] = [\mathcal{L}_2]$. That is, the following diagram commutes

$$\begin{array}{ccc} K({}_{\mathfrak{i}}\mathcal{Q}) & \xrightarrow{{}_{\mathfrak{i}}j_!} & \bigoplus_\nu K(\mathcal{D}_{G_{\mathbf{V}}}(E_{\mathbf{V}})) \\ \uparrow {\scriptstyle {}_{\mathfrak{i}}\hat{\lambda}_{\mathcal{A}}^{-1}} & & \uparrow \\ & & K(\mathcal{Q}) \\ & & \uparrow {\scriptstyle \hat{\lambda}_{\mathcal{A}}^{-1}} \\ {}_{\mathfrak{i}}\hat{\mathbf{f}}_{\mathcal{A}} & \longrightarrow & \hat{\mathbf{f}}_{\mathcal{A}}. \end{array}$$

Hence ${}_{\mathfrak{i}}j_{\mathbf{V}!}(K({}_{\mathfrak{i}}\mathcal{Q}_{\mathbf{V}})) \in K(\mathcal{Q}_{\mathbf{V}})$.

□

Lemma 4.8. *For any $\mathcal{L}' \in {}_{\mathfrak{i}}\mathcal{Q}_{\mathbf{V}'}$ and $\mathcal{L}'' \in {}_{\mathfrak{i}}\mathcal{Q}_{\mathbf{V}''}$, $[\mathcal{L}' * \mathcal{L}''] \in K({}_{\mathfrak{i}}\mathcal{Q}_{\mathbf{V}})$.*

Proof. Let $\hat{\mathcal{L}}' = {}_{\mathfrak{i}}j_{\mathbf{V}'!}(\mathcal{L}')$ and $\hat{\mathcal{L}}'' = {}_{\mathfrak{i}}j_{\mathbf{V}''!}(\mathcal{L}'')$. Lemma 4.6 implies $\mathcal{L}' * \mathcal{L}'' = {}_{\mathfrak{i}}j^*_{\mathbf{V}}{}_{\mathfrak{i}}j_{\mathbf{V}!}(\mathcal{L}' * \mathcal{L}'') = {}_{\mathfrak{i}}j^*_{\mathbf{V}}(\hat{\mathcal{L}}' * \hat{\mathcal{L}}'')$. By Lemma 4.7, $[\hat{\mathcal{L}}']$ and $[\hat{\mathcal{L}}''] \in K(\mathcal{Q}_{\mathbf{V}})$. Hence $[\hat{\mathcal{L}}' * \hat{\mathcal{L}}''] \in K(\mathcal{Q}_{\mathbf{V}})$. So $[\mathcal{L}' * \mathcal{L}''] \in K({}_{\mathfrak{i}}\mathcal{Q}_{\mathbf{V}})$.

□

Hence, we get an associative $\mathcal{A}$-bilinear multiplication

$$\begin{aligned} \circledast : K({}_{\mathfrak{i}}\mathcal{Q}_{\mathbf{V}'}) \times K({}_{\mathfrak{i}}\mathcal{Q}_{\mathbf{V}''}) &\rightarrow K({}_{\mathfrak{i}}\mathcal{Q}_{\mathbf{V}}) \\ ([\mathcal{L}'], [\mathcal{L}'']) &\mapsto [\mathcal{L}'] \circledast [\mathcal{L}''] = [\mathcal{L}' \circledast \mathcal{L}''], \end{aligned}$$

where $\mathcal{L}' \circledast \mathcal{L}'' = (\mathcal{L}' * \mathcal{L}'')[m_{\nu'\nu''}](\frac{m_{\nu'\nu''}}{2})$. Then $K({}_{\mathfrak{i}}\mathcal{Q})$ becomes an associative $\mathcal{A}$-algebra and the set $\{[\mathcal{L}] \mid \mathcal{L} \in {}_{\mathfrak{i}}\mathcal{P}_{\mathbf{V}}\}$ is a basis of $K({}_{\mathfrak{i}}\mathcal{Q}_{\mathbf{V}})$.

Proposition 4.9. *We have the following commutative diagram*

$$\begin{array}{ccc} K({}_{\mathfrak{i}}\mathcal{Q}) & \xrightarrow{{}_{\mathfrak{i}}j_!} & K(\mathcal{Q}) \\ \downarrow {\scriptstyle {}_{\mathfrak{i}}\hat{\lambda}_{\mathcal{A}}} & & \downarrow {\scriptstyle \hat{\lambda}_{\mathcal{A}}} \\ {}_{\mathfrak{i}}\hat{\mathbf{f}}_{\mathcal{A}} & \longrightarrow & \hat{\mathbf{f}}_{\mathcal{A}}. \end{array}$$

Moreover, ${}_{\mathfrak{i}}\hat{\lambda}_{\mathcal{A}} : K({}_{\mathfrak{i}}\mathcal{Q}) \rightarrow {}_{\mathfrak{i}}\hat{\mathbf{f}}_{\mathcal{A}}$ is an isomorphism of algebras.

Proof. By the proof of Lemma 4.7, we have the following commutative diagram

$$\begin{array}{ccccc} K({}_{\mathfrak{i}}\mathcal{Q}) & \xrightarrow{{}_{\mathfrak{i}}j_!} & K(\mathcal{Q}) & \xrightarrow{{}_{\mathfrak{i}}j^*} & K({}_{\mathfrak{i}}\mathcal{Q}) \\ \downarrow {\scriptstyle {}_{\mathfrak{i}}\hat{\lambda}_{\mathcal{A}}} & & \downarrow {\scriptstyle \hat{\lambda}_{\mathcal{A}}} & & \downarrow {\scriptstyle {}_{\mathfrak{i}}\hat{\lambda}_{\mathcal{A}}} \\ {}_{\mathfrak{i}}\hat{\mathbf{f}}_{\mathcal{A}} & \longrightarrow & \hat{\mathbf{f}}_{\mathcal{A}} & \xrightarrow{{}_{\mathfrak{i}}\pi_{\mathcal{A}}} & {}_{\mathfrak{i}}\hat{\mathbf{f}}_{\mathcal{A}}. \end{array}$$

Lemma 4.6 implies ${}_{\mathfrak{i}}j_! : K({}_{\mathfrak{i}}\mathcal{Q}) \rightarrow K(\mathcal{Q})$ is a monomorphism of algebras. Since $\hat{\lambda}_{\mathcal{A}} : K(\mathcal{Q}) \rightarrow \hat{\mathbf{f}}_{\mathcal{A}}$ is an isomorphism of algebras, ${}_{\mathfrak{i}}\hat{\lambda}_{\mathcal{A}} : K({}_{\mathfrak{i}}\mathcal{Q}) \rightarrow {}_{\mathfrak{i}}\hat{\mathbf{f}}_{\mathcal{A}}$ is also an isomorphism of algebras.

□

4.3. **Geometric realization of ${}_{i}\mathbf{f}$.**

4.3.1. Let $\tilde{Q} = (Q, a)$ be a quiver with automorphism, where $Q = (\mathbf{I}, H, s, t)$. Fix $i \in I = \mathbf{I}^a$ and assume that $\mathbf{i}$ is a sink for any $\mathbf{i} \in i$.

For any $\nu \in \mathbb{N}I = \mathbb{N}\mathbf{I}^a$ and $\mathbf{V} \in \tilde{\mathcal{C}}_\nu$, $\mathbf{V}$ can be viewed as an object in $\mathcal{C}'_\nu$. Hence ${}_iE_{\mathbf{V}}$ is defined in Section 4.2. The morphism $a : E_{\mathbf{V}} \to E_{\mathbf{V}}$ satisfies that $a({}_iE_{\mathbf{V}}) = {}_iE_{\mathbf{V}}$. Hence we have a functor $a^* : \mathcal{D}_{G_{\mathbf{V}}}({}_iE_{\mathbf{V}}) \to \mathcal{D}_{G_{\mathbf{V}}}({}_iE_{\mathbf{V}})$.

Lemma 4.10. *It holds that $a^*({}_i\mathcal{Q}_{\mathbf{V}}) = {}_i\mathcal{Q}_{\mathbf{V}}$.*

Proof. Note that $a \circ {}_ij_{\mathbf{V}} = {}_ij_{\mathbf{V}} \circ a$. Hence $a^*{}_ij^*_{\mathbf{V}} = {}_ij^*_{\mathbf{V}}a^*$. Since $a^*(\mathcal{Q}_{\mathbf{V}}) = \mathcal{Q}_{\mathbf{V}}$ and ${}_ij^*_{\mathbf{V}}(\mathcal{Q}_{\mathbf{V}}) = {}_i\mathcal{Q}_{\mathbf{V}}$, $a^*({}_i\mathcal{Q}_{\mathbf{V}}) = {}_i\mathcal{Q}_{\mathbf{V}}$.

