Acute Triangulation of Rectangles

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Abstract

We proved in this paper that 14 triangles are necessary to triangulate a square with every angle no more than 72°, answering an unsolved problem proposed by Professor D.Eppstein. At first, computer is used to search for topologically feasible solutions of triangulations of a rectangle. Then the geometric feasibility is analyzed case by case by plane geometry. We also proved that any rectangle can be 72° triangulated if the number of triangles is not limited.

1 Introduction

1.1 Preface

On the website the Geometry Junkyard [1] for mathematical problems, D.Eppstein proposed a question: How many triangles are needed at least to subdivide a square (the biggest angle of each triangle should be equal to or less than 72°)? This problem was raised again in the review article [2] presented by C.Zamfirescu. Literature [1] has given the example in which a square is subdivided into 14 triangles (Figure 2), and has assumed that this is the least number of triangles to subdivide a square. In the later parts, this paper will prove that 14 is minimal.

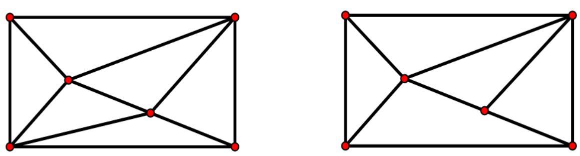
Triangulation of an arbitrary polygon is an important practice in computation geometry or engineering calculation. If any triangle used in the subdivision has no angle of extreme degrees, the computational stability will be better. In [3], Y. Burago and V. Zalgaller have proved the feasibility of subdividing any polygon area into acute triangles, but not all angles are smaller than 72°. Literature [2] has presented several problems related to 72° triangulation or acute triangulation, and has given corresponding results of these problems.

A data structure describing any triangle subdivision in planar polygonal regions is proposed in this paper to keep track of the degrees of new apexes of the formed triangle. Based on this data structure, corresponding computer algorithm is designed to search for the topology types of possible triangle subdivisions in rectangles in which the maximum number of subdivided triangles is 14. Any angle of the formed triangle should be no greater than 72° and there are restrictions on the degree of each apex. Plane geometry is used to deduce the feasibility of the topological structure of triangle subdivisions. It is found that partial topological structures are infeasible and some can be formed by rectangles of fixed length-width ratios. Only one topological structure can achieve triangle subdivision with 72° apex which can make up a square or almost a square. There are exactly 14 such constituent triangles. It is then proved that 14 is the minimum number of such triangles that can form above-mentioned topology structure. As a deduction of above research, any rectangle can be subdivided into triangles with 72° apex when the number of such triangles is not limited.

In our team, Yibin Zhang is responsible for the argumentation of related mathematical problems; Xiaoyang Sun sees to the construction of mathematical graphs and proof in plane geometry; Zhiyuan Fan shoulders algorithm design and computer programming. The whole paper is completed under the supervisors guidance and the collective efforts of the three students.

1.2 Mathematical Definition

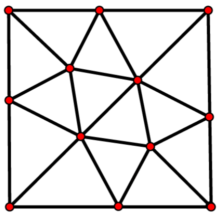
Triangulation of a plane polygon area *A* refers to representing *A* by a union of some triangles that are disjoint or share a vertex or an entire common edge. In Figure 1, a) is a triangulation; b) is not a triangulation as a vertex of one triangle falls on the edge of another triangle.



a) Triangulation of a rectangle　　　b) not triangulation

Figure 1

If any triangle in triangulation is an acute one, the triangulation is called acute triangulation, of which if the greatest angle is no greater than 72°, the triangulation is called a 72° triangulation. 72° triangulation of a square is shown in Figure 2, where the three angles of each triangle having common edges with the square are 45°, 63° and 72°; the three angles of any other triangle are 54°, 54° and 72°. In [1], D. Eppstein proposes a question that whether 14 is the least number of triangles used in 72-degree-triangulation of a square, but in [2] this problem is regarded as an unsolved problem.



