A PARAMETERIZATION OF THE CANONICAL BASES OF AFFINE MODIFIED QUANTIZED ENVELOPING ALGEBRAS

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ABSTRACT. For a symmetrizable Kac-Moody Lie algebra $\mathfrak{g}$, Lusztig introduced the corresponding modified quantized enveloping algebra $\dot{\mathbf{U}}$ and its canonical basis $\dot{\mathbf{B}}$ in [13]. In this paper, in case $\mathfrak{g}$ is a symmetric Kac-Moody Lie algebra of finite or affine type, we define a set $\tilde{\mathcal{M}}$ which depends only on the root category $\mathcal{R}$ and prove that there is a bijection between $\tilde{\mathcal{M}}$ and $\dot{\mathbf{B}}$, where $\mathcal{R}$ is the T^2-orbit category of the bounded derived category of corresponding Dynkin or tame quiver. Our method is based on a result of Lin, Xiao and Zhang in [10], which gives a PBW-type basis of $\mathbf{U}^+$.

1. INTRODUCTION

Let $\mathbf{U}^+$ be the positive part of the quantized enveloping algebra $\mathbf{U}$ associated with a Cartan datum. In the case of finite type, Lusztig gave two approachs to construct the canonical basis $\mathbf{B}$ of $\mathbf{U}^+$ ([11]). The first one is an elementary algebraic construction. By using Ringel-Hall algebra realization of $\mathbf{U}^+$, the isomorphism classes of representations of the corresponding Dynkin quiver form a PBW-type basis of $\mathbf{U}^+$ and there is an order on this basis. Under this order, the transition matrix between this basis and a monomial basis is a unipotent lower triangular matrix. By a standard linear algebra method one can get a bar-invariant basis, which is the canonical basis $\mathbf{B}$. The second one is a geometric construction. Lusztig constructed the canonical basis $\mathbf{B}$ by using perverse sheaves and intersection cohomology. The geometric construction of $\mathbf{B}$ was generalized to the cases of all types in [12]. In the case of affine type, Lin, Xiao and Zhang in [10] provided a process to construct a PBW-type basis of $\mathbf{U}^+$ and the canonical basis $\mathbf{B}$ by using Ringel-Hall algebra approach ([10]).

Let $\dot{\mathbf{U}}$ be the modified quantized enveloping algebra obtained from $\mathbf{U}$ by modifying the Cartan part $\mathbf{U}^0$ to $\bigoplus_{\lambda\in P}\mathbb{Q}(v)\mathbf{1}_\lambda$, where P is the weight lattice. $\dot{\mathbf{U}}$ can be considered as the limit of tensor products of highest weight modules and lowest weight modules. Lusztig introduced the canonical bases of the tensor products and then the canonical basis $\dot{\mathbf{B}}$ of $\dot{\mathbf{U}}$ ([13, 14]). Kashiwara also studied the algebra $\dot{\mathbf{U}}$ and its canonical basis $\dot{\mathbf{B}}$ ([9]).

Happel studied the bounded derived category $D^b(\Lambda)$ of a finite dimensional algebra Λ in [6, 7]. In case Λ is hereditary and representation-finite, he proved that there is a bijection between the isomorphism classes of indecomposable objects in

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$\mathcal{R} = D^b(\Lambda)/T^2$ and all the roots of the corresponding Lie algebra, where T is the translation functor in the triangulated category $D^b(\Lambda)$. Hence $\mathcal{R}$ is called a root category. It was proved in [15] that $\mathcal{R}$ is still a triangulated category. In [15, 16], Peng and Xiao gave a realization of all symmetrizable Kac-Moody Lie algebras via the root categories of finite dimensional hereditary algebras.

Note that the construction of canonical basis $\dot{\mathbf{B}}$ is abstract and depends on the construction of canonical basis $\mathbf{B}$ of $\mathbf{U}^+$. Inspired by the method of Peng and Xiao, we want to study the relations between the canonical basis $\dot{\mathbf{B}}$ and the corresponding root category $\mathcal{R}$. In this paper, first we associate a set $\tilde{\mathcal{M}}$ to $\mathcal{R}$. In [10], Lin, Xiao, and Zhang associated a set $\mathcal{M}$ to a hereditary category and the definition of $\tilde{\mathcal{M}}$ is based on that of $\mathcal{M}$. However, $\tilde{\mathcal{M}}$ is independent of the embedding of the hereditary category to $\mathcal{R}$. Fixing an embedding of the hereditary category to $\mathcal{R}$, we can get a bijection between $\tilde{\mathcal{M}}$ and the canonical basis $\dot{\mathbf{B}}_\lambda$ of $\dot{\mathbf{U}}\mathbf{1}_\lambda$ for every $\lambda \in P$. Hence we say that the set $\tilde{\mathcal{M}}$ provides a parameterization of the canonical basis $\dot{\mathbf{B}}$.

Since [21], it has been an open problem: how to realize the whole quantized enveloping algebras by using Hall algebras from derived categories or root categories. A lot of efforts have been paid on the progress ([3, 8, 20, 22]) and the most recent progress is given by Bridgeland in [1]. We hope that the main result in the present paper can provide a strong evidence for the connection between canonical bases and root categories.

In Section 2, we first give some notations of quantized enveloping algebras and modified quantized enveloping algebras. Then we review the definitions of Ringel-Hall algebras and root categories. In Section 3, we study the case of finite type, which is simpler and can reflect the idea clearly. In Section 4, we study the case of affine type. We first review the construction of the PBW-type basis of $\mathbf{U}^+$ in [10]. Then we define a set $\tilde{\mathcal{M}}$ depending on the corresponding root category $\mathcal{R}$ and a PBW-type basis of $\dot{\mathbf{U}}\mathbf{1}_\lambda$ with $\tilde{\mathcal{M}}$ as an index. By a standard linear algebra method, we get a bar-invariant basis and prove that each element in it is the leading term of an element in $\dot{\mathbf{B}}_\lambda$. At last, we prove that there is a bijection between $\tilde{\mathcal{M}}$ and $\dot{\mathbf{B}}_\lambda$.

2. Preliminaries

2.1. **Quantized enveloping algebras.** Let $\mathbb{Q}$ be the field of rational numbers and $\mathbb{Z}$ be the ring of integers. Let I be a finite index set with $|I| = n$ and $A = (a_{ij})_{i,j\in I}$ be a generalized Cartan matrix. Denote by $r(A)$ the rank of A. Let $P^\vee$ be a free abelian group of rank $2n - r(A)$ with a $\mathbb{Z}$-basis $\{h_i \mid i \in I\} \cup \{d_s \mid s = 1, \dots, n - r(A)\}$ and $\mathfrak{h} = \mathbb{Q} \otimes_{\mathbb{Z}} P^\vee$ be the $\mathbb{Q}$-linear space spanned by $P^\vee$. We call $P^\vee$ the dual weight lattice and $\mathfrak{h}$ the Cartan subalgebra. We also define the weight lattice to be $P = \{\lambda \in \mathfrak{h}^* \mid \lambda(P^\vee) \subset \mathbb{Z}\}$.

Set $\Pi^\vee = \{h_i \mid i \in I\}$ and choose a linearly independent subset $\Pi = \{\alpha_i \mid i \in I\} \subset \mathfrak{h}^*$ satisfying $\alpha_j(h_i) = a_{ij}$ and $\alpha_j(d_s) = 0$ or 1 for all $i, j \in I$, $s = 1, \dots, n - \mathrm{rank} A$. The elements of Π are called simple roots, and the elements of $\Pi^\vee$ are called simple coroots. The quintuple $(A, \Pi, \Pi^\vee, P, P^\vee)$ is called a Cartan datum associated with the generalized Cartan matrix A.

We shall review the definition of quantized enveloping algebras ([14]). From now on, assume that the generalized Cartan matrix $A = (a_{ij})_{i,j\in I}$ is symmetric.

Fix an indeterminate v. For any $n \in \mathbb{Z}$, set

$$[n]_v = \frac{v^n - v^{-n}}{v - v^{-1}}.$$

Set $[0]_v! = 1$ and $[n]_v! = [n]_v[n-1]_v \cdots [1]_v$ for any $n \in \mathbb{Z}_{>0}$. For any two nonnegative integers $m \geq n$, the analogue of binomial coefficient is given by

$$\begin{bmatrix} m \\ n \end{bmatrix}_v = \frac{[m]_v!}{[n]_v![m-n]_v!}.$$

Note that $[n]_v$ and $\begin{bmatrix} m \\ n \end{bmatrix}_v$ are elements of the field $\mathbb{Q}(v)$.

The quantized enveloping algebra $\mathbf{U}$ associated with a Cartan datum $(A, \Pi, \Pi^\vee, P, P^\vee)$ is an associative algebra over $\mathbb{Q}(v)$ with $\mathbf{1}$ generated by the elements E_i, $F_i (i \in I)$ and $K_\mu (\mu \in P^\vee)$ subject to the following relations:

$$K_0 = \mathbf{1}, K_\mu K_{\mu'} = K_{\mu+\mu'} \text{ for all } \mu, \mu' \in P^\vee;$$
$$K_\mu E_i K_{-\mu} = v^{\alpha_i(\mu)} E_i \text{ for all } i \in I,\ \mu \in P^\vee;$$
$$K_\mu F_i K_{-\mu} = v^{-\alpha_i(\mu)} F_i \text{ for all } i \in I,\ \mu \in P^\vee;$$
$$E_i F_j - F_j E_i = \delta_{ij} \frac{K_i - K_{-i}}{v - v^{-1}} \text{ for all } i, j \in I;$$

for all $i \neq j$, setting $b = 1 - a_{ij}$,

$$\sum_{k=0}^{b} (-1)^k E_i^{(k)} E_j E_i^{(b-k)} = 0; \qquad \sum_{k=0}^{b} (-1)^k F_i^{(k)} F_j F_i^{(b-k)} = 0.$$

Here, $K_i = K_{h_i}$ and $E_i^{(n)} = E_i^n/[n]_v!$, $F_i^{(n)} = F_i^n/[n]_v!$.

Let $\mathbf{U}^+$ (resp. $\mathbf{U}^-$) be the subalgebra of $\mathbf{U}$ generated by the elements E_i (resp. F_i) for all $i \in I$, and $\mathbf{U}^0$ be the subalgebra of $\mathbf{U}$ generated by K_μ for all $\mu \in P^\vee$. The quantized enveloping algebra $\mathbf{U}$ has the following triangular decomposition

$$\mathbf{U} \cong \mathbf{U}^- \otimes \mathbf{U}^0 \otimes \mathbf{U}^+.$$

Denote by $\bar{()}$ the unique $\mathbb{Q}$-algebra automorphism of $\mathbf{U}$ given by

$$\bar{E}_i = E_i, \bar{F}_i = F_i, \bar{K}_\mu = K_{-\mu} \text{ for all } i \in I, \mu \in P^\vee,$$
$$\overline{fx} = \bar{f}\bar{x}, \text{ for all } f \in \mathbb{Q}(v), x \in \mathbf{U},$$

where $\bar{f}(v) = f(v^{-1})$.

Let $\mathbf{f}$ be the associative algebra defined by Lusztig in [14]. $\mathbf{f}$ is generated by $\theta_i (i \in I)$ subject to the following relations:

$$\sum_{k=0}^{b} (-1)^k \theta_i^{(k)} \theta_j \theta_i^{(b-k)} = 0 \text{ for all } i \neq j,$$

where $b = 1 - a_{ij}$ and $\theta_i^{(n)} = \theta_i^n/[n]_v!$.

There are two well-defined $\mathbb{Q}(v)$-algebra homomorphisms ${}^+ : \mathbf{f} \to \mathbf{U}$ and ${}^- : \mathbf{f} \to \mathbf{U}$ satisfying $E_i = \theta_i^+$ and $F_i = \theta_i^-$ for all $i \in I$. The images of ${}^+$ and ${}^-$ are $\mathbf{U}^+$ and $\mathbf{U}^-$ respectively.

Denote by $\bar{()}$ the unique $\mathbb{Q}$-algebra automorphism of $\mathbf{f}$ given by

$$\bar{\theta}_i = \theta_i \qquad \text{for all } i \in I,$$
$$\overline{fx} = \bar{f}\bar{x} \qquad \text{for all } f \in \mathbb{Q}(v),\ x \in \mathbf{f}.$$

Note that, for all $x \in \mathbf{f}$, $\overline{x^\pm} = \bar{x}^\pm$.

Let $\mathcal{A} = \mathbb{Q}[v, v^{-1}]$ and $\mathcal{Z} = \mathbb{Z}[v, v^{-1}]$. Denote by $\mathbf{U}_\mathcal{Z}^\pm$ the $\mathcal{Z}$-subalgebras of $\mathbf{U}^\pm$ generated by $E_i^{(s)}$ and $F_i^{(s)}$ for all $i \in I$ and $s \in \mathbb{N}$ respectively. Also, denote by $\mathbf{U}_\mathcal{Z}$ the $\mathcal{Z}$-subalgebra of $\mathbf{U}$ generated by $E_i^{(s)}$, $F_i^{(s)}$ and K_μ for all $i \in I$, $s \in \mathbb{N}$

and $\mu \in P^\vee$. Let $\mathbf{U}_\mathcal{A}^\pm = \mathbf{U}_\mathcal{Z}^\pm \otimes_\mathcal{Z} \mathcal{A}$ and $\mathbf{U}_\mathcal{A} = \mathbf{U}_\mathcal{Z} \otimes_\mathcal{Z} \mathcal{A}$. Similarly, let $\mathbf{f}_\mathcal{Z}$ be the $\mathcal{Z}$-subalgebra of $\mathbf{f}$ generated by $\theta_i^{(s)}$ for all $i \in I$ and $s \in \mathbb{N}$. At last, let $\mathbf{f}_\mathcal{A} = \mathbf{f}_\mathcal{Z} \otimes_\mathcal{Z} \mathcal{A}$.

In [11, 12, 14], Lusztig defined the canonical basis $\mathbf{B}$ of $\mathbf{f}$.

2.2. **Modified quantized enveloping algebras.** Let us review the definition of the modified form $\dot{\mathbf{U}}$ of $\mathbf{U}$ ([13, 14]).

For any $\lambda', \lambda'' \in P$, set

$$_{\lambda'}\mathbf{U}_{\lambda''} = \mathbf{U}/\left(\sum_{\mu \in P^\vee}(K_\mu - q^{\lambda'(\mu)})\mathbf{U} + \sum_{\mu \in P^\vee}\mathbf{U}(K_\mu - q^{\lambda''(\mu)})\right).$$

Let $\pi_{\lambda',\lambda''} : \mathbf{U} \to {}_{\lambda'}\mathbf{U}_{\lambda''}$ be the canonical projection and

$$\dot{\mathbf{U}} = \bigoplus_{\lambda',\lambda'' \in P} {}_{\lambda'}\mathbf{U}_{\lambda''}.$$

Consider the weight space decomposition $\mathbf{U} = \bigoplus_{\beta \in \mathbb{Z}I} \mathbf{U}(\beta)$, where

$$\mathbf{U}(\beta) = \{x \in \mathbf{U} \mid K_\mu x K_\mu^{-1} = v^{\beta(\mu)}x \text{ for all } \mu \in P^\vee\}.$$

Here, the set I is viewed as a subset of P and i is identified with α_i for each $i \in I$. The images of summands $\mathbf{U}(\beta)$ under $\pi_{\lambda',\lambda''}$ form the weight space decomposition

$$_{\lambda'}\mathbf{U}_{\lambda''} = \bigoplus_{\beta \in \mathbb{Z}I} {}_{\lambda'}\mathbf{U}_{\lambda''}(\beta).$$

Note that $_{\lambda'}\mathbf{U}_{\lambda''}(\beta) = 0$ unless $\lambda' - \lambda'' = \beta$.

