

Heterotic String Compactification and New Vector Bundles

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Abstract

We propose a construction of Kähler and non-Kähler Calabi-Yau manifolds by branched double covers of twistor spaces. In this construction we use the twistor spaces of four-manifolds with self-dual conformal structures, with the examples of connected sum of n \mathbb{P}^2 s. We also construct $K3$ -fibered Calabi-Yau manifolds from the branched double covers of the blow-ups of the twistor spaces. These manifolds can be used in heterotic string compactifications to four dimensions. We also construct stable and polystable vector bundles. Some classes of these vector bundles can give rise to supersymmetric grand unified models with three generations of quarks and leptons in four dimensions.

1 Introduction

Superstring compactifications from heterotic string theory to four dimensions give a promising approach to find realistic Standard Model like particle physics with three generations of quarks and leptons. In the heterotic string compactification, the higher dimensional spacetime is a product of the Minkowski four-manifold and the internal six dimensional manifold, see for example [1, 2, 3]. One of the standard approaches has been the compactification of the heterotic string on smooth Calabi-Yau three-folds with holomorphic vector bundles. The question whether the number of Calabi-Yau manifolds in each dimension is finite has been put forward in [4]. These bundles break the E_8 gauge theory down to E_6 , $SO(10)$ and $SU(5)$ grand unified theories. Many classes of models using general holomorphic vector bundles on the internal manifold can lead to any of the above three grand unified groups. These unified gauge groups can further break down to the Standard Model gauge group, for example with Wilson line turned on as a usual method. In the context of the heterotic string compactifications, progresses of building phenomenologically viable models have been made in for example [5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16] and references therein.

The heterotic string theory contains the vector bundle degree of freedom, and the first and second Chern classes of the vector bundle on the Calabi-Yau manifolds need to satisfy nontrivial constraints. In the usual Calabi-Yau compactifications, the first Chern class of the vector bundle vanishes due to the Hermitian Yang-Mills equations, and the second Chern class of the vector bundle equals the second Chern class of the tangent bundle, up to a total effective curve class from fivebranes.

Heterotic string compactification on a simply connected Calabi-Yau manifold can also be considered [15, 17, 18] with the examples of elliptic Calabi-Yau manifolds. The vector bundles on elliptically fibered Calabi-Yau can be constructed by means of spectral cover construction, see for example [19, 20]. In some cases, freely acting involutions in elliptically fibered Calabi-Yau threefolds with two sections, were also proposed, see for example [21, 22, 23] and references therein.

In this paper we construct Kähler and non-Kähler Calabi-Yau manifolds which can be used in the heterotic string compactifications. We construct them from branched double covers of twistor spaces of four-manifolds with self-dual conformal structures. These twistor spaces have balanced metrics. The manifolds as the branched covers solve the conformally balanced equation. We also construct stable and polystable vector bundles on these manifolds, which satisfy the anomaly cancellation condition and the Hermitian-Yang-Mills equations. Among the vector bundles we construct, there are those which give supersymmetric grand unified models with three generations of quarks and leptons.

The non-Kähler Calabi-Yau spaces here are complex three-folds with trivial canonical bundle. They may play an important role in the compactification of heterotic string theory to four dimensions. The construction of these non-Kähler Calabi-Yau manifolds

opens up more possibilities in finding the vacuum corresponding to the Standard Model from superstring theory.

Our approach include both Kähler Calabi-Yau spaces and non-Kähler Calabi-Yau spaces. More specifically, we constructed the Kähler and non-Kähler Calabi-Yau spaces by branched double covers of twistor spaces, or branched double covers of the blow-ups of twistor spaces. We consider the twistor spaces of four-manifolds M^4 , for example, the connected sum of n copies of \mathbb{P}^2 s, which have self-dual conformal structures. The branch locus of the double cover of the twistor space $\text{Tw}(M^4)$ can be either smooth or singular. In the smooth case, the branch locus is a divisor whose divisor class is twice of the divisor class of $K3$ surface. In the singular case, the branch locus is a union of two $K3$ surfaces, and after blowing up along the singular locus, the resulting manifolds are Calabi-Yau manifolds and are also $K3$ fibrations over \mathbb{P}^1 .

We consider the exploration of a specific Standard Model gauge group or GUT gauge group with three generations of quarks and leptons. To obtain $SU(5)$ and $SO(10)$ GUT groups as the subgroups of the E_8 group, we need to construct rank 5 and rank 4 vector bundles, respectively. These vector bundles are stable or polystable. The polystability or stability of the vector bundle guarantees the existence of the solution to the Hermitian-Yang-Mills equations, see for example [24, 25].

We also propose a new approach to construct stable and polystable bundles. We constructed rank 5 bundles with nonzero c_3 , which include the rank 5 bundles with $c_3 = 6$, corresponding to supersymmetric $SU(5)$ GUT models with three generations of chiral fermions. After a GUT symmetry breaking, some of the models can give rise to models with Standard Model gauge groups and three generations of chiral fermions. These examples have relevance to model buildings for obtaining phenomenologically viable four-dimensional theory. This is also an example where the anomaly cancellation condition is satisfied without adding fivebranes or effective curve class. In another example, we constructed rank 4 bundles with zero c_3 , in which fivebranes or effective curve class are introduced in the anomaly cancellation condition.

The organization of this paper is as follows. In Section 2, we describe the physical constraints of the relevant manifolds and the Chern classes of the vector bundles. In Section 3, we propose a general procedure for constructing the Kähler and non-Kähler Calabi-Yau manifolds by using branched double covers of twistor spaces of four-manifolds, with the examples of the connected sum of n copies of \mathbb{P}^2 s, that is, $n\mathbb{P}^2$. Afterwards in Section 4, we construct $K3$ -fibered Calabi-Yau manifolds from the branched double covers of the blow ups of the twistor spaces, and in particular the Kähler Calabi-Yau manifolds for the Kähler twistor spaces with the example of $S^4 = 0\mathbb{P}^2$. Then in Section 5, we construct the case of non-Kähler Calabi-Yau manifolds for the non-Kähler twistor spaces with the example of $2\mathbb{P}^2$. Finally we briefly discuss our results and make some conclusions in Section 6.

2 Physical constraints

The heterotic superstring theory can be compactified on a warped product of Minkowski four-manifold and an internal complex three-fold. The complex three-fold is endowed with a holomorphic $(3, 0)$ three-form Ω and a Hermitian $(1, 1)$ form ω associated with the Hermitian metric. The background contains a vector bundle V , and the gauge fields F of the vector bundle satisfy the Hermitian Yang-Mills equations. The background also satisfy the conformally balanced condition and the anomaly cancellation condition. These above three conditions are described by

$$d(\|\Omega\| \omega^2) = 0, \quad (2.1)$$

$$F^{1,1} \wedge \omega^2 = 0 \quad \text{and} \quad F^{(2,0)} = F^{(0,2)} = 0, \quad (2.2)$$

$$dH = \frac{\alpha'}{4} (\text{tr}(R \wedge R) - \text{tr}(F \wedge F)) - [W]. \quad (2.3)$$

The anomaly cancellation condition (2.3) can also be characterized as a modified Bianchi identity for the H -flux, in which the $[W]$ term corresponds to the fivebrane source term to the H -flux. These equations are analyzed in details by for example [3, 26, 27, 28, 29]. The norm $\|\Omega\|$ is defined by $\Omega \wedge \bar{\Omega} = -i\frac{4}{3} \|\Omega\|^2 \omega^3$. The physical fields are related to the holomorphic three-form and the Hermitian form of the manifolds by

$$H = i(\bar{\partial} - \partial)\omega, \quad e^{-2\phi} = \|\Omega\| \quad (2.4)$$

and $dH = 2i\partial\bar{\partial}\omega$. These complex three-folds also preserve $N = 1$ supersymmetry in four dimensions.

