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# A note on standard equivalences

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Abstract

We prove that any derived equivalence between triangular algebras is standard, that is, it is isomorphic to the derived tensor functor given by a two-sided tilting complex.

## 1. *Introduction*

Let $k$ be a field. We require that all categories and functors we are discussing are $k$-linear. Let $A$ be a finite-dimensional $k$-algebra. We denote by $A$-mod the category of finite-dimensional left $A$-modules and by $\mathbf{D}^b(A\text{-mod})$ its bounded derived category.

Let $B$ be another finite-dimensional $k$-algebra. We will require that $k$ acts centrally on any $B$-$A$-bimodule. Recall that a *two-sided tilting complex* is a bounded complex $X$ of $B$-$A$-bimodules such that the derived tensor functor gives an equivalence $X \otimes^{\mathbf{L}}_A -\colon \mathbf{D}^b(A\text{-mod}) \to \mathbf{D}^b(B\text{-mod})$.

A triangle equivalence $F\colon \mathbf{D}^b(A\text{-mod}) \to \mathbf{D}^b(B\text{-mod})$ is said to be *standard* if it is isomorphic, as a triangle functor, to $X \otimes^{\mathbf{L}}_A -$ for some two-sided tilting complex $X$. It is an open question whether all triangle equivalences are standard; see the remarks before [**7**, Corollary 3.5]. We mention that the answer to this question is yes for hereditary algebras in [**6**, Theorem 1.8], and for algebras with ample or anti-ample canonical bundles in [**5**, Theorem 4.5].

The aim of this note is to answer the above question affirmatively in another special case, which contains hereditary algebras.

Recall that an algebra $A$ is *triangular* provided that the Ext-quiver of $A$ has no oriented cycles. There are explicit examples of algebras $A$ and $B$, which are derived equivalent such that $A$ is triangular, but $B$ is not; the reader is referred to the top of [**2**, p. 21]. It makes sense to have the following notion: an algebra $A$ is *derived-triangular* if it is derived equivalent to a triangular algebra.

Theorem 1.1. *Let $A$ be a derived-triangular algebra. Then any triangle equivalence $F\colon \mathbf{D}^b(A\text{-mod}) \to \mathbf{D}^b(B\text{-mod})$ is standard.*

We observe that a derived-triangular algebra has finite global dimension. The converse is not true in general. Indeed, let $A$ be a non-triangular algebra with two simple modules that has finite global dimension; for an example, one may take the Schur algebra $S(2,2)$ in characteristic two. Then $A$ is not derived-triangular. Indeed, any triangular algebra $B$ that is derived equivalent to $A$ has two simple modules and thus is hereditary. This forces that the algebra $A$ is triangular, yielding a contradiction.

We recall that a piecewise hereditary algebra is triangular. In particular, Theorem 1.1 implies that the assumption on the standardness of the autoequivalence in [**4**, Section 4] is superfluous.

The proof of Theorem 1.1 is a rather immediate application of [**1**, Theorem 4.7], which characterizes certain triangle functors between the bounded homotopy categories of Orlov

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categories. Here, we observe that the category of projective modules over a triangular algebra is naturally an Orlov category.

We refer the reader to [**8**, **9**] for unexplained notions in the representation theory of algebras.

## 2. *The bounded homotopy category of an Orlov category*

Let $\mathcal{A}$ be a $k$-linear additive category, which is Hom-finite and has split idempotents. Here, the Hom-finiteness means that all the Hom spaces are finite-dimensional. It follows that $\mathcal{A}$ is a Krull–Schmidt category; see [**3**, Corollary A.2].

We denote by $\operatorname{Ind}\mathcal{A}$ a complete set of representatives of indecomposable objects in $\mathcal{A}$. The category $\mathcal{A}$ is called *bricky* if the endomorphism algebra of each indecomposable object is a division algebra.

