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FINITISTIC DIMENSIONS AND PIECEWISE HEREDITARY PROPERTY OF SKEW GROUP ALGEBRAS

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Abstract. Let Λ be a finite-dimensional algebra and G be a finite group whose elements act on Λ as algebra automorphisms. Assume that Λ has a complete set E of primitive orthogonal idempotents, closed under the action of a Sylow p-subgroup $S \leqslant G$. If the action of S on E is free, we show that the skew group algebra ΛG and Λ have the same finitistic dimension, and have the same strong global dimension if the fixed algebra Λ^S is a direct summand of the Λ^S-bimodule Λ. Using a homological characterization of piecewise hereditary algebras proved by Happel and Zacharia, we deduce a criterion for ΛG to be piecewise hereditary.

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1. Introduction. Throughout this note let Λ be a finite-dimensional k-algebra, where k is an algebraically closed field with characteristic $p \geqslant 0$, and let G be a finite group whose elements act on Λ as algebra automorphisms. The *skew group algebra* ΛG is the vector space $\Lambda \otimes_k kG$ equipped with a bilinear product $\cdot$ determined by the following rule: for $\lambda, \mu \in \Lambda$, $g, h \in G$, $(\lambda \otimes g) \cdot (\mu \otimes h) = \lambda g(\mu) \otimes gh$, where $g(\mu)$ is the image of μ under the action of g. We write λg rather than $\lambda \otimes g$ to simplify the notation. Correspondingly, the product can be written as $\lambda g \cdot \mu h = \lambda g(\mu) gh$. Denote the identity of Λ and the identity of G by 1_Λ and 1_G respectively.

It has been observed that when $|G|$, the order of G, is invertible in k, ΛG and Λ share many common properties [**4**, **11**, **13**]. We wonder for arbitrary groups G, under what conditions this phenomenon still happens. This problem is considered in [**9**], where under the hypothesis that Λ has a complete set E of primitive orthogonal idempotents closed under the action of a Sylow p-subgroup $S \leqslant G$, we show that Λ and ΛG share certain properties such as finite global dimension, finite representation type, etc., if and only if the action of S on E is free. Clearly, this answer generalizes results in [**13**] since if $|G|$ is invertible in k, the only Sylow p-subgroup of G is the trivial group.

In this note, we continue to study representations and homological properties of modular skew group algebras. Using the ideas and techniques described in [**9**], we show that Λ and ΛG share more common properties under the same hypothesis and condition. Explicitly, we have:

THEOREM 1.1. *Let Λ and G be as above, and let $S \leqslant G$ be a Sylow p-subgroup. Suppose that Λ has a complete set E of primitive orthogonal idempotents closed under the action of S. Then:*

(1) *If the action of S on E is free, then ΛG, Λ, and Λ^S (the fixed algebra by S) have the same finitistic dimension.*
(2) *ΛG has finite strong global dimension if and only if Λ^S has finite strong global dimension and S acts freely on E. In this situation, ΛG and Λ^S have the same strong global dimension; moreover, if $\Lambda \cong \Lambda^S \oplus B$ as Λ^S-bimodules, then ΛS and Λ have the same strong global dimension.*

In [**4**], it has been proved that if Λ is a piecewise hereditary algebra (defined in Section 3) and $|G|$ is invertible in k, then ΛG is piecewise hereditary as well. The second part of the above theorem generalizes this result by using the homological characterization of piecewise hereditary algebras by Happel and Zacharia in [**6**].

We introduce some notations and conventions here. Throughout this note all modules are finitely generated left modules. Composition of maps and morphisms is from right to left. For an algebra A, gl. dim A, fin. dim A, and sgl. dim A are the global dimension, finitistic dimension, and strong global dimension (defined in Section 4) of A respectively. For an A-module M, $\mathrm{pd}_A M$ is the projective dimension of M. We use A-mod to denote the category of finitely generated A-modules. Its bounded homotopy category and bounded derived category are denoted by $K^b(A)$ and $D^b(A)$ respectively.

2. Projective dimensions and finitistic dimensions. We first describe some background knowledge and elementary results. Most of them can be found in literature. We suggest the reader to refer to [**1**–**3**, **9**–**14**] for more details.

For every subgroup $H \leqslant G$, elements in H also act on Λ as algebra automorphism, so we can define a skew group algebra ΛH, which is a subalgebra of ΛG. The induction functor and the restriction functor are defined in the usual way. For a ΛH module V, the induced module is $V\uparrow_H^G = \Lambda G \otimes_{\Lambda H} V$, where ΛG acts on the left side. Every ΛG-module M can be viewed as a ΛH-module, denoted by $M\downarrow_H^G$. Observe that ΛG is a free (both left and right) ΛH-module. Therefore, these two functors are exact, and perverse projective modules.