There is a natural associative $\mathbb{Q}(v)$-algebra structure on $\dot{\mathbf{U}}$ inherited from that of $\mathbf{U}$. It is defined as follow: for any $\lambda_1', \lambda_1'', \lambda_2', \lambda_2'' \in P$, $\beta_1, \beta_2 \in \mathbb{Z}I$ such that $\lambda_1' - \lambda_1'' = \beta_1, \lambda_2' - \lambda_2'' = \beta_2$ and any $x \in \mathbf{U}(\beta_1), y \in \mathbf{U}(\beta_2)$,

$$\pi_{\lambda_1',\lambda_1''}(x)\pi_{\lambda_2',\lambda_2''}(y) = \begin{cases} \pi_{\lambda_1',\lambda_2''}(xy) & \text{if } \lambda_1'' = \lambda_2' \\ 0 & \text{otherwise} \end{cases}.$$

For any $\lambda \in P$, let $\mathbf{1}_\lambda = \pi_{\lambda,\lambda}(\mathbf{1})$, where $\mathbf{1}$ is the unit element of $\mathbf{U}$. They satisfy the following relations:

$$\mathbf{1}_\lambda \mathbf{1}_{\lambda'} = \delta_{\lambda,\lambda'}\mathbf{1}_\lambda.$$

In general, there is no unit element in the algebra $\dot{\mathbf{U}}$. However the family $(\mathbf{1}_\lambda)_{\lambda \in P}$ can be regarded locally as the unit element in $\dot{\mathbf{U}}$.

Note that $_{\lambda'}\mathbf{U}_{\lambda''} = \mathbf{1}_{\lambda'}\dot{\mathbf{U}}\mathbf{1}_{\lambda''}$. Define $\dot{\mathbf{U}}\mathbf{1}_\lambda = \bigoplus_{\lambda' \in P}\mathbf{1}_{\lambda'}\dot{\mathbf{U}}\mathbf{1}_\lambda$. Then $\dot{\mathbf{U}} = \bigoplus_{\lambda \in P}\dot{\mathbf{U}}\mathbf{1}_\lambda$.

The $\mathbb{Q}$-algebra automorphism $\bar{()} : \mathbf{U} \to \mathbf{U}$ induces a linear isomorphism $_{\lambda'}\mathbf{U}_{\lambda''} \to {}_{\lambda'}\mathbf{U}_{\lambda''}$ for any $\lambda', \lambda'' \in P$. Taking direct sums, we obtain an algebra automorphism $\bar{()} : \dot{\mathbf{U}} \to \dot{\mathbf{U}}$. Note that $\overline{\mathbf{1}_\lambda} = \mathbf{1}_\lambda$ for any $\lambda \in P$.

The elements $b^+b'^-\mathbf{1}_\lambda$ for all $b, b' \in \mathbf{B}$ form a basis of the $\mathbb{Q}(v)$-vector space $\dot{\mathbf{U}}\mathbf{1}_\lambda$ ([14], 23.2.1). Denote by $\dot{\mathbf{U}}_\mathcal{Z}$ the subalgebra of $\dot{\mathbf{U}}$ generated by the elements $E_i^{(n)}\mathbf{1}_\lambda$ and $F_i^{(n)}\mathbf{1}_\lambda$ over $\mathcal{Z}$ for all $i \in I$, $n \in \mathbb{N}$ and $\lambda \in P$. The set $\{b^+b'^-\mathbf{1}_\lambda \mid b, b' \in \mathbf{B}, \lambda \in P\}$ is a $\mathcal{Z}$-basis of $\dot{\mathbf{U}}_\mathcal{Z}$.

As the notations in [14], the canonical basis of $\dot{\mathbf{U}}$ is denoted by

$$\dot{\mathbf{B}} = \{b\diamondsuit_\lambda b' \mid b, b' \in \mathbf{B}, \lambda \in P\}.$$

Note that $\{b\diamondsuit_\lambda b' \mid b, b' \in \mathbf{B}\}$ is also a $\mathcal{Z}$-basis of $\dot{\mathbf{U}}_{\mathcal{Z}}\mathbf{1}_\lambda$. According to the proof of Theorem 25.2.1 in [14]

$$b\diamondsuit_\lambda b' \equiv b^+ b'^- \mathbf{1}_\lambda \mod P(\mathrm{tr}|b|-1, \mathrm{tr}|b'|-1). \tag{1}$$

Here $P(\mathrm{tr}|b|-1, \mathrm{tr}|b'|-1)$ is the $\mathbb{Q}(v)$-submodule of $\dot{\mathbf{U}}$ spanned by the set

$$\{b_1^+ b_2^- 1_\lambda \mid b_1, b_2 \in \mathbf{B} \text{ such that } \mathrm{tr}|b_1| \le \mathrm{tr}|b|-1,\ \mathrm{tr}|b_2| \le \mathrm{tr}|b'|-1 \text{ and } |b_1|-|b_2| = |b|-|b'|\},$$

where $|b|$ is the weight of b and $\mathrm{tr}\mu = \sum a_i$ if $\mu = \sum a_i \alpha_i$.

2.3. **Ringel-Hall algebras.** In this subsection, we shall review the definition of Ringel-Hall algebras ([5, 10, 18]).

A quiver $Q = (I, H, s, t)$ consists of a vertex set I, an arrow set H, and two maps $s, t : H \to I$ such that an arrow $\rho \in H$ starts at $s(\rho)$ and terminates at $t(\rho)$.

Let k be a field and $\Lambda = kQ$ be the path algebra of Q over k. Denote by mod-Λ the category of finite dimensional left Λ-modules and rep-Q the category of finite dimensional representations of Q over k. It is well-known that mod-Λ is equivalent to rep-Q. We shall identify Λ-modules with representations of Q under this equivalence.

Let $\mathcal{P}$ be the set of isomorphism classes of finite dimensional nilpotent Λ-modules and ind$(\mathcal{P})$ be the set of isomorphism classes of indecomposable finite dimensional nilpotent Λ-modules. For any $\alpha \in \mathcal{P}$, fix a Λ-module $M(\alpha)$ in the isomorphism class α.

The set of isomorphism classes of nilpotent simple Λ-modules is indexed by the set I and the Grothendieck group $G(\Lambda)$ of mod-Λ is the free abelian group $\mathbb{Z}I$. For any Λ-module M, the dimension vector $\underline{\dim}M$ of M is an element in $G(\Lambda) = \mathbb{Z}I$.

The Euler form $\langle -, - \rangle$ on $G(\Lambda) = \mathbb{Z}I$ is defined by

$$\langle \alpha, \beta \rangle = \sum_{i\in I} a_i b_i - \sum_{\rho \in H} a_{s(\rho)} b_{t(\rho)}$$

where $\alpha = \sum_{i\in I} a_i i, \beta = \sum_{i \in I} b_i i \in \mathbb{Z}I$. For any Λ-modules M and N, one has

$$\langle \underline{\dim}M, \underline{\dim}N \rangle = \dim_k \mathrm{Hom}_\Lambda(M,N) - \dim_k \mathrm{Ext}_\Lambda(M,N).$$

The symmetric Euler form is defined by $(\alpha, \beta) = \langle \alpha, \beta\rangle + \langle \beta, \alpha \rangle$ for all $\alpha, \beta \in \mathbb{Z}I$. This gives rise to a symmetric generalized Cartan matrix $A = (a_{ij})_{i,j\in I}$ where $a_{ij} = (i,j)$. The generalized Cartan matrix A depends only on the underlying graph of quiver Q.

From now on, let k be a finite field with q elements. Given three modules L, M and N in mod-Λ, let g^L_{MN} be the number of Λ-submodules W of L such that $W \simeq N$ and $L/W \simeq M$ in mod-Λ. Let $v = \sqrt{q} \in \mathbb{C}$. By definition, the Ringel-Hall algebra $\mathcal{H}_q(\Lambda)$ of Λ is the $\mathbb{Q}(v)$-vector space with basis $\{u_{[M]} \mid [M] \in \mathcal{P}\}$ whose multiplication is given by

$$u_{[M]} u_{[N]} = \sum_{[L]\in\mathcal{P}} g^L_{MN} u_{[L]}.$$

It is easily seen that $\mathcal{H}_q(\Lambda)$ is an associative $\mathbb{Q}(v)$-algebra with unit $u_{[0]}$, where 0 denotes the zero module. Note that, the Ringel-Hall algebra $\mathcal{H}_q(\Lambda)$ is a $\mathbb{N}I$-graded algebra by dimension vectors of modules.

The twisted Ringel-Hall algebra $\mathcal{H}_q^*(\Lambda)$ is defined as follow. Set $\mathcal{H}_q^*(\Lambda)=\mathcal{H}_q(\Lambda)$ as $\mathbb{Q}(v)$-vector space and define the multiplication by

$$u_{[M]}*u_{[N]}=v^{\langle\underline{\dim}M,\underline{\dim}N\rangle}\sum_{[L]\in\mathcal{P}}g_{MN}^{L}u_{[L]}.$$

Let S_i be the nilpotent simple module corresponding to $i\in I$ and define $u_i=u_{[S_i]}$. The composition algebra $\mathcal{C}_q^*(\Lambda)$ is a subalgebra of $\mathcal{H}_q^*(\Lambda)$ generated by u_i for all $i\in I$. For any Λ-module M, denote

$$\langle M\rangle=v^{-\dim_k M+\dim_k \mathrm{End}_\Lambda(M)}u_{[M]}.$$

Note that $\{\langle M(\alpha)\rangle\mid\alpha\in\mathcal{P}\}$ is a $\mathbb{Q}(v)$-basis of $\mathcal{H}_q^*(\Lambda)$.

Let Q be a finite quiver. Then consider the generic Ringel-Hall algebra associated with Q. Let k be a finite field and $\Lambda_k=kQ$. Denote by $\mathcal{H}_q^*(\Lambda_k)$ the corresponding twisted Ringel-Hall algebra. Let $\mathcal{K}$ be a set of some finite fields k such that the set $\{q_k=|k|\mid k\in\mathcal{K}\}$ is an infinite set. Let R be an integral domain containing $\mathbb{Q}$ and v_{q_k}, where $v_{q_k}=\sqrt{q_k}$ for any $k\in\mathcal{K}$. For each $k\in\mathcal{K}$, the composition algebra $\mathcal{C}_q^*(\Lambda_k)$ is the R-subalgebra of $\mathcal{H}_q^*(\Lambda_k)$ generated by the elements $u_i(k)$ for all $i\in I$. Consider the direct product

$$\mathcal{H}^*(Q)=\prod_{k\in\mathcal{K}}\mathcal{H}_q^*(\Lambda_k)$$

and the elements $v=(v_{q_k})_{k\in\mathcal{K}}$, $v^{-1}=(v_{q_k}^{-1})_{k\in\mathcal{K}}$ and $u_i=(u_i(k))_{k\in\mathcal{K}}$. By $\mathcal{C}^*(Q)_{\mathcal{A}}$ we denote the subalgebra of $\mathcal{H}^*(Q)$ generated by v, v^{-1} and u_i over $\mathbb{Q}$. We may regard it as an $\mathcal{A}$-algebra generated by u_i, where v is viewed as an indeterminate. Finally, define $\mathcal{C}^*(Q)=\mathbb{Q}(v)\otimes\mathcal{C}^*(Q)_{\mathcal{A}}$, which is called the generic composition algebra of Q.

Then we have the following well-known result of Green and Ringel ([5, 18]).

Theorem 2.1. *Let Q be a connected quiver, A be the corresponding generalized Cartan matrix, and $\mathbf{f}$ be the Lusztig's algebra of type A. Then there is an isomorphism of algebras:*

$$\begin{array}{rcl}\mathcal{C}^*(Q) & \cong & \mathbf{f}\\ u_i & \mapsto & \theta_i.\end{array}$$

Hence, we always identify $\mathcal{C}^*(Q)$ with $\mathbf{f}$.

2.4. **Root categories.** A triangulated category $(\mathcal{C},T)$ is called 2-periodic if the translation functor T satisfies $T^2\simeq\mathrm{id}$.

Let k be a field. Given a finite dimensional hereditary k-algebra Λ, denote by $D^b(\Lambda)$ the bounded derived category of the abelian category mod-Λ and T the translation functor in this triangulated category. Consider the orbit category $\mathcal{R}(\Lambda)=D^b(\Lambda)/T^2$ of $D^b(\Lambda)$ under the equivalent functor T^2. Let $F:D^b(\Lambda)\to\mathcal{R}(\Lambda)$ be the canonical functor. The translation functor T of $D^b(\Lambda)$ induces an equivalent functor in $\mathcal{R}(\Lambda)$ of order 2, which is still denoted by T. By [15], $(\mathcal{R}(\Lambda),T)$ is also a triangulated category and the functor $F:D^b(\Lambda)\to\mathcal{R}(\Lambda)$ sends each triangle in $D^b(\Lambda)$ to a triangle in $\mathcal{R}(\Lambda)$. It is clear that the root category $\mathcal{R}=\mathcal{R}(\Lambda)$ is a 2-periodic triangulated category.

Let Q be a connected quiver and $\mathcal{R}(Q)=D^b(kQ)/T^2$. Denote by $\tilde{\mathcal{P}}$ the set of isomorphism classes of objects in $\mathcal{R}(Q)$ and $\mathrm{ind}(\tilde{\mathcal{P}})$ the set of isomorphism classes of indecomposable objects in $\mathcal{R}(Q)$. Note that mod-kQ can be embeded into $\mathcal{R}(Q)$ as a full subcategory and $\text{ind}(\tilde{\mathcal{P}}) = \text{ind}(\mathcal{P})\dot{\cup}\text{ind}(T(\mathcal{P}))$, where $\dot{\cup}$ means disjoint union.

3. Finite type

3.1. **PBW-type basis of $\mathbf{U}^+$.** In this section, let Q be a connected Dynkin quiver, k be a finite field and $\Lambda = kQ$. Denote by Φ^+ (resp. Φ^-) the set of positive (resp. negative) roots of the Dynkin quiver Q. Note that Φ^+ and Φ^- can be viewed as subsets of $\mathbb{Z}I$. By Gabriel's Theorem, the map $\underline{\dim}$ induces a bijection between $\text{ind}(\mathcal{P})$ and Φ^+. Given a positive root α, the corresponding isomorphism class is also denoted by α.

Since Q is representation-directed, we can define a total order on the set

$$\Phi^+ = \{\alpha_1, \alpha_2, \cdots, \alpha_n\}$$

such that the corresponding indecomposable Λ-modules

$$\{M(\alpha_1), M(\alpha_2), \cdots, M(\alpha_n)\}$$

satisfy the following conditions

$$\text{Hom}(M(\alpha_i), M(\alpha_j)) \neq 0 \Rightarrow i \leq j.$$

Define

$$\mathbb{N}^{\text{ind}(\mathcal{P})} = \{\mathbf{a} : \Phi^+ \to \mathbb{N}\}.$$

For any $\mathbf{a} \in \mathbb{N}^{\text{ind}(\mathcal{P})}$, we can define a representation

$$M(\mathbf{a}) = \bigoplus_{\alpha \in \Phi^+} \mathbf{a}(\alpha) M(\alpha)$$

and any representation can be written in this form.

Since the Hall polynomials exist in this case, we can consider the generic composition algebra $\mathcal{C}^*(Q)$ directly. Note that the set $\mathbb{N}^{\text{ind}(\mathcal{P})}$ is independent of the choice of the finite field and $\langle M(\mathbf{a}) \rangle$ can be viewed as an element in $\mathcal{C}^*(Q)$ for any $\mathbf{a} \in \mathbb{N}^{\text{ind}(\mathcal{P})}$.