We slightly generalize [**1**, Definition 4.1]. A bricky category $\mathcal{A}$ is called an *Orlov category* provided that there is a degree function $\deg\colon \operatorname{Ind}\mathcal{A} \to \mathbb{Z}$ with the following property: for any indecomposable objects $P, P'$ having $\operatorname{Hom}_{\mathcal{A}}(P, P') \neq 0$, we have that $P \simeq P'$ or $\deg(P) > \deg(P')$. An object $X$ in $\mathcal{A}$ is homogeneous of degree $n$ if it is isomorphic to a finite direct sum of indecomposables of degree $n$. An additive functor $F\colon \mathcal{A} \to \mathcal{A}$ is *homogeneous* if it sends homogeneous objects to homogeneous objects and preserves their degrees.

Let $A$ be a finite-dimensional $k$-algebra. We denote by $\{S_1, S_2, \dots, S_n\}$ a complete set of representatives of simple $A$-modules. Denote by $P_i$ the projective cover of $S_i$. We recall that the *Ext-quiver* $Q_A$ of $A$ is defined as follows. The vertex set of $Q_A$ equals $\{1, 2, \dots, n\}$, and there is a unique arrow from $i$ to $j$ provided that $\operatorname{Ext}^1_A(S_i, S_j) \neq 0$. The algebra $A$ is *triangular* provided that $Q_A$ has no oriented cycles.

Let $A$ be a triangular algebra. We denote by $Q^0_A$ the set of sources in $Q_A$. Here, a vertex is a source if there is no arrow ending at it. For each $d \geqslant 1$, we define the set $Q^d_A$ inductively, such that a vertex $i$ belongs to $Q^d_A$ if and only if any arrow ending at $i$ necessarily starts at $\bigcup_{0\leqslant m\leqslant d-1} Q^m_A$. It follows that $Q^0_A \subseteq Q^1_A \subseteq Q^2_A \subseteq \cdots$ and that $\bigcup_{d\geqslant 0} Q^d_A = \{1, 2, \dots, n\}$. We mention that this construction can be found in [**8**, p. 42].

We denote by $A$-proj the category of finite-dimensional projective $A$-modules. Then $\{P_1, P_2, \dots, P_n\}$ is a complete set of representatives of indecomposables in $A$-proj. For each $1 \leqslant i \leqslant n$, we define $\deg(P_i) = d$ such that $i \in Q^d_A$ and $i \notin Q^{d-1}_A$.

The following example of an Orlov category seems to be well known.

LEMMA 2.1. *Let $A$ be a triangular algebra. Then $A$-proj is an Orlov category with the above degree function. Moreover, any equivalence $F\colon A$-proj $\to A$-proj is homogeneous.*

*Proof.* Since $A$ is triangular, it is well known that $\operatorname{End}_A(P_i)$ is isomorphic to $\operatorname{End}_A(S_i)$, which is a division algebra. Then $A$-proj is bricky. We recall that for $i \neq j$ with $\operatorname{Hom}_A(P_i, P_j) \neq 0$, there is a path from $j$ to $i$ in $Q_A$. From the very construction, we infer that, for an arrow $\alpha\colon a \to b$ with $b \in Q^d_A$, we have $a \in Q^{d-1}_A$. Then we are done by the following consequence: if there is a path from $j$ to $i$ in $Q_A$, then $\deg(P_j) < \deg(P_i)$.

For the final statement, we observe that the equivalence $F$ extends to an autoequivalence on $A$-mod, and thus induces an automorphism of $Q_A$. The automorphism preserves the subsets $Q^d_A$. Consequently, the equivalence $F$ preserves degrees, and is homogeneous. □

Let $\mathcal{A}$ be a $k$-linear additive category as above. We denote by $\mathbf{K}^b(\mathcal{A})$ the homotopy category of bounded complexes in $\mathcal{A}$. Here, a complex $X$ is visualized as $\cdots \to X^{n-1} \xrightarrow{d^{n-1}_X} X^n \xrightarrow{d^n_X} X^{n+1} \to \cdots$, where the differentials satisfy $d^n_X \circ d^{n-1}_X = 0$. The translation functor on $\mathbf{K}^b(\mathcal{A})$ is denoted by $[1]$, whose $n$th power is denoted by $[n]$.

We view an object $A$ in $\mathcal{A}$ as a stalk complex concentrated at degree zero, which is still denoted by $A$. In this way, we identify $\mathcal{A}$ as a full subcategory of $\mathbf{K}^b(\mathcal{A})$.

