Motion in Quantum-Classical Transition via ODM

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Abstract: In this paper, we firstly re-present the results of Bondar et al. (2013). Then, following the approach taken in it, we use the method of Operational Dynamic Modeling (ODM) to deal with the situation where the background mechanics is varying between quantum and classical mechanics.

Key Words: Operational Dynamic Modeling (ODM); Variable background mechanics; Quantum-classical transition

0 Introduction

In quantum mechanics, it is always said that by doing the $\hbar \to 0$ approximation, the classical mechanics can be acquired, but there is a problem, which is that $\hbar$ is a fundamental physical constant with dimension. Letting $\hbar \to 0$ without really arriving at 0 is only changing the system of units. This makes the unification of quantum and classical mechanics hard to achieve. However, in Bondar et al. (2013), an approach via Operational Dynamic Modeling (ODM) is taken, and has gained great success. This approach works as following:

First, ODM axioms are introduced, which include: i) the states of physical systems are represented by normalized vectors in a complex Hilbert space; ii) observables are Hermitian operators acting on this space[1]; iii) the expectation value of a system Ψ's

[1] Here, the use of terminology "Hermitian" is a subject of Griffiths (2004).

observable $\hat{Q}$ at time t, $\overline{Q}(t)$, is $\langle\Psi(t)|\hat{Q}|\Psi(t)\rangle$; iv) the probability[1] that a measurement of Q would yield Q_0 is $|\langle f_{Q_0}|\Psi(t)\rangle|^2$, where $\hat{Q}|f_{Q_0}\rangle = Q_0|f_{Q_0}\rangle$ and $|f_{Q_0}\rangle$ is normalized. As can be seen, these axioms are just the same with ordinary quantum mechanics axioms. However, the Schrödinger equation, of course, is not going to be included. Instead, the following two axioms are used:

v) Ehrenfest Theorems:

$$m\frac{d}{dt}\bar{x}(t) = \bar{p}(t)$$

$$\frac{d}{dt}\bar{p}(t) = \overline{-U'(\hat{x})}(t)$$

vi) Stone's Theorem:

$$i\hbar\left|\frac{d\Psi}{dt}\right\rangle = \hat{H}|\Psi\rangle$$

where it shall be noticed that $\hat{H}$ does not necessarily represent Hamiltonian in quantum mechanics, but rather an unknown Hermitian operator acting as the motion generator. The next axiom is where the difference between quantum and classical mechanics lies. For the quantum one, it goes as following:

vii) $[\hat{x},\hat{p}] = i\hbar$

For the classical one:

[1] For cases when $\hat{Q}$'s eigenvalues form a continuous range, $|\langle f_{Q_0}|\Psi(t)\rangle|^2$ would be the probability density instead of the probability.

vii) $[\hat{x}, \hat{p}] = 0$

Thus, it can be seen that the fundamental difference between quantum and classical mechanics is the commutator of position and momentum operator. Setting this to $i\hbar$ and 0 will yield quantum and classical mechanics, respectively. Therefore, accepting the following unified axiom:

vii) $[\hat{x}, \hat{p}] = i\hbar k$

where k is a real number satisfying $0 \leq k \leq 1$, an unified mechanics can be obtained.

The last axiom is not explicitly included in Bondar et al. (2013). Nonetheless, it is in fact used:

viii) $\hat{x}$ and $\hat{p}$ are independent of one another. Formally, $\frac{\partial \hat{x}}{\partial p} = 0$ and $\frac{\partial \hat{p}}{\partial x} = 0$.

This makes sense in both quantum and classical cases, under the former of which $\hat{p} = \frac{\hbar}{i}\frac{\partial}{\partial x}$, and under the latter of which x and p are two independent variables. In the following part of the current paper, this axiom is not going to be explicitly mentioned, as it is quite unnecessary to do so.

The detailed physical background of these axioms and several notes related to them will be given in **Chapter 1**. Following that, in **Chapter 2**, how the unified mechanics is acquired in Bondar et al. (2013) is going to be re-presented. Also, several slight differences between our approach and Bondar et al. (2013) will be illustrated in footnotes.

In Bondar et al. (2013), only the case where k is a constant is discussed, leaving the case where the background mechanics is “changing” uncovered. That is, $k = k(t)$. This is not just changing k’s in the final equations of motion, but rather, several steps towards these equations need to be reconsidered. One of this paper’s main aims is to obtain the equation of motion in this changing-k case, and briefly discuss some example situations. This will be done in **Chapter 3 & 4**.

