AN APPLICATION OF NAKAYAMA FUNCTOR IN REPRESENTATION STABILITY THEORY

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Abstract. Using the Nakayama functor, we construct an equivalence from a Serre quotient category of a category of finitely generated modules to a category of finite-dimensional modules. We then apply this result to the categories FI_G and VI_q, and answer positively an open question of Nagpal on representation stability theory.

1. Introduction

In a recent paper [9], Nagpal has proved that quite a few representation theoretic and homological properties of the category FI, whose objects are finite sets and morphisms are injections (see [2]), hold for the category VI_q, whose objects are finite-dimensional vector spaces over a finite field $\mathbb{F}_q$ and morphisms are linear injections. In that paper, he also asked a question of whether one can establish an equivalence

$$\mathrm{VI}_q\text{-mod}/\mathrm{VI}_q\text{-fdmod} \xrightarrow{\sim} \mathrm{VI}_q\text{-fdmod},$$

where VI_q-mod is the category of finitely generated VI_q-modules over a field of characteristic zero, and VI_q-fdmod is its full subcategory of finite-dimensional VI_q-modules; see [9, Question 1.11]. An analogue of this equivalence for FI-modules was proved by Sam and Snowden in [11, Theorem 3.2.1].

The purpose of the present paper is to prove a general result in an abstract setting from which this kind of equivalences can be deduced. In particular, applying our general result, the above equivalence can be obtained for both FI_G and VI_q with G an arbitrary finite group (see Section 4 for definition). Therefore, we give not only an affirmative answer to the above-mentioned open question in [9, Question 1.11], but also a new proof for the case of the category FI when taking G to be trivial in FI_G. Our approach only relies on several abstract homological properties of representations, and hence works for a wider class of categories including FI_G and VI_q as specific examples. Furthermore, via the Nakayama functor, the above equivalence for FI becomes transparent in our approach, compared with the one in [11].

We briefly describe the essential idea of our approach. Let $\mathcal{C}$ be a small EI-category (EI means that every endomorphism is an isomorphism) satisfying certain finiteness conditions. One can define the *Nakayama functor* ν and *inverse Nakayama functor* ν^{-1}

$$\mathcal{C}\text{-mod} \underset{\nu^{-1}}{\overset{\nu}{\rightleftarrows}} \mathcal{C}\text{-fdmod}$$

between the category $\mathcal{C}$-mod of finitely generated $\mathcal{C}$-modules and the category $\mathcal{C}$-fdmod of finite-dimensional $\mathcal{C}$-modules. Note that ν and ν^{-1} form a pair of adjoint functors, and furthermore, they give rise to an equivalence between the category of finitely generated projective $\mathcal{C}$-modules and

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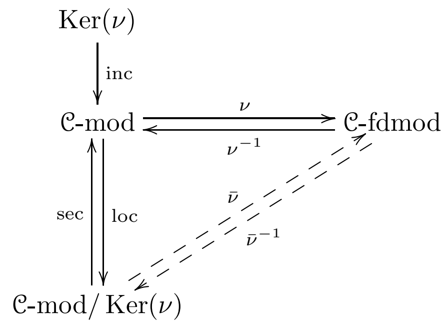
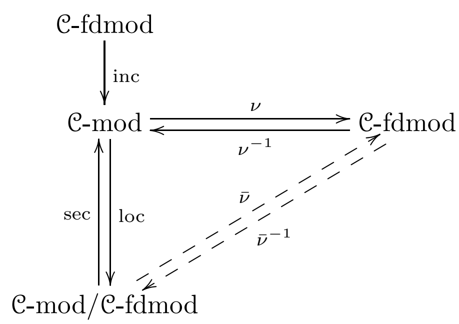
the category of finite-dimensional injective modules. Under the assumption that $\mathcal{C}$ is *locally self-injective* (that is, every finitely generated projective $\mathcal{C}$-module is also injective), ν is an exact functor, and $\nu \circ \nu^{-1}$ is isomorphic to the identity functor on $\mathcal{C}$-fdmod. Therefore, by a classical result of Gabriel ([4, Proposition III.2.5]), the kernel of ν is a localizing subcategory of $\mathcal{C}$-mod, and one obtains the following commutative diagram in which $\bar{\nu}$ and $\bar{\nu}^{-1}$ are quasi-inverse to each other, loc is the localization functor, and sec is a section functor:



We then consider $\mathrm{Ker}(\nu)$. If every morphism in $\mathcal{C}$ is a monomorphism, then $\mathcal{C}$-fdmod $\subseteq \mathrm{Ker}(\nu)$. We also give two equivalent characterizations such that $\mathrm{Ker}(\nu) \subseteq \mathcal{C}$-fdmod. When the category $\mathcal{C}$ satisfies these conditions (for instances, $\mathcal{C}$ is a skeleton of FI_G or VI_q), the above commutative diagram becomes



Thus we get what we want.