□

Similarly to $\tilde{\mathcal{Q}}_{\mathbf{V}}$, we can get a category ${}_i\tilde{\mathcal{Q}}_{\mathbf{V}}$. The objects in ${}_i\tilde{\mathcal{Q}}_{\mathbf{V}}$ are pairs $(\mathcal{L}, \phi)$, where $\mathcal{L} \in {}_i\mathcal{Q}_{\mathbf{V}}$ and $\phi : a^*\mathcal{L} \to \mathcal{L}$ is an isomorphism such that

$$a^{*\mathbf{n}}\mathcal{L} \to a^{*(\mathbf{n}-1)}\mathcal{L} \to \ldots \to a^*\mathcal{L} \to \mathcal{L}$$

is the identity map of $\mathcal{L}$. A morphism in $\mathrm{Hom}_{{}_i\tilde{\mathcal{Q}}_{\mathbf{V}}}((\mathcal{L}, \phi), (\mathcal{L}', \phi'))$ is a morphism $f \in \mathrm{Hom}_{{}_i\mathcal{Q}_{\mathbf{V}}}(\mathcal{L}, \mathcal{L}')$ such that

$$\begin{array}{ccc} a^*\mathcal{L} & \xrightarrow{\phi} & \mathcal{L} \\ \downarrow{\scriptstyle a^*f} & & \downarrow{\scriptstyle f} \\ a^*\mathcal{L}' & \xrightarrow{\phi'} & \mathcal{L}'. \end{array}$$

For any $(\mathcal{L}, \phi) \in \tilde{\mathcal{Q}}_{\mathbf{V}}$, the map ${}_i\phi = {}_ij^*_{\mathbf{V}}\phi : a^*{}_ij^*_{\mathbf{V}}\mathcal{L} = {}_ij^*_{\mathbf{V}}a^*\mathcal{L} \to {}_ij^*_{\mathbf{V}}\mathcal{L}$ is also an isomorphism. Hence we get a functor ${}_ij^*_{\mathbf{V}} : \tilde{\mathcal{Q}}_{\mathbf{V}} \to {}_i\tilde{\mathcal{Q}}_{\mathbf{V}}$. Similarly, $K({}_i\tilde{\mathcal{Q}}_{\mathbf{V}})$ has a natural $\mathcal{O}'$-module structure.

For $\nu, \nu', \nu'' \in \mathbb{N}I = \mathbb{N}\mathbf{I}^a$ such that $\nu = \nu' + \nu''$, fix $\mathbf{V} \in \tilde{\mathcal{C}}_\nu$, $\mathbf{V}' \in \tilde{\mathcal{C}}_{\nu'}$, $\mathbf{V}'' \in \tilde{\mathcal{C}}_{\nu''}$. Similarly to Lemma 3.5, the induction functor

$$* : \mathcal{D}_{G_{\mathbf{V}'}}({}_iE_{\mathbf{V}'}) \times \mathcal{D}_{G_{\mathbf{V}''}}({}_iE_{\mathbf{V}''}) \to \mathcal{D}_{G_{\mathbf{V}}}({}_iE_{\mathbf{V}})$$

is compatible with a^*. By Lemma 4.8, we have an associative $\mathcal{O}'$-bilinear multiplication

$$\begin{array}{rcl} \circledast : K({}_i\tilde{\mathcal{Q}}_{\mathbf{V}'}) \times K({}_i\tilde{\mathcal{Q}}_{\mathbf{V}''}) & \to & K({}_i\tilde{\mathcal{Q}}_{\mathbf{V}}) \\ ([\mathcal{L}', \phi'], [\mathcal{L}'', \phi'']) & \mapsto & [\mathcal{L}, \phi], \end{array}$$

where $(\mathcal{L}, \phi) = (\mathcal{L}', \phi') \circledast (\mathcal{L}'', \phi'') = ((\mathcal{L}', \phi') * (\mathcal{L}'', \phi''))[m_{\nu'\nu''}](\frac{m_{\nu'\nu''}}{2})$. Then $K({}_i\tilde{\mathcal{Q}})$ becomes an associative $\mathcal{O}'$-algebra.

4.3.2. Fix a nonzero element $\nu \in \mathbb{N}\mathbf{I}^a$ and $\mathbf{V} \in \tilde{\mathcal{C}}_\nu$. For any element $\mathbf{y} \in Y^a_\nu$, the automorphism $a : {}_iF_{\mathbf{y}} \to {}_iF_{\mathbf{y}}$ is defined as

$$a(\phi) = (\mathbf{V} = a(\mathbf{V}^k) \supset a(\mathbf{V}^{k-1}) \supset \cdots \supset a(\mathbf{V}^0) = 0)$$

for any

$$\phi = (\mathbf{V} = \mathbf{V}^k \supset \mathbf{V}^{k-1} \supset \cdots \supset \mathbf{V}^0 = 0) \in {}_iF_{\mathbf{y}}.$$

We also have an automorphism $a : {}_i\tilde{F}_{\mathbf{y}} \to {}_i\tilde{F}_{\mathbf{y}}$, defined as $a((x, \phi)) = (a(x), a(\phi))$ for any $(x, \phi) \in {}_i\tilde{F}_{\mathbf{y}}$.

By the natural isomorphism $a^*(\mathbf{1}_{{}_i\tilde{F}_{\mathbf{y}}}) \cong \mathbf{1}_{{}_i\tilde{F}_{\mathbf{y}}}$ induced by the automorphism $a : {}_i\tilde{F}_{\mathbf{y}} \to {}_i\tilde{F}_{\mathbf{y}}$, there exists an isomorphism

$${}_i\phi_0 : a^*{}_i\mathcal{L}_{\mathbf{y}} = a^*(\pi_{\mathbf{y}})_!(\mathbf{1}_{{}_i\tilde{F}_{\mathbf{y}}})[d_{\mathbf{y}}] = (\pi_{\mathbf{y}})_!a^*(\mathbf{1}_{\tilde{F}_{\mathbf{y}}})[d_{\mathbf{y}}] \cong (\pi_{\mathbf{y}})_!(\mathbf{1}_{{}_i\tilde{F}_{\mathbf{y}}})[d_{\mathbf{y}}] = {}_i\mathcal{L}_{\mathbf{y}}.$$

Then $({}_i\mathcal{L}_{\mathbf{y}}, {}_i\phi_0)$ is an object in ${}_i\tilde{\mathcal{Q}}_{\mathbf{V}}$.

Let ${}_i\mathbf{k}_\nu$ be the $\mathcal{A}$-submodule of $K({}_i\tilde{\mathcal{Q}}_{\mathbf{V}})$ spanned by $({}_i\mathcal{L}_{\mathbf{y}}, {}_i\phi_0)$ for all $\mathbf{y} \in Y_\nu^a$. Let ${}_i\mathbf{k} = \bigoplus_{\nu\in\mathbb{N}I} {}_i\mathbf{k}_\nu$.

4.3.3. Since $({}_i\mathcal{L}_{\mathbf{y}}, {}_i\phi_0) = {}_ij^*_{\mathbf{V}}(\mathcal{L}_{\mathbf{y}}, \phi_0)$, the functor ${}_ij^*_{\mathbf{V}} : \tilde{\mathcal{Q}}_{\mathbf{V}} \to {}_i\tilde{\mathcal{Q}}_{\mathbf{V}}$ induces a map ${}_ij^* : \mathbf{k} \to {}_i\mathbf{k}$.

Theorem 4.11. *There is an isomorphism of vector spaces*

$$ {}_i\lambda_{\mathcal{A}} : {}_i\mathbf{k} \to {}_i\mathbf{f}_{\mathcal{A}} $$

such that the following diagram commutes

$$ \begin{array}{ccc} \mathbf{k} & \xrightarrow{{}_ij^*} & {}_i\mathbf{k} \\ \downarrow{\scriptstyle \lambda_{\mathcal{A}}} & & \downarrow{\scriptstyle {}_i\lambda_{\mathcal{A}}} \\ \mathbf{f}_{\mathcal{A}} & \xrightarrow{{}_i\pi_{\mathcal{A}}} & {}_i\mathbf{f}_{\mathcal{A}}. \end{array} $$

For the proof of Theorem 4.11, we need the following lemmas.

Lemma 4.12. *There exists an embedding ${}_i\delta : {}_i\mathbf{f} \to {}_i\hat{\mathbf{f}}$ such that the following diagram commutes*

$$ \begin{array}{ccc} \mathbf{f}_{\mathcal{A}} & \xrightarrow{{}_i\pi_{\mathcal{A}}} & {}_i\mathbf{f}_{\mathcal{A}} \\ \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle {}_i\delta} \\ \hat{\mathbf{f}}_{\mathcal{A}} & \xrightarrow{{}_i\pi_{\mathcal{A}}} & {}_i\hat{\mathbf{f}}_{\mathcal{A}}, \end{array} $$

where $\delta : \mathbf{f}_{\mathcal{A}} \to \hat{\mathbf{f}}_{\mathcal{A}}$ is defined in Section 3.3.2.

Proof. Consider the following diagram

$$ \begin{array}{ccccccccc} 0 & \longrightarrow & \theta_i\mathbf{f}_{\mathcal{A}} & \longrightarrow & \mathbf{f}_{\mathcal{A}} & \xrightarrow{{}_i\pi_{\mathcal{A}}} & {}_i\mathbf{f}_{\mathcal{A}} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle \delta} & & \vdots{\scriptstyle {}_i\delta} & & \\ 0 & \longrightarrow & \sum_{\mathbf{i}\in i}\theta_{\mathbf{i}}\hat{\mathbf{f}}_{\mathcal{A}} & \longrightarrow & \hat{\mathbf{f}}_{\mathcal{A}} & \xrightarrow{{}_i\pi_{\mathcal{A}}} & {}_i\hat{\mathbf{f}}_{\mathcal{A}} & \longrightarrow & 0. \end{array} \tag{3} $$

Note that $\theta_i\mathbf{f}_{\mathcal{A}} \cap \mathbf{B}$ is a basis of $\theta_i\mathbf{f}_{\mathcal{A}}$, $\sum_{\mathbf{i}\in i}\theta_{\mathbf{i}}\hat{\mathbf{f}}_{\mathcal{A}} \cap \hat{\mathbf{B}}$ is a basis of $\sum_{\mathbf{i}\in i}\theta_{\mathbf{i}}\hat{\mathbf{f}}_{\mathcal{A}}$ and $\delta(\mathbf{B}) = \hat{\mathbf{B}}^a$. For any $[\mathcal{L}, \phi] \in \theta_i\mathbf{f}_{\mathcal{A}} \cap \mathbf{B}$, $\mathcal{L} \in \mathcal{P}_{i,\gamma_i}$ and $[\mathcal{L}] \in \hat{\mathbf{B}}^a$, where $\gamma_i = \sum_{\mathbf{i}\in i}\mathbf{i}$. Since $\mathcal{P}_{i,\gamma_i} \in \mathcal{P}_{\mathbf{i},\mathbf{i}}$, $\mathcal{L} \in \mathcal{P}_{\mathbf{i},\mathbf{i}}$ for any $\mathbf{i} \in i$. Hence $\delta([\mathcal{L}, \phi]) = [\mathcal{L}] \in \sum_{\mathbf{i}\in i}\theta_{\mathbf{i}}\hat{\mathbf{f}}_{\mathcal{A}} \cap \hat{\mathbf{B}}^a$. So there exists an map ${}_i\delta : {}_i\mathbf{f} \to {}_i\hat{\mathbf{f}}$ such that the Diagram (3) commutes.

