The Extension of Euler's Formula and Its Applications

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Contents

ABSTRACT

In this paper we first introduce a fractional form formula among a number of Euler's formulas. We then extend the formula and with mathematical induction prove the case when the number of terms increases and the exponent is integer. Afterwards, we study the connection between Euler's formula and Lagrange interpolating polynomial and use the latter to prove part of the extended formula. We then obtain a new formula from this connection. At last, we derive a set of new equations from the extended formula.

Keywords: Euler's Formula, Lagrange Interpolating Polynomial, Mathematical Induction

Chapter 1 Introduction

Euler discovered the following equation in 18th century, and named it as Euler's Formula, which is the name for many of his formulas, making them sometimes confusing.

$$\frac{a^r}{(a-b)(a-c)}+\frac{b^r}{(b-a)(b-c)}+\frac{c^r}{(c-a)(c-b)}=\begin{cases}0, r=0,1\\ 1, r=2\\ a+b+c, r=3\end{cases} \quad (1.1)$$

However, this formula only includes four cases with $r=0,1,2,3$ and three terms. Thus, this formula draws attention of the people who love math. Someone (reference [2]) has already succeeded in expanding the formula to the cases when *r* is any non-negative integers.

In this paper, we will furthermore systematically expand the formula to

$$f(n,r)=\sum_{i=1}^{n}\frac{a_i^r}{(a_i-a_1)\cdots(a_i-a_{i-1})(a_i-a_{i+1})\cdots(a_i-a_n)} \quad (1.2)$$

where $a_1,a_2,\cdots,a_n$ are distinct and $n\geq 2$, $r\in\mathbb{Z}$, that is when r is any integer and the number of terms is greater than one.

At first, we spent a month to solve and prove the case when $r=0$ using mathematical induction, which is often the tool used in our proofs. Then, from this case we got the recursion formula about the relation between $f(n,r)$ and $f(n-1,i)$, $(i=0,1,\cdots,r-1)$. Using the new formula, we solved the cases when $r-n+1<0$ and $r-n+1\geq 0$ separately$(r>0)$. After that, we solved the cases when $r<0$ through similar approach.

Subsequently, we studied the connection between Euler's Formula and Lagrange Interpolating Polynomial, and proved part of the expanded formula of $f(n,r)$. Through this connection we also discovered a new equation.

At last, we derived some new formulas as well as new thoughts from the expansion of Euler's Formula.

Chapter 2 The expansion of Euler's Formula

2.1 The case when $r=0$

We used a C++ program to choose random a_i and calculate $f(n,0)$. We found out that the outcome is always very close to 0.

n	**r**	a_i	$f(n,0)$
3	0	0,-6,2	0
4	0	-2,-7,3,1	0
5	0	-6,-1,5,9,-9	1.05879×10^{-22}
6	0	0,-3,1,7,-1,-4	-4.23516×10^{-22}
7	0	-6,-7,-3,0,-4,6,3	1.32349×10^{-23}

Chart 1

(The source code and the outcome of a $f(7,0)$ and a $f(11,0)$ are in the appendix)

Hence, we guess that $f(n,0)=0$.

Lemma (1): $f(n,0)=0$

Proof:

First we have $\frac{1}{a-b}+\frac{1}{b-a}=0$

Suppose $f(k,0)=0$, that is

$$\frac{1}{(a_1-a_2)(a_1-a_3)\cdots(a_1-a_{k-1})(a_1-a_k)}+\frac{1}{(a_2-a_1)(a_2-a_3)\cdots(a_2-a_{k-1})(a_2-a_k)}$$
$$+\cdots+\frac{1}{(a_k-a_1)(a_k-a_2)\cdots(a_k-a_{k-2})(a_k-a_{k-1})}=0$$