The paper is organized as follows. In Section 2, we collect basic results on Nakayama functor. In Section 3 and Section 4, we prove our main result and consider its application to both FI_G and VI_q in representation stability theory, respectively.

2. Preliminaries

The Nakayama functor is well known in the representation theory of finite-dimensional algebras (see, for example, [1], [13]). Most of the proofs in this section are standard, so we leave the details to the reader.

2.1. **Notations.** An EI-category is a small category in which every endomorphism is an isomorphism. Let $\mathcal{C}$ be a skeletal EI-category, and I be the set of objects of $\mathcal{C}$. For any $i, j \in I$, we write $\mathcal{C}(i,j)$ for the set of morphisms in $\mathcal{C}$ from i to j. Recall that there is a partial order on I defined by $i \leqslant j$ if $\mathcal{C}(i,j)$ is nonempty.

We fix a field $\Bbbk$. A left (respectively, right) $\mathcal{C}$-module is a covariant (respectively, contravariant) functor W from $\mathcal{C}$ to the category of $\Bbbk$-vector spaces.

For any $i, j \in I$, denote by $\Bbbk\mathcal{C}(i,j)$ the vector space with basis $\mathcal{C}(i,j)$. Denote by $A = \bigoplus_{i,j\in I} \Bbbk\mathcal{C}(i,j)$ the category algebra of $\mathcal{C}$. The associative algebra A is non-unital if I is an infinite set. Denote by

$e_i \in \mathcal{C}(i,i)$ the identity endomorphism of i. We say that a left (respectively, right) A-module V is *graded* if $V = \bigoplus_{i\in I} e_i V$ (respectively, $V = \bigoplus_{i\in I} V e_i$). If W is a left (respectively, right) $\mathcal{C}$-module, then $\bigoplus_{i\in I} W(i)$ has a natural structure of a graded left (respectively, right) A-module. Conversely, any graded left (respectively, right) A-module naturally defines a left (respectively, right) $\mathcal{C}$-module. Thus, we shall not distinguish left (respectively, right) $\mathcal{C}$-modules from graded left (respectively, right) A-modules.

A (left or right) $\mathcal{C}$-module is finitely generated (respectively, finite-dimensional) if it is finitely generated (respectively, finite-dimensional) as a graded A-module. We write $\mathcal{C}$-mod (respectively, $\mathcal{C}^{\mathrm{op}}$-mod) for the category of finitely generated left (respectively, right) $\mathcal{C}$-modules; and $\mathcal{C}$-fdmod (respectively, $\mathcal{C}^{\mathrm{op}}$-fdmod) for the full subcategory of $\mathcal{C}$-mod (respectively, $\mathcal{C}^{\mathrm{op}}$-mod) whose objects are the finite-dimensional left (respectively, right) $\mathcal{C}$-modules. On the categories of finite-dimensional modules, there is the *standard duality functor* $D := \mathrm{Hom}_{\Bbbk}(-,\Bbbk)$:

$$\mathcal{C}\text{-fdmod} \underset{D}{\overset{D}{\rightleftarrows}} \mathcal{C}^{\mathrm{op}}\text{-fdmod}$$

2.2. **Finiteness conditions and Nakayama functor.** We first recall from [10, Section 2] that the category $\mathcal{C}$ is said to be *inwards finite* if, for each $j \in I$, there are only finitely many $i \in I$ such that $\mathcal{C}(i,j)$ is nonempty; and *hom-finite* if $\mathcal{C}(i,j)$ is a finite set for every $i,j \in I$. Further, the category $\mathcal{C}$ is called *locally noetherian* if every $\mathcal{C}$-submodule of each finitely generated left $\mathcal{C}$-module is also finitely generated.

From now on, we always assume that $\mathcal{C}$ is an inwards finite, hom-finite, and locally noetherian EI-category.

Since $\mathcal{C}$ is inwards finite and hom-finite, the right projective $\mathcal{C}$-module $e_j A$ is finite-dimensional for each $j \in I$. Hence every finitely generated right $\mathcal{C}$-module is finite-dimensional.

Next, we introduce the Nakayama functor on $\mathcal{C}$-modules.

The category algebra A of $\mathcal{C}$ is an A-bimodule which is graded as both a left A-module and a right A-module. If V (respectively, W) is a left (respectively, right) $\mathcal{C}$-module, then $\mathrm{Hom}_{\mathcal{C}}(V,A)$ (respectively, $\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(W,A)$) is a right (respectively, left) A-module. If, moreover, V (respectively, W) is *finitely generated*, then $\mathrm{Hom}_{\mathcal{C}}(V,A)$ (respectively, $\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(W,A)$) is graded, that is,

$$\mathrm{Hom}_{\mathcal{C}}(V,A) \cong \bigoplus_{i\in I} \mathrm{Hom}_{\mathcal{C}}(V, Ae_i) \quad \Big(\text{respectively, } \mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(W,A) \cong \bigoplus_{j\in I} \mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(W, e_j A)\Big); \qquad (2.1)$$

see [8, VIII.1.15]. (The proof of the claim below (1) in [8, VIII.1.15] remains valid for our algebra A even though A might be non-unital.)