For the sake of convenience, all the physical systems in this paper will be assumed to consist of only one particle and only move one-dimensionally.

1 Notes on the Axioms

The development of quantum mechanics introduced a new way of doing physics, which is via Hilbert space and using operators to represent observables. Using Hilbert space, a function space, is the combination of the wave formulation and the matrix formulation of quantum mechanics, since by doing so, the wave *functions* can also be *vectors*, and the observables can be both *functionals* acting on functions and *matrices* acting on vectors. This gave birth to the first two axioms. The need for the observables’ representing operators being Hermitian is crucial for the expectation values to be real.

The next two axioms are the matter of generalized statistical interpretation, in which it shall be noticed that the inner product $\langle f|g\rangle$, following Griffiths (2004), is defined to be:

$$\int_{-\infty}^{\infty} f^* g \, dx$$

The latter of whom actually implies the former. This can be proved as following:

First of all, for any observable $\hat{Q}$, eigenvectors with distinct eigenvalues are orthogonal. Suppose we have two eigenvectors of this kind and let them be denoted by f and g, respectively. Also, we denote their associated eigenvalues with Q_f and Q_g, which are both real since $\hat{Q}$ is Hermitian. Then $\langle f|\hat{Q}|g\rangle = \langle \hat{Q}f|g\rangle$, which leads to $Q_g\langle f|g\rangle = Q_f^*\langle f|g\rangle = Q_f\langle f|g\rangle$, and ultimately $\langle f|g\rangle = 0$ since Q_f and Q_g are distinct. When two or more eigenvectors with an identical associated eigenvalue Q exist, we can still choose an orthonormal basis for the space spanned by these eigenvectors, whose elements are all eigenvectors with the same eigenvalue Q.

Moreover, for any observable $\hat{Q}$, its eigenvectors form a basis. Since if it is not the case, an normalized Ψ cannot be written as a linear combination of $\hat{Q}$'s eigenvectors exists. Formally:

$$\Psi = F + \sum c_n f_n$$

where f_n's are orthonormal eigenvectors and F is a vector orthogonal to all of them. Then the sum of the probabilities mentioned in axiom iv shall be examined, which is $\sum |c_n|^2$. We further notice that:

$$\langle \sum c_n f_n \mid \sum c_n f_n \rangle = \sum_n \sum_m c_n c_m \langle f_n | f_m \rangle = \sum |c_n|^2$$

Since this is the sum of the probabilities of all possible cases, it should be 1. However, due to Ψ's normalization and F's non-triviality, $\langle \sum c_n f_n \mid \sum c_n f_n \rangle$ is in fact less than 1, indicating a contradiction.

Combining the above results shows that $\hat{\mathrm{Q}}$'s eigenvectors form an orthonormal basis. Therefore, the above *F* term can be abolished:

$$\Psi = \sum c_n f_n$$

Let the corresponding eigenvalue of f_n be Q_n. Thus, the expectation value, due to axiom iv, is:

$$\sum |c_n|^2 \mathrm{Q}_n$$

Due to axiom iii, the expectation value is $\langle \Psi | \hat{\mathrm{Q}} | \Psi \rangle$ while:

$$\langle \Psi | \hat{\mathrm{Q}} | \Psi \rangle = \langle \sum c_n f_n \mid \sum c_n \mathrm{Q}_n f_n \rangle = \sum |c_n|^2 \mathrm{Q}_n$$

It matches exactly what follows from axiom iv. Hence, it can be concluded that axiom iii is actually a corollary of axiom iv.[1] □

Axiom v governs the motion of expectation values of the position and the momentum.

Axiom vi should be paid special attention to. If it is not a non-commutative operator-function case, but a commutative real number case, separation of variables can be easily

[1] However, this proof only works for discrete-eigenvalue cases, as in continuous cases inner products may not exist and c_n's will not be probabilities but probability densities instead.

applied to solve for Ψ, as following:

$$\frac{d\Psi}{\Psi} = -\frac{i}{\hbar}\hat{H}dt$$

Hence:

$$\Psi = e^{-\frac{i}{\hbar}t\hat{H}}\Psi_0 \tag{1}$$

However, this is not the case. Nevertheless, due to this, it is proper to *assume* that (1) is our answer. Differentiating (1) shows that it implies axiom vi. For the other direction, it is needed to Taylor-expand Ψ with respect to *t*:

$$\Psi = \sum_{n=0}^{\infty} \frac{\Psi_0^{(n)}}{n!} t^n$$

Notice that, due to axiom vi, $\frac{d}{dt} = -\frac{i}{\hbar}\hat{H}$, therefore:

$$\Psi = \sum_{n=0}^{\infty} \frac{1}{n!}\left(-\frac{i}{\hbar}t\hat{H}\right)^n \Psi_0 = e^{-\frac{i}{\hbar}t\hat{H}}\Psi_0$$

This is identical with (1).