Then

$$\begin{aligned}&\frac{(a_1-a_{k+1})}{(a_1-a_2)(a_1-a_3)\cdots(a_1-a_{k+1})}+\frac{(a_2-a_{k+1})}{(a_2-a_1)(a_2-a_3)\cdots(a_2-a_{k+1})}\\&+\cdots+\frac{(a_k-a_{k+1})}{(a_k-a_1)(a_k-a_2)\cdots(a_k-a_{k-1})(a_k-a_{k+1})}=0\end{aligned} \tag{2.1}$$

Now we need to prove $f(k+1,0)=0$, that is

$$\begin{aligned}&\frac{1}{(a_1-a_2)(a_1-a_3)\cdots(a_1-a_{k+1})}+\frac{1}{(a_2-a_1)(a_2-a_3)\cdots(a_2-a_{k+1})}+\cdots+\\&\frac{1}{(a_k-a_1)\cdots(a_k-a_{k-1})(a_k-a_{k+1})}+\frac{1}{(a_{k+1}-a_1)\cdots(a_{k+1}-a_{k-1})(a_{k+1}-a_k)}=0\end{aligned} \tag{2.2}$$

(2.2) left side ×(a_1-a_{k+1}),

$$\begin{aligned}&\frac{a_1-a_{k+1}}{(a_1-a_2)(a_1-a_3)\cdots(a_1-a_{k+1})}+\frac{a_1-a_{k+1}}{(a_2-a_1)(a_2-a_3)\cdots(a_2-a_{k+1})}\\&+\cdots+\frac{a_1-a_{k+1}}{(a_k-a_1)\cdots(a_k-a_{k-1})(a_k-a_{k+1})}\end{aligned} \tag{2.3}$$

(2.1) left side - (2.3),

$$\begin{aligned}&\frac{0}{(a_1-a_2)(a_1-a_3)\cdots(a_1-a_k)(a_1-a_{k+1})}+\frac{a_2-a_1}{(a_2-a_1)(a_2-a_3)\cdots(a_2-a_k)(a_2-a_{k+1})}\\&+\cdots+\frac{a_k-a_1}{(a_k-a_1)(a_k-a_2)\cdots(a_k-a_{k-1})(a_k-a_{k+1})}+\frac{a_{k+1}-a_1}{(a_{k+1}-a_1)\cdots(a_{k+1}-a_{k-1})(a_{k+1}-a_k)}\\&=\frac{1}{(a_2-a_3)(a_2-a_4)\cdots(a_2-a_k)(a_2-a_{k+1})}+\frac{1}{(a_3-a_2)(a_3-a_4)\cdots(a_3-a_k)(a_3-a_{k+1})}\\&+\cdots+\frac{1}{(a_{k+1}-a_2)(a_{k+1}-a_3)\cdots(a_{k+1}-a_{k-1})(a_{k+1}-a_k)}\\&=0\end{aligned}$$

∴ (2.1)-(2.3)$=0$

∵ (2.1)$=0$

∴ (2.3)$=0$

Also (2.3)$=$(2.2)×(a_1-a_{k+1})

∴ (2.2)$=0$, that is if $f(k,0)=0$, $f(k+1,0)=0$

Q.E.D.

	A	B	C	D	E	F	G	H	I	J	K
1	…										
2	8										
3	7										
4	6										
5	5										
6	4										
7	3										
8	2										
9	1										
10	0	0	0	0	0	0	0	0	0	0	0
11	r/n	2	3	4	5	6	7	8	9	10	…

Figure 1

2.2 The cases when $r > 0$

2.2.1 The recursion formula of $f(n,r)$

After guessing that $f(n,0)=0$, we figured that if this hypothesis was proven true, then there would be a relation between $f(a,b)$ and $f(a-1,0), f(a-1,1), \cdots, f(a-1,b-1)$. We give the following lemma to illustrate that relation.