PROPOSITION 2.1. *Let $H \leqslant G$ be a subgroup. Then:*

(1) *Every ΛH-module V is isomorphic to a summand of $V\uparrow_H^G\downarrow_H^G$.*
(2) *If $|G : H|$ is invertible in k, then every ΛG-module M is isomorphic to a summand of $M\downarrow_H^G\uparrow_H^G$.*

Proof. This is Proposition 2.1 in [**9**]. □

The above proposition immediately implies:

COROLLARY 2.2. *Let $H \leqslant G$ be a subgroup. For every $M \in \Lambda G$-mod,* $\mathrm{pd}_{\Lambda H} M\downarrow_H^G \leqslant \mathrm{pd}_{\Lambda G} M$. *If $|G : H|$ is invertible in k, then the equality holds.*

Proof. Take a minimal projective resolution $P^\bullet \to M$ of ΛG-modules and apply the restriction functor termwise. Since this functor is exact and preserves projective modules, we get a projective resolution $P^\bullet\downarrow_H^G \to M\downarrow_H^G$ of ΛH-modules, which might not be minimal. Thus $\mathrm{pd}_{\Lambda H} M\downarrow_H^G \leqslant \mathrm{pd}_{\Lambda G} M$, which is true even for the case that either $\mathrm{pd}_{\Lambda H} M\downarrow_H^G = \infty$ or $\mathrm{pd}_{\Lambda G} M = \infty$.

Now, suppose that $|G : H|$ is invertible in k. Take a minimal projective resolution $Q^\bullet \to M\downarrow_H^G$ of ΛH-modules and apply the induction functor termwise. Since, it is exact and preserves projective modules, we get a projective resolution

$Q^\bullet \uparrow_H^G \to M \downarrow_H^G \uparrow_H^G$ of ΛG-modules, so $\operatorname{pd}_{\Lambda G} M \downarrow_H^G \uparrow_H^G \leqslant \operatorname{pd}_{\Lambda H} M \downarrow_H^G$. But M is isomorphic to a summand of $M \downarrow_H^G \uparrow_H^G$, so $\operatorname{pd}_{\Lambda G} M \leqslant \operatorname{pd}_{\Lambda H} M \downarrow_H^G$. Putting two inequalities together, we get the equality. □

Let $E = \{e_i\}_{i \in [n]}$ be a set of primitive orthogonal idempotents in Λ. We say it is *complete* if $\sum_{i \in [n]} e_i = 1_\Lambda$. Throughout this note we assume that G has a Sylow p-subgroup S such that E is closed under the action of S. That is, $g(e_i) \in E$ for all $i \in [n]$ and $g \in S$. In practice, this condition is usually satisfied. A trivial case is that $|G|$ is invertible in k, and hence S is the trivial subgroup.

We introduce some notations here. Let Λ^S be the space consisting of all elements in Λ fixed by S. Clearly, Λ^S is a subalgebra of Λ, and S acts on it trivially. For every $M \in \Lambda S$-mod, elements $v \in M$ satisfying $g(v) = v$ for every $g \in S$ form a Λ^S-module, which is denoted by M^S. Let F^S be the functor from ΛS-mod to Λ^S-mod, sending M to M^S. This is indeed a functor since for every ΛS-module homomorphism $f : M \to N$, M^S is mapped into N^S by f.

We collect in the following proposition some results taken from [**9**].

PROPOSITION 2.3. *Let $S \leqslant G$ and E be as above and suppose that E is closed under the action of S. Then:*

(1) The set E is also a complete set of primitive orthogonal idempotents of ΛS, where we identify e_i with $e_i 1_S$.
(2) $\Lambda^S = \{\sum_{g \in S} g(\mu) \mid \mu \in \Lambda\}$.
(3) The global dimension $\operatorname{gl.dim} \Lambda S < \infty$ *if and only if* $\operatorname{gl.dim} \Lambda < \infty$ *and the action of S on E is free.*

Moreover, if the action of S on E is free, then

(4) ΛS is a matrix algebra over Λ^S, and hence is Morita equivalent to Λ^S.
(5) The functor F^S is exact.
(6) The regular representation ${}_{\Lambda S}\Lambda S \cong \Lambda^{|S|}$, where Λ is the trivial ΛS-module.
(7) A ΛS-module M is projective (resp., injective) if and only if the Λ-module ${}_\Lambda M$ is projective (resp., injective).