(1) is actually closer to the original version of Stone's Theorem (Stone, 1932). Stone's Theorem states that for any strongly continuous one-parameter unitary group $U(t)$, there exists a Hermitian operator $\hat{\mathrm{Q}}$ such that $U = e^{it\hat{\mathrm{Q}}}$. In our current discussion, $U(t)$ is the time evolution of the wave function, and Stone's Theorem therefore guarantees the existence of the Hermitian motion generator $\hat{H}$ (in fact, Stone's Theorem guarantees $-\frac{\hat{H}}{\hbar}$ is Hermitian, rather than $\hat{H}$ itself, but they are equivalent

anyway).

However, a seeming problem arises here: $U(t)$ is a family of operators while $\widehat{H}$ is not, which infers that for the Stone's Theorem to hold, $\widehat{H}$ is needed to be independent of time, which is not always the case. In spite of this, it is in fact not a problem because the Stone's Theorem is about the existence of a $\widehat{H}$ independent of time satisfying (1), yet it does *not* indicate a $\widehat{H}$ *depending on time* satisfying (1) cannot exist. Moreover, the reverse of the Stone's Theorem also holds (Stone, 1932), which ensures that, for any Hermitian operator $\widehat{\mathrm{Q}}$ and any real constant t_0, $e^{it_0\widehat{\mathrm{Q}}}$ is an unitary operator. This will work even if $\widehat{\mathrm{Q}}$ is *depending on time* which will just make $e^{it\widehat{\mathrm{Q}}}$ a family of unitary operators. Following that, it is clear that whether $\widehat{H}$ depends on time or not, Ψ, evolving according to the equation of motion (1), will remain normalized. This makes certain that having a time-depending $\widehat{H}$ will not cause any inconsistency.

Axiom vii can be viewed from two aspects. Firstly, if two observables can both be measured to arbitrary precision, they must have identical eigenvectors. Hence, they cannot have non-trivial constant[1] commutator. This can be proved as following:

Suppose the two observables $\widehat{\mathrm{Q}}$ and $\widehat{\mathrm{P}}$ have an identical eigenvector *f*, and their commutator is iC where *C* is a nontrivial real number. Thus:

$$\widehat{\mathrm{Q}}f = \mathrm{Q}f;\ \widehat{\mathrm{P}}f = \mathrm{P}f$$

[1] Here, "constant" indicates that the following is a (complex) number instead of an operator rather than indicating time-independency.

for some Q and P. Hence:

$$[\hat{\mathrm{Q}},\hat{\mathrm{P}}]f=(\hat{\mathrm{Q}}\hat{\mathrm{P}}-\hat{\mathrm{P}}\hat{\mathrm{Q}})f=(\mathrm{QP}-\mathrm{PQ})f=0=iCf$$

This contradiction finalizes the proof. □

This result shows that, when $[\hat{x},\hat{p}]=i\hbar k$, the position and the momentum cannot be both measured to arbitrary precision except for the case of classical mechanics.

Secondly, following from the generalized uncertainty principle (which is a corollary of axiom i, ii, and iii), $\sigma_{\mathrm{Q}}^2\sigma_{\mathrm{P}}^2\geq\left(\frac{1}{2i}\langle[\hat{\mathrm{Q}},\hat{\mathrm{P}}]\rangle\right)^2$ (Griffiths, 2004), and axiom vii:

$$\sigma_x\sigma_p\geq\frac{\hbar k}{2}$$

Thus, *k* may be seen as a measure of uncertainty. Classical mechanics has no uncertainty while quantum mechanics has the most.