On the other hand, for any $[\mathcal{L}] \in \sum_{\mathbf{i}\in i}\theta_{\mathbf{i}}\hat{\mathbf{f}}_{\mathcal{A}} \cap \hat{\mathbf{B}}^a$, $[\mathcal{L}] \in \theta_{\mathbf{i}}\hat{\mathbf{f}}_{\mathcal{A}} \cap \hat{\mathbf{B}}^a$ for some $\mathbf{i} \in i$. Hence $\mathcal{L} \in \mathcal{P}_{\mathbf{i},\mathbf{i}}$ and $[\mathcal{L}] \in \hat{\mathbf{B}}^a$. By Lemma 12.5.1 in [14], $\mathcal{L} \in \mathcal{P}_{i,\gamma_i}$. So $[\mathcal{L}] \in \delta(\theta_i\mathbf{f}_{\mathcal{A}} \cap \mathbf{B})$. Hence ${}_i\delta : {}_i\mathbf{f} \to {}_i\hat{\mathbf{f}}$ is an embedding.

□

Lemma 4.13. *There exists an embedding ${}_i\tilde{\delta} : {}_i\mathbf{k} \to K({}_i\mathcal{Q})$ such that the following diagram commutes*

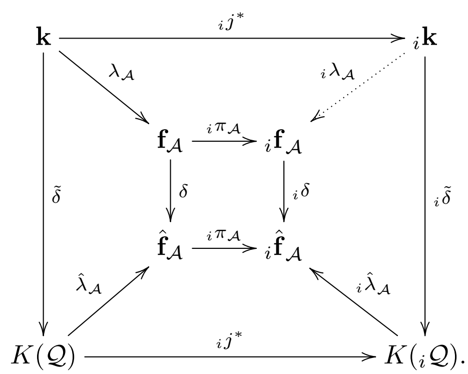
$$ \begin{array}{ccc} \mathbf{k} & \xrightarrow{{}_ij^*} & {}_i\mathbf{k} \\ \downarrow{\scriptstyle \tilde{\delta}} & & \downarrow{\scriptstyle {}_i\tilde{\delta}} \\ K(\mathcal{Q}) & \xrightarrow{{}_ij^*} & K({}_i\mathcal{Q}), \end{array} $$

where $\tilde{\delta} : \mathbf{k} \to K(\mathcal{Q})$ *is defined in Section 3.3.2.*

Proof. Let ${}_i\mathbf{B} = \{{}_ij^*([\mathcal{L}, \phi]) \mid [\mathcal{L}, \phi] \in \mathbf{B} \text{ and } {}_ij^*[\mathcal{L}] \neq 0\}$, which is an $\mathcal{A}$-basis of ${}_i\mathbf{k}$. For any $[{}_i\mathcal{L}, {}_i\phi] \in {}_i\mathbf{B}$, there exist a unique $[\mathcal{L}, \phi] \in \mathbf{B}$ such that ${}_ij^*(\mathcal{L}, \phi) = ({}_i\mathcal{L}, {}_i\phi)$. Define ${}_i\tilde{\delta}([{}_i\mathcal{L}, {}_i\phi]) = {}_ij^*\tilde{\delta}[\mathcal{L}, \phi]$. Then, we get the desired embedding ${}_i\tilde{\delta} : {}_i\mathbf{k} \to K({}_i\mathcal{Q})$.

□

Proof of Theorem 4.11. Consider the following diagram



Lemma 4.12, 4.13 and Proposition 4.5 imply that there exists a unique $\mathcal{A}$-linear isomorphism

$$ {}_i\lambda_{\mathcal{A}} : {}_i\mathbf{k} \to {}_i\mathbf{f}_{\mathcal{A}} $$

such that the following diagram commutes

$$ \begin{array}{ccc} \mathbf{k} & \xrightarrow{{}_ij^*} & {}_i\mathbf{k} \\ \downarrow{\scriptstyle \lambda_{\mathcal{A}}} & & \downarrow{\scriptstyle {}_i\lambda_{\mathcal{A}}} \\ \mathbf{f}_{\mathcal{A}} & \xrightarrow{{}_i\pi_{\mathcal{A}}} & {}_i\mathbf{f}_{\mathcal{A}}. \end{array} $$

□

4.3.4. In Proposition 4.9, we have defined a map ${}_{\mathrm{i}}j_{\mathbf{V}!} : K({}_{\mathrm{i}}\mathcal{Q}) \to K(\mathcal{Q})$. Since $a \circ {}_ij_{\mathbf{V}} = {}_ij_{\mathbf{V}} \circ a$, we have $a^*{}_ij_{\mathbf{V}!} = {}_ij_{\mathbf{V}!}a^*$. For any isomorphism $\phi : a^*\mathcal{L} \to \mathcal{L}$, the map $\phi' = {}_ij_{\mathbf{V}!}\phi : a^*{}_ij_{\mathbf{V}!}\mathcal{L} = {}_ij_{\mathbf{V}!}a^*\mathcal{L} \to {}_ij_{\mathbf{V}!}\mathcal{L}$ is also an isomorphism. Hence we get a map ${}_{\mathrm{i}}j_! : K({}_{\mathrm{i}}\tilde{\mathcal{Q}}) \to K(\tilde{\mathcal{Q}})$.

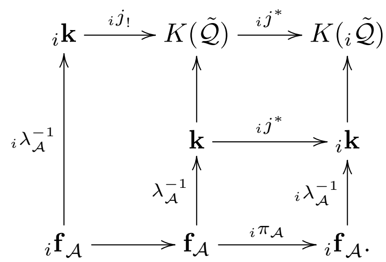
Consider Diagram (1). Since a commutes with p_1, p_2, p_3 and j_1, j_2, Lemma 4.6 implies that ${}_{\mathrm{i}}j_! : K({}_{\mathrm{i}}\tilde{\mathcal{Q}}) \to K(\tilde{\mathcal{Q}})$ is a homomorphism of algebras.

Theorem 4.14. *It holds that* ${}_ij_!({}_i\mathbf{k}) \subset \mathbf{k}$ *and we have the following commutative diagram*

$$ \begin{array}{ccc} {}_i\mathbf{k} & \xrightarrow{{}_ij_!} & \mathbf{k} \\ \downarrow{\scriptstyle {}_i\lambda_{\mathcal{A}}} & & \downarrow{\scriptstyle \lambda_{\mathcal{A}}} \\ {}_i\mathbf{f}_{\mathcal{A}} & \longrightarrow & \mathbf{f}_{\mathcal{A}}. \end{array} $$

Moreover, ${}_i\mathbf{k}$ *is a subalgebra of* $\mathbf{k}$ *and* ${}_i\lambda_{\mathcal{A}} : {}_i\mathbf{k} \to {}_i\mathbf{f}_{\mathcal{A}}$ *is an isomorphism of algebras.*

Proof. Consider the following diagram



Since the compositions

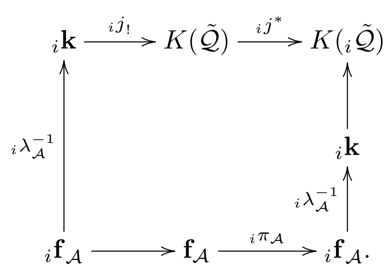
$$ {}_i\mathbf{k} \xrightarrow{{}_ij_!} K(\tilde{\mathcal{Q}}) \xrightarrow{{}_ij^*} K({}_i\tilde{\mathcal{Q}}) $$

and

$$ {}_i\mathbf{f}_{\mathcal{A}} \longrightarrow \mathbf{f}_{\mathcal{A}} \xrightarrow{{}_i\pi_{\mathcal{A}}} {}_i\mathbf{f}_{\mathcal{A}} $$

are identifies, the following diagram commutes

(4)



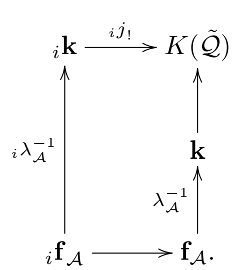
For any homogeneous $x \in {}_i\mathbf{f}_{\mathcal{A}}$, choose $(\mathcal{L}, \phi) \in \mathcal{Q}_{\mathbf{V}}$ such that $[\mathcal{L}, \phi] = {}_i\lambda_{\mathcal{A}}^{-1}(x)$. Let $(\mathcal{L}_1, \phi_1) = {}_ij_!(\mathcal{L}, \phi)$. It is clear that $\mathrm{supp}(\mathcal{L}_1) \in {}_iE_{\mathbf{V}}$.