Without any reference, we shall use the following well-known fact: for any idempotent $e \in A$, there hold:

$$\mathrm{Hom}_{\mathcal{C}}(Ae, A)_A \cong eA_A \qquad \text{and} \qquad {}_A\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(eA, A) \cong {}_A Ae. \qquad (2.2)$$

Lemma 2.1. *If V is a finitely generated left $\mathcal{C}$-module, then $\mathrm{Hom}_{\mathcal{C}}(V,A)$ is a finite-dimensional right $\mathcal{C}$-module.*

Proof. Suppose that V is generated by $v_1, \ldots, v_s$ where $v_1 \in V(j_1), \ldots, v_s \in V(j_s)$. Then $\mathrm{Hom}_{\mathcal{C}}(V, Ae_i) = 0$ if $\mathcal{C}(i,j_1), \ldots, \mathcal{C}(i,j_s)$ are all empty sets. Since $\mathcal{C}$ is inwards finite, there are only finitely many $i \in I$ such that $\mathrm{Hom}_{\mathcal{C}}(V, Ae_i) \neq 0$. Since $\mathcal{C}$ is hom-finite, each $\mathrm{Hom}_{\mathcal{C}}(V, Ae_i)$ is finite-dimensional. □

Lemma 2.2. *If W is a finite-dimensional right $\mathcal{C}$-module, then $\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(W,A)$ is a finitely generated left $\mathcal{C}$-module.*

Proof. There exists a surjective homomorphism $e_{j_1}A \oplus \cdots \oplus e_{j_s}A \to W$ for some $j_1, \ldots, j_s \in I$. Applying the functor $\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(-,A)$ and using (2.2), we obtain an injective homomorphism

$$\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(W,A) \to Ae_{j_1} \oplus \cdots \oplus Ae_{j_s}.$$

Since $\mathcal{C}$ is locally noetherian, it follows that $\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(W, A)$ is finitely generated. □

By Lemmas 2.1 and 2.2, we have a pair of contravariant functors

$$\mathcal{C}\text{-mod} \underset{\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(-,A)}{\overset{\mathrm{Hom}_{\mathcal{C}}(-,A)}{\rightleftarrows}} \mathcal{C}^{\mathrm{op}}\text{-fdmod}$$

Definition 2.3. The *Nakayama functor* ν of $\mathcal{C}$ (or A) is defined to be the following composition:

$$D \circ \mathrm{Hom}_{\mathcal{C}}(-, A) : \mathcal{C}\text{-mod} \longrightarrow \mathcal{C}\text{-fdmod}.$$

The *inverse Nakayama functor* ν^{-1} is defined to be the following composition:

$$\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(-, A) \circ D : \mathcal{C}\text{-fdmod} \longrightarrow \mathcal{C}\text{-mod}.$$

Let us remark that the functor ν is a right exact covariant functor, while the functor ν^{-1} is a left exact covariant functor. But we should warn the reader that the functor ν^{-1} is, in general, neither the inverse nor a quasi-inverse of ν.

Lemma 2.4. *The pair (ν, ν^{-1}) is an adjoint pair of functors:*

$$\mathcal{C}\text{-mod} \underset{\nu^{-1}}{\overset{\nu}{\rightleftarrows}} \mathcal{C}\text{-fdmod}.$$

Proof. Let $V \in \mathrm{Ob}(\mathcal{C}\text{-mod})$ and $U \in \mathrm{Ob}(\mathcal{C}\text{-fdmod})$. Since V is a left A-module and DU is a right A-module, the tensor product $V \otimes_{\Bbbk} DU$ is an A-bimodule. One has the following canonical isomorphisms:

$$\begin{aligned}\mathrm{Hom}_{\mathcal{C}}(D\,\mathrm{Hom}_{\mathcal{C}}(V, A), U) = \mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(DU, \mathrm{Hom}_{\mathcal{C}}(V, A)) &= \mathrm{Hom}_{A\text{-bimod}}(V \otimes_{\Bbbk} DU, A)\\ &= \mathrm{Hom}_{\mathcal{C}}(V, \mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(DU, A));\end{aligned}$$

see [7, Exercise XI.6.6]. □

2.3. **Projectives and finite-dimensional injectives.** Denote by $\mathcal{C}$-proj the full subcategory of $\mathcal{C}$-mod whose objects are the finitely generated projective left $\mathcal{C}$-modules. Denote by $\mathcal{C}$-fdinj the full subcategory of $\mathcal{C}$-fdmod whose objects are the finite-dimensional injective left $\mathcal{C}$-modules.