To have $SU(5)$ and $SO(10)$ grand unified groups as the subgroups of the E_8 group, one need to construct rank 5 and rank 4 vector bundles respectively. The commutant group of the rank 5 bundle in the E_8 group is the $SU(5)$ grand unified group, while the commutant group of the rank 4 bundle in the E_8 group is the $SO(10)$ grand unified group. The two equations (2.2) in the Hermitian Yang-Mills mean that the vector bundle V is holomorphic and $c_1(V) = 0$. These equations have solutions when the vector bundles are stable or polystable [30, 31]. The stability or polystability of the vector bundle ensures the existence of the solution to the Hermitian-Yang-Mills equations, see for example [24, 25].

Since our branched double cover construction produces Calabi-Yau manifolds that are either Kähler or non-Kähler, the first Chern class of the tangent bundle of M is zero, that is $c_1(M) = 0$. In both these cases in this paper, these manifolds are complex three-folds.

The heterotic string compactification with three generations of chiral fermions have three physical constraints on the Chern classes of the vector bundle V as

$$c_1(V) = 0, \quad (2.5)$$

$$c_2(V) = c_2(M) - [W], \quad (2.6)$$

$$c_3(V) = 6. \quad (2.7)$$

The first two conditions (2.5) and (2.6) are necessary conditions for consistent configurations. The $[W]$ is a total effective curve class coming from fivebranes. The third condition (2.7) is not a necessary condition for general configurations, but is a condition for having three generations of chiral fermions when reducing the model to four dimensions. The configurations that satisfy all other constraints except the one for $c_3(V)$ are consistent configurations in the heterotic string theory, though not having three generations of chiral fermions. If the manifold M has a freely acting involution γ , of order $|\gamma|$, then by the index theorem and the Riemann-Roch formula, the number of generations of the chiral fermions are

$$N = \int \text{ch}(V) \text{Td}(M) = \frac{c_3(V)}{2|\gamma|} = 3. \quad (2.8)$$

Here, the $\text{ch}(V)$ is the Chern character and $\text{Td}(M)$ is the Todd class. Two relevant cases in our approach are $c_3(V) = 6$, $|\gamma| = 1$ and $c_3(V) = 12$, $|\gamma| = 2$.

Two common grand unified groups are $SU(5)$ and $SO(10)$. We will focus on the construction of rank 5 bundles V corresponding to supersymmetric $SU(5)$ GUT models. In this case we have that

$$\wedge^5 V \cong \mathcal{O}_M. \quad (2.9)$$

The Higgs particles in these models can be in the representations $\mathbf{5}_H$ or $\bar{\mathbf{5}}_H$. The matter fields can be in the the representations $\bar{\mathbf{5}}$ or $\mathbf{10}$. The representations $\bar{\mathbf{5}}$, $\mathbf{5}$ of the $SU(5)$ GUT model correspond to $H^1(M, \wedge^2 V)$ and $H^1(M, \wedge^2 V^\vee)$ respectively, while the representation $\mathbf{10}$ corresponds to $H^1(M, V)$. There exist several types of couplings between the Higgs particles and the matter particles, the $\bar{\mathbf{5}} \bar{\mathbf{5}} \mathbf{10}$, which corresponds to the nonzero pairing

$$H^1(M, \wedge^2 V) \otimes H^1(M, \wedge^2 V) \otimes H^1(M, V) \longrightarrow \mathbb{C}, \quad (2.10)$$

and the $\mathbf{10} \mathbf{10} \mathbf{5}$, which corresponds to the nonzero pairing

$$H^1(M, V) \otimes H^1(M, V) \otimes H^1(M, \wedge^2 V^\vee) \longrightarrow \mathbb{C}, \quad (2.11)$$

where we have used $H^3(M, \wedge^5 V) \cong H^3(M, \mathcal{O}_M) \cong \mathbb{C}$. In addition, higher dimensional representations of the Higgs particles in $SU(5)$ grand unified models are possible.

There are similar conditions for the rank 4 vector bundles V , corresponding to the supersymmetric $SO(10)$ GUT models. These have been discussed in detail in [32]. In this case

$$\wedge^4 V \cong \mathcal{O}_M. \quad (2.12)$$

The Higgs particles in these models can be in the representations $\mathbf{10}_H$. The matter fields can be in the the representations $\mathbf{16}$. The representation $\mathbf{10}$ of the $SO(10)$

GUT model corresponds to $H^1(M, \wedge^2 V)$, and the representation **16** corresponds to $H^1(M, V)$. There exist couplings between the Higgs particles and the matter particles, the **10 16 16**, which corresponds to the nonzero pairing

$$H^1(M, \wedge^2 V) \otimes H^1(M, V) \otimes H^1(M, V) \longrightarrow \mathbb{C}, \quad (2.13)$$

where we have used $H^3(M, \wedge^4 V) \cong H^3(M, \mathcal{O}_M) \cong \mathbb{C}$. Again, higher dimensional representations of the Higgs particles in $SO(10)$ grand unified models are possible.

3 Construction of Calabi-Yau threefolds from twistor spaces

There are many ways to construct Calabi-Yau manifolds. Here we shall describe general ideas of constructing Calabi-Yau threefolds as double cover of twistor spaces. The resulting Calabi-Yau is often non-Kähler. The advantage of working on twistor spaces is that we can find natural balanced metrics on such Calabi-Yau threefolds. This section is devoted to general aspects of this approach. In the next two sections we shall describe specific examples and application to heterotic superstring theory.

3.1 Twistor spaces of connected sum of \mathbb{P}^2 s

We start by recalling some basic notions of twistor spaces of self-dual four-manifolds. For any oriented four-manifold M^4 , its twistor space is defined as

$$\text{Tw}(M^4) = P \times_{SO(4)} SO(4)/U(2), \quad (3.1)$$

where P is the $SO(4)$ principal bundle of M^4 . It was proved [33] that M^4 admits a self-dual conformal structure, that is, $W_- = 0$ for Weyl tensor W , if and only if the natural almost complex structure on $\text{Tw}(M^4)$ is integrable. Taubes [34] showed that for any compact oriented four manifold, after taking connected sum with sufficiently many \mathbb{P}^2 s, the resulting four manifold admits a self-dual conformal structure.

In the sequel we assume M^4 is self-dual. The twistor space has a natural differentiable map $\text{Tw}(M^4) \rightarrow M^4$ which is an S^2 -fibration. Each fiber is a holomorphic \mathbb{P}^1 with the induced holomorphic structure. In addition, there is a real structure, an anti-holomorphic map $\tau : \text{Tw}(M^4) \rightarrow \text{Tw}(M^4)$, preserving the fibration and induces the antipodal map on each fiber S^2 .

Next we shall focus on four-manifolds $M^4 = n\mathbb{P}^2$, the connected sum of n copies of \mathbb{P}^2 s. Floer [35] proved the existence of self-dual metrics on $n\mathbb{P}^2$ by perturbation arguments. Shortly after that, Donaldson and Friedman [36] gave an algebraic proof of a more general result by constructing its twistor space using deformation theory which will be sketched later.

The simplest example is $n = 0$, that is, the 4-sphere S^4 . It is well-known that $\text{Tw}(S^4) = \mathbb{P}^3$. Hitchin [37] showed that in the compact case, the twistor space is Kähler if and only if it is \mathbb{P}^3 or the complete flag manifold $\mathbb{F} = \mathbb{P}(T_{\mathbb{P}^2})$, which are twistor spaces of standard conformal structures on S^4 and \mathbb{P}^2 respectively.