72°triangulation of a square into 14 triangles
Figure 2

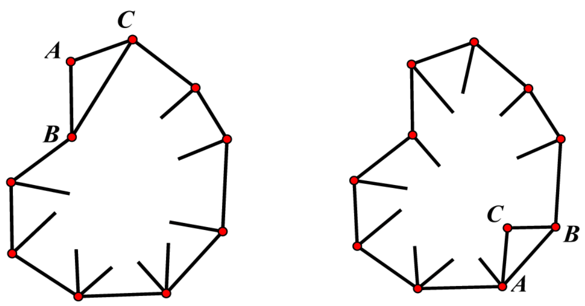
As to triangulation of any rectangle, the triangle vertex inside the rectangle is called inner vertex; the triangle vertex on rectangle edge, excluding the four corners, is called edge vertex; the triangle vertices on the four corners of a rectangle are called corner vertices. Numbers of the three kinds of vertex above in triangulation (in the above order) are recorded as N, B and J. By calculating the sum of interior angles of triangles, it can be inferred that the number of triangles in the triangulation is $2N+B+2$. In 72° triangulation, each angle of the triangle is within the range of $[36°, 72°]$. Therefore, in 72° triangulation, each inner vertex belongs to at least 5 triangles and at most 10 triangles; each edge vertex belongs to 3 to 5 triangles; each corner vertex belongs to 2 triangles.

1.3 Topology of triangulation

For triangulation of a polygon area, we specify the formation sequence of triangles, and regain the triangulation in a special method. We hope to start from polygon edges, and after each formation of a triangle, ensure that the un-subdivided area is still a polygon. For each vertex of the polygon mentioned in this process, its degree of freedom is defined as the number of edges not connecting this point to formed vertices. For a polygon, its freedom sequence is defined as the sequence of degree freedom of all vertices of the polygon in counter clockwise order, starting from one marked vertex of the polygon.

The following two methods are used to increase triangles:

1. If the freedom degree of a polygon vertex A is 0, the freedom degrees of the two adjacent vertices, B and C, must be positive, thus forming a triangle. At this time, vertex A will be removed from the vertex sequence of the polygon (Figure 3a).
2. If the freedom degrees of all vertices of a polygon are positive, in the triangulation there must be inner vertices. Take an edge AB of the polygon to increase a triangle ABC in which C is a new vertex with an assigned degree of freedom. Vertex C is then inserted between A and B in the vertex sequence of the polygon (Figure 3b).



a) Delete vertex with 0 degree of freedom   b) Add a new internal vertex

Figure 3

Obviously, the two methods of increasing triangles do not change the number of polygon (only 1), so changing the sequence of freedom degree can be used to describe the process of triangulation. To facilitate programming, three operations are used:

1. Merging: if there is any 0 in the sequence, we can remove it from the sequence and subtract 1 respectively from the two adjacent numbers of the deleted number.
2. Insertion: for a sequence, we can insert a new number x before the first number, and subtract 1 respectively from the first number and the last number.

3. Rotation: put the last number of the sequence before the first number.

The above sequence of freedom degree terminates at $(0,0,0)$ and we call it a terminator sequence which corresponds to a polygon with three vertices with all degrees of freedom being 0. This is the last formed triangle in the triangulation.

If an initial sequence of freedom degree is set, we can develop it into a sequence in which all numbers are natural numbers, and this sequence of operation is permissible. It can be proved that any triangulation can be achieved by a permissible sequence operation of freedom degree.

In order to remove the equivalent permissible operation gained by operation 3 triangulation, we require that, in the final permissible sequence gained by rotation, the first vertex, the last vertex and some other boundary vertex (not internal) form a triangle before the rotation. If this is unattainable, this permissible sequence will not be considered.

As to the sequence of degree of freedom applicable to 72°triangulation, the degree of freedom for each newly added vertex ranges from 3 to 8. The initial freedom sequence has four 1s in it which correspond to corner vertices; other degree of freedom ranges from 2 to 4, and corresponds to the edge vertex.