By [19], we have

Proposition 3.1. *The set* $\{\langle M(\mathbf{a}) \rangle \mid \mathbf{a} \in \mathbb{N}^{ind(\mathcal{P})}\}$ *is an* $\mathcal{A}$*-basis of* $\mathcal{C}^*_{\mathcal{A}}(Q)$.

3.2. **PBW-type basis of $\dot{\mathbf{U}}\mathbf{1}_\lambda$.** Let $\mathcal{R}(Q)$ be the root category corresponding to a connected Dynkin quiver Q over some finite field k. Remember that $\text{ind}(\tilde{\mathcal{P}})$ is the set of isomorphism classes of indecomposable objects in $\mathcal{R}(Q)$. Let $\Phi = \{\underline{\dim}(M) \mid M \in \text{ind}(\tilde{\mathcal{P}})\}$. Then Φ is the root system of the corresponding Lie algebra and there is a bijection between $\text{ind}(\tilde{\mathcal{P}})$ and Φ by Gabriel's Theorem. Note that $\Phi = \Phi^+ \dot{\cup} \Phi^-$. For any element $\alpha \in \Phi$, we also denote by $M(\alpha)$ the corresponding object in $\mathcal{R}(Q)$.

Define

$$\mathbb{N}^{\text{ind}(\tilde{\mathcal{P}})} = \{\tilde{\mathbf{a}} : \Phi \to \mathbb{N}\}.$$

For any $\tilde{\mathbf{a}} \in \mathbb{N}^{\text{ind}(\tilde{\mathcal{P}})}$, we can define an object

$$M(\tilde{\mathbf{a}}) = \bigoplus_{\alpha \in \Phi} \tilde{\mathbf{a}}(\alpha) M(\alpha)$$

and any object in $\mathcal{R}(Q)$ can be written in this form.

Note that the category $\mathcal{R}(Q)$, so the set $\mathbb{N}^{\mathrm{ind}(\tilde{\mathcal{P}})}$, depends only on the underlying graph of Q. If Q' is another quiver such that $D^b(kQ) \simeq D^b(kQ')$, they give the same set $\mathbb{N}^{\mathrm{ind}(\tilde{\mathcal{P}})}$.

Given any symmetric generalized Cartan matrix $A = (a_{ij})_{n\times n}$ of finite type, consider a quiver Q, the quantum enveloping algebra $\mathbf{U}$ and the modified quantized enveloping algebra $\dot{\mathbf{U}}$ corresponding to A.

Remember that mod-kQ can be embedded into $\mathcal{R}(Q)$ as a full subcategory and

$$\mathrm{ind}(\tilde{\mathcal{P}}) = \mathrm{ind}(\mathcal{P})\dot{\cup}\mathrm{ind}(T(\mathcal{P})).$$

For any $\tilde{\mathbf{a}} \in \mathbb{N}^{\mathrm{ind}(\tilde{\mathcal{P}})}$, let $\mathbf{a}_1 = \tilde{\mathbf{a}}|_{\mathrm{ind}(\mathcal{P})}$ and $\mathbf{a}_2 = \tilde{\mathbf{a}}|_{\mathrm{ind}(T(\mathcal{P}))}$, which is denoted by $\tilde{\mathbf{a}} = (\mathbf{a}_1, \mathbf{a}_2)$. Since we always identify $\mathcal{C}^*(Q)$ with $\mathbf{f}$, the elements in the following set

$$\{\langle M(\tilde{\mathbf{a}})\rangle_\lambda = \langle M(\mathbf{a}_1)\rangle^+ \cdot \langle M(\mathbf{a}_2)\rangle^- \mathbf{1}_\lambda \mid \tilde{\mathbf{a}} \in \mathbb{N}^{\mathrm{ind}(\tilde{\mathcal{P}})}\}$$

can be regarded as elements in $\dot{\mathbf{U}}\mathbf{1}_\lambda$.

We have the following proposition.

Proposition 3.2. *The set* $\{\langle M(\tilde{\mathbf{a}})\rangle_\lambda \mid \tilde{\mathbf{a}} \in \mathbb{N}^{ind(\tilde{\mathcal{P}})}\}$ *is a* $\mathbb{Q}(v)$*-basis of* $\dot{\mathbf{U}}\mathbf{1}_\lambda$.

Proof The modified quantized enveloping algebra $\dot{\mathbf{U}}$ is a free $\mathbf{f}\otimes\mathbf{f}^{\mathrm{opp}}$-module with basis $(\mathbf{1}_\lambda)_{\lambda\in P}$ ([14], 23.2.1). So the set

$$\{\langle M(\mathbf{a}_1)\rangle^+ \cdot \langle M(\mathbf{a}_2)\rangle^- \mathbf{1}_\lambda \mid \mathbf{a}_1 \in \mathrm{ind}(\mathcal{P}), \mathbf{a}_2 \in \mathrm{ind}(T(\mathcal{P}))\}$$

is a basis of $\dot{\mathbf{U}}\mathbf{1}_\lambda$. By $\tilde{\mathbf{a}} = (\mathbf{a}_1, \mathbf{a}_2)$, the set $\{\langle M(\tilde{\mathbf{a}})\rangle_\lambda \mid \tilde{\mathbf{a}} \in \mathbb{N}^{\mathrm{ind}(\tilde{\mathcal{P}})}\}$ is a $\mathbb{Q}(v)$-basis of $\dot{\mathbf{U}}\mathbf{1}_\lambda$.

□

Denote by $B_Q(\dot{\mathbf{U}}\mathbf{1}_\lambda)$ the PBW-type basis $\{\langle M(\tilde{\mathbf{a}})\rangle_\lambda \mid \tilde{\mathbf{a}} \in \mathbb{N}^{\mathrm{ind}(\tilde{\mathcal{P}})}\}$. Note that this PBW-type basis depends on the embedding of mod-kQ into $\mathcal{R}(Q)$.

3.3. **A bar-invariant basis of $\dot{\mathbf{U}}\mathbf{1}_\lambda$.** As before, let Q be a connected Dynkin quiver and $\mathcal{R}(Q)$ be the corresponding root category. Remember that the set of positive roots $\Phi^+ = \{\alpha_1, \ldots, \alpha_n\}$. For any $\mathbf{a}, \mathbf{b} : \Phi^+ \to \mathbb{N}$, define $\mathbf{b} \prec \mathbf{a}$ if and only if there exists some $1 \le j \le n$ such that $\mathbf{b}(\alpha_i) = \mathbf{a}(\alpha_i)$ for all $i < j$ and $\mathbf{b}(\alpha_j) > \mathbf{a}(\alpha_j)$. For any $\tilde{\mathbf{a}}, \tilde{\mathbf{b}} : \Phi \to \mathbb{N}$, define $\tilde{\mathbf{a}} \prec \tilde{\mathbf{b}}$ if and only if $\mathbf{a}_1 \preceq \mathbf{b}_1$ and $\mathbf{a}_2 \preceq \mathbf{b}_2$ but $\tilde{\mathbf{a}} \neq \tilde{\mathbf{b}}$, where $\tilde{\mathbf{a}} = (\mathbf{a}_1, \mathbf{a}_2)$ and $\tilde{\mathbf{b}} = (\mathbf{b}_1, \mathbf{b}_2)$.

For any $\mathbf{c} : \Phi^+ \to \mathbb{N}$, there exists a monomial $w_*(\mathbf{c})$ on Chevalley generators u_i satisfying

$$w_*(\mathbf{c}) = \langle M(\mathbf{c})\rangle + \sum_{\mathbf{c}'\prec\mathbf{c}} a_{\mathbf{c}\mathbf{c}'}\langle M(\mathbf{c}')\rangle,$$

where $a_{\mathbf{c}\mathbf{c}'} \in \mathcal{A}$ ([19]). Note that the transition matrix $a = (a_{\mathbf{c}\mathbf{c}'})$ from $\{\langle M(\mathbf{c})\rangle \mid \mathbf{c} : \Phi^+ \to \mathbb{N}\}$ to $\{w_*(\mathbf{c}) \mid \mathbf{c} : \Phi^+ \to \mathbb{N}\}$ satisfies that $a_{\mathbf{c}\mathbf{c}} = 1$ and $a_{\mathbf{c}\mathbf{c}'} = 0$ unless $\mathbf{c}' \prec \mathbf{c}$. That is, a is a unipotent lower triangular matrix.

Let $\bar{a} = (\overline{a_{\mathbf{c}\mathbf{c}'}})$. Since $\overline{w_*(\mathbf{c})} = w_*(\mathbf{c})$, we have

$$w_*(\mathbf{c}) = \overline{w_*(\mathbf{c})} = \sum_{\mathbf{c}'} \bar{a}_{\mathbf{c}\mathbf{c}'}\overline{\langle M(\mathbf{c}')\rangle},$$

thus

$$\overline{\langle M(\mathbf{c})\rangle} = \sum_{\mathbf{c}'} \bar{a}^{-1}_{\mathbf{c}\mathbf{c}'} w_*(\mathbf{c}') = \sum_{\mathbf{c}'}\sum_{\mathbf{c}''} \bar{a}^{-1}_{\mathbf{c}\mathbf{c}'} a_{\mathbf{c}'\mathbf{c}''}\langle M(\mathbf{c}'')\rangle.$$

Let $h = \bar{a}^{-1}a$. The matrix h is again a unipotent lower triangular matrix and $\bar{h} = h^{-1}$. There exists a unique unipotent lower triangular matrix $d = (d_{\mathbf{c}\mathbf{c}'})$ with off-diagonal entries in $v^{-1}\mathbb{Q}[v^{-1}]$, such that $d = \bar{d}h$. Then the canonical basis of $\mathbf{f}$ is

$$\mathcal{E}^{\mathbf{c}} = \langle M(\mathbf{c})\rangle + \sum_{\mathbf{c}'\prec\mathbf{c}} d_{\mathbf{c}\mathbf{c}'}\langle M(\mathbf{c}')\rangle,$$

with $d_{\mathbf{c}\mathbf{c}'} \in v^{-1}\mathbb{Q}[v^{-1}]$ ([19]).

Similarly, we can get a bar-invariant basis of $\dot{\mathbf{U}}\mathbf{1}_\lambda$ from

$$B_Q(\dot{\mathbf{U}}\mathbf{1}_\lambda) = \{\langle M(\mathbf{c}_1)\rangle^+ \cdot \langle M(\mathbf{c}_2)\rangle^-\mathbf{1}_\lambda \mid \tilde{\mathbf{c}} : \Phi \to \mathbb{N}, \tilde{\mathbf{c}} = (\mathbf{c}_1, \mathbf{c}_2)\}$$

and

$$\{w_*(\mathbf{c}_1)^+ \cdot w_*(\mathbf{c}_2)^-\mathbf{1}_\lambda \mid \tilde{\mathbf{c}} : \Phi \to \mathbb{N}, \tilde{\mathbf{c}} = (\mathbf{c}_1, \mathbf{c}_2)\}$$

under the order $\prec$ on $\mathbb{N}^{\mathrm{ind}(\tilde{\mathcal{P}})}$ defined above.

First define $w_*(\tilde{\mathbf{c}})_\lambda = w_*(\mathbf{c}_1)^+ \cdot w_*(\mathbf{c}_2)^-\mathbf{1}_\lambda$, where $\tilde{\mathbf{c}} = (\mathbf{c}_1, \mathbf{c}_2)$. Since

$$w_*(\mathbf{c}) = \langle M(\mathbf{c})\rangle + \sum_{\mathbf{c}'\prec\mathbf{c}} a_{\mathbf{c}\mathbf{c}'}\langle M(\mathbf{c}')\rangle,$$

we have

$$w_*(\mathbf{c}_1)^+ = \langle M(\mathbf{c}_1)\rangle^+ + \sum_{\mathbf{c}_1'\prec\mathbf{c}_1} a_{\mathbf{c}_1\mathbf{c}_1'}\langle M(\mathbf{c}_1')\rangle^+$$

and

$$w_*(\mathbf{c}_2)^- = \langle M(\mathbf{c}_2)\rangle^- + \sum_{\mathbf{c}_2'\prec\mathbf{c}_2} a_{\mathbf{c}_2\mathbf{c}_2'}\langle M(\mathbf{c}_2')\rangle^-$$

in $\mathbf{U}^\pm$ respectively. Hence, we have

$$\begin{aligned}
w_*(\tilde{\mathbf{c}})_\lambda &= w_*(\mathbf{c}_1)^+ \cdot w_*(\mathbf{c}_2)^-\mathbf{1}_\lambda \\
&= (\langle M(\mathbf{c}_1)\rangle + \sum_{\mathbf{c}_1'\prec\mathbf{c}_1} a_{\mathbf{c}_1\mathbf{c}_1'}\langle M(\mathbf{c}_1')\rangle)^+ \cdot (\langle M(\mathbf{c}_2)\rangle + \sum_{\mathbf{c}_2'\prec\mathbf{c}_2} a_{\mathbf{c}_2\mathbf{c}_2'}\langle M(\mathbf{c}_2')\rangle)^-\mathbf{1}_\lambda \\
&= \langle M(\mathbf{c}_1)\rangle^+ \cdot \langle M(\mathbf{c}_2)\rangle^-\mathbf{1}_\lambda + \langle M(\mathbf{c}_1)\rangle^+ \cdot \sum_{\mathbf{c}_2'\prec\mathbf{c}_2} a_{\mathbf{c}_2\mathbf{c}_2'}\langle M(\mathbf{c}_2')\rangle^-\mathbf{1}_\lambda + \\
&\quad \sum_{\mathbf{c}_1'\prec\mathbf{c}_1} a_{\mathbf{c}_1\mathbf{c}_1'}\langle M(\mathbf{c}_1')\rangle^+ \cdot \langle M(\mathbf{c}_2)\rangle^-\mathbf{1}_\lambda + \sum_{\mathbf{c}_1'\prec\mathbf{c}_1} a_{\mathbf{c}_1\mathbf{c}_1'}\langle M(\mathbf{c}_1')\rangle^+ \cdot \sum_{\mathbf{c}_2'\prec\mathbf{c}_2} a_{\mathbf{c}_2\mathbf{c}_2'}\langle M(\mathbf{c}_2')\rangle^-\mathbf{1}_\lambda \\
&= \langle M(\tilde{\mathbf{c}})\rangle_\lambda + \sum_{\tilde{\mathbf{c}}'\prec\tilde{\mathbf{c}}} \tilde{a}_{\tilde{\mathbf{c}}\tilde{\mathbf{c}}'}\langle M(\tilde{\mathbf{c}}')\rangle_\lambda,
\end{aligned}$$

where $\tilde{\mathbf{c}} = (\mathbf{c}_1, \mathbf{c}_2)$, $\tilde{\mathbf{c}}' = (\mathbf{c}_1', \mathbf{c}_2')$ and $\tilde{a}_{\tilde{\mathbf{c}}\tilde{\mathbf{c}}'} = a_{\mathbf{c}_1\mathbf{c}_1'}a_{\mathbf{c}_2\mathbf{c}_2'} \in \mathcal{A}$.

As before, the transition matrix $\tilde{a} = (\tilde{a}_{\tilde{\mathbf{c}}\tilde{\mathbf{c}}'})$ from $\{\langle M(\tilde{\mathbf{c}})\rangle_\lambda \mid \tilde{\mathbf{c}} : \Phi \to \mathbb{N}\}$ to $\{w_*(\tilde{\mathbf{c}})_\lambda \mid \tilde{\mathbf{c}} : \Phi \to \mathbb{N}\}$ satisfies that $\tilde{a}_{\tilde{\mathbf{c}}\tilde{\mathbf{c}}} = 1$ and $a_{\tilde{\mathbf{c}}\tilde{\mathbf{c}}'} = 0$ unless $\tilde{\mathbf{c}}' \prec \tilde{\mathbf{c}}$. That is, $\tilde{a}$ is also a unipotent lower triangular matrix with off-diagonal entries in $\mathcal{A}$.