We are interested in triangle functors on $\mathbf{K}^b(\mathcal{A})$. We recall that a *triangle functor* $(F, \theta)$ consists of an additive functor $F\colon \mathbf{K}^b(\mathcal{A}) \to \mathbf{K}^b(\mathcal{A})$ and a natural isomorphism $\theta\colon [1]F \to F[1]$, which preserves triangles. More precisely, for any triangle $X \to Y \to Z \xrightarrow{h} X[1]$ in $\mathbf{K}^b(\mathcal{A})$, the sequence $FX \to FY \to FZ \xrightarrow{\theta_X \circ F(h)} (FX)[1]$ is a triangle. We refer to $\theta$ as the *connecting isomorphism* for $F$. A natural transformation between triangle functors is required to respect the two connecting isomorphisms.

For a triangle functor $(F, \theta)$, the connecting isomorphism $\theta$ is *trivial* if $[1]F = F[1]$ and $\theta$ is the identity transformation. In this case, we suppress $\theta$ and write $F$ for the triangle functor.

Any additive functor $F\colon \mathcal{A} \to \mathcal{A}$ gives rise to a triangle functor $\mathbf{K}^b(F)\colon \mathbf{K}^b(\mathcal{A}) \to \mathbf{K}^b(\mathcal{A})$, which acts on complexes componentwise. The connecting isomorphism for $\mathbf{K}^b(F)$ is trivial. Similarly, any natural transformation $\eta\colon F \to F'$ extends to a natural transformation $\mathbf{K}^b(\eta)\colon \mathbf{K}^b(F) \to \mathbf{K}^b(F')$ between triangle functors.

The following fundamental result is due to [**1**, Theorem 4.7].

PROPOSITION 2.2. *Let $\mathcal{A}$ be an Orlov category and let $(F, \theta)\colon \mathbf{K}^b(\mathcal{A}) \to \mathbf{K}^b(\mathcal{A})$ be a triangle functor such that $F(\mathcal{A}) \subseteq \mathcal{A}$. We assume further that $F|_{\mathcal{A}}\colon \mathcal{A} \to \mathcal{A}$ is homogeneous. Let $F_1, F_2\colon \mathcal{A} \to \mathcal{A}$ be two homogeneous functors.*

(i) *Then there is a unique natural isomorphism $(F, \theta) \to \mathbf{K}^b(F|_{\mathcal{A}})$ of triangle functors, which is the identity on the full subcategory $\mathcal{A}$.*

(ii) *Any natural transformation $\mathbf{K}^b(F_1) \to \mathbf{K}^b(F_2)$ of triangle functors is of the form $\mathbf{K}^b(\eta)$ for a unique natural transformation $\eta\colon F_1 \to F_2$.*

*Proof.* The existence of the natural isomorphism in (1) is due to [**1**, Theorem 4.7]; cf. [**1**, Remark 4.8]. The uniqueness follows from the commutative diagram (4.10) and Lemma 4.5(2) in [**1**], by induction on the support of a complex in the sense of [**1**, Subsection 4.1]. Here, we emphasize that the connecting isomorphism $\theta$ is used in the construction of the natural isomorphism on stalk complexes; compare the second paragraph in [**1**, p. 1541].

The statement (2) follows by the same uniqueness reasoning as above. More precisely, in the notation of [**1**, Theorem 4.7], the extension of $\theta^0$ therein to $\theta$ is unique. □

Recall that $\mathbf{D}^b(A\text{-mod})$ denotes the bounded derived category of $A$-mod. We identify $A$-mod as the full subcategory of $\mathbf{D}^b(A\text{-mod})$ formed by stalk complexes concentrated at degree zero. We denote by $H^n(X)$ the $n$th cohomology of a complex $X$.

The following observation is immediate.