The subalgebra ${}_i\mathbf{f}$ of $\mathbf{f}$ is generated by $f(i,j;m)$ for all $j \neq i$ and $m \leq -a_{ij}$. Let $\nu^{(m)} = m\gamma_i + \gamma_j \in \mathbb{N}\mathbf{I}^a$. Fix an object $\mathbf{V}^{(m)} \in \tilde{\mathcal{C}}$ such that $\underline{\dim}\mathbf{V}^{(m)} = \nu^{(m)}$. Denote by $\mathbf{1}_{{}_iE_{\mathbf{V}^{(m)}}} \in \mathcal{D}_{G_{\mathbf{V}^{(m)}}}({}_iE_{\mathbf{V}^{(m)}})$ the constant sheaf on ${}_iE_{\mathbf{V}^{(m)}}$. Define

$$ \mathcal{E}^{(m)} = j_{\mathbf{V}^{(m)}!}(v^{-mN}\mathbf{1}_{{}_iE_{\mathbf{V}^{(m)}}}) \in \mathcal{D}_{G_{\mathbf{V}^{(m)}}}(E_{\mathbf{V}^{(m)}}). $$

In Section 5.2, it will be proved that $[\mathcal{E}^{(m)}, \mathrm{id}] = \lambda_{\mathcal{A}}^{-1}(f(i,j;m))$. Since $\mathrm{supp}(\mathcal{E}^{(m)}) \in {}_iE_{\mathbf{V}^{(m)}}$, there exists $(\mathcal{L}_2, \phi_2)$ such that $[\mathcal{L}_2, \phi_2] = \lambda_{\mathcal{A}}^{-1}(x)$ and $\mathrm{supp}(\mathcal{L}_2) \in {}_iE_{\mathbf{V}}$.

The Diagram (4) implies $[{}_ij^*\mathcal{L}_1, {}_ij^*\phi_1] = [{}_ij^*\mathcal{L}_2, {}_ij^*\phi_2]$. Hence $[\mathcal{L}_1, \phi_1] = [\mathcal{L}_2, \phi_2]$. That is the following diagram commutes



Hence ${}_ij_!({}_i\mathbf{k}) \subset \mathbf{k}$ and we have the following commutative diagram

$$\begin{array}{ccccc}
{}_i\mathbf{k} & \xrightarrow{{}_ij_!} & \mathbf{k} & \xrightarrow{{}_ij^*} & {}_i\mathbf{k} \\
\downarrow {}_i\lambda_{\mathcal{A}} & & \downarrow \lambda_{\mathcal{A}} & & \downarrow {}_i\lambda_{\mathcal{A}} \\
{}_i\mathbf{f}_{\mathcal{A}} & \longrightarrow & \mathbf{f}_{\mathcal{A}} & \xrightarrow{{}_i\pi_{\mathcal{A}}} & {}_i\mathbf{f}_{\mathcal{A}}.
\end{array}$$

Since ${}_{\mathfrak{i}}j_! : K({}_{\mathfrak{i}}\tilde{\mathcal{Q}}) \to K(\tilde{\mathcal{Q}})$ is a homomorphism of algebras, the first row of the commutative diagram above implies that ${}_i\mathbf{k}$ is a subalgebra of $\mathbf{k}$ and ${}_ij_! : {}_i\mathbf{k} \to \mathbf{k}$ is a monomorphism of algebras. Since ${}_i\mathbf{f}_{\mathcal{A}}$ is a subalgebra of $\mathbf{f}_{\mathcal{A}}$, the $\mathcal{A}$-algebra isomorphism $\lambda_{\mathcal{A}} : \mathbf{f}_{\mathcal{A}} \to \mathbf{k}$ induces that ${}_i\lambda_{\mathcal{A}} : {}_i\mathbf{f}_{\mathcal{A}} \to {}_i\mathbf{k}$ is also an isomorphism of algebras.

□

4.4. Geometric realization of ${}^i\mathbf{f}$.

4.4.1. Let $\mathfrak{i}$ be a subset of $\mathbf{I}$ satisfying the following conditions: (1) for any $\mathbf{i} \in \mathfrak{i}$, $\mathbf{i}$ is a source; (2) for any $\mathbf{i}, \mathbf{j} \in \mathfrak{i}$, there are no arrows between them.

For any $\mathbf{V} \in \mathcal{C}'_\nu$, consider a subvariety ${}^{\mathfrak{i}}E_{\mathbf{V}}$ of $E_{\mathbf{V}}$

$${}^{\mathfrak{i}}E_{\mathbf{V}} = \{x \in E_{\mathbf{V}} \mid \bigoplus_{h\in H, s(h)=\mathbf{i}} x_h : V_{\mathbf{i}} \to \bigoplus_{h\in H, s(h)=\mathbf{i}} V_{t(h)} \text{ is injective for any } \mathbf{i} \in \mathfrak{i}\}.$$

Denote by ${}^{\mathfrak{i}}j_{\mathbf{V}} : {}^{\mathfrak{i}}E_{\mathbf{V}} \to E_{\mathbf{V}}$ the canonical embedding.

Similarly to the notations in Section 4.2, the categories ${}^{\mathfrak{i}}\mathcal{P}_{\mathbf{V}}$ and ${}^{\mathfrak{i}}\mathcal{Q}_{\mathbf{V}}$ can be defined. Let $K({}^{\mathfrak{i}}\mathcal{Q}_{\mathbf{V}})$ be the Grothendieck group.

The canoniccal embedding ${}^{\mathfrak{i}}j_{\mathbf{V}} : {}^{\mathfrak{i}}E_{\mathbf{V}} \to E_{\mathbf{V}}$ induces ${}^{\mathfrak{i}}j^*_{\mathbf{V}} : \mathcal{Q}_{\mathbf{V}} \to {}^{\mathfrak{i}}\mathcal{Q}_{\mathbf{V}}$ and ${}^{\mathfrak{i}}j_{\mathbf{V}!} : K({}^{\mathfrak{i}}\mathcal{Q}_{\mathbf{V}}) \to K(\mathcal{Q}_{\mathbf{V}})$. Considering all dimension vectors, we have ${}^{\mathfrak{i}}j^* : K(\mathcal{Q}) \to K({}^{\mathfrak{i}}\mathcal{Q})$ and ${}^{\mathfrak{i}}j_! : K({}^{\mathfrak{i}}\mathcal{Q}) \to K(\mathcal{Q})$.

Proposition 4.15. *There exists an isomorphism of algebras* ${}^{\mathfrak{i}}\hat{\lambda}_{\mathcal{A}} : K({}^{\mathfrak{i}}\mathcal{Q}) \to {}^{\mathfrak{i}}\hat{\mathbf{f}}_{\mathcal{A}}$ *such that the following diagram commutes*

$$\begin{array}{ccccc}
K({}^{\mathfrak{i}}\mathcal{Q}) & \xrightarrow{{}^{\mathfrak{i}}j_!} & K(\mathcal{Q}) & \xrightarrow{{}^{\mathfrak{i}}j^*} & K({}^{\mathfrak{i}}\mathcal{Q}) \\
\downarrow {}^{\mathfrak{i}}\hat{\lambda}_{\mathcal{A}} & & \downarrow \hat{\lambda}_{\mathcal{A}} & & \downarrow {}^{\mathfrak{i}}\hat{\lambda}_{\mathcal{A}} \\
{}^{\mathfrak{i}}\hat{\mathbf{f}}_{\mathcal{A}} & \longrightarrow & \hat{\mathbf{f}}_{\mathcal{A}} & \xrightarrow{{}^{\mathfrak{i}}\pi_{\mathcal{A}}} & {}^{\mathfrak{i}}\hat{\mathbf{f}}_{\mathcal{A}}.
\end{array}$$

4.4.2. Let $\tilde{Q} = (Q, a)$ be a quiver with automorphism, where $Q = (\mathbf{I}, H, s, t)$. Fix $i \in I = \mathbf{I}^a$ and assume that $\mathbf{i}$ is a source for any $\mathbf{i} \in i$.

Similarly to the notations in Section 4.3, the category ${}^i\tilde{\mathcal{Q}}_{\mathbf{V}}$ can be defined. Let $K({}^i\tilde{\mathcal{Q}}_{\mathbf{V}})$ be the Grothendieck group. We also have ${}^ij^*_{\mathbf{V}} : \tilde{\mathcal{Q}}_{\mathbf{V}} \to {}^i\tilde{\mathcal{Q}}_{\mathbf{V}}$ and ${}^ij_! : K({}^i\tilde{\mathcal{Q}}_{\mathbf{V}}) \to K(\tilde{\mathcal{Q}}_{\mathbf{V}})$.

Consider the subalgebra ${}^i\mathbf{k}$ of $K({}^i\tilde{\mathcal{Q}})$. The functor ${}^ij^*_{\mathbf{V}} : \tilde{\mathcal{Q}}_{\mathbf{V}} \to {}^i\tilde{\mathcal{Q}}_{\mathbf{V}}$ induces a map ${}^ij^* : \mathbf{k} \to {}^i\mathbf{k}$. The map ${}^ij_! : K({}^i\tilde{\mathcal{Q}}) \to K(\tilde{\mathcal{Q}})$ induces a map ${}^ij_! : {}^i\mathbf{k} \to \mathbf{k}$.