For $n = 2$ and 3 , Poon [38] analyzed in details the structure of the twistor spaces $\text{Tw}(n\mathbb{P}^2)$. They are non-Kähler but turn out to be Moishezon, and there is a moduli of such self-dual conformal structures.

When $n \geq 4$, it becomes complicated. The twistor space can have algebraic dimension zero when n is large. However, LeBrun [39] gave an explicit conformal structure for all $n\mathbb{P}^2$. In addition, the twistor spaces of these conformal structures are Moishezon and can be described explicitly. For other related work on twistor spaces of $n\mathbb{P}^2$, see for example [40, 41, 42, 43, 44].

3.2 Donaldson-Friedman construction

For later use, we recall briefly Donaldson-Friedman's construction of self-dual four-manifolds from deformation of singular spaces. We shall focus on the case of $M^4 = 2\mathbb{P}^2$.

Recall that the twistor space of \mathbb{P}^2 with its Fubini-Study metric is the flag manifold $\mathbb{F} = \mathbb{P}(T_{\mathbb{P}^2})$. Let $\pi : \tilde{\mathbb{F}} \rightarrow \mathbb{F}$ be the blowup of \mathbb{F} along a real twistor line. Then the exceptional divisor is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ with normal line bundle $\mathcal{O}(1, -1)$.

Take two identical such flag manifolds \mathbb{F} , labeled by \mathbb{F}_1 and \mathbb{F}_2 . After blowing-up we obtain $\tilde{\mathbb{F}}_1$ and $\tilde{\mathbb{F}}_2$ with exceptional divisors E_1 and E_2 , both isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. Note that $\mathbb{P}^1 \times \mathbb{P}^1$ has an automorphism u switching the two factors. We glue $\tilde{\mathbb{F}}_1$ and $\tilde{\mathbb{F}}_2$ along E_1 and E_2 via such automorphism u , and obtain

$$Z_0 = \tilde{\mathbb{F}}_1 \cup_{\mathbb{P}^1 \times \mathbb{P}^1} \tilde{\mathbb{F}}_2 \quad (3.2)$$

which is simple normal crossing with singularity along $D = \mathbb{P}^1 \times \mathbb{P}^1$. Furthermore, Z_0 admits a natural real structure.

On each component $\tilde{\mathbb{F}}_i$ of Z_0 , we denote the normal bundle of D by N_i for $i = 1, 2$. Then $N_1 = \mathcal{O}(1, -1)$ and $N_2 = \mathcal{O}(-1, 1)$. Therefore $N_1 \otimes N_2 = \mathcal{O}_D$; in other words, Z_0 is d -semistable.

Now we consider the general theory of global smoothing, and suppose we have a d -semistable space $Y = Y_1 \cup_D Y_2$. Let N_i be normal bundles of D in Y_i . If the following spaces

1. $H^1(\mathcal{O}_D), H^2(T_{Y_i})$,
2. $H^p(N_i)$ for all p ,
3. $H^p(T_D), p = 1, 2$

are vanishing, then there is a global smoothing of Y [36]. Applying this criterion to the case $Z_0 = \tilde{\mathbb{F}}_1 \cup_{\mathbb{P}^1 \times \mathbb{P}^1} \tilde{\mathbb{F}}_2$, we can verify that Z_0 admits a smoothing. In addition, there is a smoothing which preserves the real structure so that such a smoothing gives a twistor space Z . The corresponding four-manifold is then a self-dual structure on $2\mathbb{P}^2$.

3.3 Calabi-Yau as double cover of twistor spaces

The double cover construction of Calabi-Yau works for any Fano threefold, or more generally, any smooth threefold X so that $|-2K_X|$ admits a smooth divisor.

Given such a threefold X , a nontrivial section $s \in H^0(-2K_X)$ defines a homomorphism $m : K_X \otimes K_X \rightarrow \mathcal{O}_X$, which in turn induces an algebraic structure on the sheaf $\mathcal{O}_X \oplus K_X$. Denote

$$M = \mathbf{Spec}(\mathcal{O}_X \oplus K_X). \quad (3.3)$$

The natural map $f : M \rightarrow X$ is a double cover of X . Now suppose the section $s \in H^0(-2K_X)$ defines a smooth divisor $B = s^{-1}(0)$, then M is smooth and it is straightforward to show that the canonical bundle K_M is trivial using adjunction formula. Moreover, one can verify that $B \subset X$ is the branch divisor of $f : M \rightarrow X$.

We shall also work on the case B is not smooth. Suppose B_i are smooth divisors linearly equivalent to $-K_X$, and $B = B_1 \cup B_2$ with simple normal crossing singularity. Let $C = B_1 \cap B_2$. Then we can still construct a double cover $f : M' \rightarrow X$ as before. However, M' is not smooth. It has ordinary double point singularity along $f^{-1}(C)$. It is easy to verify that the dualizing sheaf of M' is trivial, and a small resolution $M \rightarrow M'$ gives a smooth threefold M with trivial canonical bundle.

The following is an equivalent point of view for this singular case. Let $\tilde{X} \rightarrow X$ be the blowup of X along $C = B_1 \cap B_2$. Then the anti-canonical divisor of \tilde{X} is base point free and defines a fibration $\tilde{X} \rightarrow \mathbb{P}^1$. Let \tilde{B}_i be the proper transform of B_i . Then $\tilde{B} = \tilde{B}_1 \cup \tilde{B}_2$ is a disjoint union and is the branch locus that we use to define a double cover of \tilde{X} . It turns out in this way we obtain the same manifold M by the previous construction.

The singular case has an extra nice structure. Because B_i is anti-canonical divisor, adjunction formula implies that it has trivial canonical bundle. If furthermore it is simply connected, then it is a $K3$ surface. Therefore we get a $K3$ fibration structure on the resulting double cover manifold.

We now focus on the case when X are the spaces $\mathrm{Tw}(n\mathbb{P}^2)$ with LeBrun's conformal structure on $n\mathbb{P}^2$. Explicit description of X shows that $-2K_X$ has a section defines a smooth divisor, or simple normal crossing divisor of the type discussed above. One therefore obtains Calabi-Yau threefolds from double cover construction. These Calabi-Yau manifolds are Moishezon. For small n , particularly less than 4, the projective model for X is well known. In the next two sections of the paper, we shall work out more geometric structures for the case $n = 0$ and $n = 2$.

3.4 Hermitian forms and conformally balanced metrics

We denote $Z = \text{Tw}(n\mathbb{P}^2)$ for LeBrun's conformal structure on $n\mathbb{P}^2$. Then $n\mathbb{P}^2$ has positive scalar curvature. There is a natural family of balanced Hermitian metrics on Z with associated positive $(1, 1)$ -forms ω_Z .

Let $f : M \rightarrow Z$ be a double cover branched along a smooth divisor B . We construct a balanced metric on the resulting double cover M as follows. Recall that the map $\omega \rightarrow \omega^2$ defines a bijection between the cone of positive $(1, 1)$ -forms and positive $(2, 2)$ -forms [45]. It suffices to find a closed positive $(2, 2)$ -form on M .

Note that the pull-back $(2, 2)$ -form $f^*(\omega_Z^2)$ is positive away from the ramification divisor $f^{-1}(B)$, and it is closed. We modify it to a positive closed form in the following way. Let L be the line bundle over M with a section $s \in H^0(L)$ so that its zero locus is $f^{-1}(B)$. Since $f^{-1}(B)$ is projective and $L|_{f^{-1}(B)}$ is positive, one can find an Hermitian metric h on L so that the Chern form $c_1(L, h)$ is positive on a neighbourhood U of the ramification divisor $f^{-1}(B)$. It follows that

$$c_1(L, h)^2|_U > 0. \quad (3.4)$$

On the other hand, $c_1(L, h)^2$ is bounded and

$$f^*(\omega_Z^2)|_{M \setminus U} > 0. \quad (3.5)$$

We can find a sufficiently large constant $C > 0$ so that

$$\omega_M^2 := C \cdot f^*(\omega_Z^2) + c_1(L, h)^2 \quad (3.6)$$

is a positive closed $(2, 2)$ -form.