2 Deduction of the Unified Mechanics

Applying chain rule to axiom v (assuming $\hat{x}$ and $\hat{p}$ do not depend explicitly on time):

$$m\left\langle\frac{d\Psi}{dt}\middle|\hat{x}\middle|\Psi\right\rangle+m\left\langle\Psi\middle|\hat{x}\middle|\frac{d\Psi}{dt}\right\rangle=\langle\Psi|\hat{p}|\Psi\rangle$$

$$\left\langle\frac{d\Psi}{dt}\middle|\hat{p}\middle|\Psi\right\rangle+\left\langle\Psi\middle|\hat{p}\middle|\frac{d\Psi}{dt}\right\rangle=\langle\Psi|-U'(\hat{x})|\Psi\rangle$$

Applying axiom vi:

$$im\langle\Psi|[\hat{H},\hat{x}]|\Psi\rangle=\hbar\langle\Psi|\hat{p}|\Psi\rangle$$

$$i\langle\Psi|[\hat{H},\hat{p}]|\Psi\rangle = \hbar\langle\Psi|-U'(\hat{x})|\Psi\rangle$$

Since this applies to all states Ψ, the average may just be dropped. Thus:

$$im[\hat{H},\hat{x}] = \hbar\hat{p} \tag{2}$$

$$i[\hat{H},\hat{p}] = -\hbar U'(\hat{x}) \tag{3}$$

We are now going to apply axiom vii. For the quantum case, assuming $\hat{H} = f(\hat{x},\hat{p})$[1]:

$$m\frac{\partial f}{\partial p} = p$$

$$\frac{\partial f}{\partial x} = U'(x)$$

The quantum Hamiltonian is subsequently acquired[2]:

$$\hat{H} = \frac{\hat{p}^2}{2m} + U(\hat{x}) \tag{4}$$

For the classical case, it would no longer work to just assume $\hat{H} = f(\hat{x},\hat{p})$, since that would lead to the result that $[\hat{H},\hat{x}]$ and $[\hat{H},\hat{p}]$ both vanish. Therefore, we utilize two new auxiliary operators $\widehat{\sigma_x}$ and $\widehat{\sigma_p}$ which satisfy (C is an arbitrary real constant[3]):

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[1] The following equations are corollaries of a result in noncommutative analysis, which states that $[f(\widehat{A_1},\widehat{A_2},\cdots,\widehat{A_n}),\hat{B}] = \sum_{k=1}^{n}[\widehat{A_k},\hat{B}]\frac{\partial f}{\partial \widehat{A_k}}$ whenever all $[\widehat{A_k},\hat{B}]$'s commute with all $\widehat{A_k}$'s and $\hat{B}$ (Bondar et al., 2013). Here and in the following use of this result, the requirement above is satisfied since $[\hat{x},\hat{p}]$'s are (complex) numbers (though they may depend on time in some cases).

[2] From the preceding deduction, only a version of (4) with an arbitrary constant term can be obtained, but changing it is just changing the reference level when calculating potential energy, so for convenience, we can just set it to be 0. (This problem exists because the Schrödinger equation *does rely* on the reference level.)

[3] In Bondar et al. (2013), C is set to be 0, but it is not actually required in the deduction, and C being 0 can be a problem when retrieving quantum mechanics from the unified mechanics later since in this paper the Schrödinger equation formulation will be used instead of the phase space formulation in Bondar et al. (2013) when retrieving quantum mechanics. Therefore, we abandon this original assumption.

$$[\hat{x},\widehat{\sigma_x}]=i,[\hat{p},\widehat{\sigma_p}]=i,[\widehat{\sigma_x},\widehat{\sigma_p}]=iC$$

and all other commutators vanish. Now it is time to assume that $\widehat{H}=f(\hat{x},\hat{p},\widehat{\sigma_x},\widehat{\sigma_p})$.

Then these follow as corollaries:

$$m\frac{\partial f}{\partial \sigma_x}=\hbar p$$

$$\frac{\partial f}{\partial \sigma_p}=-\hbar U'(x)$$

The classical Liouvillian is now retrieved:

$$\hat{L}=\frac{1}{\hbar}\widehat{H}=\frac{1}{m}\hat{p}\widehat{\sigma_x}-U'(\hat{x})\widehat{\sigma_p}+F(\hat{x},\hat{p}) \quad (5)$$

where *F* is an arbitrary function (real-valued, due to the assumption $\hat{L}$ is Hermitian).