Theorem 4.16. *There is a unique $\mathcal{A}$-algebra isomorphism*

$${}^i\lambda_{\mathcal{A}} : {}^i\mathbf{k} \to {}^i\mathbf{f}_{\mathcal{A}}$$

such that the following diagram commute

$$\begin{array}{ccccc} {}^i\mathbf{k} & \xrightarrow{{}^ij_!} & \mathbf{k} & \xrightarrow{{}^ij^*} & {}^i\mathbf{k} \\ \downarrow {}^i\lambda_{\mathcal{A}} & & \downarrow \lambda_{\mathcal{A}} & & \downarrow {}^i\lambda_{\mathcal{A}} \\ {}^i\mathbf{f}_{\mathcal{A}} & \longrightarrow & \mathbf{f}_{\mathcal{A}} & \xrightarrow{{}^i\pi_{\mathcal{A}}} & {}^i\mathbf{f}_{\mathcal{A}}. \end{array}$$

5. Geometric realization of $T_i : {}_i\mathbf{f} \to {}^i\mathbf{f}$

5.1. **Geometric realization.**

5.1.1. Let $Q = (\mathbf{I}, H, s, t)$ be a quiver. Let $\mathfrak{i}$ be a subset of $\mathbf{I}$ satisfying the following conditions: (1) for any $\mathbf{i} \in \mathfrak{i}$, $\mathbf{i}$ is a sink; (2) for any $\mathbf{i}, \mathbf{j} \in \mathfrak{i}$, there are no arrows between them.

Let $Q' = \sigma_{\mathfrak{i}} Q = (\mathbf{I}, H', s, t)$ be the quiver by reversing the directions of all arrows in Q containing $\mathbf{i} \in \mathfrak{i}$. So for any $\mathbf{i} \in \mathfrak{i}$, $\mathbf{i}$ is a source of Q'.

For any $\nu, \nu' \in \mathbb{N}\mathbf{I}$ such that $\nu' = s_{\mathfrak{i}}\nu = \nu - \nu(h_i)i$ and $\mathbf{V} \in \mathcal{C}'_\nu$, $\mathbf{V}' \in \mathcal{C}'_{\nu'}$, consider the following correspondence ([13, 5])

$$ {}_{\mathfrak{i}}E_{\mathbf{V},Q} \xleftarrow{\alpha} Z_{\mathbf{V}\mathbf{V}'} \xrightarrow{\beta} {}^{\mathfrak{i}}E_{\mathbf{V}',Q'} , \tag{5}$$

where

(1) $Z_{\mathbf{V}\mathbf{V}'}$ is the subset of $E_{\mathbf{V},Q} \times E_{\mathbf{V}',Q'}$ consisting of all (x, y) satisfying the following conditions:
  (a) for any $h \in H$ such that $t(h) \notin \mathfrak{i}$, $x_h = y_h$;
  (b) for any $\mathbf{i} \in \mathfrak{i}$, the following sequence is exact

$$0 \longrightarrow V'_{\mathbf{i}} \xrightarrow{\bigoplus_{h\in H', s(h)=\mathbf{i}} y_h} \bigoplus_{h\in H, t(h)=\mathbf{i}} V_{s(h)} \xrightarrow{\bigoplus_{h\in H, t(h)=\mathbf{i}} x_h} V_{\mathbf{i}} \longrightarrow 0 \ ;$$

(2) $\alpha(x, y) = x$ and $\beta(x, y) = y$.

From now on, ${}_{\mathfrak{i}}E_{\mathbf{V},Q}$ is denoted by ${}_{\mathfrak{i}}E_{\mathbf{V}}$ and ${}^{\mathfrak{i}}E_{\mathbf{V}',Q'}$ is denoted by ${}^{\mathfrak{i}}E_{\mathbf{V}'}$. Let

$$\begin{aligned} G_{\mathbf{V}\mathbf{V}'} &= \prod_{\mathbf{i}\in\mathfrak{i}} GL(V_{\mathbf{i}}) \times \prod_{\mathbf{i}\in\mathfrak{i}} GL(V'_{\mathbf{i}}) \times \prod_{\mathbf{j}\notin\mathfrak{i}} GL(V_{\mathbf{j}}) \\ &\cong \prod_{\mathbf{i}\in\mathfrak{i}} GL(V_{\mathbf{i}}) \times \prod_{\mathbf{i}\in\mathfrak{i}} GL(V'_{\mathbf{i}}) \times \prod_{\mathbf{j}\notin\mathfrak{i}} GL(V'_{\mathbf{j}}), \end{aligned}$$

which acts on $Z_{\mathbf{V}\mathbf{V}'}$ naturally.

By (5), we have

$$\mathcal{D}_{G_{\mathbf{V}}}({}_{\mathfrak{i}}E_{\mathbf{V}}) \xrightarrow{\alpha^*} \mathcal{D}_{G_{\mathbf{V}\mathbf{V}'}}(Z_{\mathbf{V}\mathbf{V}'}) \xleftarrow{\beta^*} \mathcal{D}_{G_{\mathbf{V}'}}({}^{\mathfrak{i}}E_{\mathbf{V}'}) \ .$$

Since α and β are principal bundles with fibers $\prod_{\mathbf{i}\in\mathfrak{i}} \mathrm{Aut}(V'_{\mathbf{i}})$ and $\prod_{\mathbf{i}\in\mathfrak{i}} \mathrm{Aut}(V_{\mathbf{i}})$ respectively, α^* and β^* are equivalences of categories (Section 2.2.5 in [2]).

Hence, for any $\mathcal{L} \in \mathcal{D}_{G_{\mathbf{V}}}({}_{\mathfrak{i}}E_{\mathbf{V}})$ there exists a unique $\mathcal{L}' \in \mathcal{D}_{G_{\mathbf{V}'}}({}^{\mathfrak{i}}E_{\mathbf{V}'})$ such that $\alpha^*(\mathcal{L}) = \beta^*(\mathcal{L}')$. Define

$$\begin{aligned} \tilde{\omega}_{\mathfrak{i}} : \mathcal{D}_{G_{\mathbf{V}}}({}_{\mathfrak{i}}E_{\mathbf{V}}) &\to \mathcal{D}_{G_{\mathbf{V}'}}({}^{\mathfrak{i}}E_{\mathbf{V}'}) \\ \mathcal{L} &\mapsto \mathcal{L}'[-s(\mathbf{V})](-\frac{s(\mathbf{V})}{2}), \end{aligned}$$

where $s(\mathbf{V})=\sum_{\mathbf{i}\in\mathfrak{i}}(\dim \mathrm{GL}(V_{\mathbf{i}})-\dim \mathrm{GL}(V'_{\mathbf{i}}))$. Since α^* and β^* are equivalences of categories, $\tilde{\omega}_i$ is also an equivalence of categories.

Proposition 5.1. *It holds that* $\tilde{\omega}_{\mathfrak{i}}({}_{\mathfrak{i}}\mathcal{Q}_{\mathbf{V}})={}^{\mathfrak{i}}\mathcal{Q}_{\mathbf{V}'}$.

For the proof of Proposition 5.1, we give some new notations. Let $\mathfrak{i}_1$ be a subset of $\mathfrak{i}$ and $\mathfrak{i}_2=\mathfrak{i}-\mathfrak{i}_1$. Consider the quiver $\sigma_{\mathfrak{i}_1}Q$. For any $\mathbf{V}\in\mathcal{C}'_\nu$, consider a subvariety ${}^{\mathfrak{i}_1}_{\mathfrak{i}_2}E_{\mathbf{V},\sigma_{\mathfrak{i}_1}Q}$ of $E_{\mathbf{V},\sigma_{\mathfrak{i}_1}Q}$

$$\begin{aligned}{}^{\mathfrak{i}_1}_{\mathfrak{i}_2}E_{\mathbf{V},\sigma_{\mathfrak{i}_1}Q}=\{x\in E_{\mathbf{V},\sigma_{\mathfrak{i}_1}Q}\mid &\bigoplus_{h\in H,t(h)=\mathbf{i}}x_h:\bigoplus_{h\in H,t(h)=\mathbf{i}}V_{s(h)}\to V_{\mathbf{i}}\text{ is surjective}\\ &\text{for any }\mathbf{i}\in\mathfrak{i}_2\text{ and }\bigoplus_{h\in H,s(h)=\mathbf{i}}x_h:V_{\mathbf{i}}\to\bigoplus_{h\in H,s(h)=\mathbf{i}}V_{t(h)}\text{ is injective for any }\mathbf{i}\in\mathfrak{i}_1\}.\end{aligned}$$

Denote by ${}^{\mathfrak{i}_1}_{\mathfrak{i}_2}j_{\mathbf{V},\sigma_{\mathfrak{i}_1}Q}:{}^{\mathfrak{i}_1}_{\mathfrak{i}_2}E_{\mathbf{V},\sigma_{\mathfrak{i}_1}Q}\to E_{\mathbf{V},\sigma_{\mathfrak{i}_1}Q}$ the canonical embedding.