LEMMA 2.3. *Let $A$ be a finite-dimensional algebra and let $F\colon \mathbf{D}^b(A\text{-mod}) \to \mathbf{D}^b(A\text{-mod})$ be a triangle equivalence with $F(A) \simeq A$. Then we have $F(A\text{-mod}) = A\text{-mod}$, and thus the restricted equivalence $F|_{A\text{-mod}}\colon A\text{-mod} \to A\text{-mod}$.*

*Proof.* We use the canonical isomorphisms $H^n(X) \simeq \mathrm{Hom}_{\mathbf{D}^b(A\text{-mod})}(A[-n], X)$. It follows that both $F$ and its quasi-inverse send stalk complexes to stalk complexes. Then we are done. □

We assume that we are given an equivalence $F\colon A\text{-mod} \to A\text{-mod}$ with $F(A) \simeq A$. Then there is an algebra automorphism $\sigma\colon A \to A$ such that $F$ is isomorphic to ${}_\sigma A_1 \otimes_A -$. Here, the $A$-bimodule ${}_\sigma A_1$ is given by the regular right $A$-module, where the left $A$-module is twisted by $\sigma$. This bimodule is invertible and thus viewed as a two-sided tilting complex. We refer the reader to [7,9, Subsection 6.5] for details on two-sided tilting complexes and standard equivalences.

We now combine the above results.

Proposition 2.4. *Let $A$ be a triangular algebra and let $(F,\theta)\colon \mathbf{D}^b(A\text{-mod}) \to \mathbf{D}^b(A\text{-mod})$ be a triangle equivalence with $F(A)\simeq A$. We recall the algebra automorphism $\sigma$ given by the restricted equivalence $F|_{A\text{-mod}}$, and the $A$-bimodule ${}_\sigma A_1$. Then there is a natural isomorphism $(F,\theta) \to {}_\sigma A_1 \otimes^{\mathbf{L}}_A -$ of triangle functors. In particular, the triangle equivalence $(F,\theta)$ is standard.*

*Proof.* Since the algebra $A$ is triangular, it has finite global dimension. The natural functor $\mathbf{K}^b(A\text{-proj}) \to \mathbf{D}^b(A\text{-mod})$ is a triangle equivalence. We identify these two categories. Therefore, the triangle functor $(F,\theta)\colon \mathbf{K}^b(A\text{-proj}) \to \mathbf{K}^b(A\text{-proj})$ restricts to an equivalence $F|_{A\text{-proj}}$, which is isomorphic to ${}_\sigma A_1 \otimes_A -$. By Lemma 2.1, the statements in Proposition 2.2 apply in our situation. Consequently, we have an isomorphism between $(F,\theta)$ and $\mathbf{K}^b({}_\sigma A_1 \otimes_A -)$. Here, we identify the functors $\mathbf{K}^b({}_\sigma A_1 \otimes_A -)$ and ${}_\sigma A_1 \otimes^{\mathbf{L}}_A -$. Then we are done. □

## 3. *The proof of Theorem* 1.1

We now prove Theorem 1.1. In what follows, for simplicity, when writing a triangle functor, we suppress its connecting isomorphism.

We first assume that the algebra $A$ is triangular. The complex $F(A)$ is a one-sided tilting complex. By [9, Theorem 6.4.1], there is a two-sided tilting complex $X$ of $B$-$A$-bimodules with an isomorphism $X \to F(A)$ in $\mathbf{D}^b(B\text{-mod})$. Denote by $G$ a quasi-inverse of the standard equivalence $X \otimes^{\mathbf{L}}_A -\colon \mathbf{D}^b(A\text{-mod}) \to \mathbf{D}^b(B\text{-mod})$. Then the triangle functor $GF\colon \mathbf{D}^b(A\text{-mod}) \to \mathbf{D}^b(A\text{-mod})$ satisfies $GF(A)\simeq A$. Proposition 2.4 implies that $GF$ is standard, and thus $F$ is isomorphic to the composition of $X \otimes^{\mathbf{L}}_A -$ and a standard equivalence. Then we are done in this case by the well-known fact that the composition of two standard equivalences is standard.

In general, let $A$ be derived-triangular. Assume that $A'$ is a triangular algebra that is derived equivalent to $A$. By [9, Proposition 6.5.5], there is a standard equivalence $F'\colon \mathbf{D}^b(A'\text{-mod}) \to \mathbf{D}^b(A\text{-mod})$. The above argument implies that the composition $FF'$ is standard. Recall from [9, Proposition 6.5.6] that a quasi-inverse $F'^{-1}$ of $F'$ is standard. We are done by observing that $F$ is isomorphic to the composition $(FF')F'^{-1}$, a composition of two standard equivalences.

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