Now it is ready to apply the unified axiom vii, for consistency with classical case, $\widehat{\sigma_x}$ and $\widehat{\sigma_p}$ are still needed:

$$mk\frac{\partial H}{\partial p}+\frac{m}{\hbar}\frac{\partial H}{\partial \sigma_x}=p$$

$$k\frac{\partial H}{\partial x}-\frac{1}{\hbar}\frac{\partial H}{\partial \sigma_p}=U'(x)$$

Therefore:

$$\widehat{H}=\frac{1}{k}\left[\frac{\hat{p}^2}{2m}+U(\hat{x})\right]+F\left(\hat{p}-\hbar k\widehat{\sigma_x},\hat{x}+\hbar k\widehat{\sigma_p}\right) \quad (6)$$

where *F* is again an arbitrary function (real-valued, and also required to be differentiable this time). In Bondar et al. (2013), a quite complicated method is used to show that $F(p,x)$, under suitable assumptions, must be of form:

$$F(p,x)=-\frac{1}{k}\left[\frac{p^2}{2m}+U(x)\right]+O(1) \tag{7}$$

However, we are not going to prove it here.

Plugging (7) into (6)[1]:

$$\hat{H}=\frac{\hbar}{m}\widehat{\sigma_x}\left(\hat{p}-\frac{\hbar k}{2}\widehat{\sigma_x}\right)+\frac{1}{k}\left[U(\hat{x})-U\left(\hat{x}+\hbar k\widehat{\sigma_p}\right)\right]+O(1)$$

Let $k=1$ and then notice that $\widehat{\sigma_x}$ can be set $\frac{1}{\hbar}\hat{p}$ to satisfy all the assumptions, and similarly $\widehat{\sigma_p}$ can be set $-\frac{1}{\hbar}\hat{x}$:

$$\hat{H}=\frac{\hat{p}^2}{2m}+U(\hat{x})-U(0)+O(1) \tag{8}$$

Comparing this with (4), the conclusion could be reached that this $O(1)$ should be $U(0)$. Also, it could be easily tested that this would be consistent with (5) (when k tends to 0). Consequently, we now arrive at the final unified Hamiltonian:

$$\hat{H}=\frac{\hbar}{m}\widehat{\sigma_x}\left(\hat{p}-\frac{\hbar k}{2}\widehat{\sigma_x}\right)+\frac{1}{k}\left[U(\hat{x})-U\left(\hat{x}+\hbar k\widehat{\sigma_p}\right)\right]+U(0) \tag{9}$$

3 Background Mechanics as a Function of Time

In **Chapter 1**, how the approach in Bondar et al. (2013) has achieved an unified

[1] In Bondar et al. (2013), the term $O(1)$ is directly set to be 0, but here, instead of the phase space formulation, we are trying to retrieve the Schrödinger equation formulation, and setting $O(1)$ to be 0 would be unsuitable for this task unless (as we will see) $U(0)=0$ (though in most cases this does hold). Therefore, we keep this term and test with $k=1$ to see what actually this $O(1)$ term should be.

mechanics has been re-presented, but from its very beginning, we have assumed that $\hat{x}$ and $\hat{p}$ do not depend explicitly on time, which implies that the characteristic of background mechanics (i.e., the commutator of $\hat{x}$ and $\hat{p}$) does not change. In this chapter the variable case will be tested, and its equation of motion will be deduced.

Going back to axiom v and applying chain rule (noticing that this time, $\frac{\partial \hat{x}}{\partial t}$ and $\frac{\partial \hat{p}}{\partial t}$ may not be trivial) once again, we see that:

$$m\left\langle \frac{d\Psi}{dt} \middle| \hat{x} \middle| \Psi \right\rangle + m\left\langle \Psi \middle| \frac{\partial \hat{x}}{\partial t} \middle| \Psi \right\rangle + m\left\langle \Psi \middle| \hat{x} \middle| \frac{d\Psi}{dt} \right\rangle = \langle \Psi | \hat{p} | \Psi \rangle$$

$$\left\langle \frac{d\Psi}{dt} \middle| \hat{p} \middle| \Psi \right\rangle + \left\langle \Psi \middle| \frac{\partial \hat{p}}{\partial t} \middle| \Psi \right\rangle + \left\langle \Psi \middle| \hat{p} \middle| \frac{d\Psi}{dt} \right\rangle = \langle \Psi | -U'(\hat{x}) | \Psi \rangle$$