Lemma 2.5. *(1) Every finitely generated projective left $\mathcal{C}$-module is a finite direct sum of indecomposable projective left $\mathcal{C}$-modules.*

(2) Let V be a finitely generated left $\mathcal{C}$-module. Then V is an indecomposable projective left $\mathcal{C}$-module if and only if V is isomorphic to Ae for some primitive idempotent $e \in e_iAe_i$ with $i \in I$.

Proof. The two statement follows from [3, Theorem I.11.18] and [13, Proposition I.8.2], respectively. □

Definition 2.6. (1) A full subcategory $\mathcal{C}'$ of $\mathcal{C}$ is said to be *right-closed* if, for any $i, j \in \mathrm{Ob}(\mathcal{C})$, we have $i \in \mathrm{Ob}(\mathcal{C}')$ whenever $i \leqslant j$ for some $j \in \mathrm{Ob}(\mathcal{C}')$.

(2) The *support* of a (left or right) $\mathcal{C}$-module V is the set of all $i \in I$ such that $V(i)$ is nonzero, where $V(i)$ is the image of i under the functor V.

By definition, we have the following trivial observation.

Lemma 2.7. *Let $\mathcal{C}'$ be a right-closed subcategory of $\mathcal{C}$. If V is an injective left $\mathcal{C}$-module whose support is contained in $\mathrm{Ob}(\mathcal{C}')$, then restricting V to $\mathcal{C}'$ gives an injective left $\mathcal{C}'$-module. If V' is an injective left $\mathcal{C}'$-module, then extending V' to $\mathcal{C}$ by zero on $\mathrm{Ob}(\mathcal{C}) \setminus \mathrm{Ob}(\mathcal{C}')$ gives an injective left $\mathcal{C}$-module.*

Lemma 2.8. *Let U be a finite-dimensional left $\mathcal{C}$-module. Then U is an indecomposable injective left $\mathcal{C}$-module if and only if U is isomorphic to $D(eA)$ for some primitive idempotent $e \in e_iAe_i$ with $i \in I$.*

Proof. This follows from Lemma 2.7 and [13, Proposition I.8.19]. □

Corollary 2.9. *The functor ν gives an equivalence of categories*

$$\mathcal{C}\text{-proj} \xrightarrow{\sim} \mathcal{C}\text{-fdinj}$$

with a quasi-inverse given by the functor ν^{-1}.

Proof. This follows immediately from (2.2), Lemmas 2.5 and 2.8. $\square$

Definition 2.10. We say that $\mathcal{C}$ is *locally self-injective* if Ae_i is an injective left $\mathcal{C}$-module for every $i \in I$.

Clearly, $\mathcal{C}$ is locally self-injective if and only if every finitely generated projective left $\mathcal{C}$-module is injective.

Lemma 2.11. *Suppose that $\mathcal{C}$ is locally self-injective. Then the Nakayama functor $\nu : \mathcal{C}\text{-mod} \longrightarrow \mathcal{C}\text{-fdmod}$ is exact.*

Proof. Since Ae_i is an injective left $\mathcal{C}$-module for each $i \in I$, the functor $\bigoplus_{i\in I} \mathrm{Hom}_{\mathcal{C}}(-, Ae_i)$ is exact. It follows from (2.1) and the exactness of D that $\nu : \mathcal{C}\text{-mod} \longrightarrow \mathcal{C}\text{-fdmod}$ is exact. $\square$

3. Main result

3.1. **Injective resolutions of finite-dimensional modules.** The partial order on I induces a partial order on the set of objects of any full subcategory of $\mathcal{C}$.

Lemma 3.1. *Suppose that the characteristic of $\Bbbk$ is zero. Let $\mathcal{C}'$ be a right-closed subcategory of $\mathcal{C}$ such that $\mathrm{Ob}(\mathcal{C}')$ is a finite set. Let $\mathcal{C}''$ be the full subcategory of $\mathcal{C}'$ on the objects which are not maximal in $\mathrm{Ob}(\mathcal{C}')$. Let W be a finite-dimensional right $\mathcal{C}$-module whose support is contained in $\mathrm{Ob}(\mathcal{C}')$. Then:*

(1) The subcategory $\mathcal{C}''$ of $\mathcal{C}$ is right-closed.
(2) There exists a short exact sequence

$$0 \to W' \to P \to W \to 0$$

where P is a finite direct sum of right $\mathcal{C}$-modules of the form eA with $e^2 = e \in e_iAe_i$ and $i \in \mathrm{Ob}(\mathcal{C}')$, and where W' is a finite-dimensional right $\mathcal{C}$-module whose support is contained in $\mathrm{Ob}(\mathcal{C}'')$.

Proof. (1) Suppose $i \in \mathrm{Ob}(\mathcal{C})$ and $i \leqslant j$ for some $j \in \mathrm{Ob}(\mathcal{C}'')$. Then $j \in \mathrm{Ob}(C')$, and j is not a maximal object in $\mathrm{Ob}(\mathcal{C}')$. Hence $i \in \mathrm{Ob}(C')$, and i is not a maximal object in $\mathrm{Ob}(\mathcal{C}')$.