Let $\bar{\tilde{a}} = (\overline{\tilde{a}_{\tilde{\mathbf{c}}\tilde{\mathbf{c}}'}})$. Since $\overline{w_*(\tilde{\mathbf{c}})_\lambda} = w_*(\tilde{\mathbf{c}})_\lambda$, we have

$$w_*(\tilde{\mathbf{c}})_\lambda = \overline{w_*(\tilde{\mathbf{c}})_\lambda} = \sum_{\tilde{\mathbf{c}}'} \bar{\tilde{a}}_{\tilde{\mathbf{c}}\tilde{\mathbf{c}}'}\overline{\langle M(\tilde{\mathbf{c}}')\rangle_\lambda},$$

thus

$$\overline{\langle M(\tilde{\mathbf{c}})\rangle_\lambda} = \sum_{\tilde{\mathbf{c}}'} \bar{\tilde{a}}^{-1}_{\tilde{\mathbf{c}}\tilde{\mathbf{c}}'} w_*(\tilde{\mathbf{c}}')_\lambda = \sum_{\tilde{\mathbf{c}}'}\sum_{\tilde{\mathbf{c}}''} \bar{\tilde{a}}^{-1}_{\tilde{\mathbf{c}}\tilde{\mathbf{c}}'}\tilde{a}_{\tilde{\mathbf{c}}'\tilde{\mathbf{c}}''}\langle M(\tilde{\mathbf{c}}'')\rangle_\lambda.$$

Let $\tilde{h} = \bar{\tilde{a}}^{-1}\tilde{a}$. The matrix $\tilde{h}$ is again a unipotent lower triangular matrix and $\bar{\tilde{h}} = \tilde{h}^{-1}$. There exists a unique unipotent lower triangular matrix $\tilde{d} = (\tilde{d}_{\tilde{\mathbf{c}}\tilde{\mathbf{c}}'})$ with off-diagonal entries in $v^{-1}\mathbb{Q}[v^{-1}]$, such that $\tilde{d} = \bar{\tilde{d}}\tilde{h}$. Then we get a bar-invariant basis of $\dot{\mathbf{U}}\mathbf{1}_\lambda$

$$\mathcal{E}_\lambda^{\tilde{\mathbf{c}}} = \langle M(\tilde{\mathbf{c}})\rangle_\lambda + \sum_{\tilde{\mathbf{c}}' \prec \tilde{\mathbf{c}}} \tilde{d}_{\tilde{\mathbf{c}}\tilde{\mathbf{c}}'} \langle M(\tilde{\mathbf{c}}')\rangle_\lambda,$$

with $\tilde{d}_{\tilde{\mathbf{c}}'\tilde{\mathbf{c}}} \in v^{-1}\mathbb{Q}[v^{-1}]$. We denote this basis by $\mathcal{B}_Q(\dot{\mathbf{U}}\mathbf{1}_\lambda)$.

Theorem 3.3. $\mathcal{B}_Q(\dot{\mathbf{U}}\mathbf{1}_\lambda) = \{\mathcal{E}_\lambda^{\tilde{\mathbf{c}}} \mid \tilde{\mathbf{c}} : \Phi \to \mathbb{N}\} = \{b^+ b'^- \mathbf{1}_\lambda \mid b, b' \in \mathbf{B}\}$.

We omit the proof of this theorem. The proof of Theorem 3.3 is simpler than Theorem 4.9 of affine case, which will be proved in next section.

3.4. **A parameterization of the canonical basis of $\dot{\mathbf{U}}\mathbf{1}_\lambda$.** Let $\dot{\mathbf{U}} = \bigoplus_{\lambda \in P} \dot{\mathbf{U}}\mathbf{1}_\lambda$ be the modified quantized enveloping algebra corresponding to the quiver Q and $\dot{\mathbf{B}}_\lambda$ be the canonical basis of $\dot{\mathbf{U}}\mathbf{1}_\lambda$.

Theorem 3.4. *We have a bijection*

$$\Psi_Q : \mathbb{N}^{ind(\tilde{\mathcal{P}})} \to \dot{\mathbf{B}}_\lambda$$

given by

$$\tilde{\mathbf{c}} \mapsto \mathcal{E}^{\mathbf{c}_1} \diamondsuit_\lambda \mathcal{E}^{\mathbf{c}_2},$$

which is the composition of the following two bijections

$$\begin{array}{rcl} \mathbb{N}^{ind(\tilde{\mathcal{P}})} & \to & \mathcal{B}_Q(\dot{\mathbf{U}}\mathbf{1}_\lambda) \\ \tilde{\mathbf{c}} & \mapsto & \mathcal{E}_\lambda^{\tilde{\mathbf{c}}}, \end{array}$$

and

$$\begin{array}{rcl} \mathcal{B}_Q(\dot{\mathbf{U}}\mathbf{1}_\lambda) & \to & \dot{\mathbf{B}}_\lambda \\ b^+ b'^- \mathbf{1}_\lambda & \mapsto & b \diamondsuit_\lambda b'. \end{array}$$

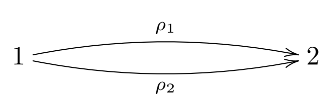
Proof The first bijection from $\mathbb{N}^{\mathrm{ind}(\tilde{\mathcal{P}})}$ to $\mathcal{B}_Q(\dot{\mathbf{U}}\mathbf{1}_\lambda)$ comes from our construction of $\mathcal{E}_\lambda^{\tilde{\mathbf{c}}}$ and the second bijection from $\mathcal{B}_Q(\dot{\mathbf{U}}\mathbf{1}_\lambda)$ to $\dot{\mathbf{B}}_\lambda$ comes from formula (1). By Theorem 3.3, $\mathcal{E}_\lambda^{\tilde{\mathbf{c}}} = \mathcal{E}^{\mathbf{c}_1 +} \mathcal{E}^{\mathbf{c}_2 -} \mathbf{1}_\lambda$. Hence, $\Psi_Q : \mathbb{N}^{\mathrm{ind}(\tilde{\mathcal{P}})} \to \dot{\mathbf{B}}_\lambda$ is a bijection.

□

4. Affine type

4.1. **PBW-type basis of $\mathbf{U}^+$.** We first review the construction of the PBW-type basis in [2, 4, 10, 23].

4.1.1. *The integral basis arising from the Kronecker quiver.* Let Q be the Kronecker quiver with $I = \{1, 2\}$ and $H = \{\rho_1, \rho_2\}$:



Let k be a finite field with q elements, $v = \sqrt{q}$ and $\Lambda = kQ$ be the path algebra of Q.

The set of dimension vectors of indecomposable Λ-modules is

$$\Phi^+ = \{(l+1, l), (m, m), (n, n+1) \mid l, m, n \in \mathbb{Z}, l \geq 0, m \geq 1, n \geq 0\}.$$

The dimension vectors $(l+1, l)$ and $(n, n+1)$ correspond to preprojective and preinjective indecomposable Λ-modules respectively.

Remember that $\mathcal{P}$ is the set of isomorphism classes of finite dimensional Λ-modules. Denote by $\mathcal{H}_q$ (resp. $\mathcal{H}_q^*$) the Ringel-Hall (resp. twisted Ringel-Hall) algebra of Λ.

Define

$$E_{(n+1,n)} = \langle M(n+1,n)\rangle \text{ and } E_{(n,n+1)} = \langle M(n,n+1)\rangle,$$

where $M(n+1,n)$ (resp. $M(n,n+1)$) is the corresponding Λ-module of dimension vector $(n+1,n)$ (resp. $(n,n+1)$) for any $n \in \mathbb{N}$. Let $\delta = (1,1)$. For any $n \geq 1$, define

$$\tilde{E}_{n\delta} = E_{(n-1,n)} * E_{(1,0)} - v^{-2} E_{(1,0)} * E_{(n-1,n)}.$$

Then, define inductively

$$E_{0\delta} = 1,\ E_{k\delta} = \frac{1}{[k]} \sum_{s=1}^{k} v^{s-k} \tilde{E}_{s\delta} * E_{(k-s)\delta} \text{ for all } k \geq 1.$$

Consider the generic composition algebra $\mathcal{C}^*$. Since $E_{k\delta}$, $E_{(m+1,m)}$ and $E_{(n,n+1)}$ are defined in each $\mathcal{H}_q^*$, they can be regarded as elements in $\prod_q \mathcal{H}_q^*$. Note that, these elements also belong to $\mathcal{C}_{\mathcal{A}}^*$.

Denote by $\mathbf{P}(m)$ the set of all partitions of m. For any partition

$$w = (w_1 \geq w_2 \geq \ldots \geq w_t) \in \mathbf{P}(m),$$

define

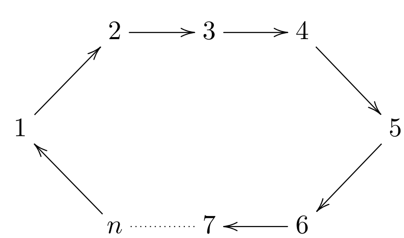
$$E_{w\delta} = E_{w_1\delta} * E_{w_2\delta} * \cdots * E_{w_t\delta}.$$

Proposition 4.1. (cf. [2, 10, 23]) *The set*

$$\{\langle P\rangle * E_{w\delta} * \langle I\rangle\}$$

is an $\mathcal{A}$-basis of $\mathcal{C}_{\mathcal{A}}^*$, where $P \in \mathcal{P}$ is preprojective, $w \in \mathbf{P}(m)$, $I \in \mathcal{P}$ is preinjective and $m \in \mathbb{N}$.

4.1.2. *The integral basis arising from a tube.* Let $\Delta = \Delta(n)$ be the cyclic quiver whose vertex set is $\Delta_0 = \mathbb{Z}/n\mathbb{Z}$={1,2,...,n} and arrow set is $\Delta_1 = \{i \to i+1 \mid i \in \Delta_0\}$:



Let k be a finite field with q elements, $v = \sqrt{q}$ and $\mathcal{T} = \mathcal{T}(n)$ be the category of finite dimensional nilpotent representations of $\Delta(n)$ over k. For any $i \in \Delta_0$, denote by S_i the corresponding simple object in $\mathcal{T}$. For any $i \in \Delta_0$ and $l \in \mathbb{N}$, denote by $S_i[l]$ the indecomposable object in $\mathcal{T}$ with top S_i and length l. Note that $S_i[l]$ is independent of the choice of finite fields. Let $\mathcal{P}$ be the set of isomorphism classes of objects in $\mathcal{T}$. Denote by $\mathcal{H}$ (resp. $\mathcal{H}^*$) the corresponding Ringel-Hall algebra (resp. twisted Ringel-Hall algebra). Since the Hall polynomials always exist in this case, they are regarded as generic forms. Denote by $\mathcal{C}^*$ the twisted composition subalgebra of $\mathcal{H}^*$.

Let Π be the set of n-tuples of partitions $\pi = (\pi^{(1)}, \pi^{(2)}, \ldots, \pi^{(n)})$, where $\pi^{(i)} = (\pi_1^{(i)} \geq \pi_2^{(i)} \geq \cdots)$ is a partition of some non-negative integer. For each $\pi \in \Pi$, define an object in $\mathcal{T}$

$$M(\pi) = \bigoplus_{i \in \Delta_0, j \geq 1} S_i[\pi_j^{(i)}].$$

In this way, we obtain a bijection between Π and $\mathcal{P}$.

An n-tuple $\pi = (\pi^{(1)}, \pi^{(2)}, \ldots, \pi^{(n)})$ of partitions in Π is called aperiodic, if for each $l \geq 1$ there exists some $i = i(l) \in \Delta_0$ such that $\pi_j^{(i)} \neq l$ for all $j \geq 1$. Denote by Π^a the set of aperiodic n-tuples of partitions. An object M in $\mathcal{T}$ is called aperiodic if $M \simeq M(\pi)$ for some $\pi \in \Pi^a$. For any dimension vector $\alpha \in \mathbb{N}\Delta_0$, define $\Pi_\alpha = \{\lambda \in \Pi \mid \underline{\dim} M(\lambda) = \alpha\}$ and $\Pi_\alpha^a = \Pi^a \cap \Pi_\alpha$.

For any objects M and N in $\mathcal{T}$, there exists a unique (up to isomorphism) extension L of M by N with minimal $\dim \mathrm{End}(L)$. The extension L is called the generic extension of M by N, which is denoted by $L = M \diamond N$.

Let Ω be the set of all words on the alphabet Δ_0. For each $w = i_1 i_2 \cdots i_m \in \Omega$, set $M(w) = S_{i_1} \diamond S_{i_2} \diamond \cdots \diamond S_{i_m}$. Then there is a unique $\mathfrak{p}(w) = \pi \in \Pi$ such that $M(\pi) \simeq M(w)$. It has been proved in [17] that $\pi = \mathfrak{p}(w) \in \Pi^a$ and $\mathfrak{p}$ induces a surjection $\mathfrak{p} : \Omega \twoheadrightarrow \Pi^a$.

For each object M in $\mathcal{T}$ and $s \geq 1$, denote by sM the direct sum of s copies of M. For any $w \in \Omega$, write w in tight form $w = j_1^{e_1} j_2^{e_2} \cdots j_t^{e_t} \in \Omega$ with $j_{r-1} \neq j_r$ for all r. Let μ_r be the element in Π such that $M(\mu_r) = e_r S_{j_r}$. For any $\lambda \in \Pi_{\sum_{r=1}^t e_r j_r}$, write g_w^λ for the Hall polynomial $g_{M(\mu_1), \ldots, M(\mu_t)}^{M(\lambda)}$. A word w is called distinguished if the Hall Polynomial $g_w^{\mathfrak{p}(w)} = 1$. For any $\pi \in \Pi^a$, there exists a distinguished word $w_\pi = j_1^{e_1} j_2^{e_2} \cdots j_t^{e_t} \in \mathfrak{p}^{-1}(\pi)$ in tight form by [4]. From now on, fix a distinguished word $w_\pi \in \mathfrak{p}^{-1}(\pi)$ for any $\pi \in \Pi^a$. Thus we have a section $\mathcal{D} = \{w_\pi \mid \pi \in \Pi^a\}$ of $\mathfrak{p}$ over Π^a. $\mathcal{D}$ is called a section of distinguished words in [4].

For each $w = j_1^{e_1} j_2^{e_2} \cdots j_t^{e_t} \in \Omega$ in tight form, define in $\mathcal{C}^*$ a monomial

$$\mathbf{m}^{(w)} = E_{j_i}^{*e_1} * \cdots * E_{j_t}^{*e_t}.$$

Then define E_π for all $\pi \in \Pi^a$ inductively by the following relations

$$E_\pi = \mathbf{m}^{(w_\pi)} \text{ if } \pi \in \Pi_\alpha^a \text{ is minimal,}$$

and

$$E_\pi = \mathbf{m}^{(w_\pi)} - \sum_{\lambda \prec \pi, \lambda \in \Pi_\alpha^a} v^d g_{w_\pi}^\lambda (v^2) E_\lambda$$

where $\alpha = \sum_{i=1}^t e_r j_r$, $d = -\dim M(\pi) + \dim \mathrm{End} M(\pi) + \dim M(\lambda) - \dim \mathrm{End} M(\lambda)$ and $\lambda \prec \mu \Leftrightarrow \dim \mathrm{Hom}(M, M(\lambda)) \leq \dim \mathrm{Hom}(M, M(\mu))$ for all objects M in $\mathcal{T}$.