Following similar procedure, we arrive at:

$$im\left[\hat{H}, \hat{x}\right] + \hbar \frac{\partial \hat{x}}{\partial t} = \hbar \hat{p} \tag{10'}$$

$$i\left[\hat{H}, \hat{p}\right] + \hbar \frac{\partial \hat{p}}{\partial t} = -\hbar U'(\hat{x}) \tag{11}$$

Axiom vii should now be rewritten as:

vii) $[\hat{x}, \hat{p}] = i\hbar k(t)$

Before continuing, we should take a look at how we obtain (8). We take $\widehat{\sigma_x}$ to be $\frac{1}{\hbar}\hat{p}$ and $\widehat{\sigma_p}$ to be $-\frac{1}{\hbar}\hat{x}$ in the case $k = 1$. Generalizing this, it could be seen that whenever $k \neq 0$, $\widehat{\sigma_x}$ can be $\frac{1}{\hbar k}\hat{p}$, and $\widehat{\sigma_p}$ can be $-\frac{1}{\hbar k}\hat{x}$. Furthermore, it is consistent taking $\hat{p}$ to be $\frac{\hbar k}{i}\frac{\partial}{\partial x}$ (in *x*-representation). Thus, except for the classical case, two independent variables are not needed to have the equation of motion written down.

However for the classical case this *is* necessary, which can be proved as following:

Assume, without loss of generality, that classical mechanics can be written in *x*-representation, which indicates that $\hat{p} = f\left(x, \frac{\partial}{\partial x}\right)$. However, due to $\hat{p}$'s independency from $\hat{x}$, it can only be a function of $\frac{\partial}{\partial x}$. Then its commutator with $\hat{x}$ shall be examined:

$$[\hat{p}, \hat{x}] = \left[\frac{\partial}{\partial x}, x\right] \frac{\partial \hat{p}}{\partial\left(\frac{\partial}{\partial x}\right)} = \frac{\partial \hat{p}}{\partial\left(\frac{\partial}{\partial x}\right)}$$

which should be 0 in classical mechanics. This indicates that $\hat{p}$ must be a (real) constant, *C*. Applying axiom v gives:

$$\overline{-U'(x)} = 0$$

It restricts the possible forms of potential functions. For example, $U = x$ cannot work since a contradiction would arise. This, however, is not allowed in a physical background and thus finishes the proof. □

There are two-variable forms of quantum mechanics, e.g. phase space formulation. However, since using two independent variables would be extra complicated for this variable-mechanics case, and it is not needed since for discrete reaches of classical mechanics when the background mechanics is changing, taking limits would be sufficient, and while for continuous reaches, it could be directly dealt with normal classical-mechanics ways, for the sake of convenience, it is better to disregard the two-variable classical cases consequently. Accordingly, we would stop use auxiliary operators, $\widehat{\sigma_x}$ and $\widehat{\sigma_p}$, since they can be taken as linear combination of $\hat{x}$ and $\hat{p}$. In addition, the results of the following deduction are assumed to only be used in *x*-representation without loss of generality, so only $\hat{p}$ would be a function of time.

Axiom vii can now be specified to:

vii) $[\hat{x}, \hat{p}(t)] = i\hbar k(t)$

Now, back to (10') and (11), we see that the second term in (10') does not exist anymore:

$$im[\hat{H}, \hat{x}] = \hbar \hat{p} \tag{10}$$

Applying axiom vii to (10) and (11), the following can be obtained:

$$mk(t)\frac{\partial H}{\partial p} = p$$

$$k(t)\frac{\partial H}{\partial x} - \frac{\partial p}{\partial t} = U'(x)$$

In order to figure out what $\frac{\partial p}{\partial t}$ is, we need to take partial derivative of axiom vii with respect to *t*:

$$\left[\hat{x}, \frac{\partial \hat{p}}{\partial t}\right] = i\hbar \frac{\partial k}{\partial t}$$

Notice that:

$$\frac{\partial\left(\frac{\partial \hat{p}}{\partial t}\right)}{\partial x} = \frac{\partial^2 \hat{p}}{\partial t \partial x} = \frac{\partial\left(\frac{\partial \hat{p}}{\partial x}\right)}{\partial t} = 0$$

$$i\hbar \frac{\partial k}{\partial t} = \left[\hat{x}, \frac{\partial \hat{p}}{\partial t}\right] = i\hbar k \frac{\partial\left(\frac{\partial \hat{p}}{\partial t}\right)}{\partial p}$$