Having a balanced metric ω_M on M , we can reduce the conformally balanced equation $d(\|\Omega\|_{\omega} \omega^2) = 0$ to $\omega_M^2 = \|\Omega\|_{\omega} \omega^2$. This is essentially a complex Monge-Ampère equation which is solvable by the method in [46].

3.5 Deformation construction

We describe a more general construction of Calabi-Yau threefolds by double cover of singular space and smoothing.

Again we let \mathbb{F} be the twistor space of \mathbb{P}^2 . Let $\pi : \tilde{\mathbb{F}} \rightarrow \mathbb{F}$ be the blow up of \mathbb{F} along a real twistor line ℓ with exceptional divisor E . Then $K_{\tilde{\mathbb{F}}} = \pi^*K_{\mathbb{F}} + E$. Hence the log canonical divisor of $\tilde{\mathbb{F}}$ is $K_{\tilde{\mathbb{F}}} + E = \pi^*K_{\mathbb{F}} + 2E$.

Recall that we defined

$$Z_0 = \tilde{\mathbb{F}}_1 \cup_{\mathbb{P}^1 \times \mathbb{P}^1} \tilde{\mathbb{F}}_2 \quad (3.7)$$

with $D = E_1 \cong E_2$ via an isomorphism u . The dualizing sheaf of Z_0 is a trivial line bundle coming from gluing of $K_{\tilde{\mathbb{F}}_i} + E_i$ along their restrictions to D .

We construct double cover of the singular space Z_0 which is again a threefold with simple normal crossing singularity. A global smoothing of it gives a Calabi-Yau threefold.

More precisely, we start with double cover $Y \rightarrow \tilde{\mathbb{F}}$ branched along $-2(K_{\tilde{\mathbb{F}}} + E)$. Note that ℓ is a twistor line, we have $K_{\mathbb{F}} \cdot \ell = -4$. Therefore, the restriction of $K_{\tilde{\mathbb{F}}} + E$ to E is isomorphic to $\mathcal{O}(-2, -2) = K_E$.

Let $S = Y \times_{\tilde{\mathbb{F}}} E$. Then the projection $S \rightarrow E$ is a double cover branched over a divisor $-2K_E$. Hence S is a $K3$ surface. In this way we obtain a pair $S \subset Y$ lying above $E \subset \tilde{\mathbb{F}}$. The normal bundle of S in Y is the pull back of $\mathcal{O}(1, -1)$.

The following is the main result of this subsection:

Proposition 1 *Let $\pi : \tilde{\mathbb{F}} \rightarrow \mathbb{F}$ be the blowup of \mathbb{F} along a real twistor line ℓ with exceptional divisor $E = \mathbb{P}^1 \times \mathbb{P}^1$. Let $\tilde{\mathbb{F}}_1$ and $\tilde{\mathbb{F}}_2$ be two copies of $\tilde{\mathbb{F}}$. Suppose we can find smooth divisors $D_i \subset \tilde{\mathbb{F}}_i$ in the class $-2K_{\tilde{\mathbb{F}}_i} - 2E_i$ so that the intersection $D_i \cap E_i$ is smooth and invariant under the automorphism of E_i switching the two factors. Then we obtain double covers Y_i of $\tilde{\mathbb{F}}_i$ and singular space $M_0 = Y_1 \cup Y_2$ gluing along a $K3$, so that there exists a smoothing of M_0 to an (often non-Kähler) Calabi-Yau threefold.*

3.6 Noncompact case

One can also consider the noncompact Calabi-Yau manifolds constructed from the twistor spaces. In [47], a different double cover was taken, in which the branch locus is a $K3$, and the resulting double cover is a positive curvature manifold. In [47] then a noncompact Calabi-Yau can be produced by deleting a divisor from this positive curvature manifold.

The double cover in this paper is different from the double cover in [47] because the branch locus is of different divisor class. In this paper, the twistor space is branched over a divisor that is in twice the divisor class of $K3$ or a union of two $K3$ s, and the twistor fiber \mathbb{P}^1 intersects the branch locus at eight points, and thus this double cover is a fibration by a genus three Riemann surface which is the double cover of the \mathbb{P}^1 fiber branched over eight points. In [47] the branch locus of the double cover is a $K3$ and the twistor fiber \mathbb{P}^1 intersects the branch locus at four points, thus that double cover is a fibration by a genus one Riemann surface which is the double cover of the \mathbb{P}^1 fiber branched over four points.¹ The noncompact Calabi-Yau can be produced by deleting a $K3$ from the double cover of the twistor space branched over $K3$. In this paper, compact Calabi-Yau are produced by the double cover branched over the divisor of a different divisor class.

¹In this way a T^2 fiber is produced and this can be connected to F-theory configurations via general heterotic/F-theory dualities for example [48, 49, 50, 51, 52, 47].

4 $K3$ fibered Calabi-Yau threefolds

4.1 The $\mathbb{P}^3 = \text{Tw}(S^4)$ example

In this subsection we shall construct a $K3$ fibered Calabi-Yau threefold M with a rank 5 stable bundle V satisfying the following constraints

1. $\wedge^5 V \cong \mathcal{O}_M$;
2. $c_2(V) = c_2(M)$;
3. $c_3(V) = 6$.

To fix the notation we first recall the construction of the $K3$ fibered Calabi-Yau threefold M . Let X_1 and X_2 be two smooth quartic $K3$ surfaces in \mathbb{P}^3 so that their intersection $C = X_1 \cap X_2$ is smooth. We shall specify the choice of these $K3$ surfaces later when we construct the stable bundle V . For now, we work with any such $K3$ surfaces. Obviously, X_1 and X_2 generates a pencil in the complete linear system $|\mathcal{O}_{\mathbb{P}^3}(4)|$ with fixed locus C . Let $\pi : \tilde{\mathbb{P}}^3 \rightarrow \mathbb{P}^3$ be the blow-up of \mathbb{P}^3 along C with exception divisor E . By blowing up the fixed locus C , we obtain a $K3$ -fibration $q : \tilde{\mathbb{P}}^3 \rightarrow \mathbb{P}^1$. Because the normal bundle of $C \subset \mathbb{P}^3$ is $\mathcal{O}_C(4) \oplus \mathcal{O}_C(4)$, we have an isomorphism $E \cong C \times \mathbb{P}^1$. We denote by \tilde{X}_i the proper transform of X_i , and $C_i = \tilde{X}_i \cap E$. See Figure 1.

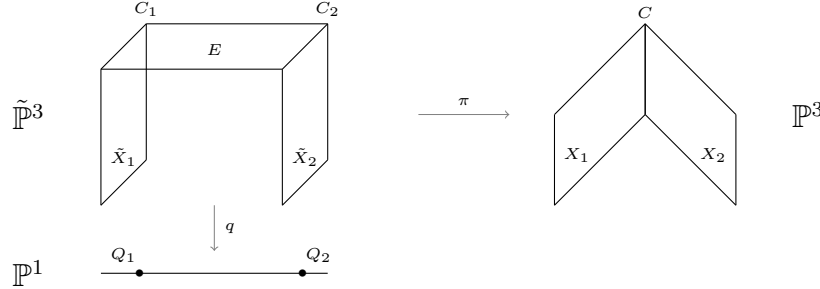


Figure 1: Blowup of \mathbb{P}^3 .