Proof. See Lemmas 3.1, 3.2, 3.6 and Propositions 3.3, 3.5 in [**9**]. □

Recall for a finite-dimensional algebra A, its *finitistic dimension*, denoted by $\operatorname{fin.dim} A$, is the supremum of projective dimensions of finitely generated indecomposable A-modules M with $\operatorname{pd}_A M < \infty$. If A has finite global dimension, then $\operatorname{fin.dim} A = \operatorname{gl.dim} A$. The famous finitistic dimension conjecture asserts that the finitistic dimension of every finite-dimensional algebra is finite. For more details, see [**15**].

An immediate consequence of Proposition 2.1 and Corollary 2.2 is:

LEMMA 2.4. *Let $H \leqslant G$ be a subgroup of G. Then* $\operatorname{fin.dim} \Lambda H \leqslant \operatorname{fin.dim} \Lambda G$. *If $|G : H|$ is invertible in k, the equality holds.*

Proof. Take an arbitrary indecomposable $V \in \Lambda H$-mod with $\operatorname{pd}_{\Lambda H} V < \infty$ and consider $\tilde{V} = V \uparrow_H^G$. We claim that $\operatorname{pd}_{\Lambda G} \tilde{V} < \infty$. Indeed, by applying the induction functor to a minimal projective resolution $P^\bullet \to V$ termwise, we get a projective resolution $\Lambda G \otimes_{\Lambda H} P^\bullet \to \tilde{V}$. The finite length of the first resolution implies the finite length of the second one. Therefore, $\operatorname{pd}_{\Lambda G} \tilde{V} < \infty$. Consequently, $\operatorname{pd}_{\Lambda G} \tilde{V} \leqslant \operatorname{fin.dim} \Lambda G$.

By Corollary 2.2, $\mathrm{pd}_{\Lambda G}\tilde{V} \geqslant \mathrm{pd}_{\Lambda H}\tilde{V}\downarrow_H^G$. Since, V is isomorphic to a summand of $\tilde{V}\downarrow_H^G$ by Proposition 2.1, we get $\mathrm{fin.\,dim}\,\Lambda G \geqslant \mathrm{pd}_{\Lambda G}\tilde{V} \geqslant \mathrm{pd}_{\Lambda H}\tilde{V}\downarrow_H^G \geqslant \mathrm{pd}_{\Lambda H} V$. In conclusion, $\mathrm{fin.\,dim}\,\Lambda G \geqslant \mathrm{fin.\,dim}\,\Lambda H$.

Now, suppose that $|G : H|$ is invertible in k. Take an arbitrary indecomposable $M \in \Lambda G$-mod with $\mathrm{pd}_{\Lambda G} M < \infty$. By applying the restriction functor $\downarrow_H^G$ to a minimal projective resolution $Q^\bullet \to M$ termwise, we get a projective resolution of $M\downarrow_H^G$, which is of finite length. Therefore, $\mathrm{pd}_{\Lambda H} M\downarrow_H^G \leqslant \mathrm{fin.\,dim}\,\Lambda H$. By Corollary 2.2, we have $\mathrm{pd}_{\Lambda G} M = \mathrm{pd}_{\Lambda H} M\downarrow_H^G \leqslant \mathrm{fin.\,dim}\,\Lambda H$. In conclusion, $\mathrm{fin.\,dim}\,\Lambda G \leqslant \mathrm{fin.\,dim}\,\Lambda H$. Combining this with the inequality in the previous paragraph, we have $\mathrm{fin.\,dim}\,\Lambda G = \mathrm{fin.\,dim}\,\Lambda H$. □

LEMMA 2.5. *If a Sylow p-subgroup $S \leqslant G$ acts freely on E, then* $\mathrm{fin.\,dim}\,\Lambda S \leqslant \mathrm{fin.\,dim}\,\Lambda$.