Lemma (2):

$$\begin{aligned} f(n,r) = {} & a_n^{\,r-1} f(n-1,0) + a_n^{\,r-2} f(n-1,1) + \cdots + \\ & a_n^{\,1} f(n-1,r-2) + a_n^{\,0} f(n-1,r-1) \end{aligned} \tag{2.4}$$

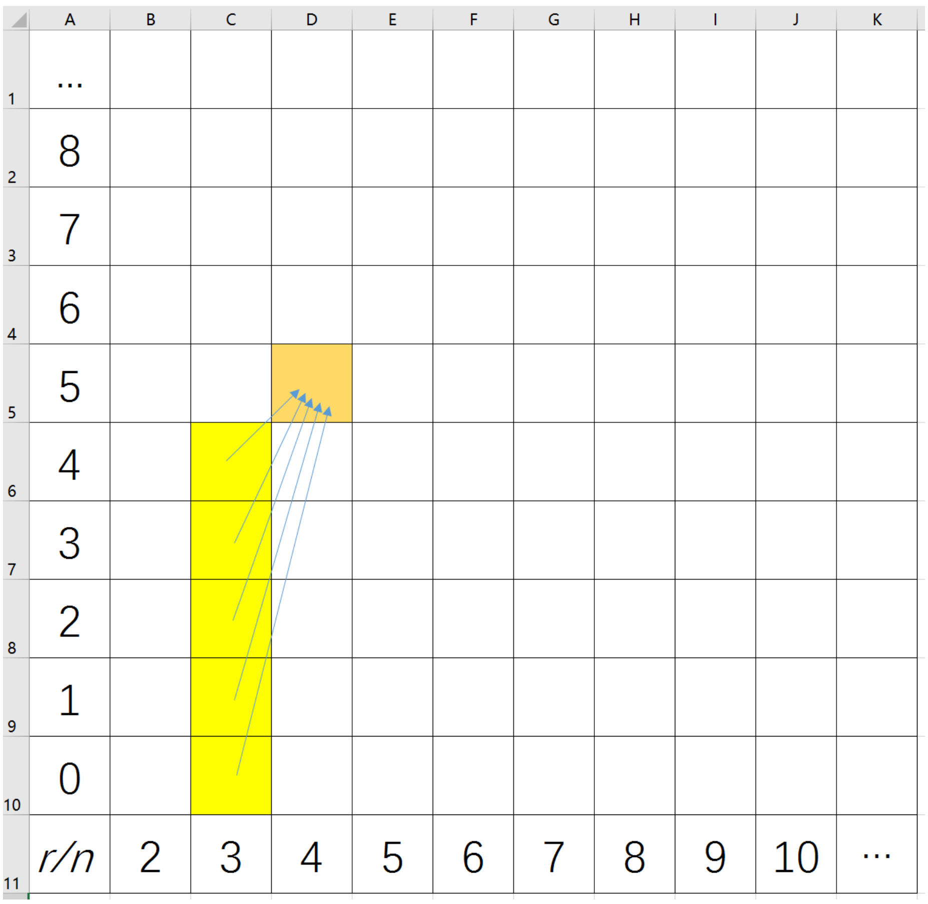

Figure 2

Proof:

$$
\begin{aligned}
f(n,r) &= \frac{a_1^{\,r}}{(a_1-a_2)(a_1-a_3)\cdots(a_1-a_n)} + \frac{a_2^{\,r}}{(a_2-a_1)(a_2-a_3)\cdots(a_2-a_n)} \\
&\quad + \cdots + \frac{a_n^{\,r}}{(a_n-a_1)(a_n-a_2)\cdots(a_n-a_{n-1})} \\
&= \frac{a_1^{\,r}-a_n^{\,r}}{(a_1-a_2)(a_1-a_3)\cdots(a_1-a_n)} + \frac{a_2^{\,r}-a_n^{\,r}}{(a_2-a_1)(a_2-a_3)\cdots(a_2-a_n)} \\
&\quad + \cdots + \frac{a_n^{\,r}-a_n^{\,r}}{(a_n-a_1)(a_n-a_2)\cdots(a_n-a_{n-1})} \\
&= \frac{a_1^{\,r-1}+a_1^{\,r-2}a_n+\cdots+a_n^{\,r-1}}{(a_1-a_2)(a_1-a_3)\cdots(a_1-a_{n-1})} + \frac{a_2^{\,r-1}+a_2^{\,r-2}a_n+\cdots+a_n^{\,r-1}}{(a_2-a_1)(a_2-a_3)\cdots(a_2-a_{n-1})} \\
&\quad + \cdots + \frac{a_{n-1}^{\,r-1}+a_{n-1}^{\,r-2}a_n+\cdots+a_n^{\,r-1}}{(a_{n-1}-a_1)(a_{n-1}-a_2)\cdots(a_{n-1}-a_{n-2})} \\
&= a_n^{\,r-1}f(n-1,0)+a_n^{\,r-2}f(n-1,1)+\cdots+a_n^{\,1}f(n-1,r-2)+a_n^{\,0}f(n-1,r-1)
\end{aligned}
$$