For any $\pi \in \Pi^a$, E_π is contained in $\mathcal{C}^*$. We have the following proposition.

Proposition 4.2. (cf. [4, 10]) *Let $\mathcal{D} = \{w_\pi \mid \pi \in \Pi^a\}$ be a section of distinguished words. Then both $\{\mathbf{m}^{(w_\pi)} \mid \pi \in \Pi^a\}$ and $\{E_\pi \mid \pi \in \Pi^a\}$ are $\mathcal{A}$-bases of $\mathcal{C}_{\mathcal{A}}^*$. And the transition matrix between these two bases is a unipotent lower triangular matrix with off-diagonal entries in $\mathcal{A}$.*

It has been proved in [4] that the basis $\{E_\pi \mid \pi \in \Pi^a\}$ is independent of the choice of the sections of distinguished words.

4.1.3. *The integral bases arising from preprojective and preinjective components.* Let k be a finite field with q elements and $v=\sqrt{q}$. Let Q be a connected tame quiver without oriented cycles and $\Lambda=kQ$ be the corresponding path algebra. Denote by $Prep$ and $Prei$ the sets of isomorphism classes of indecomposable preprojective and preinjective Λ-modules respectively, which are independent of the choice of finite fields. Let $\mathcal{H}_q$ (resp. $\mathcal{H}_q^*$) be the Ringel-Hall (resp. twisted Ringel-Hall) algebra of Λ.

Since $Prei$ is representation-directed, we can define a total order on the set

$$\Phi^+_{Prei}=\{\cdots,\beta_3,\beta_2,\beta_1\}$$

of all positive real roots appearing in $Prei$ such that the corresponding Λ-modules

$$\{\cdots,M(\beta_3),M(\beta_2),M(\beta_1)\}$$

satisfy the following conditions

$$\mathrm{Hom}(M(\beta_i),M(\beta_j))\neq 0\Rightarrow i\geq j.$$

Similarly, since $Prep$ is representation-directed, we can define a total order on the set

$$\Phi^+_{Prep}=\{\alpha_1,\alpha_2,\alpha_3,\cdots\}$$

of all positive real roots appearing in $Prep$ such that the corresponding Λ-modules

$$\{M(\alpha_1),M(\alpha_2),M(\alpha_3),\cdots\}$$

satisfy the following conditions

$$\mathrm{Hom}(M(\alpha_i),M(\alpha_j))\neq 0\Rightarrow i\leq j.$$

Define

$$\mathbb{N}_f^{Prei}=\{\mathbf{b}:\Phi^+_{Prei}\to\mathbb{N}\mid \mathbf{b}\text{ is support-finite}\}.$$

For any $\mathbf{b}\in\mathbb{N}_f^{Prei}$, we can define a preinjective representation

$$M(\mathbf{b})=\bigoplus_{\beta_i\in\Phi^+_{Prei}}\mathbf{b}(\beta_i)M(\beta_i)$$

and any preinjective representation can be written in this form.

Define

$$\mathbb{N}_f^{Prep}=\{\mathbf{a}:\Phi^+_{Prep}\to\mathbb{N}\mid \mathbf{a}\text{ is support-finite}\}.$$

For any $\mathbf{a}\in\mathbb{N}_f^{Prep}$, we can define a preprojective representation

$$M(\mathbf{a})=\bigoplus_{\alpha_i\in\Phi^+_{Prep}}\mathbf{a}(\alpha_i)M(\alpha_i)$$

and any preprojective representation can be written in this form.

For any three elements $\mathbf{b},\mathbf{b}_1,\mathbf{b}_2\in\mathbb{N}_f^{Prei}$ (resp. $\mathbf{a},\mathbf{a}_1,\mathbf{a}_2\in\mathbb{N}_f^{Prep}$), the Hall polynomial $g^{M(\mathbf{b})}_{M(\mathbf{b}_1)M(\mathbf{b}_2)}$ (resp. $g^{M(\mathbf{a})}_{M(\mathbf{a}_1)M(\mathbf{a}_2)}$) always exists.

Consider the generic composition algebra $\mathcal{C}^*$ of Q. Note that $\langle M(\mathbf{b})\rangle\in\mathcal{C}^*_{\mathcal{A}}$ (resp. $\langle M(\mathbf{a})\rangle\in\mathcal{C}^*_{\mathcal{A}}$) for any $\mathbf{b}\in\mathbb{N}_f^{Prei}$ (resp. $\mathbf{a}\in\mathbb{N}_f^{Prep}$). Denote by $\mathcal{C}^*_{Prei}$ (resp. $\mathcal{C}^*_{Prep}$) the $\mathcal{A}$-submodule of $\mathcal{C}^*_{\mathcal{A}}$ generated by $\{\langle M(\mathbf{b})\rangle\mid\mathbf{b}\in\mathbb{N}_f^{Prei}\}$ (resp. $\{\langle M(\mathbf{a})\rangle\mid\mathbf{a}\in\mathbb{N}_f^{Prep}\}$).

Proposition 4.3. (cf. [10]) *The $\mathcal{A}$-submodule $\mathcal{C}^*_{Prei}$ (resp. $\mathcal{C}^*_{Prep}$) is a subalgebra of $\mathcal{C}^*_{\mathcal{A}}$ and $\{\langle M(\mathbf{b})\rangle \mid \mathbf{b} \in \mathbb{N}_f^{Prei}\}$ (resp. $\{\langle M(\mathbf{a})\rangle \mid \mathbf{a} \in \mathbb{N}_f^{Prep}\}$) is an $\mathcal{A}$-basis of $\mathcal{C}^*_{Prei}$ (resp. $\mathcal{C}^*_{Prep}$).*

4.1.4. *The integral basis for the generic composition algebra.* Let k also be a finite field with q elements and $v = \sqrt{q}$. Let Q be a connected tame quiver without oriented cycles and $\Lambda = kQ$ be the corresponding path algebra.

First consider the embedding of the category of representations of Kronecker quiver into that of Q.

Let e be a extending vertex of Q and $P = P(e)$ be the projective cover of simple module S_e. Let $\mathfrak{p} = \underline{\dim} P(e)$ and δ be the minimal imaginary root vector. Note that $\langle \mathfrak{p}, \mathfrak{p}\rangle = 1 = \langle \mathfrak{p}, \delta\rangle$ and there exists a unique indecomposable preprojective module L such that $\underline{\dim} L = \mathfrak{p} + \delta$. Moreover, $\mathrm{Hom}_\Lambda(L, P) = 0$ and $\mathrm{Ext}_\Lambda(L, P) = 0$. Let $\mathfrak{C}(P, L)$ be the smallest full subcategory of mod-Λ which contains P and L and is closed under taking extensions, kernels of epimorphisms and cokernels of monomorphisms. The category $\mathfrak{C}(P, L)$ is equivalent to the module category of Kronecker quiver K over k. Thus we have an exact embedding F : mod-$kK \hookrightarrow$ mod-Λ. Note that the embedding F is independent of the choice of finite fields. Hence, this gives rise to a monomorphism of algebras $F : \mathcal{H}^*(K) \to \mathcal{H}^*(Q)$. In $\mathcal{H}^*(K)$, we have defined $E_{m\delta_K}$ for any $m \in \mathbb{N}$. Define $E_{m\delta} = F(E_{m\delta_K})$. Since $E_{m\delta_K} \in \mathcal{C}^*(K)$, $E_{m\delta} \in \mathcal{C}^*(Q)$.

List all non-homogeneous tubes $\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_s$ in mod-Λ (in fact $s \le 3$). For each $\mathcal{T}_i$, let r_i be the period of $\mathcal{T}_i$, $\mathcal{C}^*(\mathcal{T}_i)$ be the corresponding generic composition algebra and $\mathcal{C}^*(\mathcal{T}_i)_{\mathcal{A}}$ be its integral form as we did in Section 4.1.2. For each $\mathcal{T}_i$, denote by Π_i^a the set of aperiodic r_i-tuples of partitions. We have constructed in Section 4.1.2 the elements E_{π_i} for any $\pi_i \in \Pi_i^a$ and the set $\{E_{\pi_i} \mid \pi_i \in \Pi_i^a\}$ is an $\mathcal{A}$-basis of $\mathcal{C}^*(\mathcal{T}_i)_{\mathcal{A}}$.

Let $\mathcal{M}$ be the set of

$$\mathbf{c} = (\mathbf{a_c}, \mathbf{b_c}, \pi_{\mathbf{c}}, w_{\mathbf{c}})$$

where $\mathbf{a_c} \in \mathbb{N}_f^{Prep}$, $\mathbf{b_c} \in \mathbb{N}_f^{Prei}$, $\pi_{\mathbf{c}} = (\pi_{1\mathbf{c}}, \pi_{2\mathbf{c}}, \ldots, \pi_{s\mathbf{c}}) \in \Pi_1^a \times \Pi_2^a \times \cdots \times \Pi_s^a$ and $w_{\mathbf{c}} = (w_1 \ge w_2 \ge \cdots \ge w_t)$ is a partition of $m \in \mathbb{N}$.

For each $\mathbf{c} \in \mathcal{M}$, define

$$E^{\mathbf{c}} = \langle M(\mathbf{a_c})\rangle * E_{\pi_{1\mathbf{c}}} * E_{\pi_{2\mathbf{c}}} * \cdots * E_{\pi_{s\mathbf{c}}} * E_{w_{\mathbf{c}}\delta} * \langle M(\mathbf{b_c})\rangle,$$

where $\langle M(\mathbf{a_c})\rangle$ and $\langle M(\mathbf{b_c})\rangle$ are defined in Section 4.1.3, $E_{\pi_{i\mathbf{c}}}$ is defined in Section 4.1.2 and $E_{w_{\mathbf{c}}\delta}$ is defined in Section 4.1.1.

Note that the set $\{E^{\mathbf{c}} \mid \mathbf{c} \in \mathcal{M}\}$ is contained in $\mathcal{C}^*(Q)$. We have the following proposition.

Proposition 4.4. (cf. [10]) *The set $\{E^{\mathbf{c}} \mid \mathbf{c} \in \mathcal{M}\}$ is an $\mathcal{A}$-basis of $\mathcal{C}^*(Q)_{\mathcal{A}}$.*

From this basis we can get a bar-invariant basis. But it is not the one considered by Lusztig. Hence in [10], another PBW-type basis is constructed. Let us review its definition.

There is a bilinear form $(-,-)$ on $\mathcal{H}_q^*(\Lambda)$ defined in [5]. It is also well-defined on $\mathcal{C}^*(Q)$ which coincides with the one defined by Lusztig in [14]. Consider the $\mathbb{Q}(v)$-basis $\{E^{\mathbf{c}} \mid \mathbf{c} \in \mathcal{M}\}$. Let $R(\mathcal{C}^*(Q))$ be the $\mathbb{Q}(v)$-subspace of $\mathcal{C}^*(Q)$ with the basis $\{E_{\pi_{1\mathbf{c}}} * E_{\pi_{2\mathbf{c}}} * \cdots * E_{\pi_{s\mathbf{c}}} * E_{w_{\mathbf{c}}\delta}\}$, where $\pi_{\mathbf{c}} = (\pi_{1\mathbf{c}}, \pi_{2\mathbf{c}}, \ldots, \pi_{s\mathbf{c}}) \in \Pi_1^a \times \Pi_2^a \times \cdots \times \Pi_s^a$, and $w_{\mathbf{c}} = (w_1 \ge w_2 \ge \cdots \ge w_t)$ is a partition. $R(\mathcal{C}^*(Q))$ is a subalgebra of $\mathcal{C}^*(Q)$.

Let $R^a(\mathcal{C}^*(Q))$ be the subalgebra of $R(\mathcal{C}^*(Q))$ with the basis $\{E_{\pi_{1\mathbf{c}}} * E_{\pi_{2\mathbf{c}}} * \cdots * E_{\pi_{s\mathbf{c}}} \mid \pi_{\mathbf{c}} = (\pi_{1\mathbf{c}}, \pi_{2\mathbf{c}}, \ldots, \pi_{s\mathbf{c}}) \in \Pi_1^a \times \Pi_2^a \times \cdots \times \Pi_s^a\}$. For any $\alpha, \beta \in \mathbb{N}I$, define $\alpha \leq \beta$ if $\beta - \alpha \in \mathbb{N}I$. If $\beta < \delta$, $R(\mathcal{C}^*(Q))_\beta = R^a(\mathcal{C}^*(Q))_\beta$. Define $\mathcal{F}_\delta = \{x \mid (x, R^a(\mathcal{C}^*(Q))_\delta) = 0\}$.

In [10], it is proved that

$$R(\mathcal{C}^*(Q))_\delta = R^a(\mathcal{C}^*(Q))_\delta \oplus \mathcal{F}_\delta$$

and $\dim \mathcal{F}_\delta = 1$. By the method of Schmidt orthogonalization, we may set

$$E'_\delta = E_\delta - \sum_{M(\pi_{i\mathbf{c}}), \underline{\dim} M(\pi_{i\mathbf{c}}) = \delta, 1 \leq i \leq s} a_{\pi_{i\mathbf{c}}} E_{\pi_{i\mathbf{c}}}$$

satisfying $\mathcal{F}_\delta = \mathbb{Q}(v) E'_\delta$.

Now let $R(\mathcal{C}^*(Q))(1)$ be the subalgebra of $R(\mathcal{C}^*(Q))$ generated by $R^a(\mathcal{C}^*(Q))$ and $\mathcal{F}_\delta$. If $\beta < 2\delta$, $R(\mathcal{C}^*(Q))(1)_\beta = R(\mathcal{C}^*(Q))_\beta$. Define

$$\mathcal{F}_{2\delta} = \{x \mid (x, R(\mathcal{C}^*(Q))(1)_{2\delta}) = 0\}.$$

Then $\dim \mathcal{F}_{2\delta} = 1$ and $R(\mathcal{C}^*(Q))_{2\delta} = R(\mathcal{C}^*(Q))(1)_{2\delta} \oplus \mathcal{F}_{2\delta}$.

In general, define

$$\mathcal{F}_{n\delta} = \{x \mid (x, R(\mathcal{C}^*(Q))(n-1)_{n\delta}) = 0\}.$$

Let $R(\mathcal{C}^*(Q))(n)$ be the subalgebra of $R(\mathcal{C}^*(Q))$ generated by $R(\mathcal{C}^*(Q))(n-1)$ and $\mathcal{F}_{n\delta}$. Then $\dim \mathcal{F}_{n\delta} = 1$ and $R(\mathcal{C}^*(Q))_{n\delta} = R(\mathcal{C}^*(Q))(n-1)_{n\delta} \oplus \mathcal{F}_{n\delta}$. Similarly, choose $E'_{n\delta}$ such that $E_{n\delta} - E'_{n\delta} \in R(\mathcal{C}^*(Q))(n-1)_{n\delta}$ and $\mathcal{F}_{n\delta} = \mathbb{Q}(v) E'_{n\delta}$ for all $n \geq 1$.