In order to use the triangulation topology with no more than 14 triangles, we enumerate the distribution of freedom degree of vertices on edges. For each distribution, we enumerate operation sequences of permissible degree of freedom. By imposing special limitation on the sequence of operation (limit the degree and total number of vertices), we can achieve all the triangulation of the original polygon under the given conditions, as shown in Table 1:

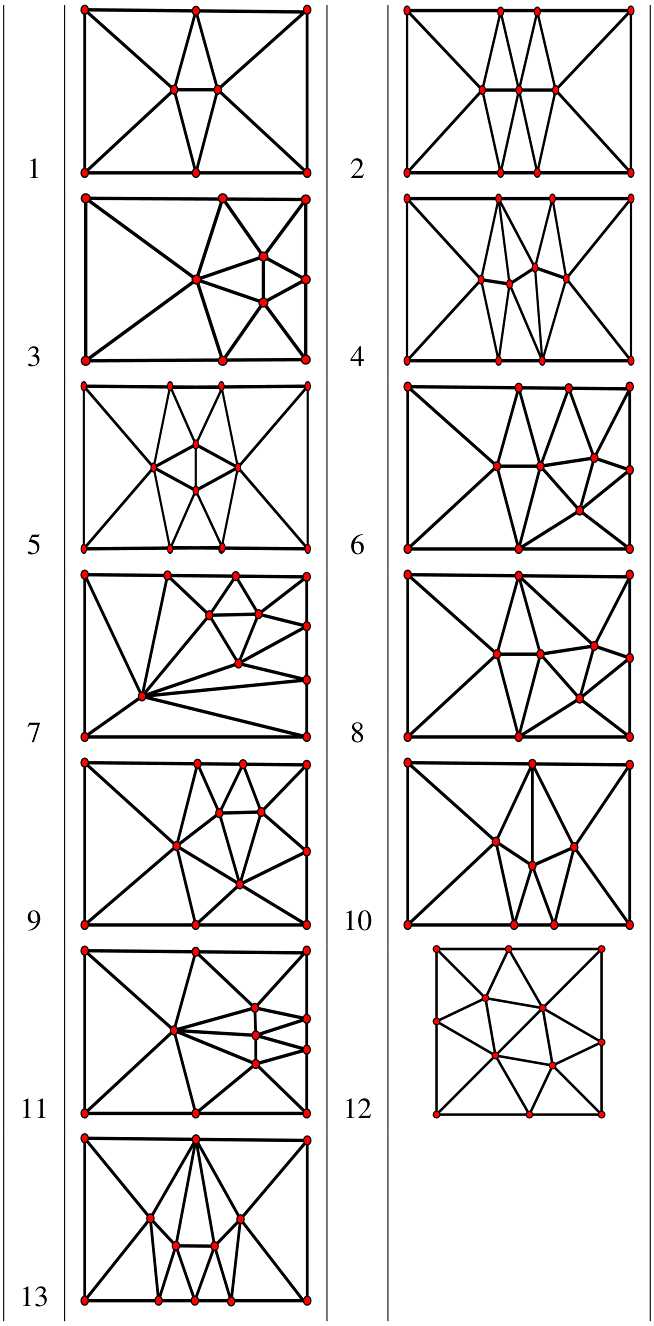


Table 1: Topology of 72°triangulation of rectangle (Less than 14 triangles)

2 Proving and analysis

2.1 Proving of several lemmas

For cases in Table 1, we study the feasibility of 72° triangulation geometrically with the following lemmas.

Lemma 1: It is impossible for a triangle with vertices of three 5-degree points to exist in any rectangle.

Proof: According to the definition of 72° triangulation, each of the five angles sharing a common vertex with the 5-degree point is 72°. If there is any triangle with each of its vertex as such, each of its three inner angles is equally 72°, and the sum of its inner angles is $72^\circ \times 3 = 216^\circ$, which is impossible.

The lemma is thus demonstrated.

Lemma 2: If there is no triangulation vertex on some edges of a rectangle except the two corner vertices, in the triangle containing this edge, the angle opposite to the edge is $72^\circ$and the other two angles are equally 54°.

Proof: in rectangle $ABCD$, if there is no triangulation vertex on edge AB, it will be demonstrated that in the sub-divided triangle $\triangle ABE$ including vertices A and B, $\angle AEB = 72^\circ$.

In fact, it can be inferred from the relationship between degrees of angles formed between parallel lines that $\angle AEB = \angle DAE + \angle CBE(*)$.