Let $\mathfrak{i}=\{\mathbf{i}_1,\mathbf{i}_2,\dots,\mathbf{i}_l\}$, $\mathfrak{i}_k=\{\mathbf{i}_k,\dots,\mathbf{i}_{l-1},\mathbf{i}_l\}$ and $\mathfrak{i}'_k=\mathfrak{i}-\mathfrak{i}_k$. Let $Q_{k+1}=\sigma_{\mathbf{i}_k}Q_k$ For any $\nu_1,\nu_2,\dots,\nu_{l+1}\in\mathbb{N}\mathbf{I}$ such that $\nu_{k+1}=s_{\mathbf{i}_k}\nu_k$ and $\mathbf{V}^k\in\mathcal{C}'_{\nu_k}$, consider the following commutative diagrams

$$\begin{array}{ccccc}
{}^{\mathfrak{i}'_k}_{\mathfrak{i}_k}E_{\mathbf{V}^k,Q_k} & \xleftarrow{\alpha_k} & \hat{Z}_{\mathbf{V}^k\mathbf{V}^{k+1}} & \xrightarrow{\beta_k} & {}^{\mathfrak{i}'_{k+1}}_{\mathfrak{i}_{k+1}}E_{\mathbf{V}^{k+1},Q_{k+1}}\\
\downarrow{\scriptstyle {}^{\mathfrak{i}'_k}_{\mathfrak{i}_k}j_{\mathbf{V}^k,Q_k}} & & \downarrow{\scriptstyle j_k} & & \downarrow{\scriptstyle {}^{\mathfrak{i}'_{k+1}}_{\mathfrak{i}_{k+1}}j_{\mathbf{V}^{k+1},Q_{k+1}}}\\
{}_{\mathfrak{i}_k}E_{\mathbf{V}^k,Q_k} & \xleftarrow{\alpha_k} & Z_{\mathbf{V}^k\mathbf{V}^{k+1}} & \xrightarrow{\beta_k} & {}^{\mathfrak{i}_k}E_{\mathbf{V}^{k+1},Q_{k+1}},
\end{array}$$

where

(1) $\hat{Z}_{\mathbf{V}^k\mathbf{V}^{k+1}}$ is the subset of ${}^{\mathfrak{i}'_k}_{\mathfrak{i}_k}E_{\mathbf{V}^k,Q_k}\times{}^{\mathfrak{i}'_{k+1}}_{\mathfrak{i}_{k+1}}E_{\mathbf{V}^{k+1},Q_{k+1}}$ consisting of all (x,y) satisfying the following conditions:
  (a) for any $h\in H$ such that $t(h)\neq\mathbf{i}_k$, $x_h=y_h$;
  (b) the following sequence is exact

$$0\longrightarrow V^{k+1}_{\mathbf{i}_k}\xrightarrow{\bigoplus_{h\in H',s(h)=\mathbf{i}_k}y_h}\bigoplus_{h\in H,t(h)=\mathbf{i}_k}V^k_{s(h)}\xrightarrow{\bigoplus_{h\in H,t(h)=\mathbf{i}_k}x_h}V^k_{\mathbf{i}_k}\longrightarrow 0\ ;$$

(2) $\alpha_k(x,y)=x$ and $\beta_k(x,y)=y$;
(3) $j_k:\hat{Z}_{\mathbf{V}^k\mathbf{V}^{k+1}}\to Z_{\mathbf{V}^k\mathbf{V}^{k+1}}$ is the canonical embedding.

The algebraic group $G_{\mathbf{V}^k\mathbf{V}^{k+1}}$ also acts on $\hat{Z}_{\mathbf{V}^k\mathbf{V}^{k+1}}$ naturally. Then we have

$$\begin{array}{ccccc}
\mathcal{D}_{G_{\mathbf{V}^k}}({}_{\mathfrak{i}_k}E_{\mathbf{V}^k,Q_k}) & \xrightarrow{\alpha_k^*} & \mathcal{D}_{G_{\mathbf{V}^k\mathbf{V}^{k+1}}}(Z_{\mathbf{V}^k\mathbf{V}^{k+1}}) & \xleftarrow{\beta_k^*} & \mathcal{D}_{G_{\mathbf{V}^{k+1}}}({}^{\mathfrak{i}_k}E_{\mathbf{V}^{k+1},Q_{k+1}})\\
\downarrow{\scriptstyle {}^{\mathfrak{i}'_k}_{\mathfrak{i}_k}j^*_{\mathbf{V}^k,Q_k}} & & \downarrow{\scriptstyle j_k^*} & & \downarrow{\scriptstyle {}^{\mathfrak{i}'_{k+1}}_{\mathfrak{i}_{k+1}}j^*_{\mathbf{V}^{k+1},Q_{k+1}}}\\
\mathcal{D}_{G_{\mathbf{V}^k}}({}^{\mathfrak{i}'_k}_{\mathfrak{i}_k}E_{\mathbf{V}^k,Q_k}) & \xrightarrow{\alpha_k^*} & \mathcal{D}_{G_{\mathbf{V}^k\mathbf{V}^{k+1}}}(\hat{Z}_{\mathbf{V}^k\mathbf{V}^{k+1}}) & \xleftarrow{\beta_k^*} & \mathcal{D}_{G_{\mathbf{V}^{k+1}}}({}^{\mathfrak{i}'_{k+1}}_{\mathfrak{i}_{k+1}}E_{\mathbf{V}^{k+1},Q_{k+1}}),
\end{array}$$

where α_k^* and β_k^* are equivalences of categories.

Hence, for any $\mathcal{L}\in\mathcal{D}_{G_{\mathbf{V}^k}}({}^{\mathfrak{i}'_k}_{\mathfrak{i}_k}E_{\mathbf{V}^k,Q_k})$, there exists a unique

$$\mathcal{L}'\in\mathcal{D}_{G_{\mathbf{V}^{k+1}}}({}^{\mathfrak{i}'_{k+1}}_{\mathfrak{i}_{k+1}}E_{\mathbf{V}^{k+1},Q_{k+1}})$$

such that $\alpha_k^*(\mathcal{L}) = \beta_k^*(\mathcal{L}')$. Define

$$\begin{aligned}\hat{\tilde{\omega}}_{\mathbf{i}_k} : \mathcal{D}_{G_{\mathbf{V}^k}}({}^{\mathbf{i}'_k}_{\mathbf{i}_k}E_{\mathbf{V}^k,Q_k}) &\to \mathcal{D}_{G_{\mathbf{V}^{k+1}}}({}^{\mathbf{i}'_{k+1}}_{\mathbf{i}_{k+1}}E_{\mathbf{V}^{k+1},Q_{k+1}}) \\ \mathcal{L} &\mapsto \mathcal{L}'[-s(\mathbf{V}^k)](-\frac{s(\mathbf{V}^k)}{2}),\end{aligned}$$

where $s(\mathbf{V}^k) = \dim \mathrm{GL}(V^k_{\mathbf{i}_k}) - \dim \mathrm{GL}(V^{k+1}_{\mathbf{i}_k})$. Since α_k^* and β_k^* are equivalences of categories, $\hat{\tilde{\omega}}_{\mathbf{i}_k}$ is also an equivalence of categories and we have the following commutative diagram

$$\begin{array}{ccc}\mathcal{D}_{G_{\mathbf{V}^k}}({}_{\mathbf{i}_k}E_{\mathbf{V}^k,Q_k}) & \xrightarrow{\tilde{\omega}_{\mathbf{i}_k}} & \mathcal{D}_{G_{\mathbf{V}^{k+1}}}({}^{\mathbf{i}_k}E_{\mathbf{V}^{k+1},Q_{k+1}}) \\ \downarrow {}^{\mathbf{i}'_k}_{\mathbf{i}_k}j^*_{\mathbf{V}^k,Q_k} & & \downarrow {}^{\mathbf{i}'_{k+1}}_{\mathbf{i}_{k+1}}j^*_{\mathbf{V}^{k+1},Q_{k+1}} \\ \mathcal{D}_{G_{\mathbf{V}^k}}({}^{\mathbf{i}'_k}_{\mathbf{i}_k}E_{\mathbf{V}^k,Q_k}) & \xrightarrow{\hat{\tilde{\omega}}_{\mathbf{i}_k}} & \mathcal{D}_{G_{\mathbf{V}^{k+1}}}({}^{\mathbf{i}'_{k+1}}_{\mathbf{i}_{k+1}}E_{\mathbf{V}^{k+1},Q_{k+1}}).\end{array}$$

In [20], it was proved that $\tilde{\omega}_{\mathbf{i}_k}({}_{\mathbf{i}_k}\mathcal{Q}_{\mathbf{V}^k,Q_k}) = {}^{\mathbf{i}_k}\mathcal{Q}_{\mathbf{V}^{k+1},Q_{k+1}}$. Hence

$$\hat{\tilde{\omega}}_{\mathbf{i}_k}({}^{\mathbf{i}'_k}_{\mathbf{i}_k}\mathcal{Q}_{\mathbf{V}^k,Q_k}) = {}^{\mathbf{i}'_{k+1}}_{\mathbf{i}_{k+1}}\mathcal{Q}_{\mathbf{V}^{k+1},Q_{k+1}}. \tag{6}$$

The proof of Proposition 5.1. Note that $\tilde{\omega}_{\mathrm{i}} = \prod_{k=1}^{l} \hat{\tilde{\omega}}_{\mathbf{i}_k}$. Formula (6) implies

$$\tilde{\omega}_{\mathrm{i}}({}_{\mathrm{i}}\mathcal{Q}_{\mathbf{V}}) = {}^{\mathrm{i}}\mathcal{Q}_{\mathbf{V}'}.$$

□

Hence, we can define $\tilde{\omega}_{\mathrm{i}} : {}_{\mathrm{i}}\mathcal{Q}_{\mathbf{V}} \to {}^{\mathrm{i}}\mathcal{Q}_{\mathbf{V}'}$ and $\tilde{\omega}_{\mathrm{i}} : K({}_{\mathrm{i}}\mathcal{Q}) \to K({}^{\mathrm{i}}\mathcal{Q})$.

5.1.2. Let $\tilde{Q} = (Q, a)$ be a quiver with automorphism, where $Q = (\mathbf{I}, H, s, t)$. Let $i \in I = \mathbf{I}^a$ and $\mathbf{i}$ is a sink for any $\mathbf{i} \in i$.