Let $P_{n\delta} = n E'_{n\delta}$. For a partition $w = (1^{r_1} 2^{r_2} \cdots t^{r_t})$ of $m \in \mathbb{N}$, let $P_{w\delta} = P_{1\delta}^{*r_1} * \cdots * P_{t\delta}^{*r_t}$. For any $\mathbf{c} \in \mathcal{M}$, let $S_{w_{\mathbf{c}}\delta}$ be the Schur function corresponding to $P_{w_{\mathbf{c}}\delta}$ and

$$F^{\mathbf{c}} = \langle M(\mathbf{a}_{\mathbf{c}}) \rangle * E_{\pi_{1\mathbf{c}}} * E_{\pi_{2\mathbf{c}}} * \cdots * E_{\pi_{s\mathbf{c}}} * S_{w_{\mathbf{c}}\delta} * \langle M(\mathbf{b}_{\mathbf{c}}) \rangle.$$

Proposition 4.5. (cf. [10]) *The set $\{F^{\mathbf{c}} \mid \mathbf{c} \in \mathcal{M}\}$ is an almost orthonormal $\mathbb{Q}(v)$-basis of $\mathcal{C}^*(Q) \simeq \mathbf{f}$.*

4.2. **PBW-type basis of $\dot{\mathbf{U}}\mathbf{1}_\lambda$.** Let Q be a connected tame quiver without oriented cycles and $\mathcal{R}(Q)$ be the corresponding root category over some finite field k. Remember that $\tilde{\mathcal{P}}$ is the set of isomorphism classes of objects in $\mathcal{R}(Q)$ and $\mathrm{ind}(\tilde{\mathcal{P}})$ is the set of isomorphism classes of indecomposable objects in $\mathcal{R}(Q)$. The set $\mathrm{ind}(\tilde{\mathcal{P}})$ can be divided into four parts as follow

$$\mathrm{ind}(\tilde{\mathcal{P}}) = \mathbb{P} \mathbin{\dot\cup} \mathbb{T} \mathbin{\dot\cup} T(\mathbb{P}) \mathbin{\dot\cup} T(\mathbb{T}).$$

Fix an embedding of mod-kQ into $\mathcal{R}(Q)$. Then $\mathbb{P} = Prep(Q) \dot\cup T(Prei(Q))$ and $\mathbb{T}$ is the set of isomorphism classes of all indecomposable regular representations of Q. $\mathbb{T}$ consists of isomorphism classes of indecomposable representations in homogeneous tubes and non-homogeneous tubes $\mathcal{T}_1, \mathcal{T}_2, \cdots, \mathcal{T}_s$ appearing in mod-kQ.

Let $\tilde{\mathcal{M}}$ be the set of

$$\tilde{\mathbf{c}} = (\mathbf{d}_{\tilde{\mathbf{c}}}, \pi_{\tilde{\mathbf{c}}}, w_{\tilde{\mathbf{c}}}, \mathbf{d}'_{\tilde{\mathbf{c}}}, \pi'_{\tilde{\mathbf{c}}}, w'_{\tilde{\mathbf{c}}})$$

where

$$\mathbf{d}_{\tilde{\mathbf{c}}} \in \mathbb{N}_f^{\mathbb{P}}, \quad \mathbf{d}'_{\tilde{\mathbf{c}}} \in \mathbb{N}_f^{T(\mathbb{P})},$$

$$\pi_{\tilde{\mathbf{c}}} = (\pi_{1\tilde{\mathbf{c}}}, \pi_{2\tilde{\mathbf{c}}}, \ldots, \pi_{s\tilde{\mathbf{c}}}) \in \Pi_1^a \times \Pi_2^a \times \cdots \times \Pi_s^a,$$
$$\pi'_{\tilde{\mathbf{c}}} = (\pi'_{1\tilde{\mathbf{c}}}, \pi'_{2\tilde{\mathbf{c}}}, \ldots, \pi'_{s\tilde{\mathbf{c}}}) \in \Pi_1^a \times \Pi_2^a \times \cdots \times \Pi_s^a,$$

and

$$w_{\tilde{\mathbf{c}}} = (w_1 \geq w_2 \geq \cdots \geq w_t),$$
$$w'_{\tilde{\mathbf{c}}} = (w'_1 \geq w'_2 \geq \cdots \geq w'_{t'}),$$

are partitions of $m \in \mathbb{N}$ and $m' \in \mathbb{N}$ respectively. $\mathbb{N}_f^{\mathbb{P}}$ is the set of all support-finite functions $\mathbf{d} : \mathbb{P} \to \mathbb{N}$ and $\mathbb{N}_f^{T(\mathbb{P})}$ is the set of all support-finite functions $\mathbf{d} : T(\mathbb{P}) \to \mathbb{N}$. Note that $\pi_{\tilde{\mathbf{c}}} = (\pi_{1\tilde{\mathbf{c}}}, \pi_{2\tilde{\mathbf{c}}}, \ldots, \pi_{s\tilde{\mathbf{c}}})$ and $w_{\tilde{\mathbf{c}}} = (w_1 \geq w_2 \geq \cdots \geq w_t)$, defined in Section 4.1, come from the set $\mathbb{T}$, while $\pi'_{\tilde{\mathbf{c}}} = (\pi'_{1\tilde{\mathbf{c}}}, \pi'_{2\tilde{\mathbf{c}}}, \ldots, \pi'_{s\tilde{\mathbf{c}}})$ and $w'_{\tilde{\mathbf{c}}} = (w'_1 \geq w'_2 \geq \cdots \geq w'_{t'})$, defined in Section 4.1, come from the set $T(\mathbb{T})$.

Note that the category $\mathcal{R}(Q)$, so the set $\tilde{\mathcal{M}}$, depends only on the underlying graph of Q. If Q' is another quiver such that $D^b(kQ) \simeq D^b(kQ')$, they give the same set $\tilde{\mathcal{M}}$.

Given any symmetric generalized Cartan matrix $A = (a_{ij})_{n\times n}$ of affine type, consider a quiver Q, the quantum enveloping algebra $\mathbf{U}$ and the modified quantized enveloping algebra $\dot{\mathbf{U}}$ corresponding to A.

Remember that mod-kQ can be embedding into $\mathcal{R}(Q)$ as a full subcategory and

$$\mathrm{ind}(\tilde{\mathcal{P}}) = \mathrm{ind}(\mathcal{P}) \dot{\cup} \mathrm{ind}(T(\mathcal{P})).$$

For any $\tilde{\mathbf{c}} = (\mathbf{d}_{\tilde{\mathbf{c}}}, \pi_{\tilde{\mathbf{c}}}, w_{\tilde{\mathbf{c}}}, \mathbf{d}'_{\tilde{\mathbf{c}}}, \pi'_{\tilde{\mathbf{c}}}, w'_{\tilde{\mathbf{c}}}) \in \tilde{\mathcal{M}}$, let $\mathbf{d}_1 = \mathbf{d}_{\tilde{\mathbf{c}}}|_{T(Prei(Q))}$ and $\mathbf{d}_2 = \mathbf{d}_{\tilde{\mathbf{c}}}|_{Prep(Q)}$, which is denoted by $\mathbf{d}_{\tilde{\mathbf{c}}} = (\mathbf{d}_1, \mathbf{d}_2)$. Also, let $\mathbf{d}'_1 = \mathbf{d}'_{\tilde{\mathbf{c}}}|_{Prei(Q)}$ and $\mathbf{d}'_2 = \mathbf{d}'_{\tilde{\mathbf{c}}}|_{T(Prep(Q))}$, which is denoted by $\mathbf{d}'_{\tilde{\mathbf{c}}} = (\mathbf{d}'_1, \mathbf{d}'_2)$. Then $\mathbf{c}_1 = (\mathbf{d}_2, \pi_{\tilde{\mathbf{c}}}, w_{\tilde{\mathbf{c}}}, \mathbf{d}'_1)$ and $\mathbf{c}_2 = (\mathbf{d}'_2, \pi'_{\tilde{\mathbf{c}}}, w'_{\tilde{\mathbf{c}}}, \mathbf{d}_1)$ can be regarded as elements in $\mathcal{M}$ and this is denoted by $\tilde{\mathbf{c}} = (\mathbf{c}_1, \mathbf{c}_2)$.

Since we always identify $\mathcal{C}^*(Q)$ with $\mathbf{f}$, the elements in the following set

$$\{F_\lambda^{\tilde{\mathbf{c}}} = \langle M(\mathbf{d}_1)\rangle^- \langle M(\mathbf{d}_2)\rangle^+ E^+_{\pi_{1\tilde{\mathbf{c}}}} E^+_{\pi_{2\tilde{\mathbf{c}}}} \cdots E^+_{\pi_{s\tilde{\mathbf{c}}}} S^+_{w_{\tilde{\mathbf{c}}}\delta} \langle M(\mathbf{d}'_1)\rangle^+ \langle M(\mathbf{d}'_2)\rangle^- E^-_{\pi'_{1\tilde{\mathbf{c}}}} E^-_{\pi'_{2\tilde{\mathbf{c}}}} \cdots E^-_{\pi'_{s\tilde{\mathbf{c}}}} S^-_{w'_{\tilde{\mathbf{c}}}\delta} \mathbf{1}_\lambda\}$$

can be regarded as elements in $\dot{\mathbf{U}}\mathbf{1}_\lambda$. Denote by $B_Q(\dot{\mathbf{U}}\mathbf{1}_\lambda)$ the set $\{F_\lambda^{\tilde{\mathbf{c}}} \mid \tilde{\mathbf{c}} \in \tilde{\mathcal{M}}\}$.

We also consider the following subset of $\dot{\mathbf{U}}\mathbf{1}_\lambda$

$$\{F_\lambda'^{\tilde{\mathbf{c}}} = F^{\mathbf{c}_1+} \cdot F^{\mathbf{c}_2-} \mathbf{1}_\lambda \mid \tilde{\mathbf{c}} = (\mathbf{c}_1, \mathbf{c}_2), \tilde{\mathbf{c}} \in \tilde{\mathcal{M}}\}.$$

This set is denoted by $B'_Q(\dot{\mathbf{U}}\mathbf{1}_\lambda)$.

Lemma 4.6. *The set $B'_Q(\dot{\mathbf{U}}\mathbf{1}_\lambda)$ is a $\mathbb{Q}(v)$-basis of $\dot{\mathbf{U}}\mathbf{1}_\lambda$.*

Proof By Proposition 4.5, $\{F^{\mathbf{c}} \mid \mathbf{c} \in \mathcal{M}\}$ is a basis of $\mathbf{f}$. Remember that $\dot{\mathbf{U}}$ is a free $\mathbf{f} \otimes \mathbf{f}^{\mathrm{opp}}$-module with basis $(\mathbf{1}_\lambda)_{\lambda \in P}$ ([14], 23.2.1). So the set

$$\{F^{\mathbf{c}_1+} \cdot F^{\mathbf{c}_2-} \mathbf{1}_\lambda \mid \mathbf{c}_1 \in \mathcal{M}, \mathbf{c}_2 \in \mathcal{M}\}$$

is a $\mathbb{Q}(v)$-basis of $\dot{\mathbf{U}}\mathbf{1}_\lambda$. By $\tilde{\mathbf{c}} = (\mathbf{c}_1, \mathbf{c}_2)$, the set $\{F_\lambda'^{\tilde{\mathbf{c}}} \mid \tilde{\mathbf{c}} \in \tilde{\mathcal{M}}\}$ is a $\mathbb{Q}(v)$-basis of $\dot{\mathbf{U}}\mathbf{1}_\lambda$.

□

For any $\tilde{\mathbf{c}} = (\mathbf{c}_1, \mathbf{c}_2), \tilde{\mathbf{c}}' = (\mathbf{c}'_1, \mathbf{c}'_2) \in \tilde{\mathcal{M}}$, if $\mathrm{tr}|F^{\mathbf{c}_1}| \neq \mathrm{tr}|F^{\mathbf{c}'_1}|$ or $\mathrm{tr}|F^{\mathbf{c}_2}| \neq \mathrm{tr}|F^{\mathbf{c}'_2}|$, we define $\tilde{\mathbf{c}} < \tilde{\mathbf{c}}'$ if and only if $\mathrm{tr}|F^{\mathbf{c}_1}| \leq \mathrm{tr}|F^{\mathbf{c}'_1}|$ and $\mathrm{tr}|F^{\mathbf{c}_2}| \leq \mathrm{tr}|F^{\mathbf{c}'_2}|$; if $\mathrm{tr}|F^{\mathbf{c}_1}| = \mathrm{tr}|F^{\mathbf{c}'_1}|$ and $\mathrm{tr}|F^{\mathbf{c}_2}| = \mathrm{tr}|F^{\mathbf{c}'_2}|$, we define $\tilde{\mathbf{c}} < \tilde{\mathbf{c}}'$ if and only if $\mathbf{c}_1 \preceq \mathbf{c}'_1$ and $\mathbf{c}_2 \preceq \mathbf{c}'_2$ but $\tilde{\mathbf{c}} \neq \tilde{\mathbf{c}}'$, where $\prec$ is the order on the set $\mathcal{M}$ defined in [10].

Lemma 4.7. *The transition matrix from $B_Q(\dot{\mathbf{U}}\mathbf{1}_\lambda)$ to $B'_Q(\dot{\mathbf{U}}\mathbf{1}_\lambda)$ under the order $<$ defined above is an invertible lower triangular matrix, whose diagonal entries are powers of v and off-diagonal entries belong to $\mathcal{A}$.*

Proof For any $x, y \in \mathbf{f}$ homogeneous, write

$$(r\otimes 1)r(x) = \sum x_1 \otimes x_2 \otimes x_3$$

with $x_k \in \mathbf{f}$ homogeneous and

$$(\bar{r}\otimes 1)\bar{r}(y) = \sum y_1 \otimes y_2 \otimes y_3$$

with $y_k \in \mathbf{f}$ homogeneous, where $r : \mathbf{f} \to \mathbf{f}\otimes\mathbf{f}$ is defined by $r(\theta_i) = \theta_i \otimes 1 + 1 \otimes \theta_i$ and $\bar{r}(x) = \overline{r(\bar{x})}$. By Proposition 3.1.7 in [14], the following equality holds in $\mathbf{U}$:

$$x^- y^+ = \sum (-1)^{\mathrm{tr}|x_1| - \mathrm{tr}|x_3|} v^{-\mathrm{tr}|x_1| + \mathrm{tr}|x_3|}(x_1, y_1) K_{|x_1|} y_2^+ x_2^- \{x_3, y_3\} K_{-|x_3|},$$

where $\{x, y\} = \overline{(\bar{x}, \bar{y})}$. Since $\mathrm{tr}|x_2| \le \mathrm{tr}|x|$ and $\mathrm{tr}|x_2| = \mathrm{tr}|x|$ if and only if $x_1 = x_3 = \mathbf{1}$, $\mathrm{tr}|y_2| \le \mathrm{tr}|y|$ and $\mathrm{tr}|y_2| = \mathrm{tr}|y|$ if and only if $y_1 = y_3 = \mathbf{1}$, we have

$$x^- y^+ \mathbf{1}_\lambda \equiv y^+ x^- \mathbf{1}_\lambda \bmod P(\mathrm{tr}|x| - 1, \mathrm{tr}|y| - 1).$$