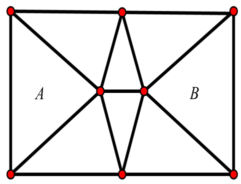
According to the definition of 72° triangulation, $\angle AEB \leq 72^\circ, \angle DAE \geq 36^\circ$ and $\angle CBE \geq 36^\circ$, so the equation $(*)$ exists. It can be inferred that $\angle AEB = 72^\circ, \angle DAE = 36^\circ$ and $\angle CBE = 36^\circ$. The lemma is thus demonstrated.

Lemma 3: If an edge vertex of a rectangle is shared by three different triangles and one of which has the angle of 36°at this vertex, the angles here for other two triangles are of 72°.

Proof: it is obviously true according to angle range of a 72° triangulation.

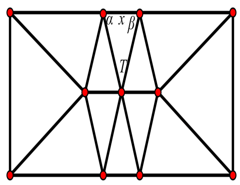
2.2 Analysis of different topological subdivision diagrams

Case 1:



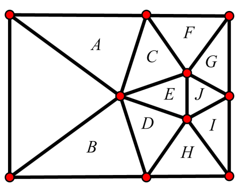
According to Lemma 2, triangle A and triangle B are isosceles triangles with apex angles equally being 72°. Other six triangles are isosceles triangles with apex angles equally being 36°. The edge length ratio corresponding to the triangulation in this case is fixed.

Case 2:



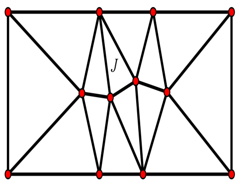
By similar derivation to case 1, the angles of the left and the right triangles are of 72°, 54° and 54°; angles of its 4 adjacent triangles are of 72°, 72° and 36°. By studying the middle triangle T on the long edge, it can be observed that the height h of triangle T on the edge x of the rectangle is fixed, and the two adjacent angles of x are expressed as α and β; the length of the edge $x = \frac{h}{cot\alpha} + \frac{h}{cot\beta}$; when $\alpha = \beta = 72°$, the minimum value can be achieved, and when $\alpha = 72°, \beta = 36°$(or $\beta = 72°$) , the maximum value can be achieved. The aspect ratio range of this type of rectangle is$[(2+\frac{\sqrt{5}-1}{2})^2, 3] \cdot \frac{sin54°}{sin72°}$.

Case 3:



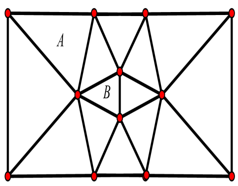
As shown in the figure, according to Lemma 1, triangle A and triangle B are isosceles triangles with apex angles of 36°; triangle C and triangle D are adjacent to an inner apex of degree 5, so their maximum angles are 72°; E and J are isosceles triangles with their apex angles being 36°; the common edge of A and C is of the same length to that of B and D, so triangle C and triangle D have two edges of the same length and a same angle (maximum angle) facing the edge. Therefore, triangle C and triangle D are congruent ones — isosceles triangles with their apex angles being 72°; by calculating the angles, it can be inferred that F and H are isosceles triangles with their apex angles being 72°; G and I are isosceles triangles with their apex angles being 36°. By calculation, we can see that the common edges of E and D, D and H as well as H and I are of the same length which is equal to the waist length of E or J. However, this is contradict with the shape of I, so sub-division under such circumstances cannot be fully achieved.

Case 4:



As shown in the figure, same as the discussion in case 1, the positions and shapes of the 5 left triangles in the rectangle are identical to those in case 1.Similarly, the positions and shapes of the 5 right triangles are identical to those in case 1. Therefore, in case 4, the four inner vertices are on the horizontal axis of symmetry of the rectangle and J is an obtuse triangle. This is a contradiction, so case 4 cannot be realized.

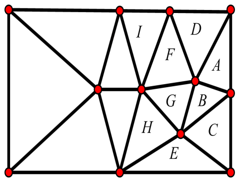
Case 5:



By repeating the discussion in case 3, we can see that the graphs have fixed shape, and are symmetrical on the horizontal and vertical center lines. Triangle A, triangle B and their symmetrical triangles are all isosceles triangles with their apex angles being 36°, and other triangles are isosceles triangles with 72°of apex angles.

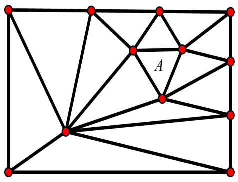
Therefore, this case is suitable for rectangles with a fixed aspect ratio.