(2) Let

$$P = \bigoplus_{i \in \mathrm{Ob}(\mathcal{C}')} We_i \otimes_{e_iAe_i} e_iA.$$

The algebra e_iAe_i is the group algebra of the finite group $\mathrm{Aut}_{\mathcal{C}}(i)$ for each $i \in I$. Since $\Bbbk$ has characteristic zero, the algebra e_iAe_i is semisimple. It follows that We_i is a finite direct sum of irreducible right e_iAe_i-modules, each of which is isomorphic to eAe_i for some primitive idempotent $e \in e_iAe_i$. One has a canonical isomorphism of right $\mathcal{C}$-modules: $eAe_i \otimes_{e_iAe_i} e_iA \cong eA$. We see that P is of the required form.

The multiplication map $\rho : P \to W$ is a homomorphism of right A-modules. Let W' be the kernel of ρ. Since P is finite-dimensional and has support contained in $\mathrm{Ob}(\mathcal{C}')$, the same is true for W'.

For each $i \in \mathrm{Ob}(\mathcal{C}')$, since ρ maps $We_i \otimes_{e_iAe_i} e_iAe_i$ bijectively to We_i, we see that ρ is surjective. Moreover, if i is maximal in $\mathrm{Ob}(\mathcal{C}')$, then $We_j \otimes_{e_jAe_j} e_jAe_i = 0$ if $j \in \mathrm{Ob}(\mathcal{C}')$ and $j \neq i$. Therefore if i is maximal in $\mathrm{Ob}(\mathcal{C}')$, then ρ maps Pe_i bijectively to We_i. It follows that W' has support contained in $\mathrm{Ob}(\mathcal{C}'')$. $\square$

Lemma 3.2. *Suppose that the characteristic of* $\Bbbk$ *is zero. Let* U *be a finite-dimensional left* $\mathcal{C}$*-module. Then there exists an exact sequence*

$$0 \to U \to I_0 \to I_1 \to \cdots \to I_n \to 0$$

of left $\mathcal{C}$*-modules such that* $I_0, I_1, \ldots, I_n$ *are finite-dimensional injective left* $\mathcal{C}$*-modules.*

Proof. Let W be the right $\mathcal{C}$-module DU. Let $\mathcal{C}_0$ be the full subcategory of $\mathcal{C}$ on the objects i such that $i \leqslant j$ for some j in the support of W. It is clear that $\mathcal{C}_0$ is right-closed. Since the support of W is a finite set and $\mathcal{C}$ is inwards finite, the set $\mathrm{Ob}(\mathcal{C}_0)$ is finite. Let $\mathcal{C}_1$ be the full subcategory of $\mathcal{C}_0$ on the objects which are not maximal in $\mathcal{C}_0$. Then $\mathcal{C}_1$ is also a right-closed subcategory of $\mathcal{C}$ with $\mathrm{Ob}(\mathcal{C}_1)$ finite. By Lemma 3.1, there is a short exact sequence

$$0 \to W_1 \to P_0 \to W \to 0$$

of right $\mathcal{C}$-modules such that:

- P_0 is a finite direct sum of right $\mathcal{C}$-modules of the form eA for some idempotent $e \in e_iAe_i$ with $i \in \mathrm{Ob}(C_0)$;
- W_1 is finite dimensional and its support is contained in $\mathrm{Ob}(\mathcal{C}_1)$.

Let $\mathcal{C}_2$ be the full subcategory of $\mathcal{C}_1$ on the objects which are not maximal in $\mathcal{C}_1$. We now apply Lemma 3.1 again to obtain a short exact sequence

$$0 \to W_2 \to P_1 \to W_1 \to 0$$

of right $\mathcal{C}$-modules such that:

- P_1 is a finite direct sum of right $\mathcal{C}$-modules of the form eA for some idempotent $e \in e_iAe_i$ with $i \in \mathrm{Ob}(C_1)$;
- W_2 is finite dimensional and its support is contained in $\mathrm{Ob}(\mathcal{C}_2)$.

Recursively, we obtain the subcategories $\mathcal{C}_0, \mathcal{C}_1, \mathcal{C}_2, \ldots$ and a projective resolution

$$\cdots \to P_n \to \cdots \to P_1 \to P_0 \to W \to 0 \tag{3.1}$$

where each P_s is a finite direct sum of right $\mathcal{C}$-modules of the form eA for some idempotent $e \in e_iAe_i$ with $i \in \mathrm{Ob}(\mathcal{C}_s)$. Since $|\,\mathrm{Ob}(\mathcal{C}_0)| > |\,\mathrm{Ob}(\mathcal{C}_1)| > |\,\mathrm{Ob}(\mathcal{C}_2)| > \cdots$, the projective resolution (3.1) is of finite length. By Lemma 2.8, we see that applying the functor D to (3.1) gives the required exact sequence. □

3.2. **An equivalence of categories.** For the adjoint pair (ν, ν^{-1}) in Lemma 2.4, we have the following result.