Proof. Take an arbitrary indecomposable $M \in \Lambda S$-mod with $\mathrm{pd}_{\Lambda S} M = n < \infty$ and a minimal projective resolution:

$$\cdots \to 0 \to P^n \to \cdots \to P^1 \to P^0 \to M \to 0.$$

Regarded as Λ-modules, we get a projective resolution:

$$\cdots \to 0 \to {}_\Lambda P^n \to \cdots \to {}_\Lambda P^1 \to {}_\Lambda P^0 \to {}_\Lambda M \to 0.$$

Clearly, for every $0 \leqslant s \leqslant n$, the syzygy $\Omega^s(M)$ viewed as a Λ-module is a direct sum of $\Omega^s({}_\Lambda M)$ and a projective Λ-module. We claim that for each $0 \leqslant s \leqslant n$, the syzygy $\Omega^s({}_\Lambda M) \neq 0$. Otherwise, $\Omega^s(M)$ viewed as a Λ-module is projective. By (7) of Proposition 2.3, $\Omega^s(M)$ is a projective ΛS-module, which must be 0. But this implies $\mathrm{pd}_{\Lambda S} M < n$, contradicting our choice of M. Therefore, for each $0 \leqslant s \leqslant n$, $\Omega^s({}_\Lambda M) \neq 0$. Consequently, ${}_\Lambda M$ has a summand with projective dimension n, so $n \leqslant \mathrm{fin.\,dim}\,\Lambda$. In conclusion, $\mathrm{fin.\,dim}\,\Lambda S \leqslant \mathrm{fin.\,dim}\,\Lambda$. □

Now, we can prove the first statement of Theorem 1.1.

PROPOSITION 2.6. *If G has a Sylow p-subgroup S acting freely on E, then ΛG and Λ have the same finitistic dimension.*

Proof. Since $|G : S|$ is invertible in k, by Lemma 3.1, $\mathrm{fin.\,dim}\,\Lambda G = \mathrm{fin.\,dim}\,\Lambda S$. Also by this lemma, $\mathrm{fin.\,dim}\,\Lambda S \geqslant \mathrm{fin.\,dim}\,\Lambda$ since S contains the trivial group. The previous lemma tells us $\mathrm{fin.\,dim}\,\Lambda S \leqslant \mathrm{fin.\,dim}\,\Lambda$. Putting all these pieces of information together, we get $\mathrm{fin.\,dim}\,\Lambda G = \mathrm{fin.\,dim}\,\Lambda$ as claimed. □

From the previous proposition we conclude immediately that if the action of S on E is free, then the finiteness of finitistic dimension of Λ implies the finiteness of finitistic dimension of ΛG. We wonder whether this conclusion is true in general. That is,

CONJECTURE 2.7. Let Λ, G, E and S be as before and suppose that E is closed under the action of S. If $\mathrm{fin.\,dim}\,\Lambda < \infty$ (or even stronger $\mathrm{gl.\,dim}\,\Lambda < \infty$), then $\mathrm{fin.\,dim}\,\Lambda G < \infty$.

Hopefully the proof of this question can shed light on the final proof of the finitistic dimension conjecture.

3. Strong global dimensions and piecewise hereditary algebras. A finite-dimensional k-algebra A is called *piecewise hereditary* if the derived category $D^b(A)$ is equivalent to the derived category $D^b(\mathcal{H})$ of a hereditary abelian category $\mathcal{H}$ as triangulated categories; see [**5–8**]. If A is piecewise hereditary, then $\operatorname{gl.dim} A < \infty$ since the property having finite global dimension is invariant under derived equivalence [**5**]. Therefore, $D^b(A)$ is triangulated equivalent to the homotopy category $K^b({}_A\mathcal{P})$, where ${}_A\mathcal{P}$ is the full subcategory of A-mod consisting of all finitely generated projective A-modules.

Now, let A be a finite-dimensional algebra with $\operatorname{gl.dim} A < \infty$. Since $D^b(A) \cong K^b({}_A\mathcal{P})$, we identify these two triangulated categories. For every indecomposable object $0 \neq P^\bullet \in K^b({}_A\mathcal{P})$, consider its preimages $\tilde{P}^\bullet \in C^b({}_A\mathcal{P})$, the category of so-called *perfect complexes*, i.e., each term of $\tilde{P}^\bullet$ is a finitely generated projective A-module, and all but finitely many terms of $\tilde{P}^\bullet$ are 0. We can choose $\tilde{P}^\bullet$ such that it is *minimal*. That is, $\tilde{P}^\bullet$ has no direct summands isomorphic to

$$\cdots \longrightarrow 0 \longrightarrow P^s \xrightarrow{id} P^{s+1} \longrightarrow 0 \longrightarrow \cdots,$$

or equivalently, each differential map in $\tilde{P}^\bullet$ sends its source into the radical of the subsequent term. Clearly, this choice is unique for each $P^\bullet \in K^b({}_A\mathcal{P})$ up to isomorphism, so in the rest of this note we identify $P^\bullet$ and $\tilde{P}^\bullet$. Hopefully this identification will not cause trouble to the reader.