Q.E.D.

2.2.2 The expanded formula when $r-n+1<0$

First, we found that since $f(2,0)=0$, $f(3,1)=a_3^{\ 0}f(2,0)=0$. Also, because $f(3,0)=f(3,1)=0$, thus $f(4,1)=a_4^{\ 0}f(3,0)=0,\ f(4,2)=a_4^{\ 1}f(3,0)+a_4^{\ 0}f(3,1)=0$.

……

Using **Lemma(2)** (2.4), it is relatively easy to find out

Theorem (1):

When $r-n+1<0$,

$$f(n,r)=0 \tag{2.5}$$

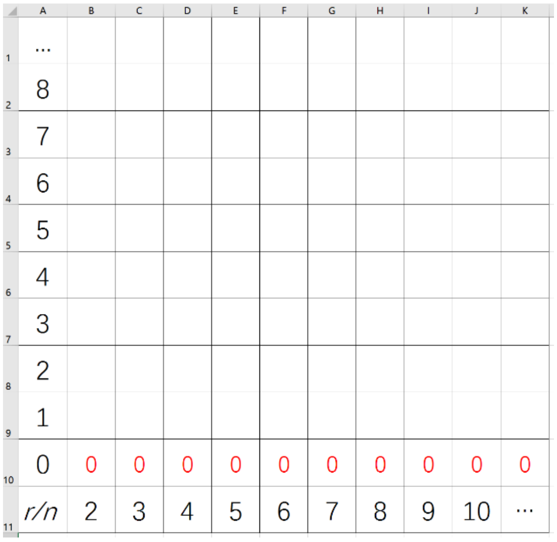

Figure 3

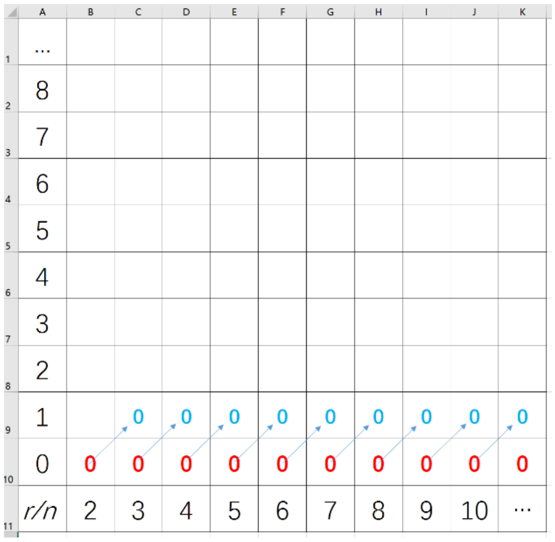

Figure 4

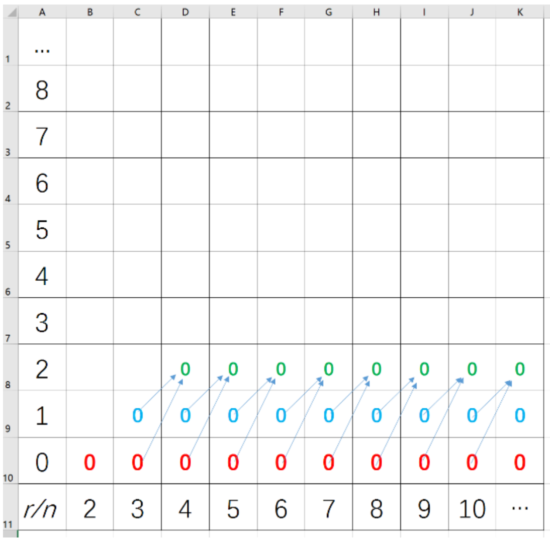

Figure 5

…

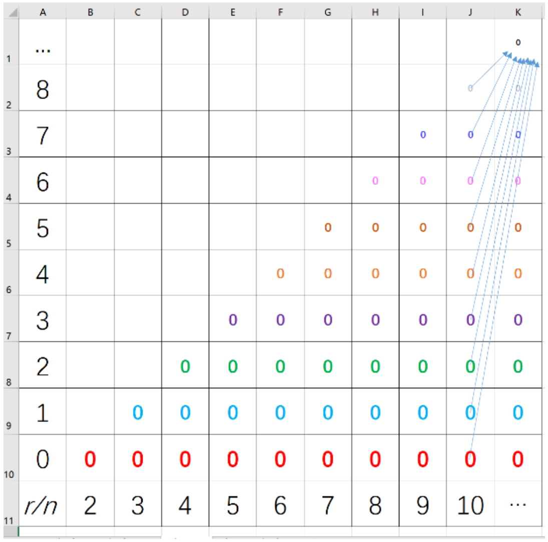

Figure 6

Proof:

Using **Lemma (2)** (2.4)

$$f(n,r) = a_n^{\,r-1} f(n-1,0) + a_n^{\,r-2} f(n-1,1) + \cdots + a_n^{\,1} f(n-1,r-2) + a_n^{\,0} f(n-1,r-1)$$

We get