Case 6:



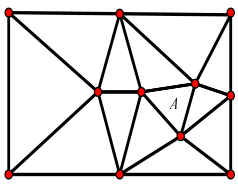
Triangle B is adjacent to two inner vertices with the degree of 5, and it is an isosceles triangle with its apex angle being 36°; the two adjacent angles of the common edge of A and B are 72°, so A is an isosceles triangle with its apex angle being 36° and C is of the same case; by calculating angles, it can be inferred that D and E are isosceles triangles with their apex angles equally being 72°; the common edge of B and G is perpendicular to the bottom edge of the rectangle. The union of triangles D, E, B and G show reflective symmetry on the common edge B and G, so the common edge of F and I is perpendicular to the bottom edge of the triangle and I is a right triangle. This is contradictory, so this case cannot be realized.

Case 7:



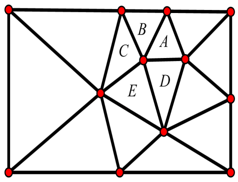
In this case, the vertices of the triangle A are all 5-degree points. Therefore, given the results in Lemma 1, this cannot be realized.

Case 8:



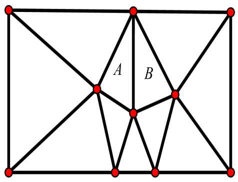
In this case, the vertices of the triangle A are all 5-degree points. Therefore, given the results in Lemma 1, this cannot be realized.

Case 9:



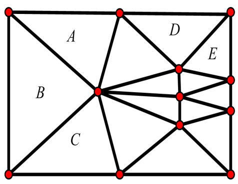
A is an isosceles triangle with its apex angle being 36°; according to Lemma 3, *B* and *A* are adjacent and their angles adjacent to the rectangle edge are 72°, so *B* an isosceles triangle with its apex angle being 36°; according to Lemma 3, *C* is also an isosceles triangle with its apex angle being 72°; The longest edges of triangle *C*, triangle *B* and triangle *A* decrease successively; *D* and *A* are congruent. Aspect ratio of the triangle *E*'s two edges adjacent to 5-degree inner vertices is $(\frac{sin72°}{sin36°})^2$ which exceeds the upper limit of the ratio of the longest edge to the shortest edge in any triangle in which no angle is larger than 72°. Therefore, this case cannot be realized.

Case 10:



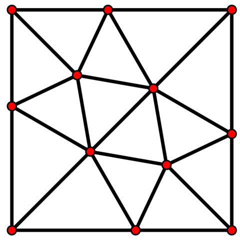
This case applies similar discussion as case 1, and it can be referred to case 4 as well. The three inner vertices are on the horizontal central line of the rectangle, so there is at least one right angle in triangle *A* and triangle *B*. This is contradictory and it cannot be realized.

Case 11:



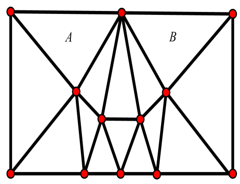
According to Lemma 2, angles of triangles *A*, *B* or *C* at the inner vertex with a degree of 7 are 72°, and angles of the other four triangles at the same point is 36°. Triangle *D* is an isosceles triangle with an apex angle of 36°and its angle at the upper right corner of the rectangle is of 72°. As a result, triangle E has an angle of 18°, which is contradictory and cannot be realized.

Case 12:



According to Figure 2, this case is geometrically feasible for the triangulation of a square.

Case 13:



According to Lemma 2, the angles of A and B at the apex of the rectangles upper edge are 72°, so the sum of the remaining three angles at this apex is 36°. This is contradictory and it cannot be realized.

In summary, the 72° triangulation with no more than 14 triangles is not applicable to that of squares, except for in case 12.

Theorem 4: There are at least 14 triangles in a 72° triangulation of a square.

2.3 Feasibility of 72° triangulation for general rectangles

Theorem 5: There exists an interval such that a rectangle with aspect ratio in this interval can be 72° triangulated according to topological diagram case 12 in Table 1.

Proof: as shown in Figure 4, triangulation is constructed with the following steps.