Let $\tilde{\mathbf{c}} = (\mathbf{c}_1, \mathbf{c}_2)$. We have

$$\begin{aligned}
& F_\lambda^{\tilde{\mathbf{c}}} \\
= \; & \langle M(\mathbf{d}_1)\rangle^- \langle M(\mathbf{d}_2)\rangle^+ E^+_{\pi_{1\tilde{\mathbf{c}}}} E^+_{\pi_{2\tilde{\mathbf{c}}}} \cdots E^+_{\pi_{s\tilde{\mathbf{c}}}} S^+_{w_{\tilde{\mathbf{c}}}\delta} \langle M(\mathbf{b}_1')\rangle^+ \langle M(\mathbf{d}_2')\rangle^- E^-_{\pi'_{1\tilde{\mathbf{c}}}} E^-_{\pi'_{2\tilde{\mathbf{c}}}} \cdots E^-_{\pi'_{s\tilde{\mathbf{c}}}} S^-_{w'_{\tilde{\mathbf{c}}}\delta} \mathbf{1}_\lambda \\
= \; & \langle M(\mathbf{d}_1)\rangle^- F^{\mathbf{c}_1 +} \langle M(\mathbf{d}_2')\rangle^- E^-_{\pi'_{1\tilde{\mathbf{c}}}} E^-_{\pi'_{2\tilde{\mathbf{c}}}} \cdots E^-_{\pi'_{s\tilde{\mathbf{c}}}} S^-_{w'_{\tilde{\mathbf{c}}}\delta} \mathbf{1}_\lambda \\
\equiv \; & F^{\mathbf{c}_1 +} \langle M(\mathbf{d}_1)\rangle^- \langle M(\mathbf{d}_2')\rangle^- E^-_{\pi'_{1\tilde{\mathbf{c}}}} E^-_{\pi'_{2\tilde{\mathbf{c}}}} \cdots E^-_{\pi'_{s\tilde{\mathbf{c}}}} S^-_{w'_{\tilde{\mathbf{c}}}\delta} \mathbf{1}_\lambda \mod P(m, n)
\end{aligned}$$

where

$$m = \mathrm{tr}|F^{\mathbf{c}_1}|$$

and

$$n = \mathrm{tr}|\langle M(\mathbf{d}_1)\rangle * \langle M(\mathbf{d}_2')\rangle * E_{\pi'_{1\tilde{\mathbf{c}}}} * E_{\pi'_{2\tilde{\mathbf{c}}}} * \cdots * E_{\pi'_{s\tilde{\mathbf{c}}}} * S_{w'_{\tilde{\mathbf{c}}}\delta}| = \mathrm{tr}|F^{\mathbf{c}_2}|.$$

Hence

$$F_\lambda^{\tilde{\mathbf{c}}} = F^{\mathbf{c}_1 +} \langle M(\mathbf{d}_1)\rangle^- \langle M(\mathbf{d}_2')\rangle^- E^-_{\pi'_{1\tilde{\mathbf{c}}}} E^-_{\pi'_{2\tilde{\mathbf{c}}}} \cdots E^-_{\pi'_{s\tilde{\mathbf{c}}}} S^-_{w'_{\tilde{\mathbf{c}}}\delta} \mathbf{1}_\lambda + \sum_{\tilde{\mathbf{c}}' < \tilde{\mathbf{c}}} \tilde{e}_{\tilde{\mathbf{c}}\tilde{\mathbf{c}}'} F_\lambda'^{\tilde{\mathbf{c}}'}$$

for some $\tilde{e}_{\tilde{\mathbf{c}}\tilde{\mathbf{c}}'} \in \mathcal{A}$.

From the definition of the order $\prec$ on $\mathcal{M}$,

$$\begin{aligned}
& F^{\mathbf{c}_1 +} \langle M(\mathbf{d}_1)\rangle^- \langle M(\mathbf{d}_2')\rangle^- E^-_{\pi'_{1\tilde{\mathbf{c}}}} E^-_{\pi'_{2\tilde{\mathbf{c}}}} \cdots E^-_{\pi'_{s\tilde{\mathbf{c}}}} S^-_{w'_{\tilde{\mathbf{c}}}\delta} \mathbf{1}_\lambda \\
= \; & v^f F^{\mathbf{c}_1 +} \left(\langle M(\mathbf{d}_2')\rangle^- E^-_{\pi'_{1\tilde{\mathbf{c}}}} E^-_{\pi'_{2\tilde{\mathbf{c}}}} \cdots E^-_{\pi'_{s\tilde{\mathbf{c}}}} S^-_{w'_{\tilde{\mathbf{c}}}\delta} \langle M(\mathbf{d}_1)\rangle^- + \sum_{\mathbf{c}_2'' \prec \mathbf{c}_2} e_{\mathbf{c}_2\mathbf{c}_2''} F^{\mathbf{c}_2'' -} \right) \mathbf{1}_\lambda \\
= \; & v^f F^{\mathbf{c}_1 +} \left(F^{\mathbf{c}_2 -} + \sum_{\mathbf{c}_2'' \prec \mathbf{c}_2} e_{\mathbf{c}_2\mathbf{c}_2''} F^{\mathbf{c}_2'' -} \right) \mathbf{1}_\lambda \\
= \; & v^f F^{\mathbf{c}_1 +} F^{\mathbf{c}_2 -} \mathbf{1}_\lambda + \sum_{\mathbf{c}_2'' \prec \mathbf{c}_2} v^f e_{\mathbf{c}_2\mathbf{c}_2''} F^{\mathbf{c}_1 +} F^{\mathbf{c}_2'' -} \mathbf{1}_\lambda \\
= \; & v^f F_\lambda'^{\tilde{\mathbf{c}}} + \sum_{\mathbf{c}_2'' \prec \mathbf{c}_2} v^f e_{\mathbf{c}_2\mathbf{c}_2''} F^{\mathbf{c}_1 +} F^{\mathbf{c}_2'' -} \mathbf{1}_\lambda \\
= \; & v^f F_\lambda'^{\tilde{\mathbf{c}}} + \sum_{\tilde{\mathbf{c}}'' < \tilde{\mathbf{c}}, \mathbf{c}_1'' = \mathbf{c}_1} \tilde{e}_{\tilde{\mathbf{c}}\tilde{\mathbf{c}}''} F_\lambda'^{\tilde{\mathbf{c}}''},
\end{aligned}$$

where $\tilde{\mathbf{c}}'' = (\mathbf{c}_1'', \mathbf{c}_2'')$, $f = (|\langle M(\mathbf{d}_1)\rangle|, |\langle M(\mathbf{d}_2')\rangle * E_{\pi'_{1\tilde{\mathbf{c}}}} * E_{\pi'_{2\tilde{\mathbf{c}}}} * \cdots * E_{\pi'_{s\tilde{\mathbf{c}}}} * S_{w'_{\tilde{\mathbf{c}}}\delta}|)$ and $\tilde{e}_{\tilde{\mathbf{c}}\tilde{\mathbf{c}}''} = v^f e_{\mathbf{c}_2\mathbf{c}_2''}$.

Hence

$$F_\lambda^{\tilde{\mathbf{c}}} = v^f F_\lambda'^{\tilde{\mathbf{c}}} + \sum_{\tilde{\mathbf{c}}' < \tilde{\mathbf{c}}} \tilde{e}_{\tilde{\mathbf{c}}\tilde{\mathbf{c}}'} F_\lambda'^{\tilde{\mathbf{c}}'},$$

where $\tilde{e}_{\tilde{\mathbf{c}}\tilde{\mathbf{c}}'} \in \mathcal{A}$.

The proof is finished.

□

Then, we have the following proposition.

Proposition 4.8. *The set* $B_Q(\dot{\mathbf{U}}\mathbf{1}_\lambda) = \{F_\lambda^{\tilde{\mathbf{c}}} \mid \tilde{\mathbf{c}} \in \tilde{\mathcal{M}}\}$ *is a* $\mathbb{Q}(v)$*-basis of* $\dot{\mathbf{U}}\mathbf{1}_\lambda$.

Proof By Lemma 4.6, $\{F_\lambda'^{\tilde{\mathbf{c}}} \mid \tilde{\mathbf{c}} \in \tilde{\mathcal{M}}\}$ is a $\mathbb{Q}(v)$-basis of $\dot{\mathbf{U}}\mathbf{1}_\lambda$. By Lemma 4.7, the transition matrix from $B_Q(\dot{\mathbf{U}}\mathbf{1}_\lambda)$ to $B'_Q(\dot{\mathbf{U}}\mathbf{1}_\lambda)$ under the order $<$ defined above is an invertible lower triangular matrix, whose diagonal entries are powers of v and off-diagonal entries belong to $\mathcal{A}$. Hence, the set $\{F_\lambda^{\tilde{\mathbf{c}}} \mid \tilde{\mathbf{c}} \in \tilde{\mathcal{M}}\}$ is also a $\mathbb{Q}(v)$-basis of $\dot{\mathbf{U}}\mathbf{1}_\lambda$.

□

4.3. **A bar-invariant basis of $\dot{\mathbf{U}}\mathbf{1}_\lambda$.** As before, let Q be a connected tame quiver without oriented cycles and $\mathcal{R}(Q)$ be the corresponding root category. There is an order $\prec$ on the set $\mathcal{M}$ in [10]. For any $\tilde{\mathbf{c}}, \tilde{\mathbf{c}}' \in \tilde{\mathcal{M}}$, define $\tilde{\mathbf{c}} \prec \tilde{\mathbf{c}}'$ if and only if $\mathbf{c}_1 \preceq \mathbf{c}_1'$ and $\mathbf{c}_2 \preceq \mathbf{c}_2'$ but $\tilde{\mathbf{c}} \neq \tilde{\mathbf{c}}'$, where $\tilde{\mathbf{c}} = (\mathbf{c}_1, \mathbf{c}_2)$ and $\tilde{\mathbf{c}}' = (\mathbf{c}_1', \mathbf{c}_2')$.

For any $\mathbf{c} \in \mathcal{M}$, there exists a monomial $\mathbf{m}_\mathbf{c}$ on Chevalley generators u_i satisfying

$$\mathbf{m}_\mathbf{c} = F^\mathbf{c} + \sum_{\mathbf{c}' \prec \mathbf{c}} a_{\mathbf{c}\mathbf{c}'} F^{\mathbf{c}'},$$

where $a_{\mathbf{c}\mathbf{c}'} \in \mathcal{A}$ ([10]). Note that the transition matrix $a = (a_{\mathbf{c}\mathbf{c}'})$ from $\{F^\mathbf{c} \mid \mathbf{c} \in \mathcal{M}\}$ to $\{\mathbf{m}_\mathbf{c} \mid \mathbf{c} \in \mathcal{M}\}$ satisfies that $a_{\mathbf{c}\mathbf{c}} = 1$ and $a_{\mathbf{c}\mathbf{c}'} = 0$ unless $\mathbf{c}' \prec \mathbf{c}$. That is, a is a unipotent lower triangular matrix.

Let $\bar{a} = (\overline{a_{\mathbf{c}\mathbf{c}'}})$. Since $\overline{\mathbf{m}_\mathbf{c}} = \mathbf{m}_\mathbf{c}$, we have

$$\mathbf{m}_\mathbf{c} = \overline{\mathbf{m}_\mathbf{c}} = \sum_{\mathbf{c}'} \bar{a}_{\mathbf{c}\mathbf{c}'} \overline{F^{\mathbf{c}'}},$$

thus

$$\overline{F^\mathbf{c}} = \sum_{\mathbf{c}'} \bar{a}^{-1}_{\mathbf{c}\mathbf{c}'} \mathbf{m}_{\mathbf{c}'} = \sum_{\mathbf{c}'} \sum_{\mathbf{c}''} \bar{a}^{-1}_{\mathbf{c}\mathbf{c}'} a_{\mathbf{c}'\mathbf{c}''} F^{\mathbf{c}''}.$$

Let $h = \bar{a}^{-1} a$. The matrix h is again a unipotent lower triangular matrix and $\bar{h} = h^{-1}$. Similarly to the case of finite type, there exists a unique unipotent lower triangular matrix $d = (d_{\mathbf{c}\mathbf{c}'})$ with off-diagonal entries in $v^{-1}\mathbb{Q}[v^{-1}]$, such that $d = \bar{d}h$. Then the canonical basis of $\mathbf{f}$ is

$$\mathcal{E}^\mathbf{c} = F^\mathbf{c} + \sum_{\mathbf{c}' \prec \mathbf{c}} d_{\mathbf{c}\mathbf{c}'} F^{\mathbf{c}'},$$

with $d_{\mathbf{c}\mathbf{c}'} \in v^{-1}\mathbb{Q}[v^{-1}]$ ([10]).

Similarly, we can get a bar-invariant basis of $\dot{\mathbf{U}}\mathbf{1}_\lambda$ from

$$B'_Q(\dot{\mathbf{U}}\mathbf{1}_\lambda) = \{F^{\mathbf{c}_1+} \cdot F^{\mathbf{c}_2-}\mathbf{1}_\lambda \mid \tilde{\mathbf{c}} \in \tilde{\mathcal{M}}, \tilde{\mathbf{c}} = (\mathbf{c}_1, \mathbf{c}_2)\}$$

and

$$\{\mathbf{m}^+_{\mathbf{c}_1} \cdot \mathbf{m}^-_{\mathbf{c}_2}\mathbf{1}_\lambda \mid \tilde{\mathbf{c}} \in \tilde{\mathcal{M}}, \tilde{\mathbf{c}} = (\mathbf{c}_1, \mathbf{c}_2)\}$$

under the order $\prec$ on $\tilde{\mathcal{M}}$ defined above.

First, define $\mathbf{m}_{\tilde{\mathbf{c}}\lambda} = \mathbf{m}^+_{\mathbf{c_1}} \cdot \mathbf{m}^-_{\mathbf{c_2}} \mathbf{1}_\lambda$ where $\tilde{\mathbf{c}} = (\mathbf{c}_1, \mathbf{c}_2)$. Since

$$\mathbf{m}_{\mathbf{c}} = F^{\mathbf{c}} + \sum_{\mathbf{c}' \prec \mathbf{c}} a_{\mathbf{c}\mathbf{c}'} F^{\mathbf{c}'},$$

we have

$$\mathbf{m}^+_{\mathbf{c}_1} = F^{\mathbf{c}_1 +} + \sum_{\mathbf{c}_1' \prec \mathbf{c}_1} a_{\mathbf{c}_1\mathbf{c}_1'} F^{\mathbf{c}_1' +}$$

and

$$\mathbf{m}^-_{\mathbf{c}_2} = F^{\mathbf{c}_2 -} + \sum_{\mathbf{c}_2' \prec \mathbf{c}_2} a_{\mathbf{c}_2\mathbf{c}_2'} F^{\mathbf{c}_2' -}$$

in $\mathbf{U}^\pm$ respectively. Hence, we have

$$\begin{aligned}
\mathbf{m}_{\tilde{\mathbf{c}}\lambda} &= \mathbf{m}^+_{\mathbf{c}_1} \cdot \mathbf{m}^-_{\mathbf{c}_2} \mathbf{1}_\lambda \\
&= (F^{\mathbf{c}_1} + \sum_{\mathbf{c}_1' \prec \mathbf{c}_1} a_{\mathbf{c}_1\mathbf{c}_1'} F^{\mathbf{c}_1'})^+ \cdot (F^{\mathbf{c}_2} + \sum_{\mathbf{c}_2' \prec \mathbf{c}_2} a_{\mathbf{c}_2\mathbf{c}_2'} F^{\mathbf{c}_2'})^- \mathbf{1}_\lambda \\
&= F^{\mathbf{c}_1 +} \cdot F^{\mathbf{c}_2 -} \mathbf{1}_\lambda + F^{\mathbf{c}_1 +} \cdot \sum_{\mathbf{c}_2' \prec \mathbf{c}_2} a_{\mathbf{c}_2\mathbf{c}_2'} F^{\mathbf{c}_2' -} \mathbf{1}_\lambda + \\
&\quad \sum_{\mathbf{c}_1' \prec \mathbf{c}_1} a_{\mathbf{c}_1\mathbf{c}_1'} F^{\mathbf{c}_1' +} \cdot F^{\mathbf{c}_2 -} \mathbf{1}_\lambda + \sum_{\mathbf{c}_1' \prec \mathbf{c}_1} a_{\mathbf{c}_1\mathbf{c}_1'} F^{\mathbf{c}_1' +} \cdot \sum_{\mathbf{c}_2' \prec \mathbf{c}_2} a_{\mathbf{c}_2\mathbf{c}_2'} F^{\mathbf{c}_2' -} \mathbf{1}_\lambda \\
&= F'^{\tilde{\mathbf{c}}}_\lambda + \sum_{\tilde{\mathbf{c}}' \prec \tilde{\mathbf{c}}} \tilde{a}_{\tilde{\mathbf{c}}\tilde{\mathbf{c}}'} F'^{\tilde{\mathbf{c}}'}_\lambda,
\end{aligned}$$

where $\tilde{\mathbf{c}} = (\mathbf{c}_1, \mathbf{c}_2)$, $\tilde{\mathbf{c}}' = (\mathbf{c}_1', \mathbf{c}_2')$ and $\tilde{a}_{\tilde{\mathbf{c}}\tilde{\mathbf{c}}'} = a_{\mathbf{c}_1\mathbf{c}_1'} a_{\mathbf{c}_2\mathbf{c}_2'} \in \mathcal{A}$.