Take an indecomposable $P^\bullet \in K^b({}_A\mathcal{P})$ and identify it with $\tilde{P}^\bullet$. Therefore, there exist $r \leqslant s \in \mathbb{Z}$ such that $P^r \neq 0 \neq P^s$, and $P^n = 0$ for $n > s$ or $n < r$. We define its length $l(P^\bullet)$ to be $s - r$. [1] The *strong global dimension* of A, denoted by $\operatorname{sgl.dim} A$, is defined as $\sup\{l(P^\bullet) \mid P^\bullet \in K^b({}_A\mathcal{P}) \text{ is indecomposable}\}$. By taking minimal projective resolutions of simple modules, it is easy to see that $\operatorname{sgl.dim} A \geqslant \operatorname{gl.dim} A$. Moreover, if A is hereditary, then $\operatorname{sgl.dim} A = \operatorname{gl.dim} A \leqslant 1$ (see [**5**]). It is not clear for what algebras of finite global dimension, $\operatorname{sgl.dim} A = \operatorname{gl.dim} A$.

Happel and Zacharia proved the following result, characterizing piecewise hereditary algebras.

THEOREM 3.1 (Theorem 3.2 in [**7**]). *A finite-dimensional k-algebra is piecewise hereditary if and only if* $\operatorname{sgl.dim} A < \infty$.

In [**4**] Dionne, Lanzilotta, and Smith show that if Λ is a piecewise hereditary algebra, and $|G|$ is invertible in k, then the skew group algebra ΛG is also piecewise hereditary. This result motivates us to characterize general piecewise hereditary skew group algebras. Using the above characterization, we take a different approach by showing that $\operatorname{sgl.dim} \Lambda G = \operatorname{sgl.dim} \Lambda$ under suitable conditions.

We first prove a result similar to Lemma 2.4.

LEMMA 3.2. *Let H be a subgroup of G and suppose that* $\operatorname{gl.dim} \Lambda G < \infty$. *Then* $\operatorname{sgl.dim} \Lambda H \leqslant \operatorname{sgl.dim} \Lambda G$. *The equality holds if $|G : H|$ is invertible in k.*

Proof. Note that $\operatorname{gl.dim} \Lambda H \leqslant \operatorname{gl.dim} \Lambda G < \infty$ by (2) of Corollary 2.2 in [**9**]. Take an indecomposable object $P^\bullet \in K^b({}_{\Lambda H}\mathcal{P})$. Up to isomorphism, its minimal preimage

[1] By this definition, the length of a minimal object $X^\bullet \in K^b({}_A P)$ counts the number of arrows between the first nonzero term and the last nonzero term in $X^\bullet$, rather than the number of nonzero terms in $X^\bullet$. In particular, a projective Λ-module, when viewed as a stalk complex in $K^b({}_A P)$, has length 0.

in $C^b({}_{\Lambda H}\mathcal{P})$ can be written as:

$$\cdots \to 0 \to P^r \xrightarrow{d^r} \cdots \longrightarrow P^{s-1} \xrightarrow{d^{s-1}} P^s \to 0 \to \cdots .$$

Applying the exact functor $\Lambda G \otimes_{\Lambda H} -$ termwise to the above complex, we get an object $\tilde{P}^\bullet = P^\bullet \uparrow_H^G \in C^b({}_{\Lambda G}\mathcal{P})$ as follows:

$$\cdots \to 0 \to \Lambda G \otimes P^r \xrightarrow{1 \otimes d^r} \cdots \longrightarrow \Lambda G \otimes P^{s-1} \xrightarrow{1 \otimes d^{s-1}} \Lambda G \otimes P^s \to 0 \to \cdots .$$

Applying the restriction functor termwise, the second complex gives an object $\tilde{P}^\bullet \downarrow_H^G \in C^b({}_{\Lambda H}\mathcal{P})$. We claim that $P^\bullet$ is isomorphic to a direct summand of $\tilde{P}^\bullet \downarrow_H^G$.

Indeed, there is a family of maps $(\iota^i)_{i\in\mathbb{Z}}$ defined as follows: for $i > s$ or $i < r$, $\iota_i = 0$; for $r \leqslant i \leqslant s$, ι^i sends $v \in P^i$ to $1 \otimes v \in \Lambda G \otimes_{\Lambda H} P^i$. The reader can check that $(\iota^i)_{i\in\mathbb{Z}}$ defined in this way indeed gives rise to a chain map $\iota^\bullet : P^\bullet \to \tilde{P}^\bullet \downarrow_H^G$.