Consider the following commutative diagram

$$\begin{array}{ccccc}{}_iE_{\mathbf{V}} & \xleftarrow{\alpha} & Z_{\mathbf{V}\mathbf{V}'} & \xrightarrow{\beta} & {}^iE_{\mathbf{V}'} \\ \downarrow a & & \downarrow a & & \downarrow a \\ {}_iE_{\mathbf{V}} & \xleftarrow{\alpha} & Z_{\mathbf{V}\mathbf{V}'} & \xrightarrow{\beta} & {}^iE_{\mathbf{V}'},\end{array}$$

where $a : Z_{\mathbf{V}\mathbf{V}'} \to Z_{\mathbf{V}\mathbf{V}'}$ is defined by $a(x, y) = (a(x), a(y))$ for any $(x, y) \in Z_{\mathbf{V}\mathbf{V}'}$. Hence we have $a^*\tilde{\omega}_i = \tilde{\omega}_i a^*$. So we have a functor $\tilde{\omega}_i : {}_i\tilde{\mathcal{Q}}_{\mathbf{V}} \to {}^i\tilde{\mathcal{Q}}_{\mathbf{V}'}$ and a map $\tilde{\omega}_i : K({}_i\tilde{\mathcal{Q}}_{\mathbf{V}}) \to K({}^i\tilde{\mathcal{Q}}_{\mathbf{V}'})$.

Proposition 5.2. *The map $\tilde{\omega}_i : K({}_i\tilde{\mathcal{Q}}_{\mathbf{V}}) \to K({}^i\tilde{\mathcal{Q}}_{\mathbf{V}'})$ is an isomorphism of algebras.*

Proof. Consider the following commutative diagram

$$\begin{array}{ccccccc}{}_iE_{\mathbf{V}_1} \times {}_iE_{\mathbf{V}_2} & \xleftarrow{p_1} & {}_iE' & \xrightarrow{p_2} & {}_iE'' & \xrightarrow{p_3} & {}_iE_{\mathbf{V}} \\ \uparrow \alpha & & \uparrow \alpha_1 & & \uparrow \alpha_2 & & \uparrow \alpha \\ Z_{\mathbf{V}_1\mathbf{V}'_1} \times Z_{\mathbf{V}_2\mathbf{V}'_2} & \xleftarrow{p_1} & Z' & \xrightarrow{p_2} & Z'' & \xrightarrow{p_3} & Z_{\mathbf{V}\mathbf{V}'} \\ \downarrow \beta & & \downarrow \beta_1 & & \downarrow \beta_2 & & \downarrow \beta \\ {}^iE_{\mathbf{V}'_1} \times {}^iE_{\mathbf{V}'_2} & \xleftarrow{p_1} & {}^iE' & \xrightarrow{p_2} & {}^iE'' & \xrightarrow{p_3} & {}^iE_{\mathbf{V}'},\end{array}$$

where

(1) Z'' is a subset of ${}_iE'' \times {}^iE''$ consisting of the elements $(x, \mathbf{W}; y, \mathbf{W}')$ satisfying the following conditions:
   (a) $(x, y) \in Z_{\mathbf{V}\mathbf{V}'}$;
   (b) for any $h \in H$ such that $t(h) \notin i$, $(x|_{\mathbf{W}})_h = (y|_{\mathbf{W}'})_h$;
   (c) for any $\mathbf{i} \in i$, the following sequence is exact

$$0 \longrightarrow W'_{\mathbf{i}} \xrightarrow{\bigoplus_{h\in H', s(h)=\mathbf{i}}(y|_{\mathbf{W}'})_h} \bigoplus_{h\in H, t(h)=\mathbf{i}} W_{s(h)} \xrightarrow{\bigoplus_{h\in H, t(h)=\mathbf{i}}(x|_{\mathbf{W}})_h} W_{\mathbf{i}} \longrightarrow 0\ ;$$

(2) Z' is a subset of ${}_iE' \times {}^iE'$ consisting of the elements $(x, \mathbf{W}, R_2, R_1; y, \mathbf{W}', R'_2, R'_1)$ satisfying the following conditions:
   (a) $(x, \mathbf{W}; y, \mathbf{W}') \in Z''$;
   (b) $(x_1, y_1) \in Z_{\mathbf{V}_1\mathbf{V}'_1}$ and $(x_2, y_2) \in Z_{\mathbf{V}_2\mathbf{V}'_2}$, where $(x_1, x_2) = p_1(x, \mathbf{W}, R_2, R_1)$ and $(y_1, y_2) = p_1(y, \mathbf{W}', R'_2, R'_1)$;
(3) $p_1(x, \mathbf{W}, R'', R', ; y, \mathbf{W}', R'_2, R'_1) = ((x_1, x_2), (y_1, y_2))$;
(4) $p_2(x, \mathbf{W}, R'', R', ; y, \mathbf{W}', R'_2, R'_1) = (x, \mathbf{W}; y, \mathbf{W}')$;
(5) $p_3(x, \mathbf{W}; y, \mathbf{W}') = (x, y)$.

For any two complexes $\mathcal{L}_1 \in \mathcal{D}_{G_{\mathbf{V}'}}({}_iE_{\mathbf{V}'})$ and $\mathcal{L}_2 \in \mathcal{D}_{G_{\mathbf{V}''}}({}_iE_{\mathbf{V}''})$, $\mathcal{L} = \mathcal{L}_1 * \mathcal{L}_2$ is defined as follows. Let $\mathcal{L}_3 = \mathcal{L}_1 \otimes \mathcal{L}_2$ and $\mathcal{L}_4 = p_1^*\mathcal{L}_3$. There exists a complex $\mathcal{L}_5$ on ${}_iE'$ such that $p_2^*(\mathcal{L}_5) = \mathcal{L}_4$. $\mathcal{L}$ is defined as $(p_3)_!\mathcal{L}_5$.

Let $\mathcal{L}'_1 \in \mathcal{D}_{G_{\mathbf{V}'}}({}_iE_{\mathbf{V}'})$ be the unique complex such that $\alpha^*\mathcal{L}_1 = \beta^*\mathcal{L}'_1$ and $\mathcal{L}'_2 \in \mathcal{D}_{G_{\mathbf{V}''}}({}_iE_{\mathbf{V}''})$ be the unique complex such that $\alpha^*\mathcal{L}_2 = \beta^*\mathcal{L}'_2$. $\mathcal{L}' = \mathcal{L}'_1 * \mathcal{L}'_2$ is defined as follows. Let $\mathcal{L}'_3 = \mathcal{L}'_1 \otimes \mathcal{L}'_2$ and $\mathcal{L}'_4 = p_1^*\mathcal{L}'_3$. There exists a complex $\mathcal{L}'_5$ on ${}_iE'$ such that $p_2^*(\mathcal{L}'_5) = \mathcal{L}'_4$. $\mathcal{L}'$ is defined as $(p_3)_!\mathcal{L}'_5$.

Since $\alpha_1^*\mathcal{L}_4 = \alpha_1^*p_1^*\mathcal{L}_3 = p_1^*\alpha^*\mathcal{L}_3$ and $\beta_1^*\mathcal{L}'_4 = \beta_1^*p_1^*\mathcal{L}'_3 = p_1^*\beta^*\mathcal{L}'_3$, we have $\alpha_1^*\mathcal{L}_4 = \beta_1^*\mathcal{L}'_4$. Since p_2^* are equivalences of categories, we have $\alpha_2^*\mathcal{L}_5 = \beta_2^*\mathcal{L}'_5$. Since

$$\begin{array}{ccc} {}_iE'' & \xrightarrow{p_3} & {}_iE_{\mathbf{V}} \\ \uparrow{\scriptstyle \alpha_2} & & \uparrow{\scriptstyle \alpha} \\ Z'' & \xrightarrow{p_3} & Z_{\mathbf{V}\mathbf{V}'} \end{array}$$

and

$$\begin{array}{ccc} Z'' & \xrightarrow{p_3} & Z_{\mathbf{V}\mathbf{V}'} \\ \downarrow{\scriptstyle \beta_2} & & \downarrow{\scriptstyle \beta} \\ {}^iE'' & \xrightarrow{p_3} & {}^iE_{\mathbf{V}'} \end{array}$$

are fiber products, $\alpha^*\mathcal{L} = \alpha^*(p_3)_!\mathcal{L}_5 = (p_3)_!\alpha_2^*\mathcal{L}_5 = (p_3)_!\beta_2^*\mathcal{L}'_5 = \beta^*(p_3)_!\mathcal{L}'_5 = \beta^*\mathcal{L}'$.

Hence we have $\tilde{\omega}_i(\mathcal{L}_1 \circledast \mathcal{L}_2) = \tilde{\omega}_i\mathcal{L}_1 \circledast \tilde{\omega}_i\mathcal{L}_2$ and the map $\tilde{\omega}_i : K({}_i\tilde{\mathcal{Q}}_{\mathbf{V}}) \to K({}^i\tilde{\mathcal{Q}}_{\mathbf{V}'})$ is an isomorphism of algebras.

□

Proposition 5.3. *For all $i \neq j \in I$ and $-a_{ij} \geq m \in \mathbb{N}$, $\tilde{\omega}_i({}_i\lambda_{\mathcal{A}}^{-1}(f(i,j,m))) = {}^i\lambda_{\mathcal{A}}^{-1}(f'(i,j,-a_{ij}-m))$.*

Proposition 5.3 will be proved in Section 5.2.