Proposition 3.3. *Suppose that the characteristic of* $\Bbbk$ *is zero and* $\mathcal{C}$ *is locally self-injective. Then the counit* $\nu \circ \nu^{-1} \to \mathrm{id}$ *is an isomorphism.*

Proof. We need to prove that the homomorphism $\nu(\nu^{-1}(U)) \to U$ is an isomorphism for each object U of $\mathcal{C}$-fdmod. By Lemma 3.2, there is a finite injective resolution of U by finite-dimensional injective left $\mathcal{C}$-modules; we shall use induction on the minimal length n among all such resolutions. For $n = 0$, the proposition follows from Corollary 2.9.

Let $0 \to U \to I_0 \to I_1 \to \cdots \to I_n \to 0$ be a resolution of U by finite-dimensional injective left $\mathcal{C}$-modules and n be minimal length with this property. Let U' be the cokernel of $U \to I_0$. One gets a short exact sequence

$$0 \to U \to I_0 \to U' \to 0.$$

Since ν^{-1} is left exact and ν is exact by Lemma 2.11, the following diagram commutes and has exact rows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \nu(\nu^{-1}(U)) & \longrightarrow & \nu(\nu^{-1}(I_0)) & \longrightarrow & \nu(\nu^{-1}(U')) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & U & \longrightarrow & I_0 & \longrightarrow & U'. \end{array}$$

Observe that the middle vertical map is an isomorphism by Corollary 2.9 and the right vertical map is an isomorphism by induction hypothesis. Therefore the left vertical map is also an isomorphism. □

The kernel $\operatorname{Ker}(\nu)$ of the functor $\nu : \mathcal{C}\text{-mod} \to \mathcal{C}\text{-fdmod}$ is the full subcategory of $\mathcal{C}$-mod on the objects V such that $\nu(V) = 0$. When ν is an exact functor, $\operatorname{Ker}(\nu)$ is a Serre subcategory of $\mathcal{C}$-mod (that is, closed under submodules, quotients and extensions) and its Serre quotient category is denoted by $\mathcal{C}\text{-mod}/\operatorname{Ker}(\nu)$.

To establish the main result of this paper, we recall the definition of section functors and a classical result of Gabriel.

Definition 3.4. We say an exact functor $F : \mathcal{A} \to \mathcal{B}$ between two abelian categories admits a *section functor* if F has a right adjoint $S : \mathcal{B} \to \mathcal{A}$ such that the counit $F \circ S \to \mathrm{id}_{\mathcal{B}}$ is an isomorphism.

Lemma 3.5. [4, Proposition III.2.5] *Let $F : \mathcal{A} \to \mathcal{B}$ be an exact functor between abelian categories which admits a section functor. Then the kernel* $\operatorname{Ker}(F)$ *is a localizing subcategory of $\mathcal{A}$, and F induces an equivalence $\bar{F} : \mathcal{A}/\operatorname{Ker}(F) \to \mathcal{B}$.*

Now we can prove the following main theorem.

Theorem 3.6. *Suppose that the characteristic of $\mathbb{k}$ is zero, and suppose that $\mathcal{C}$ is locally self-injective. Then the functor ν induces an equivalence of categories*

$$\overline{\nu} : \mathcal{C}\text{-mod}/\operatorname{Ker}(\nu) \xrightarrow{\sim} \mathcal{C}\text{-fdmod}.$$

Proof. Lemma 2.11 and Proposition 3.3 tell us that the Nakayama functor ν is exact and admits a section functor ν^{-1}. The conclusion now follows from Lemma 3.5. □

3.3. **Observations on** $\operatorname{Ker}(\nu)$**.** In this subsection we compare the subcategory $\mathcal{C}$-fdmod with the subcategory $\operatorname{Ker}(\nu)$, and prove that they coincide under certain conditions. In this case, Theorem 3.6 says that the Serre quotient category $\mathcal{C}$-mod/$\mathcal{C}$-fdmod is equivalent to $\mathcal{C}$-fdmod. Therefore the Serre quotient category has enough injective objects, and every finitely generated object in it has finite length and finite injective dimension. Moreover, $\mathcal{C}$-mod can be regarded as an extension of $\mathcal{C}$-fdmod by itself.

Firstly, we give a sufficient condition such that $\mathcal{C}\text{-fdmod} \subseteq \operatorname{Ker}(\nu)$, which should be easy to check in practice.

Definition 3.7. We say that a left $\mathcal{C}$-module V is *torsion-free* if, for every morphism f in $\mathcal{C}$, say $f \in \mathcal{C}(i,j)$, the induced map $f_* : V(i) \to V(j)$ is injective.

Lemma 3.8. *The following statements are equivalent:*

(1) Every morphism in $\mathcal{C}$ is a monomorphism, that is, if one has $f \in \mathcal{C}(j,k)$ and $g, h \in \mathcal{C}(i,j)$ such that $fg = fh$, then $g = h$.