We define another family of maps $(\delta^i)_{i\in\mathbb{Z}}$ as follows: for $i > s$ or $i < r$, $\delta^i = 0$; for $r \leqslant i \leqslant s$, δ^i sends $h \otimes v \in \Lambda H \otimes_{\Lambda H} P^i$ to $hv \in P^i$ and sends all vectors in $g \otimes v \in \bigoplus_{1\neq g\in G/H} g \otimes P^i$ to 0. Again, the reader can check that $(\delta^i)_{i\in\mathbb{Z}}$ gives rise to a chain map $\delta^\bullet : \tilde{P}^\bullet \downarrow_H^G \to P^\bullet$. Moreover, we have $\delta^\bullet \circ \iota^\bullet$ is the identity map. Therefore, $P^\bullet$ is isomorphic to a direct summand of $\tilde{P}^\bullet \downarrow_H^G$.

This claim has the following consequence: for every indecomposable object $X^\bullet \in K^b({}_{\Lambda H}\mathcal{P})$ (identified with a minimal perfect complex in $C^b({}_{\Lambda H}\mathcal{P})$), there is an indecomposable object $\tilde{X}^\bullet \in K^b({}_{\Lambda G}\mathcal{P})$ such that $X^\bullet$ is isomorphic to a direct summand of $\tilde{X}^\bullet \downarrow_H^G$. Clearly, we have $l(\tilde{X}^\bullet) \geqslant l(X^\bullet)$. Therefore, by definition, $\operatorname{sgl.dim} \Lambda G \geqslant \operatorname{sgl.dim} \Lambda H$.

Now, suppose that $|G : H|$ is invertible in k. Take an indecomposable object $P^\bullet \in K^b({}_{\Lambda G}\mathcal{P})$. Up to isomorphism, its minimal preimage in $C^b({}_{\Lambda G}\mathcal{P})$ can be written as:

$$\cdots \to 0 \to P^r \xrightarrow{d^r} \cdots \longrightarrow P^{s-1} \xrightarrow{d^{s-1}} P^s \to 0 \to \cdots .$$

Applying the restriction functor and the induction functor termwise to the above complex, we get another object in $\tilde{P}^\bullet = P^\bullet \downarrow_H^G \uparrow_H^G \in C^b({}_{\Lambda G}\mathcal{P})$ as follows:

$$\cdots \to 0 \to \Lambda G \otimes P^r \xrightarrow{1 \otimes d^r} \cdots \longrightarrow \Lambda G \otimes P^{s-1} \xrightarrow{1 \otimes d^{s-1}} \Lambda G \otimes P^s \to 0 \to \cdots .$$

We claim that $P^\bullet$ is isomorphic to a direct summand of $\tilde{P}^\bullet$.

Define a family of maps $(\theta^i)_{i\in\mathbb{Z}}$ as follows: for $i > s$ or $i < r$, $\theta^i = 0$; for $r \leqslant i \leqslant s$, θ^i sends $v \in P^i$ to $\frac{1}{|G:H|} \sum_{g\in G/H} g \otimes g^{-1}v \in \Lambda G \otimes_{\Lambda H} P^i$. We check that $(\theta^i)_{i\in\mathbb{Z}}$ defined in this way gives rise to a chain map $\theta^\bullet : P^\bullet \to \tilde{P}^\bullet$. Indeed, for $r \leqslant i \leqslant s-1$ and $v \in P^i$, we have

$$\begin{aligned}(\theta^{i+1} \circ d^i)(v) = \theta^{i+1}(d^i(v)) &= \frac{1}{|G : H|} \sum_{g\in G/H} g \otimes g^{-1} d^i(v) \\ &= \frac{1}{|G : H|} \sum_{g\in G/H} g \otimes d^i(g^{-1}v) = (1 \otimes d^i)(\theta^{i+1}(v)).\end{aligned}$$

Define another family of maps $(\rho^i)_{i\in\mathbb{Z}}$ as follows: for $i > s$ or $i < r$, $\rho^i = 0$; for $r \leqslant i \leqslant s$, ρ^i sends $g \otimes v \in \Lambda G \otimes_{\Lambda H} P^i$ to $gv \in P^i$. Again, we check that $(\rho^i)_{i\in\mathbb{Z}}$ gives rise to a chain map $\rho^\bullet : \tilde{P}^\bullet \to P^\bullet$ as shown by:

$$(\rho^{i+1} \circ (1 \otimes d^i))(g \otimes v) = \rho^{i+1}(g \otimes d^i(v))) = gd^i(v) = d^i(gv) = d^i(\rho^i(g \otimes v)).$$

Moreover, $\rho^\bullet \circ \theta^\bullet$ is the identity map. Therefore, as claimed, $P^\bullet$ is isomorphic to a summand of $\tilde{P}^\bullet$.