$$\begin{aligned} f(n,r) &= 0 + c_1 f(n-2,0) + \cdots + c_{r-3} f(n-2,r-3) + c_{r-2} f(n-2,r-2) \qquad (c_i \text{ are constants}) \\ &= 0 + 0 + d_1 f(n-3,0) + \cdots + d_{r-4} f(n-3,r-4) + d_{r-3} f(n-3,r-3) \quad (d_i \text{ are constants}) \\ &\cdots \\ &= 0 + 0 + \cdots + kf(n-r,0) \quad (k \text{ is a constant}) \end{aligned}$$

Since $n-r \geq 2$, $f(n,r) = 0$.

Q.E.D.

2.2.3 The general formula $g(n,r)$ when $r-n+1 \geq 0$

Apparently,

$$\begin{aligned} f(2,r) &= a_1^{r-1} + a_1^{r-2}a_2 + \cdots + a_1a_2^{r-2} + a_2^{r-1} \quad (r>0,\ \text{because } f(2,0)=0) \\ &= \sum_{k_1+k_2=r-1} a_1^{k_1}a_2^{k_2} \qquad (k_1,\ k_2 \geq 0) \end{aligned}$$

According to Lemma (2) (2.4)

$$\begin{aligned} f(n,r) = {} & a_n^{r-1} f(n-1,0) + a_n^{r-2} f(n-1,1) + \cdots + \\ & a_n^{1} f(n-1,r-2) + a_n^{0} f(n-1,r-1) \end{aligned}$$

We get

$$f(3,r) = a_3^{r-1} f(2,0) + a_3^{r-2} f(2,1) + \cdots + a_3^{1} f(2,r-2) + a_3^{0} f(2,r-1)$$