(2) Every finitely generated projective left $\mathcal{C}$-module is torsion-free.

Proof. (1)⇒(2) It suffices to prove that Ae_i is torsion-free for each $i \in I$. Suppose $f \in \mathcal{C}(j,k)$. Since f is a monomorphism, the map $f_* : e_jAe_i \to e_kAe_i$ sends the basis $\mathcal{C}(i,j)$ of e_jAe_i bijectively onto a subset of the basis $\mathcal{C}(i,k)$ of e_kAe_i.

(2)⇒(1) Suppose one has $f \in \mathcal{C}(j,k)$ and $g, h \in \mathcal{C}(i,j)$ such that $fg = fh$. Since Ae_i is torsion-free, the map $f_* : e_jAe_i \to e_kAe_i$ is injective. But $f_*(g-h) = 0$, so $g = h$. □

Corollary 3.9. *Suppose that the partially ordered set I has no maximal element. If every morphism in $\mathcal{C}$ is a monomorphism, then* $\mathcal{C}$-fdmod $\subseteq \mathrm{Ker}(\nu)$.

Proof. Let V be a finite-dimensional left $\mathcal{C}$-module. We need to show that $\mathrm{Hom}_{\mathcal{C}}(V, Ae_i) = 0$ for every $i \in I$. By Lemma 3.8, the $\mathcal{C}$-module Ae_i is torsion-free, so it does not contain any nonzero finite-dimensional $\mathcal{C}$-submodules. Hence, the image of any homomorphism from V to Ae_i is zero. □

The following proposition gives two equivalent characterizations such that the reverse inclusion $\mathrm{Ker}(\nu) \subseteq \mathcal{C}$-fdmod holds. Surprisingly, the answer to this question is closely related to classification of injective modules in $\mathcal{C}$-mod.

Proposition 3.10. *Suppose that the characteristic of $\Bbbk$ is zero and $\mathcal{C}$ is locally self-injective. Then the following statements are equivalent:*

(1) $\mathrm{Ker}(\nu) \subseteq \mathcal{C}$-fdmod.
(2) If V is a finitely generated, infinite-dimensional left $\mathcal{C}$-module, then there exists $i \in I$ such that $\mathrm{Hom}_{\mathcal{C}}(V, Ae_i) \neq 0$.
(3) The category $\mathcal{C}$-mod *has enough injectives, and every finitely generated injective left $\mathcal{C}$-module is isomorphic to a direct sum of a finite-dimensional injective left $\mathcal{C}$-module and a finitely generated projective left $\mathcal{C}$-module.*

Proof. The equivalence between (1) and (2) is clear. Note that if $\mathcal{C}$ has only finitely many objects, then $\mathcal{C}$-fdmod = $\mathcal{C}$-mod, and there does not exist finitely generated, infinite-dimensional left $\mathcal{C}$-modules. Therefore, in this case both (1) and (2) hold trivially.

(3) ⇒ (2) Let V be a finitely generated, infinite-dimensional left $\mathcal{C}$-module. By the assumption, there exists an injection $V \to P \oplus T$, where P is a finitely generated projective $\mathcal{C}$-module, and T is a finite-dimensional injective $\mathcal{C}$-module. Note that P is nonzero since V is infinite-dimensional. Furthermore, the composition map $V \to P \oplus T \to P$ cannot be 0 where the second one is the projection because of the same reason. This observation implies (2).

(2) ⇒ (3) Suppose that (2) (and hence (1)) holds. Let us first prove the following statement:

($\star$) If F is a finitely generated left $\mathcal{C}$-module which has no nonzero finite-dimensional $\mathcal{C}$-submodules, then there is an injective homomorphism $F \to P$, where P is a finitely generated projective left $\mathcal{C}$-module.

As in [6, Proposition 7.5], we use induction on the dimension of $\nu(F)$. If $\nu(F) = 0$, then, by (1), F is finite-dimensional and so $F = 0$. Now, suppose $\nu(F) \neq 0$. By assumption, F is infinite-dimensional. It then follows from (2) that there exists $i \in I$ and a nonzero homomorphism $f : F \to Ae_i$. Let W be the image of F under f. Then we get a short exact sequence of finitely generated left $\mathcal{C}$-modules

$$0 \to U \to F \to W \to 0,$$

which induces another short exact sequence of finite-dimensional left $\mathcal{C}$-modules:

$$0 \to \nu(U) \to \nu(F) \to \nu(W) \to 0.$$

Note that $\mathrm{Hom}_{\mathcal{C}}(W, Ae_i) \neq 0$ since there is an inclusion from W into Ae_i. In particular, $\nu(W) \neq 0$. Thus the dimension of $\nu(U)$ is strictly less than that of $\nu(F)$. By induction hypothesis, ($\star$) holds for U. Hence ($\star$) holds for F.