This claim tells us that for every indecomposable object $\tilde{X}^\bullet \in K^b({}_{\Lambda G}\mathcal{P})$ (identified with a minimal perfect complex in $C^b({}_{\Lambda G}\mathcal{P})$), there is an indecomposable object $X^\bullet \in K^b({}_{\Lambda H}\mathcal{P})$ such that $\tilde{X}^\bullet$ is isomorphic to a direct summand of $X^\bullet \uparrow_H^G$. Therefore, $l(X^\bullet) \geqslant l(\tilde{X}^\bullet)$, and $\operatorname{sgl.dim} \Lambda H \geqslant \operatorname{sgl.dim} \Lambda G$. The two inequalities force $\operatorname{sgl.dim} \Lambda G = \operatorname{sgl.dim} \Lambda H$. □

REMARK 3.3. In this lemma we actually proved a stronger conclusion. That is, the induction and restriction functor induce an 'induction' functor and a 'restriction' functor between the homotopy categories of perfect complexes. Moreover, every indecomposable object $X^\bullet \in K^b({}_{\Lambda H}\mathcal{P})$ can be obtained by applying the 'restriction' functor to an indecomposable object in $K^b({}_{\Lambda G}\mathcal{P})$ and taking a direct summand. When $|G : H|$ is invertible, every indecomposable object $\tilde{X}^\bullet \in K^b({}_{\Lambda G}\mathcal{P})$ can be obtained by applying the 'induction' functor to an indecomposable object in $K^b({}_{\Lambda H}\mathcal{P})$ and taking a direct summand.

Now, we are ready to prove the second part of our main theorem.

PROPOSITION 3.4. *Let Λ, G, S, and E as before. Then ΛG has finite strong global dimension if and only if Λ^S has finite strong global dimension and S acts freely on E. In this situation, ΛG and Λ^S have the same strong global dimension.*

Proof. Since the strong global dimension of ΛG equals that of ΛS, without loss of generality we assume that $G = S$. Note that the strong global dimension is always greater than or equal to the global dimension. Therefore, ΛS has finite global dimension, so S must act freely on E by (3) of Proposition 2.3. Then by (4) of this proposition, ΛS is Morita equivalent to Λ^S, so they have the same finite strong global dimension. Conversely, if the action of S on E is free, then again ΛS and Λ^S are Morita equivalent, so they have the same strong global dimension. □

An immediate corollary of this proposition and Theorem 3.1 is:

COROLLARY 3.5. *Let Λ, G, S, and E as before. Then ΛG is piecewise hereditary if and only if the action of S on E is free and Λ^S is piecewise hereditary.*

Proof. By Theorem 3.1, a finite-dimensional algebra is piecewise hereditary if and only if its strong global dimension is finite. The conclusion follows from the above proposition. □

In the rest of this section, we try to get a get a criterion such that the strong global dimension of Λ equals that of ΛS when the action of S on E is free. Since, we already know that ΛS is Morita equivalent to Λ^S (Proposition 2.3), instead we show that $\operatorname{sgl.dim} \Lambda = \operatorname{sgl.dim} \Lambda^S$ under a certain condition. Note that Λ^S is a subalgebra of Λ, so we can define the corresponding restriction functor from Λ-mod to Λ^S-mod and the induction functor $\Lambda \otimes_{\Lambda^S} -$ from Λ^S-mod to Λ-mod.

By **[9]**, Λ is both a left free and a right free Λ^S-module, but might not be a free bimodule; see Example 3.6 in that paper. Now, assume that $\Lambda = \Lambda^S \oplus B$ as

Λ^S-bimodule. For instance, if Λ^S is commutative, this condition is satisfied. Under this assumption, we have a split bimodule homomorphism $\zeta : \Lambda \to \Lambda^S$.