As before, the transition matrix $\tilde{a} = (\tilde{a}_{\tilde{\mathbf{c}}\tilde{\mathbf{c}}'})$ from $\{F'^{\tilde{\mathbf{c}}}_\lambda \mid \tilde{\mathbf{c}} \in \tilde{\mathcal{M}}\}$ to $\{\mathbf{m}_{\tilde{\mathbf{c}}\lambda} \mid \tilde{\mathbf{c}} \in \tilde{\mathcal{M}}\}$ satisfies that $\tilde{a}_{\tilde{\mathbf{c}}\tilde{\mathbf{c}}} = 1$ and $a_{\tilde{\mathbf{c}}\tilde{\mathbf{c}}'} = 0$ unless $\tilde{\mathbf{c}}' \prec \tilde{\mathbf{c}}$. That is, $\tilde{a}$ is a unipotent lower triangular matrix with off-diagonal entries in $\mathcal{A}$.

Let $\bar{\tilde{a}} = (\overline{\tilde{a}_{\tilde{\mathbf{c}}\tilde{\mathbf{c}}'}})$. Since $\overline{\mathbf{m}_{\tilde{\mathbf{c}}\lambda}} = \mathbf{m}_{\tilde{\mathbf{c}}\lambda}$, we have

$$\mathbf{m}_{\tilde{\mathbf{c}}\lambda} = \overline{\mathbf{m}_{\tilde{\mathbf{c}}\lambda}} = \sum_{\tilde{\mathbf{c}}'} \bar{a}_{\tilde{\mathbf{c}}\tilde{\mathbf{c}}'} \overline{F'^{\tilde{\mathbf{c}}'}_\lambda},$$

thus

$$\overline{F'^{\tilde{\mathbf{c}}}_\lambda} = \sum_{\tilde{\mathbf{c}}'} \bar{\tilde{a}}^{-1}_{\tilde{\mathbf{c}}\tilde{\mathbf{c}}'} \mathbf{m}_{\tilde{\mathbf{c}}'\lambda} = \sum_{\tilde{\mathbf{c}}'} \sum_{\tilde{\mathbf{c}}''} \bar{\tilde{a}}^{-1}_{\tilde{\mathbf{c}}\tilde{\mathbf{c}}'} \tilde{a}_{\tilde{\mathbf{c}}'\tilde{\mathbf{c}}''} F'^{\tilde{\mathbf{c}}''}_\lambda.$$

Let $\tilde{h} = \bar{\tilde{a}}^{-1}\tilde{a}$. The matrix $\tilde{h}$ is again a unipotent lower triangular matrix and $\bar{\tilde{h}} = \tilde{h}^{-1}$. There exists a unique unipotent lower triangular matrix $\tilde{d} = (\tilde{d}_{\tilde{\mathbf{c}}\tilde{\mathbf{c}}'})$ with off-diagonal entries in $v^{-1}\mathbb{Q}[v^{-1}]$, such that $\tilde{d} = \bar{\tilde{d}}\tilde{h}$. Then we get a bar-invariant basis of $\dot{\mathbf{U}}\mathbf{1}_\lambda$

$$\mathcal{E}^{\tilde{\mathbf{c}}}_\lambda = F'^{\tilde{\mathbf{c}}}_\lambda + \sum_{\tilde{\mathbf{c}}' \prec \tilde{\mathbf{c}}} \tilde{d}_{\tilde{\mathbf{c}}\tilde{\mathbf{c}}'} F'^{\tilde{\mathbf{c}}'}_\lambda,$$

with $\tilde{d}_{\tilde{\mathbf{c}}'\tilde{\mathbf{c}}} \in v^{-1}\mathbb{Q}[v^{-1}]$. We denote this basis by $\mathcal{B}_Q(\dot{\mathbf{U}}\mathbf{1}_\lambda)$.

Theorem 4.9. $\mathcal{B}_Q(\dot{\mathbf{U}}\mathbf{1}_\lambda) = \{\mathcal{E}^{\tilde{\mathbf{c}}}_\lambda \mid \tilde{\mathbf{c}} \in \tilde{\mathcal{M}}\} = \{b^+ b'^- \mathbf{1}_\lambda \mid b, b' \in \mathbf{B}\}$.

Proof We use above notations.

First, by the definition of $\tilde{a}$, we have

$$\tilde{a}_{\tilde{\mathbf{c}}\tilde{\mathbf{c}}'} = a_{\mathbf{c}_1\mathbf{c}_1'} a_{\mathbf{c}_2\mathbf{c}_2'}$$

where $\tilde{\mathbf{c}} = (\mathbf{c}_1, \mathbf{c}_2)$, $\tilde{\mathbf{c}}' = (\mathbf{c}'_1, \mathbf{c}'_2)$. Hence, we have

$$\bar{\tilde{a}}_{\tilde{\mathbf{c}}\tilde{\mathbf{c}}'} = \bar{a}_{\mathbf{c}_1\mathbf{c}'_1}\bar{a}_{\mathbf{c}_2\mathbf{c}'_2}.$$

Note that

$$\sum_{\mathbf{c}'} \bar{a}^{-1}_{\mathbf{c}\mathbf{c}'}\bar{a}_{\mathbf{c}'\mathbf{c}} = 1.$$

We have

$$\begin{aligned}
\sum_{\tilde{\mathbf{c}}'}(\bar{a}^{-1}_{\mathbf{c}_1\mathbf{c}'_1}\bar{a}^{-1}_{\mathbf{c}_2\mathbf{c}'_2})(\bar{\tilde{a}}_{\tilde{\mathbf{c}}'\tilde{\mathbf{c}}}) &= \sum_{\tilde{\mathbf{c}}'}(\bar{a}^{-1}_{\mathbf{c}_1\mathbf{c}'_1}\bar{a}^{-1}_{\mathbf{c}_2\mathbf{c}'_2})(\bar{a}_{\mathbf{c}'_1\mathbf{c}_1}\bar{a}_{\mathbf{c}'_2\mathbf{c}_2}) \\
&= \sum_{\mathbf{c}'_1}(\bar{a}^{-1}_{\mathbf{c}_1\mathbf{c}'_1})(\bar{a}_{\mathbf{c}'_1\mathbf{c}_1})\sum_{\mathbf{c}'_2}(\bar{a}^{-1}_{\mathbf{c}_2\mathbf{c}'_2})(\bar{a}_{\mathbf{c}'_2\mathbf{c}_2}) \\
&= 1.
\end{aligned}$$

Hence, we have

$$\bar{\tilde{a}}^{-1}_{\tilde{\mathbf{c}}\tilde{\mathbf{c}}'} = \bar{a}^{-1}_{\mathbf{c}_1\mathbf{c}'_1}\bar{a}^{-1}_{\mathbf{c}_2\mathbf{c}'_2}.$$

Then

$$\begin{aligned}
\tilde{h}_{\tilde{\mathbf{c}}\tilde{\mathbf{c}}''} &= \sum_{\tilde{\mathbf{c}}'}\bar{\tilde{a}}^{-1}_{\tilde{\mathbf{c}}\tilde{\mathbf{c}}'}\tilde{a}_{\tilde{\mathbf{c}}'\tilde{\mathbf{c}}''} \\
&= \sum_{\tilde{\mathbf{c}}'}\bar{a}^{-1}_{\mathbf{c}_1\mathbf{c}'_1}\bar{a}^{-1}_{\mathbf{c}_2\mathbf{c}'_2}a_{\mathbf{c}'_1\mathbf{c}''_1}a_{\mathbf{c}'_2\mathbf{c}''_2} \\
&= \sum_{\mathbf{c}'_1}\bar{a}^{-1}_{\mathbf{c}_1\mathbf{c}'_1}a_{\mathbf{c}'_1\mathbf{c}''_1}\sum_{\mathbf{c}'_2}\bar{a}^{-1}_{\mathbf{c}_2\mathbf{c}'_2}a_{\mathbf{c}'_2\mathbf{c}''_2} \\
&= h_{\mathbf{c}_1\mathbf{c}''_1}h_{\mathbf{c}_2\mathbf{c}''_2}.
\end{aligned}$$

Next, we will check that

$$\tilde{d}_{\tilde{\mathbf{c}}\tilde{\mathbf{c}}'} = d_{\mathbf{c}_1\mathbf{c}'_1}d_{\mathbf{c}_2\mathbf{c}'_2}.$$

By the uniqueness of $\tilde{d}$, we only need to show

$$d_{\mathbf{c}_1\mathbf{c}''_1}d_{\mathbf{c}_2\mathbf{c}''_2} = \sum_{\tilde{\mathbf{c}}'}\overline{d_{\mathbf{c}_1\mathbf{c}'_1}d_{\mathbf{c}_2\mathbf{c}'_2}}\tilde{h}_{\tilde{\mathbf{c}}'\tilde{\mathbf{c}}''}.$$

We can calculate it directly:

$$\begin{aligned}
\sum_{\tilde{\mathbf{c}}'}\overline{d_{\mathbf{c}_1\mathbf{c}'_1}d_{\mathbf{c}_2\mathbf{c}'_2}}\tilde{h}_{\tilde{\mathbf{c}}'\tilde{\mathbf{c}}''} &= \sum_{\tilde{\mathbf{c}}'}\bar{d}_{\mathbf{c}_1\mathbf{c}'_1}\bar{d}_{\mathbf{c}_2\mathbf{c}'_2}h_{\mathbf{c}'_1\mathbf{c}''_1}h_{\mathbf{c}'_2\mathbf{c}''_2} \\
&= \sum_{\mathbf{c}'_1}\bar{d}_{\mathbf{c}_1\mathbf{c}'_1}h_{\mathbf{c}'_1\mathbf{c}''_1}\sum_{\mathbf{c}'_2}\bar{d}_{\mathbf{c}_2\mathbf{c}'_2}h_{\mathbf{c}'_2\mathbf{c}''_2} \\
&= d_{\mathbf{c}_1\mathbf{c}''_1}d_{\mathbf{c}_2\mathbf{c}''_2}.
\end{aligned}$$

Hence, we have

$$\tilde{d}_{\tilde{\mathbf{c}}\tilde{\mathbf{c}}'} = d_{\mathbf{c}_1\mathbf{c}'_1}d_{\mathbf{c}_2\mathbf{c}'_2}.$$

Now, by definition,

$$\begin{aligned}
\mathcal{E}_\lambda^{\tilde{\mathbf{c}}} &= F_\lambda'^{\tilde{\mathbf{c}}} + \sum_{\tilde{\mathbf{c}}' \prec \tilde{\mathbf{c}}} \tilde{d}_{\tilde{\mathbf{c}}\tilde{\mathbf{c}}'} F_\lambda'^{\tilde{\mathbf{c}}'} \\
&= F_\lambda'^{\tilde{\mathbf{c}}} + \sum_{\tilde{\mathbf{c}}' \prec \tilde{\mathbf{c}}} d_{\mathbf{c}_1\mathbf{c}_1'} d_{\mathbf{c}_2\mathbf{c}_2'} F_\lambda'^{\tilde{\mathbf{c}}'} \\
&= F^{\mathbf{c}_1 +} \cdot F^{\mathbf{c}_2 -}\mathbf{1}_\lambda + \sum_{\tilde{\mathbf{c}}' \prec \tilde{\mathbf{c}}} d_{\mathbf{c}_1\mathbf{c}_1'} d_{\mathbf{c}_2\mathbf{c}_2'} F^{\mathbf{c}_1' +} \cdot F^{\mathbf{c}_2' -}\mathbf{1}_\lambda \\
&= (F^{\mathbf{c}_1} + \sum_{\mathbf{c}_1' \prec \mathbf{c}_1} d_{\mathbf{c}_1\mathbf{c}_1'} F^{\mathbf{c}_1'})^+ \cdot (F^{\mathbf{c}_2} + \sum_{\mathbf{c}_2' \prec \mathbf{c}_2} d_{\mathbf{c}_2\mathbf{c}_2'} F^{\mathbf{c}_2'})^- \mathbf{1}_\lambda \\
&= \mathcal{E}^{\mathbf{c}_1 +} \cdot \mathcal{E}^{\mathbf{c}_2 -}\mathbf{1}_\lambda.
\end{aligned}$$

The proof is finished.

□

Remark 4.10. *Although we use the embedding of mod-kQ into $\mathcal{R}(Q)$ to construct the basis $\mathcal{B}_Q(\dot{\mathbf{U}}\mathbf{1}_\lambda)$, this theorem shows that this basis is independent of the choice of the orientation of Q in fact.*

4.4. **A parameterization of the canonical basis of $\dot{\mathbf{U}}\mathbf{1}_\lambda$.** Let $\dot{\mathbf{U}} = \bigoplus_{\lambda \in P} \dot{\mathbf{U}}\mathbf{1}_\lambda$ be the modified quantized enveloping algebra corresponding to the quiver Q and $\dot{\mathbf{B}}_\lambda$ be the canonical basis of $\dot{\mathbf{U}}\mathbf{1}_\lambda$.

Theorem 4.11. *We have a bijection*

$$\Psi_Q : \tilde{\mathcal{M}} \to \dot{\mathbf{B}}_\lambda$$

given by

$$\tilde{\mathbf{c}} \mapsto \mathcal{E}^{\mathbf{c}_1} \diamondsuit_\lambda \mathcal{E}^{\mathbf{c}_2},$$

which is the composition of the following two bijections

$$\begin{aligned}
\tilde{\mathcal{M}} &\to \mathcal{B}_Q(\dot{\mathbf{U}}\mathbf{1}_\lambda) \\
\tilde{\mathbf{c}} &\mapsto \mathcal{E}_\lambda^{\tilde{\mathbf{c}}},
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{B}_Q(\dot{\mathbf{U}}\mathbf{1}_\lambda) &\to \dot{\mathbf{B}}_\lambda \\
b^+ b'^- \mathbf{1}_\lambda &\mapsto b \diamondsuit_\lambda b'.
\end{aligned}$$

Proof The first bijection from $\tilde{\mathcal{M}}$ to $\mathcal{B}_Q(\dot{\mathbf{U}}\mathbf{1}_\lambda)$ comes from our construction of $\mathcal{E}_\lambda^{\tilde{\mathbf{c}}}$ and the second bijection from $\mathcal{B}_Q(\dot{\mathbf{U}}\mathbf{1}_\lambda)$ to $\dot{\mathbf{B}}_\lambda$ comes from formula (1). By Theorem 4.9, $\mathcal{E}_\lambda^{\tilde{\mathbf{c}}} = \mathcal{E}^{\mathbf{c}_1 +} \mathcal{E}^{\mathbf{c}_2 -}\mathbf{1}_\lambda$. Hence, $\Psi_Q : \tilde{\mathcal{M}} \to \dot{\mathbf{B}}_\lambda$ is a bijection.

□

Note that the set $\tilde{\mathcal{M}}$ depends only on the root category $\mathcal{R}(Q)$, not depends on the embedding of mod-kQ into $\mathcal{R}(Q)$. Then all elements in $\tilde{\mathcal{M}}$ give a parameterization of the canonical basis of the modified quantized enveloping algebra by Theorem 4.11.

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