Let $f : M \rightarrow \tilde{\mathbb{P}}^3$ be the double cover of $\tilde{\mathbb{P}}^3$ branched along $\tilde{X}_1 \cup \tilde{X}_2$. Then it is straightforward to verify that M is a $K3$ fibered Calabi-Yau threefold. We denote the inverse image of \tilde{X}_i by \tilde{X}'_i . See Figure 2.

Having this Calabi-Yau threefold M , we compute its second Chern class $c_2(M)$. Let ℓ be the class of a line on \mathbb{P}^3 . Then $c_2(T_{\mathbb{P}^3}) = 6\ell$. Consider the blowing-up $\pi : \tilde{\mathbb{P}}^3 \rightarrow \mathbb{P}^3$. Let $\alpha \in H_2(\tilde{\mathbb{P}}^3, \mathbb{Z})$ be the class of proper transform of C . Then $\alpha = [C_1] = [C_2]$. Consider the short exact sequence

$$0 \longrightarrow \pi^* \Omega_{\mathbb{P}^3} \longrightarrow \Omega_{\tilde{\mathbb{P}}^3} \longrightarrow \Omega_{\tilde{\mathbb{P}}^3/\mathbb{P}^3} \longrightarrow 0. \quad (4.1)$$

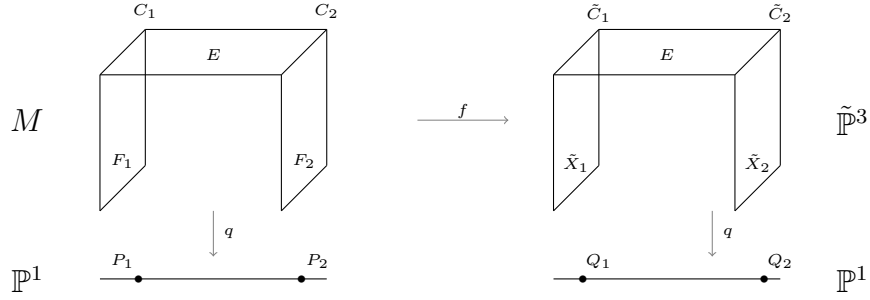


Figure 2: Double cover of $\tilde{\mathbb{P}}^3$.

We obtain

$$c_2(\Omega_{\tilde{\mathbb{P}}^3}) = c_2(\pi^*\Omega_{\mathbb{P}^3}) + c_1(\pi^*\Omega_{\mathbb{P}^3}) \cdot c_1(\Omega_{\tilde{\mathbb{P}}^3/\mathbb{P}^3}) + c_2(\Omega_{\tilde{\mathbb{P}}^3/\mathbb{P}^3}). \quad (4.2)$$

Before we compute the expression of $c_2(\Omega_{\tilde{\mathbb{P}}^3})$, we can get a rough picture on what it looks like by the following argument. Because $\Omega_{\tilde{\mathbb{P}}^3/\mathbb{P}^3}$ is a sheaf supported on E , its Chern classes can be localized on E . On the other hand, we have the following intersection numbers in cohomology of $\tilde{\mathbb{P}}^3$

1. $\pi^*\ell \cdot [\tilde{X}_1] = 4$;
2. $\alpha \cdot [\tilde{X}_1] = 0$.

Now we compute the intersection numbers of $[\tilde{X}_1]$ with both sides of (4.2). Noticing that

$$c_2(\Omega_{\tilde{\mathbb{P}}^3}) \cdot \tilde{X}_1 = c_2(\Omega_{\tilde{\mathbb{P}}^3}|_{\tilde{X}_1}) = c_2(\Omega_{\tilde{X}_1}) = 24, \quad (4.3)$$

and

$$c_2(\pi^*\Omega_{\mathbb{P}^3}) \cdot \tilde{X}_1 = \pi^*(6\ell) \cdot [\tilde{X}_1] = 24, \quad (4.4)$$

we know that $c_1(\pi^*\Omega_{\mathbb{P}^3}) \cdot c_1(\Omega_{\tilde{\mathbb{P}}^3/\mathbb{P}^3}) + c_2(\Omega_{\tilde{\mathbb{P}}^3/\mathbb{P}^3})$ is a class supported on E and has zero intersection with \tilde{X}_1 , which implies that it is equal to $\rho\alpha$ for some integer ρ .

Now we compute this number ρ . Recall that E is the exceptional divisor of $\pi : \tilde{\mathbb{P}}^3 \rightarrow \mathbb{P}^3$ and $E \cong C \times \mathbb{P}^1$. Let $\iota : E \rightarrow \tilde{\mathbb{P}}^3$ be the natural immersion and $q_1 : E \rightarrow C$ and $q_2 : E \rightarrow \mathbb{P}^1$ be the projections. Then we have

$$\Omega_{\tilde{\mathbb{P}}^3/\mathbb{P}^3} = \iota_*\Omega_{E/C} = \iota_*q_2^*\mathcal{O}_{\mathbb{P}^1}(-2) = \iota_*\mathcal{O}_E(-2C_1). \quad (4.5)$$

From the short exact sequence

$$0 \longrightarrow \mathcal{O}_{\tilde{\mathbb{P}}^3}(-E) \longrightarrow \mathcal{O}_{\tilde{\mathbb{P}}^3} \longrightarrow \iota_*\mathcal{O}_E \longrightarrow 0, \quad (4.6)$$

we obtain

$$\begin{aligned}\mathrm{ch}(\iota_*\mathcal{O}_E) &= \mathrm{ch}(\mathcal{O}_{\tilde{\mathbb{P}}^3}) - \mathrm{ch}(\mathcal{O}_{\tilde{\mathbb{P}}^3}(-E)) \\ &= 0 + E - \frac{1}{2}E^2 + \frac{1}{6}E^3.\end{aligned}$$

It follows from (4.5) that

$$\begin{aligned}\mathrm{ch}(\Omega_{\tilde{\mathbb{P}}^3/\mathbb{P}^3}) &= \mathrm{ch}(\iota_*\mathcal{O}_E(-2C_1)) \\ &= \mathrm{ch}(\iota_*\mathcal{O}_E)\mathrm{ch}(\mathcal{O}_{\tilde{\mathbb{P}}^3}(-2\tilde{X}_1)) \\ &= (0 + E - \frac{1}{2}E^2 + \frac{1}{6}E^3)(1 - 2\tilde{X}_1 + \frac{1}{2}(-2\tilde{X}_1)^2 + \frac{1}{6}(-2\tilde{X}_1)^3) \\ &= 0 + E - \frac{1}{2}E^2 - 2E\tilde{X}_1 + \frac{1}{6}E^3 + E^2\tilde{X}_1 + 2E\tilde{X}_1^2.\end{aligned}$$

Therefore

$$c_1(\Omega_{\tilde{\mathbb{P}}^3/\mathbb{P}^3}) = E, \quad \text{and} \quad c_2(\Omega_{\tilde{\mathbb{P}}^3/\mathbb{P}^3}) = E^2 + 2E\tilde{X}_1. \quad (4.7)$$

Noticing that $c_1(\pi^*\Omega_{\mathbb{P}^3}) = -\tilde{X}_1 - E$, by (4.2) we obtain

$$c_2(\Omega_{\tilde{\mathbb{P}}^3}) = c_2(\pi^*\Omega_{\mathbb{P}^3}) + (-\tilde{X}_1 - E)E + E^2 + 2E\tilde{X}_1 = \pi^*(6\ell) + E\tilde{X}_1. \quad (4.8)$$