Now let V be any finitely generated left $\mathcal{C}$-module. Let E be the maximal finite-dimensional $\mathcal{C}$-submodule of V. By Lemma 3.2, there is an injection $E \to T$ where T is a finite-dimensional injective left $\mathcal{C}$-module. Let $F = V/E$. Then F is a finitely generated left $\mathcal{C}$-module which has no nonzero finite-dimensional $\mathcal{C}$-submodules, so by ($\star$), there is an injection $F \to P$ where P is a finitely generated projective left $\mathcal{C}$-module. It follows that there is an injection $V \to T \oplus P$. Therefore the category $\mathcal{C}$-mod has enough injectives.

As in the proof of [9, Lemma 2.4], we suppose that the module V is injective and E' is a maximal essential extension of E in V. Then E' is injective by [7, Proposition X.5.4]. Moreover, E' must be finite-dimensional, for otherwise it cannot be an essential extension of E. Hence $E = E'$, and so E is injective. Therefore, V is isomorphic to $E \oplus F$. Hence F is also injective. By ($\star$), it follows that F is projective. □

4. On an open question of Nagpal

We now discuss the application of Theorem 3.6 to the categories FI_G and VI_q studied in representation stability theory. In particular, we affirmatively answer the following open question of Nagpal [9, Question 1.11]:

Question: Can one establish an equivalence

$$\mathrm{VI}_q\text{-mod}/\mathrm{VI}_q\text{-fdmod} \xrightarrow{\sim} \mathrm{VI}_q\text{-fdmod}\ ?$$

Let G be a finite group. The category FI_G was defined independently in [5, Example 3.8] and [12]; let us first recall its definition. The objects of FI_G are the finite sets. The morphisms in FI_G from a finite set X to a finite set Y are the pairs (f, c) where $f : X \to Y$ is an injection and $c : X \to G$ is an arbitrary map. If (f, c) is a morphism from X to Y, and (f', c') is a morphism from Y to Z, then their composite is defined to be the morphism (f'', c''), where

$$f''(x) = f'(f(x)), \qquad c''(x) = c'(f(x))c(x), \qquad \text{for each } x \in X.$$

When G is the trivial group, the category FI_G is the category FI of finite sets and injections.

Now, we recall the definition of the category VI_q. Let $\mathbb{F}_q$ be a finite field with q elements. The objects of VI_q are finite-dimensional vector spaces over $\mathbb{F}_q$, and morphisms are the injective linear maps. The composite of morphisms in VI_q is just the usual composition of maps. The category VI_q has been studied by many authors; see for example [9] and the references therein.

In the following lemma we collect some important results on FI_G and VI_q, which will be used to establish an equivalence between the Serre quotient category and the category of finite-dimensional modules.

Lemma 4.1. *Suppose that $\Bbbk$ is of characteristic zero and $\mathcal{C}$ is a skeleton of FI_G or VI_q. Then*

1. *$\mathcal{C}$ is locally noetherian over $\Bbbk$.*
2. *Every finitely generated projective left $\mathcal{C}$-module is injective.*
3. *Every morphism in $\mathcal{C}$ is a monomorphism.*
4. *If V is a finitely generated, infinite-dimensional left $\mathcal{C}$-module, then there exists some $i \in I$ such that $\mathrm{Hom}_A(V, Ae_i) \neq 0$.*

Proof. (1) is established in [5, Theorem 3.7, Example 3.8, Example 3.10]. (2) is verified in [6, Theorem 1.5]. (3) follows from the definitions of FI_G and VI_q. (4) follows from both [6, Lemma 7.2, Lemma 7.3] for FI_G and from [9, Theorem 4.34] for VI_q. □

Now, we are ready to prove the following result for FI_G and VI_q, which answers positively the above-mentioned question.

Theorem 4.2. *Suppose that the characteristic of $\Bbbk$ is zero and $\mathcal{C}$ is a skeleton of FI_G or VI_q. Then the Nakayama functor ν induces an equivalence of categories*

$$\mathcal{C}\text{-mod}/\mathcal{C}\text{-fdmod} \xrightarrow{\sim} \mathcal{C}\text{-fdmod}.$$

Proof. It is clear that $\mathcal{C}$ is an EI-category which is inwards finite and hom-finite, and Lemma 4.1(1) tells us that $\mathcal{C}$ is a locally noetherian category. Clearly, the set of objects of $\mathcal{C}$ has no maximal element. By Lemma 4.1(2) and Theorem 3.6, the Nakayama functor ν induces an equivalence of categories from $\mathcal{C}\text{-mod}/\mathrm{Ker}(\nu)$ to $\mathcal{C}$-fdmod. We claim $\mathrm{Ker}(\nu) = \mathcal{C}$-fdmod. Indeed, by Corollary 3.9

and Lemma 4.1(3), one has $\mathcal{C}$-fdmod $\subseteq$ Ker(ν); and by Lemma 4.1(4) and Proposition 3.10, one gets Ker(ν) $\subseteq$ $\mathcal{C}$-fdmod. □

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