For $M \in \Lambda^S$-mod, we define two linear maps:

$$\psi : M \to (\Lambda \otimes_{\Lambda^S} M) \downarrow^{\Lambda}_{\Lambda^S}, \quad v \mapsto 1 \otimes v, \quad v \in M;$$
$$\varphi : (\Lambda \otimes_{\Lambda^S} M) \downarrow^{\Lambda}_{\Lambda^S} \to M, \quad \lambda \otimes v \mapsto \zeta(\lambda)v, \quad v \in M, \lambda \in \Lambda.$$

These two maps are well defined Λ^S-module homomorphisms (to check it, we need the assumption that A^S is a summand of Λ as Λ^S-bimodules). We also observe that when $\lambda \in \Lambda^S$,

$$\varphi(\lambda \otimes v) = \zeta(\lambda)v = \lambda v.$$

Therefore, $\varphi \circ \psi$ is the identity map.

PROPOSITION 3.6. *Suppose that a Sylow p-subgroup $S \leqslant G$ acts freely on E, and $\Lambda \cong \Lambda^S \oplus B$ as Λ^S-bimodules. Then* $\operatorname{sgl.dim} \Lambda = \operatorname{sgl.dim} \Lambda S$.

Proof. By Lemma 3.2 and Proposition 3.4, $\operatorname{sgl.dim} \Lambda \leqslant \operatorname{sgl.dim} \Lambda S = \operatorname{sgl.dim} \Lambda^S$. Therefore, it suffices to show $\operatorname{sgl.dim} \Lambda \geqslant \operatorname{sgl.dim} \Lambda^S$. The proof is similar to that of Lemma 3.2, using the corresponding induction and restriction functors for the pair (Λ^S, Λ) and the two maps ψ, φ defined above.

Take an indecomposable object $P^\bullet \in K^b({}_{\Lambda^S}\mathcal{P})$ and write its minimal preimage in $C^b({}_{\Lambda^S}\mathcal{P})$ as follows:

$$\cdots \to 0 \to P^r \xrightarrow{d^r} \cdots \longrightarrow P^{s-1} \xrightarrow{d^{s-1}} P^s \to 0 \to \ldots.$$

Applying $\Lambda \otimes_{\Lambda^S} -$ and the restriction functor termwise, we get another object in $\tilde{P}^\bullet = P^\bullet \uparrow^{\Lambda}_{\Lambda^S} \downarrow^{\Lambda}_{\Lambda^S} \in C^b({}_{\Lambda^S}\mathcal{P})$:

$$\cdots \to 0 \to \Lambda \otimes P^r \xrightarrow{1 \otimes d^r} \cdots \longrightarrow \Lambda \otimes P^{s-1} \xrightarrow{1 \otimes d^{s-1}} \Lambda \otimes P^s \to 0 \to \cdots.$$

We claim that $P^\bullet$ is isomorphic to a direct summand of $\tilde{P}^\bullet$.

As in the proof of the previous lemma, $(\psi_i)_{i \in \mathbb{Z}}$ gives rise to a chain map $\psi^\bullet : P^\bullet \to \tilde{P}^\bullet$. We check the commutativity: for $r \leqslant i \leqslant s-1$ and $v \in P^i$,

$$(\psi_{i+1} \circ d^i)(v) = \psi_{i+1}(d^i(v)) = 1 \otimes d^i(v) = (1 \otimes d^i)(1 \otimes v) = (1 \otimes d^i)(\psi_i(v)).$$

Similarly, the family of maps $(\varphi_i)_{i \in \mathbb{Z}}$ gives rise to a chain map $\varphi^\bullet : \tilde{P}^\bullet \to P^\bullet$ as shown by:

$$\begin{aligned}(\varphi_{i+1} \circ (1 \otimes d^i))(\lambda \otimes v) &= \varphi_{i+1}(\lambda \otimes d^i(v)) = \zeta(\lambda) d^i(v) \\ &= d^i(\zeta(\lambda)v) = (d^i \circ \varphi_i)(\lambda \otimes v).\end{aligned}$$

Moreover, $\varphi^\bullet \circ \psi^\bullet$ is the identity map, so as claimed, $P^\bullet$ is isomorphic to a summand of $\tilde{P}^\bullet$.

Therefore, for every indecomposable object $X^\bullet \in K^b({}_{\Lambda^S}\mathcal{P})$, there is an indecomposable object $\tilde{X}^\bullet \in K^b({}_{\Lambda}\mathcal{P})$ such that $X^\bullet$ is isomorphic to a direct summand of $\tilde{X}^\bullet \downarrow^{\Lambda}_{\Lambda^S}$. Consequently, $l(X^\bullet) \leqslant l(\tilde{X}^\bullet)$, and $\operatorname{sgl.dim} \Lambda^S \leqslant \operatorname{sgl.dim} \Lambda$. □

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