$(r>1,$ because $r-1>0$ otherwise all the terms would be 0 according to Theorem (1))

$$\begin{aligned} &= 0 + a_3^{r-2} \sum_{k_1+k_2=0} a_1^{k_1}a_2^{k_2} + a_3^{r-3} \sum_{k_1+k_2=1} a_1^{k_1}a_2^{k_2} \\ &\quad + \cdots + a_3^{1} \sum_{k_1+k_2=r-3} a_1^{k_1}a_2^{k_2} + a_3^{0} \sum_{k_1+k_2=r-2} a_1^{k_1}a_2^{k_2} \\ &= \sum_{k_1+k_2+k_3=r-2} a_1^{k_1}a_2^{k_2}a_3^{k_3} \end{aligned}$$

By analogy, the following formula can be deduced.

Theorem (2):

When $r-n+1 \geq 0$,

$$f(n,r) = g(n,r) = \sum_{\sum_{i=1}^{n} k_i = r-n+1} \left(\prod_{j=1}^{n} a_j^{k_j} \right) \quad (k_i \geq 0) \qquad (2.6)$$

Proof:

Firstly, $f(2,r) = g(2,r)$

Suppose for integer u and any constant integer $v (v-u+1 \geq 0, u \geq 2, v \geq 0)$, $f(u,v) = g(u,v)$.

Then

$$
\begin{aligned}
f\left(u+1,v\right) &= {a_{u+1}}^{v-1} f\left(u,0\right) + {a_{u+1}}^{v-2} f\left(u,1\right) + \cdots + {a_{u+1}}^{0} f\left(u,v-1\right) \\
&= 0+0+\cdots+0+{a_{u+1}}^{v-u} f\left(u,u-1\right) + \cdots {a_{u+1}}^{0} f\left(u,v-1\right) \\
&= 0+0+\cdots+0+{a_{u+1}}^{v-u} g\left(u,u-1\right) + \cdots {a_{u+1}}^{0} g\left(u,v-1\right) \\
&= {a_{u+1}}^{v-u} \sum_{\sum_{i=1}^{u} k_i = 0} \left(\prod_{j=1}^{u} {a_j}^{k_j} \right) + {a_{u+1}}^{v-u-1} \sum_{\sum_{i=1}^{u} k_i = 1} \left(\prod_{j=1}^{u} {a_j}^{k_j} \right) \\
&\quad + \cdots + {a_{u+1}}^{0} \sum_{\sum_{i=1}^{u} k_i = v-u} \left(\prod_{j=1}^{u} {a_j}^{k_j} \right) \\
&= \sum_{h=0}^{v-u} \left\{ \sum_{\sum_{i=1}^{u} k_i = v-u-h} [{a_{u+1}}^{h} (\prod_{j=1}^{u} {a_j}^{k_j})] \right\} \\
&= \sum_{\sum_{i=1}^{u+1} k_i = v-u} \left(\prod_{j=1}^{u+1} {a_j}^{k_j} \right) \\
&= g\left(u+1, v-w\right)
\end{aligned}
$$

$\therefore$ While $r-n+1 \geq 0$,

$$
f\left(n,r\right) = g\left(n,r\right) = \sum_{\sum_{i=1}^{n} k_i = r-n+1} \left(\prod_{j=1}^{n} {a_j}^{k_j} \right) \quad \left(k_i \geq 0\right)
$$

Q.E.D.

2.2.4 Generalization of the cases when $r > 0$

When $r > 0$,

$$f(n,r)=\begin{cases} \displaystyle\sum_{\sum_{i=1}^{n} k_i = r-n+1} \left(\prod_{j=1}^{n} a_j^{\,k_j} \right) \ \left(k_i \ge 0\right),\ r-n \ge -1 \\ 0,\ r-n < -1 \end{cases} \tag{2.7}$$

2.3 The cases when $r<0$

2.3.1 The recursion formula when $r<0$

We predicted that there is still a recursion formula of $f(n,r)$ when $r<0$, and provided the following lemma.