Since $E\tilde{X}_1 = [C_1] = \alpha$, we get $\rho = 1$ and

$$c_2(\Omega_{\tilde{\mathbb{P}}^3}) = 6\pi^*\ell + \alpha. \quad (4.9)$$

To compute $c_2(M)$, we consider the double cover $f : M \rightarrow \tilde{\mathbb{P}}^3$. Using the short exact sequence

$$0 \longrightarrow f^*\Omega_{\tilde{\mathbb{P}}^3} \longrightarrow \Omega_M \longrightarrow \Omega_{M/\tilde{\mathbb{P}}^3} \longrightarrow 0, \quad (4.10)$$

we have

$$c_2(\Omega_M) = c_2(f^*\Omega_{\tilde{\mathbb{P}}^3}) + c_1(f^*\Omega_{\tilde{\mathbb{P}}^3}) \cdot c_1(\Omega_{M/\tilde{\mathbb{P}}^3}) + c_2(\Omega_{M/\tilde{\mathbb{P}}^3}). \quad (4.11)$$

Because $\Omega_{M/\tilde{\mathbb{P}}^3}$ supports at $\tilde{X}'_1 \cup \tilde{X}'_2$, simple computation shows that

$$c_1(\Omega_{M/\tilde{\mathbb{P}}^3}) = 2[\tilde{X}'_1], \quad c_2(\Omega_{M/\tilde{\mathbb{P}}^3}) = 0. \quad (4.12)$$

It follows that $c_1(f^*\Omega_{\tilde{\mathbb{P}}^3}) \cdot c_1(\Omega_{M/\tilde{\mathbb{P}}^3}) = 0$. Hence

$$c_2(\Omega_M) = c_2(f^*\Omega_{\tilde{\mathbb{P}}^3}). \quad (4.13)$$

In conclusion, we obtain

$$c_2(M) = c_2(\Omega_M) = c_2(f^*\Omega_{\tilde{\mathbb{P}}^3}) = 6f^*\pi^*\ell + f^*\alpha. \quad (4.14)$$

Next we construct a rank 5 bundle V over M satisfying the conditions listed at the beginning of this section. We shall construct V as a direct sum

$$V = V_2 \oplus V_3 \quad (4.15)$$

so that V_2 is a rank 2 stable bundle with

1. $\wedge^2 V_2 \cong \mathcal{O}_M$;
2. $c_2(V_2) = 6f^*\pi^*\ell$;
3. $c_3(V_2) = 0$,

and V_3 is a rank 3 stable bundle with

1. $\wedge^3 V_3 \cong \mathcal{O}_M$;
2. $c_2(V_3) = f^*\alpha$;
3. $c_3(V_3) = 6$.

Once can verify easily that any such V satisfies the required conditions.

To construct V_2 , we recall that for any $d > 0$, one can find a rank 2 instanton bundle W_d over \mathbb{P}^3 which are stable with

$$\wedge^2 W_d = \mathcal{O}_{\mathbb{P}^3}, \quad c_2(W_d) = d\ell, \quad \text{and} \quad c_3(W_d) = 0. \quad (4.16)$$

In our case, we take $V_2 = f^*\pi^*W_6$.

For V_3 , we shall use the construction of stable bundles on Calabi-Yau threefolds in [53]. For convenience, we state the main theorem in [53] for the special case of rank 3 bundles over $K3$ fibered Calabi-Yau threefolds as follows

Theorem 1 [53] *Let $M \rightarrow \mathbb{P}^1$ be a $K3$ -fibered Calabi-Yau threefold. Let $\{Y_i\}$ be disjoint irreducible curves in distinct fibers of M . Suppose $g(Y_i) \geq 1$. Then there exists a rank 3 stable bundle W over M with*

1. $\wedge^3 W \cong \mathcal{O}_M$;
2. $c_2(W) = \sum[Y_i]$;
3. $c_3(W) = \sum(2g(Y_i) - 2)$.

To apply this theorem to our case, we need to choose $K3$ surfaces X_1 and X_2 in \mathbb{P}^3 carefully and find curves Y_i satisfying conditions

1. $\sum[Y_i] = f^*\alpha$;
2. $\sum(2g(Y_i) - 2) = 6$.

To achieve this, we use the following theorem of Mori [54]

Theorem 2 [54] *There exists a non-singular curve of degree $d > 0$ and genus $g \geq 0$ on a non-singular quartic surface in \mathbb{P}^3 if and only if (1) $g = \frac{d^2}{8} + 1$ or (2) $g < \frac{d^2}{8}$ and $(d, g) \neq (5, 3)$.*

Now we let $d = 4$ and $g = 1$. They satisfy condition (2) in the above theorem. So we can find a smooth quartic $K3$ surface X_1 and a smooth curve $C'_1 \subset X_1$ with $\deg C'_1 = 4$ and $g(C'_1) = 1$. Similarly, taking $d = 8$ and $g = 4$, we get another smooth quartic $K3$ surface X_2 and curve $C'_2 \subset X_2$ with $\deg C'_2 = 8$ and $g(C'_2) = 4$. We can also choose such X_1 and X_2 so that they intersect along a smooth curve C .

Since $H_2(\mathbb{P}^3, \mathbb{Z}) = \mathbb{Z}$ and C has degree 16, we know that $[C] = [4C'_1] = 2[C'_2]$. By theorem 1, we can find a rank 3 stable bundle V_3 that fulfills the requirement.

4.2 Physical interpretations

In this subsection we make some physical interpretations of the model after compactification to four dimensions. The commutant of this rank 5 bundle in E_8 is the $SU(5)$ grand unified group. Thus this gives rise to a supersymmetric $SU(5)$ GUT model with three generations of chiral fermions.

In type IIB and F-theory, the GUT symmetry breaking can be obtained by an internal gauge field flux. In these duality frames, the gauge theory degrees of freedom can be packaged onto the worldvolume of seven-branes wrapping a divisor inside the base of the elliptic Calabi-Yau four-folds. There are examples in which the base is \mathbb{P}^3 and the divisor is $\mathbb{P}^1 \times \mathbb{P}^1$ [55, 56, 57, 58]. There is only one generator H for the second homology of \mathbb{P}^3 while there are two generators σ_1 and σ_2 for the two \mathbb{P}^1 s of $\mathbb{P}^1 \times \mathbb{P}^1$. This means that the two-cycle $\sigma = \sigma_1 - \sigma_2$ is homological to zero and trivializes in \mathbb{P}^3 . We can turn on an abelian internal gauge field flux along the linear combination of the two \mathbb{P}^1 s of the divisor $\mathbb{P}^1 \times \mathbb{P}^1$ in \mathbb{P}^3 , which breaks the GUT symmetry group to Standard Model group, see for example [57, 55, 56].

Now we consider the scenario under the blow up and double cover map. The preimage of the two-cycle $(\pi \circ f)^*(\sigma_1 - \sigma_2)$ trivializes in M . We can turn on the internal gauge field flux along this non-homological two-cycle to break the symmetry group to the Standard Model group. This is reminiscent to the scenarios in for example [57, 55, 56].

The model-dependent axions come from the B -field along the homologically non-trivial two-cycles, while the internal gauge field flux is along the homologically trivial two-cycle, hence their topological coupling vanishes after integration on the internal manifold M . Due to this topological mechanism, all couplings to the bulk axions automatically vanish. This includes both the universal axion of the heterotic compactification and the model-dependent axions from the harmonic two-forms of M .

The $c_3 = 6, g = 4$ case is the example of three generation models with rank 5 bundles. This corresponds to supersymmetric $SU(5)$ GUT models with three generations of chiral fermions in four dimensions. After a GUT symmetry breaking, some of the models can give rise to models with Standard Model gauge groups and three generations of chiral fermions.