Theorem 5.4. *It holds that* $\tilde{\omega}_i({}_i\mathbf{k}) = {}^i\mathbf{k}$ *and we have the following commutative diagram*

$$\begin{array}{ccc} {}_i\mathbf{k} & \xrightarrow{\tilde{\omega}_i} & {}^i\mathbf{k} \\ \downarrow {}_i\lambda_{\mathcal{A}} & & \downarrow {}^i\lambda_{\mathcal{A}} \\ {}_i\mathbf{f}_{\mathcal{A}} & \xrightarrow{T_i} & {}^i\mathbf{f}_{\mathcal{A}}. \end{array}$$

Proof. Since ${}_i\mathbf{k}$ and ${}^i\mathbf{k}$ are generated by ${}_i\lambda_{\mathcal{A}}^{-1}(f(i,j,m))$ and ${}^i\lambda_{\mathcal{A}}^{-1}(f'(i,j,m'))$ respectively, Proposition 5.3 implies this theorem. □

5.2. **The proof of Proposition 5.3.**

5.2.1. Let $\tilde{Q} = (Q, a)$ be a quiver with automorphism, where $Q = (\mathbf{I}, H, s, t)$. Fix $i, j \in I$ such that there are no arrows from $\mathbf{i}$ to $\mathbf{j}$ for any $\mathbf{i} \in i$ and $\mathbf{j} \in j$. Let $N = |\{\mathbf{j} \to \mathbf{i} \mid \mathbf{j} \in j, \mathbf{i} \in i\}|$ and m be a non-negative integer such that $m \leq N$. Let $\gamma_i = \sum_{\mathbf{i}\in i} \mathbf{i}$, $\gamma_j = \sum_{\mathbf{j}\in j} \mathbf{j}$ and $\nu^{(m)} = m\gamma_i + \gamma_j \in \mathbb{N}\mathbf{I}$. Fix an object $\mathbf{V}^{(m)} \in \tilde{\mathcal{C}}$ such that $\underline{\dim}\mathbf{V}^{(m)} = \nu^{(m)}$.

Denote by $\mathbf{1}_{{}_iE_{\mathbf{V}^{(m)}}} \in \mathcal{D}_{G_{\mathbf{V}^{(m)}}}({}_iE_{\mathbf{V}^{(m)}})$ the constant sheaf on ${}_iE_{\mathbf{V}^{(m)}}$. Define

$$\mathcal{E}^{(m)} = j_{\mathbf{V}^{(m)}!}(v^{-mN}\mathbf{1}_{{}_iE_{\mathbf{V}^{(m)}}}) \in \mathcal{D}_{G_{\mathbf{V}^{(m)}}}(E_{\mathbf{V}^{(m)}}).$$

For convenience, the complex $j_{\mathbf{V}^{(m)}!}(\mathbf{1}_{{}_iE_{\mathbf{V}^{(m)}}}) \in \mathcal{D}_{G_{\mathbf{V}^{(m)}}}(E_{\mathbf{V}^{(m)}})$ is also denoted by $\mathbf{1}_{{}_iE_{\mathbf{V}^{(m)}}}$. Note that there exists a natural isomorphism ${}_i\psi : a^*(\mathcal{E}^{(m)}) \cong \mathcal{E}^{(m)}$.

For each $m \geq p \in \mathbb{N}$, consider the following variety

$$\begin{aligned} \tilde{S}_p^{(m)} &= \{(x, W) \mid x \in E_{\mathbf{V}^{(m)}},\, W = \oplus_{\mathbf{i}\in i} W_{\mathbf{i}} \subset \oplus_{\mathbf{i}\in i} V_{\mathbf{i}}, \\ &\quad a(W) = W,\, \dim(W_{\mathbf{i}}) = p,\, \mathrm{Im} \bigoplus_{h\in H, t(h)=\mathbf{i}} x_h \subset W_{\mathbf{i}}\}. \end{aligned}$$

Let $\pi_p : \tilde{S}_p^{(m)} \to E_{\mathbf{V}^{(m)}}$ be the projection taking (x, W) to x and $S_p^{(m)} = \mathrm{Im}\pi_p$.

By the definitions of $S_p^{(m)}$, we have

$$E_{\mathbf{V}^{(m)}} = S_m^{(m)} \supset S_{m-1}^{(m)} \supset S_{m-2}^{(m)} \supset \cdots \supset S_0^{(m)}.$$

For each $1 \leq p \leq m$, let

$$\mathcal{N}_p^{(m)} = S_p^{(m)} \backslash S_{p-1}^{(m)}.$$

Denote by $i_p^{(m)} : S_{p-1}^{(m)} \to S_p^{(m)}$ the close embedding and $j_p^{(m)} : \mathcal{N}_p^{(m)} \to S_p^{(m)}$ the open embedding.

Define

$$I_p^{(m)} = (\pi_p)_!(\mathbf{1}_{\tilde{S}_p^{(m)}})[\dim \tilde{S}_p^{(m)}].$$

By the natural isomorphism $a^*(\mathbf{1}_{\tilde{S}_p^{(m)}}) \cong \mathbf{1}_{\tilde{S}_p^{(m)}}$ induced by a, we have an isomorphism $\phi_0 : a^*(I_p^{(m)}) \cong I_p^{(m)}$.

The following theorem is the main result in this section.

Theorem 5.5. *For* $\mathcal{E}^{(m)}$, *there exists* $s_m \in \mathbb{N}$. *For each* $s_m \geq p \in \mathbb{N}$, *there exists* $\mathcal{E}_p^{(m)} \in \mathcal{D}_{G_{\mathbf{V}^{(m)}}}(E_{\mathbf{V}^{(m)}})$ *such that*

(1) $\mathcal{E}_{s_m}^{(m)} = \mathcal{E}^{(m)}$ *and* $\mathcal{E}_0^{(m)}$ *is the direct sum of some semisimple complexes of the form* $I_{p'}^{(m)}[l]$;

(2) *for each $p \geq 1$, there exists a distinguished triangle*

$$\mathcal{E}_p^{(m)} \longrightarrow \mathcal{G}_p^{(m)} \longrightarrow \mathcal{E}_{p-1}^{(m)} \longrightarrow ,$$

where $\mathcal{G}_p^{(m)}$ is the direct sum of some semisimple complexes of the form $I_{p'}^{(m)}[l]$.

The proof of Theorem 5.5 is as same as that of Theorem 5.3 in [20].

Corollary 5.6. *For each $N \geq m \in \mathbb{N}$, we have the following formula*

$$\lambda_{\mathcal{A}}([\mathcal{E}^{(m)}, {}_i\psi]) = \sum_{p=0}^{m} (-1)^p {v_i}^{-p(1+N-m)} \theta_i^{(p)} \theta_j \theta_i^{(m-p)} = f(i,j;m).$$

Proof. Since $\lambda_{\mathcal{A}}([I_p^{(m)}, \phi_0]) = \theta_i^{(m-p)} \theta_j \theta_i^{(p)}$ for each $m \geq p \in \mathbb{N}$, we have the desired result.

□

5.2.2. Let m be a non-negative integer such that $m \leq N$ and $m' = N - m$. Let $\nu = m\gamma_i + \gamma_j \in \mathbb{N}\mathbf{I}$ and $\nu' = s_i\nu = m'\gamma_i + \gamma_j \in \mathbb{N}\mathbf{I}$. Fix two objects $\mathbf{V} \in \tilde{\mathcal{C}}_\nu$ and $\mathbf{V}' \in \tilde{\mathcal{C}}_{\nu'}$.

Denote by $\mathbf{1}_{{}_iE_{\mathbf{V}}} \in \mathcal{D}_{G_{\mathbf{V}}}({}_iE_{\mathbf{V}})$ the constant sheaf on ${}_iE_{\mathbf{V}}$ and $\mathbf{1}_{{}^iE_{\mathbf{V}'}} \in \mathcal{D}_{G_{\mathbf{V}'}}({}^iE_{\mathbf{V}'})$ the constant sheaf on ${}^iE_{\mathbf{V}'}$. Note that there exist natural isomorphisms ${}_i\psi : a^*(\mathbf{1}_{{}_iE_{\mathbf{V}}}) \cong \mathbf{1}_{{}_iE_{\mathbf{V}}}$ and ${}^i\psi : a^*(\mathbf{1}_{{}^iE_{\mathbf{V}'}}) \cong \mathbf{1}_{{}^iE_{\mathbf{V}'}}$.

Proposition 5.7. *For any $N \geq m \in \mathbb{N}$,*

$$\tilde{\omega}_i([{v_i}^{-mN}\mathbf{1}_{{}_iE_{\mathbf{V}}}, {}_i\psi]) = [{v_i}^{-m'N}\mathbf{1}_{{}^iE_{\mathbf{V}'}}, {}^i\psi].$$

Proof. By the definitions of α and β in the diagram (5),

$$\alpha^*(\mathbf{1}_{{}_iE_{\mathbf{V}}}) = \mathbf{1}_{Z_{\mathbf{V}\mathbf{V}'}} = \beta^*(\mathbf{1}_{{}^iE_{\mathbf{V}'}}).$$

Hence

$$\tilde{\omega}_i(\mathbf{1}_{{}_iE_{\mathbf{V}}}) = {v_i}^{(m-m')N}\mathbf{1}_{{}^iE_{\mathbf{V}'}}.$$

That is

$$\tilde{\omega}_i({v_i}^{-mN}\mathbf{1}_{{}_iE_{\mathbf{V}}}) = {v_i}^{-m'N}\mathbf{1}_{{}^iE_{\mathbf{V}'}}.$$

□

Corollary 5.6 implies

$${}_i\lambda_{\mathcal{A}}[{v_i}^{-mN}\mathbf{1}_{{}_iE_{\mathbf{V}}}, {}_i\psi] = f(i,j;m).$$

Similarly, we have

$${}^i\lambda_{\mathcal{A}}[{v_i}^{-m'N}\mathbf{1}_{{}^iE_{\mathbf{V}'}}, {}^i\psi] = f'(i,j;m').$$

Hence Proposition 5.7 implies Proposition 5.3.

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