Therefore, $\frac{\partial p}{\partial t} = \frac{\partial k/\partial t}{k} p$. Utilizing this, we see that[1]:

$$\hat{H} = \frac{\hat{p}^2}{2mk} + \frac{\partial k/\partial t}{k^2}\hat{p}\hat{x} + \frac{U(\hat{x})}{k} + iC$$

In order to make this constant term clear, the Hermitian conjugate of this Hamiltonian needs to be examined:

$$\hat{H}^\dagger = \frac{\hat{p}^2}{2mk} + \frac{\partial k/\partial t}{k^2}\hat{x}\hat{p} + \frac{U(\hat{x})}{k} - iC$$

Therefore:

$$\hat{H} - \hat{H}^\dagger = -\frac{\partial k/\partial t}{k^2}[\hat{x}, \hat{p}] + 2iC = -\frac{i\hbar \cdot \partial k/\partial t}{k} + 2iC$$

Since this difference should be 0, the constant term can now be known and the final form of the Hamiltonian could be obtained:

$$\hat{H} = \frac{\hat{p}^2}{2mk} + \frac{\partial k/\partial t}{k^2}\hat{p}\hat{x} + \frac{U(\hat{x})}{k} + \frac{i\hbar \cdot \partial k/\partial t}{2k} \tag{12}$$

$$= \frac{\hat{p}^2}{2mk} + \frac{\partial k/\partial t}{k^2}\left(\frac{\hat{p}\hat{x} + \hat{x}\hat{p}}{2}\right) + \frac{U(\hat{x})}{k}$$

We now move on to give the final *x*-representation form of the equation of motion. Substituting $\hat{p}$ with $\frac{\hbar k}{i}\frac{\partial}{\partial x}$ in (12) and using axiom vi gives:

$$\frac{d\Psi}{dt} = \frac{i\hbar k}{2m}\frac{\partial^2\Psi}{\partial x^2} - \frac{1}{k}\frac{\partial k}{\partial t}\frac{\partial\Psi}{\partial x}x - \frac{1}{2k}\frac{\partial k}{\partial t}\Psi - \frac{i}{\hbar k}U(x)\Psi \tag{13}$$

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[1] Due to the same reference-level reason, we have ignored the real part of the constant term. However, this Hamiltonian is impossible to be Hermitian unless a nontrivial imaginary part is utilized.

4 Motion of Free Particles in Varying Background Mechanics

We write out the equation of motion for free particles (in *x*-representation) explicitly at first:

$$\frac{d\Psi}{dt}=\frac{i\hbar k}{2m}\frac{\partial^2\Psi}{\partial x^2}-\frac{1}{k}\frac{\partial k}{\partial t}\frac{\partial\Psi}{\partial x}x-\frac{1}{2k}\frac{\partial k}{\partial t}\Psi$$

We can go no further till $k=k(t)$ is specified. Therefore, we choose two commonly-seen functions and write down their corresponding specific equations of motion.

1. Linear Functions

In this case, $k=\lambda t\ \left(0<t\leq\frac{1}{\lambda}\right)$, where λ is a real positive constant. The equation of motion would therefore be:

$$\frac{d\Psi}{dt}=\frac{i\hbar\lambda t}{2m}\frac{\partial^2\Psi}{\partial x^2}-\frac{x}{t}\frac{\partial\Psi}{\partial x}-\frac{1}{2t}\Psi$$

2. Sine Functions

In this case, $k=\frac{1}{2}[\sin(\omega t+\varphi)+1]$. The equation of motion would therefore be:

$$\frac{d\Psi}{dt}=\frac{i\hbar[\sin(\omega t+\varphi)+1]}{4m}\frac{\partial^2\Psi}{\partial x^2}-\frac{\omega x\cos(\omega t+\varphi)}{\sin(\omega t+\varphi)+1}\frac{\partial\Psi}{\partial x}-\frac{\omega\Psi\cos(\omega t+\varphi)}{2\sin(\omega t+\varphi)+2}$$

For these two cases, the specific equations of motion have been given, but due to the complexity of these equations, the author is not capable of giving precise solutions to them at this point. However, it may be possible that these equations could be solved numerically using a computer.

Acknowledgements: The author wishes to thank my advisor, Mr. Sang, my instructor in English academic writing, Ms. Tan, other teachers who gave me suggestions, my fellow students and friends, and my parents. Their understanding, support and suggestions are vital for the completion of this paper.

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