We have also constructed rank 5 bundles V with $c_3 = 12$. For the \mathbb{P}^3 case, it is

possible to have freely acting \mathbb{Z}_2 involution, so that the $c_3 = 12, g = 7$ case may also be an example of a three generation model, after \mathbb{Z}_2 involution. The classification of the automorphism groups of the $K3$ -fibered Calabi-Yau here is an interesting future direction.

This construction can potentially be generalized for other twistor spaces $\text{Tw}(n\mathbb{P}^2)$, with $n = 1, 2, 3$. In this way, the double cover of the blow up of $\text{Tw}(n\mathbb{P}^2)$, with $n = 0, 1, 2, 3$, are the $K3$ fibrations over \mathbb{P}^1 . We can construct the rank 2 stable instanton bundle V_2 on the $K3$ fibration over \mathbb{P}^1 in similar ways. Once we identify the appropriate curves in the $K3$ fibers, we can construct the rank 3 stable bundle V_3 in an analogous way [53]. The details of the construction of V_3 for general n is beyond the scope of the present paper and is an interesting direction for future investigation.

5 Non-Kähler Calabi-Yau spaces

5.1 Double cover of $\text{Tw}(2\mathbb{P}^2)$

It is known that [38] the twistor space $\text{Tw}(2\mathbb{P}^2)$ is a crepant resolution of the intersection of two quadrics in \mathbb{P}^5 . Explicitly, we let

$$\begin{aligned} Q &= \{z \in \mathbb{P}^5 : z_0^2 + z_1^2 + z_2^2 + z_3^2 + z_4^2 + z_5^2 = 0\} \\ Q_\lambda &= \{z \in \mathbb{P}^5 : 2z_0^2 + 2z_1^2 + \lambda z_2^2 + \frac{3}{2}z_3^2 + z_4^2 + z_5^2 = 0\} \end{aligned} \quad (5.1)$$

for real number $\frac{3}{2} < \lambda < 2$. Let $Z_0 = Q \cap Q_\lambda$. Then Z_0 is a projective threefold with 4 ordinary double point singularities. These 4 double points are

$$(0, 0, 0, 0, 1, \sqrt{-1}), (0, 0, 0, 0, 1, -\sqrt{-1}), (1, \sqrt{-1}, 0, 0, 0, 0), (1, -\sqrt{-1}, 0, 0, 0, 0).$$

The twistor space $Z = \text{Tw}(2\mathbb{P}^2)$ is a small resolution $Z \rightarrow Z_0$ of Z_0 at these 4 ordinary double points.

To understand better the geometry of the double cover Calabi-Yau threefold, we consider the following general phenomenon. Let $D \subset X$ be a very ample divisor of a smooth variety X . Let S be a general member in the linear system $|\mathcal{O}_X(2D)|$. Consider the pencil generated by the nonreduced divisor $2D$ and S . We denote the tautological family by $Y' \rightarrow X \times \mathbb{P}^1$ so that $Y'_0 = 2D$ and $Y'_1 = S$. Applying semistable reduction, we can replace the nonreduced central fiber Y'_0 by a reduced one. Precisely, we can find a base change $b : T \rightarrow \mathbb{P}^1$ and take the normalization of the pullback family, so that the resulting family $Y \rightarrow X \times T$ satisfies Y_0 is reduced. In fact Y_0 is a double cover of D .

To see this, we consider the degeneration of X as follows. Let $\tilde{X} \rightarrow X \times \mathbb{P}^1$ be the blow-up of $X \times \mathbb{P}^1$ along $D \times 0$. Then as a family parameterized by \mathbb{P}^1 , $\tilde{X}_t \cong X$ for $t \neq 0$ and \tilde{X}_0 is the union $X \cup_D P(\mathcal{O}_D \oplus N_D)$, where N_D is the normal bundle of D . Since $S \subset X$ is normal (or transverse) to D , that is, the natural homomorphism

$\mathcal{I}_S \otimes_{\mathcal{O}_X} \mathcal{O}_D \rightarrow \mathcal{O}_D$ is injective, applying the argument in [59], we can modify the family $Y \rightarrow X \times \mathbb{P}^1$ to obtain $Y \subset \tilde{X}$, possibly after a base change, so that $Y_0 \subset P(\mathcal{O}_D \oplus N_D)$ and it is normal to D . Hence, it is a double cover of D . See Figure 3.

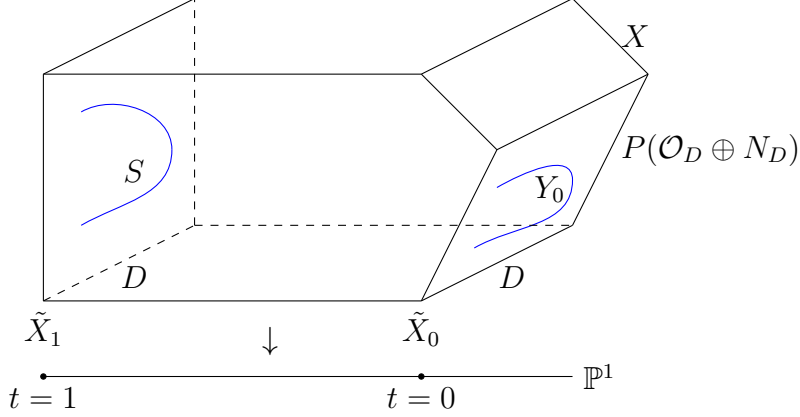


Figure 3: Double cover.

To apply this construction to the twistor space case, we let $X = Q_\lambda$, $D = Q \cap Q_\lambda$ and $S = S_4 \cap Q_\lambda$ for a general quartic hypersurface $S_4 \subset \mathbb{P}^5$. Then we can find a family Y of Calabi-Yau threefolds parameterized by \mathbb{P}^1 , so that the fiber $Y_1 = S$ is a type $(2, 4)$ Calabi-Yau threefold in \mathbb{P}^5 , and Y_0 is a Calabi-Yau threefold which is a double cover of D . A small resolution $M \rightarrow Y_0$ gives a smooth Calabi-Yau threefold M .

It follows that M and Y_t are related by a conifold transition. Here M is a non-Kähler Calabi-Yau, while Y_t is a Kähler Calabi-Yau.

We obtain the following:

Proposition 2 *There exists a family of Calabi-Yau threefolds Y parameterized by \mathbb{P}^1 , so that Y_t is a type $(2, 4)$ smooth projective Calabi-Yau threefold in \mathbb{P}^5 , and Y_0 is a singular Calabi-Yau threefold with 8 ordinary double points. The resolution of these double points gives a Moishezon (non-Kähler) Calabi-Yau threefold which is a double cover of the twistor space $\text{Tw}(2\mathbb{P}^2)$.*

5.2 Vector bundles over non-Kähler Calabi-Yau

In this subsection we consider certain rank 4 bundles over the double cover Calabi-Yau threefold constructed above.

We first discuss the tangent bundle of the double cover M and its Chern classes. The intersection $Z_0 = Q \cap Q_\lambda$ has 4 ordinary double points, taking double covering, the resulting Y_0 has 8 ordinary double points. Let $M \rightarrow Y_0$ be a crepant resolution of these 8 points with exceptional \mathbb{P}^1 s.

Let $H = O_{\mathbb{P}^5}(1)$. The Chern class of type (2, 4) Calabi-Yau threefold family $Y_t \subset \mathbb{P}^5$ follows from expanding to third order in the divisor class,

$$c(Y_t) = \frac{(1 + H)^6}{(1 + 2H)(1 + 4H)} = 1 + 7H^2 - 22H^3. \quad (5.2)$$

The Chern invariants are

$$c_1(Y_t) = 0, \quad c_2(Y_t) = 7H^2, \quad c_3(Y_t) = -22H^3. \quad (5.3)$$

The integrated Chern class is

$$\chi(Y_t) = \int c_3(Y_t) = -176, \quad (5.4)$$

since $H^3|_{Y_t} = 8$.