1. Create a parallelogram $ABCD$ and ensure $\angle BAD = 72^\circ$, $BC = 1$ and $CD = y$, and the center of the parallelogram is marked as O;

2. Construct ten angles by taking A and C(both are 5-degree points) as their vertices;

3. Intercept line segment CC' with the length of r from ray CP;

4. Draw a circle, with O being its center and OC' being its radius. The circle cuts line BD and ray AQ by B' , D' and A'.

5. $A'B'C'D'$ form a rectangle. Several other rays starting from A and C cut the rectangle edge by E, F, G and H.

6. Connect BE, BF, DG and DH to obtain the triangulation of rectangle $A'B'C'D'$.

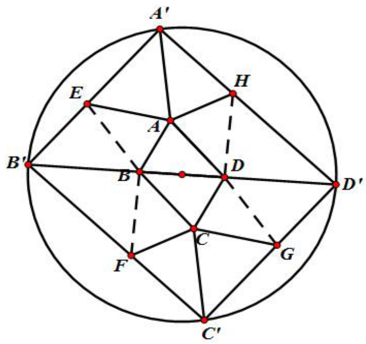


Figure 4

In the following part, the influence of different values of y and r on the triangulation is discussed and the following conclusions can be drawn consecutively:

- When $y = 1, r = r_0$, a 72° triangulation of a square can be obtained in which there are 72° angles at A, C, B and D.

- There exists ε_0, if $y = 1, r = r_0 + \varepsilon_0$, a 72° triangulation of a square can be obtained in which angles at A and C are 72° and other angles are smaller than that.

- By continuity, there exists ε, when $|y - 1| < \varepsilon, |r - (r_0 + \varepsilon_0)| < \varepsilon$, 72° triangulation of some rectangle can always be achieved.

When y is a fixed value not equal to 1, there are at most two r values to make the formed rectangle a square. If it is a square, OC' shall be perpendicular to line BD; if there is another square $A''B''C''D''$ and there are two dots on the ray CP which make both OC' and OC'' be perpendicular to BD , then CP will be perpendicular to BD. This result is contradictory to $y \neq 1$.

Therefore, as long as y is approaching to 1 as a limit but not equal to 1, for any proper r, a 72° triangulation of a rectangle can be obtained. Due to continuity, we can get the 72° triangulation of a rectangle whose aspect ratio is within a certain range. *QED*.

Theorem 6: Any rectangle with rational aspect ratio can be 72° triangulated.

Proof: for any rectangle with a rational number aspect ratio, if its aspect ratio is $\frac{p}{q}$, (wherein $(p,q) = 1$, and p and q are positive integers), the rectangle can be subdivided into $p \cdot q$ squares of the same size.

Take a 72° triangulation of a square and place it horizontally. Perform reflection transformation on the square by its right edge to gain two adjacent squares. Perform reflection transformation on the rightmost square by its right edge to gain three adjacent squares. Repeat this operation to get a row of p adjacent squares, thus forming a rectangle with its aspect ratio being p.

As to this rectangle, reflection transformation shall be performed by its upper edge, and then we can get a rectangle consisting of two rows of squares. Perform reflection transformation to the uppermost rectangle by its upper edge, and repeat this operation to get a rectangle consisting of q rows of squares. All the $p \cdot q$ squares are subdivided by congruent 72° triangulation, and the common edges of adjacent squares share the same subdivision method. Therefore, the rectangle formed by $p \cdot q$ squares are 72° triangulated. *QED*.

Theorem 7: Any rectangle can be 72° triangulated.

Proof: for a rectangle with its aspect ratio being any positive real number x, according to the denseness of rational number, there must be a rational number $\frac{p}{q}$, which makes $\frac{p}{q} \cdot x$ fall into the interval mentioned in Theorem 5.

Method in Theorem 6 can be used to perform reflection transformation to a small 72° triangulated rectangle of ratio $\frac{p}{q} \cdot x$, thus creating a larger rectangle consisting of a matrix of small rectangles of p rows and q columns. Therefore, the aspect ratio of the larger rectangle is $\frac{p}{q} \cdot \frac{q}{p} \cdot x = x.QED$.

3 Targeted research in future

There are many problems in this paper worth our studying in future. For example, similar method can be used to explore how many triangles are needed at least for subdividing a flat torus with acute triangles. In Literature [2], this is referred to as a problem unsolved.

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