We denote $p: M \rightarrow \mathbb{P}^5$ the natural map and $H_M = p^*(H_{\mathbb{P}^5})$. The Chern classes of M is therefore

$$c_1(M) = 0, \quad c_2(M) = 7H_M^2 + \sum_{i=1}^8 E_i, \quad c_3(M) = -22H_M^3 + 8\chi(\mathbb{P}^1) = -22H_M^3 + 16, \quad (5.5)$$

where E_i s are the Poincare duals of the 8 \mathbb{P}^1 s. The integrated Chern class is

$$\chi(M) = \int c_3(M) = -160. \quad (5.6)$$

Next we consider vector bundles on M . We can construct various vector bundles with either nonzero c_3 or zero c_3 . The examples with nonzero $c_3(V)$ include the tangent bundle T_M , while the examples with zero $c_3(V)$ include the instanton bundles.

We see that $c_2(M) - c_2(V)$ is the total class of the minimal effective curves $[W] = \sum_{i=1}^8 E_i$. The $c_2(M) - c_2(V)$ is the class of the minimal effective curves on M ,

$$c_2(M) - c_2(V_M) = [W]. \quad (5.7)$$

The fivebranes can wrap on these effective curves, which are the 8 \mathbb{P}^1 s in M .

There exists stable instanton bundles with rank $2l$ on \mathbb{P}^{2l+1} [60]. We are considering the $l = 2$ case, that is, the rank 4 instanton bundles V_m of quantum number $m \in \mathbb{Z}$ on \mathbb{P}^5 . The total Chern class of this vector bundle is

$$c(V_m) = (1 - H^2)^{-m}. \quad (5.8)$$

We define $V = p^*(V_m)$. The Chern class for the vector bundle of the three-fold M are expanded to third order in the divisor class,

$$c(V) = (1 - H_M^2)^{-m} = 1 + mH_M^2. \quad (5.9)$$

The Chern invariants are

$$c_1(V) = 0, \quad c_2(V) = mH_M^2, \quad c_3(V) = 0. \quad (5.10)$$

Hence we set $m = 7$, and we see that $c_2(M) - c_2(V) = [W]$ for a total minimal effective class $[W]$.

By using the theorems in [60], we expect the bundle V is stable. It suffices to show that $H^0(M, (\wedge^q V)_{\text{norm}}) = 0$ for $0 < q < 4$. Recall that [60] the instanton bundle V_m on \mathbb{P}^5 satisfies the exact sequences including

$$0 \longrightarrow \mathcal{O}(-1)^m \longrightarrow S^\vee \longrightarrow V_m \longrightarrow 0, \quad (5.11)$$

$$0 \longrightarrow \mathcal{O}(-1)^m \longrightarrow \mathcal{O}^{2l+2m} \longrightarrow S \longrightarrow 0, \quad (5.12)$$

where V_m is the vector bundle with rank $2l$, and S^\vee is a Schwarzenberger bundle of rank $2l + m$. We are in the situation with $l = 2$ and $m = 7$. So we have that

$$0 \longrightarrow \mathcal{O}(-1)^7 \longrightarrow S^\vee \longrightarrow V_7 \longrightarrow 0, \quad (5.13)$$

$$0 \longrightarrow \mathcal{O}(-1)^7 \longrightarrow \mathcal{O}^{18} \longrightarrow S \longrightarrow 0. \quad (5.14)$$

Therefore S is rank 11, and V_7 is rank 4. We also have the exact sequences,

$$0 \longrightarrow \text{Sym}^2(\mathcal{O}(-1)^7) \longrightarrow \mathcal{O}(-1)^7 \otimes S^\vee \longrightarrow \wedge^2 S^\vee \longrightarrow \wedge^2 V_7 \longrightarrow 0, \quad (5.15)$$

$$0 \longrightarrow \wedge^2 S^\vee \longrightarrow \wedge^2 \mathcal{O}^{18} \longrightarrow \mathcal{O}^{18} \otimes \mathcal{O}(1)^7 \longrightarrow \text{Sym}^2(\mathcal{O}(1)^7) \longrightarrow 0, \quad (5.16)$$

which can be used to compute $H^0(M, (\wedge^q V)_{\text{norm}})$. In the end, using the method of [60], one can show the stability of V in this way.

6 Discussion

In this paper we have constructed Kähler and non-Kähler Calabi-Yau manifolds and have proposed to use them in the heterotic string compactifications. We constructed these manifolds from branched double covers of twistor spaces of four-manifolds with self-dual conformal structures. These manifolds as the branched covers solve the conformally balanced equation. We also constructed stable and polystable vector bundles on these manifolds, which satisfy the anomaly cancellation condition and the Hermitian-Yang-Mills equations. Some classes of the vector bundles that we constructed give rise to supersymmetric grand unified models with three generations of chiral fermions.

Our construction includes both Kähler Calabi-Yau spaces and non-Kähler Calabi-Yau spaces. We constructed them by branched double covers of twistor spaces, or branched double covers of the blow-ups of the twistor spaces. In the latter case, these Calabi-Yau manifolds are also $K3$ fibrations over \mathbb{P}^1 . We considered the twistor spaces of the connected sum of n copies of \mathbb{P}^2 s. The twistor spaces $\text{Tw}(n\mathbb{P}^2)$ with $n = 0, 1$,

which are \mathbb{P}^3 and flag manifold respectively, are Kähler, while the $\text{Tw}(n\mathbb{P}^2)$ with $n = 2, 3$ are non-Kähler. The branch locus of the double cover of the twistor spaces $\text{Tw}(n\mathbb{P}^2)$ can be either smooth or singular. In the smooth case, the branch locus is a divisor whose divisor class is twice of the divisor class of $K3$ surface. In the singular case, the branch locus is a union of two $K3$ surfaces, and after blowing up along the singular locus, the resulting manifold is a $K3$ fibration over \mathbb{P}^1 . In this way, we can uniformly treat these twistor spaces $\text{Tw}(n\mathbb{P}^2)$.

We have also constructed $K3$ -fibered Calabi-Yau manifolds by the double covers of the blow-ups of the twistor spaces. In the $K3$ -fibered Calabi-Yau here, the $K3$ fiber may not be an elliptic $K3$, and hence the $K3$ -fibered Calabi-Yau here generally can be different from the elliptic Calabi-Yau with a Hirzebruch surface base.

The compactification on these non-Kähler Calabi-Yau manifolds contains both geometric moduli from the manifolds and the vector bundle moduli. Comparing to the Kähler Calabi-Yau case, the non-Kähler Calabi-Yau compactifications in some cases may have potentially fewer geometric moduli and bundle moduli than the Kähler case, and the problem to stabilize these bundle moduli to particular values may be an interesting direction.

One of the interesting aspects of the heterotic compactification is to consider the worldsheet instantons and the superpotential generated by them. Comparing to Kähler Calabi-Yau case, the non-Kähler Calabi-Yau manifolds in some cases may have potentially less rational curves. It hence may be interesting to consider the worldsheet instantons in these non-Kähler Calabi-Yau manifolds.

Our approach to use the branched double covers of twistor spaces to produce Kähler and non-Kähler Calabi-Yau manifolds and use them to compactify the heterotic string theory to four dimensions opens new possibilities to construct realistic Standard Model like physics from superstring theory. These Kähler and non-Kähler Calabi-Yau manifolds lead to new compactifications that can give rise to chiral matter with three generations and may be promising in phenomenologically viable four-dimensional theory.

Acknowledgments

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