Lemma (3):

$$
\begin{aligned}
f(n,r) &= -a_n^{\,r} f(n-1,-1) - a_n^{\,r+1} f(n-1,-2) \\
&\quad - \cdots - a_n^{-2} f(n-1,r+1) - a_n^{-1} f(n-1,r)
\end{aligned}
\tag{2.8}
$$

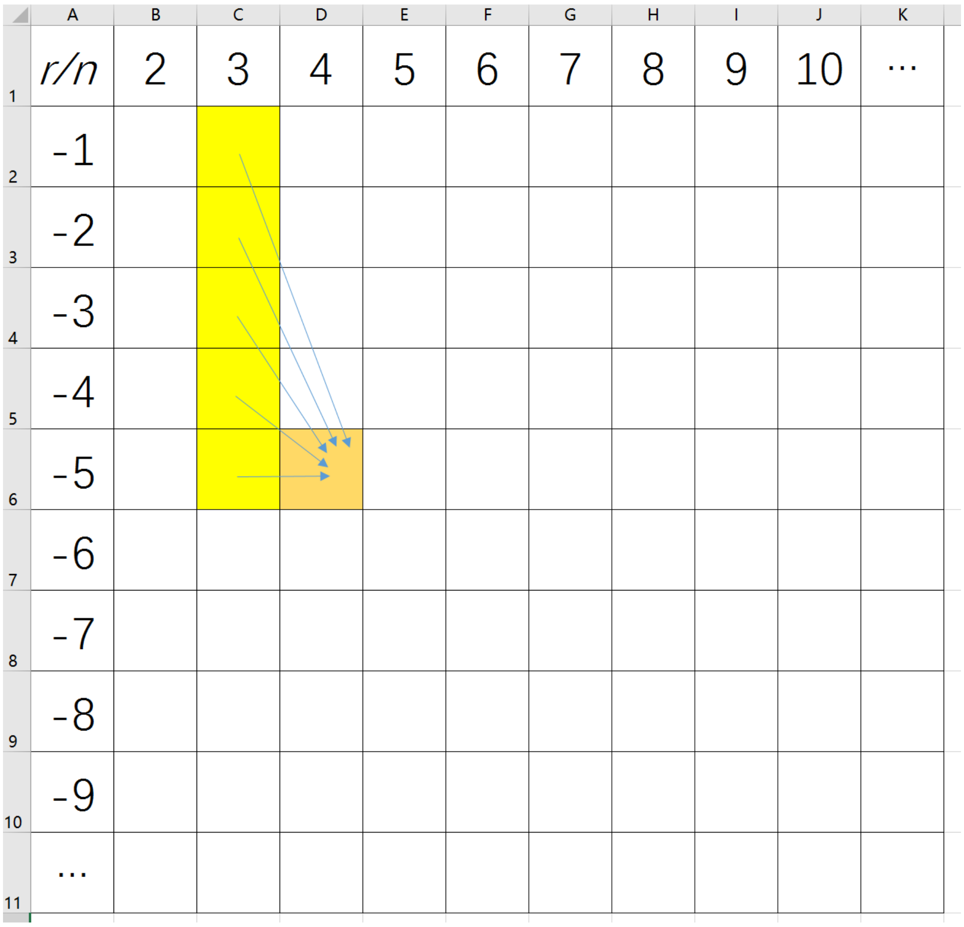

Figure 7

Proof:

$$f\left(n,r\right)=\frac{a_1^{\ r}}{\left(a_1-a_2\right)\left(a_1-a_3\right)\cdots\left(a_1-a_n\right)}+\frac{a_2^{\ r}}{\left(a_2-a_1\right)\left(a_2-a_3\right)\cdots\left(a_2-a_n\right)}$$

$$+\cdots+\frac{a_n^{\ r}}{\left(a_n-a_1\right)\left(a_n-a_2\right)\cdots\left(a_n-a_{n-1}\right)}$$

$$=\frac{a_1^{\ r}-a_n^{\ r}}{\left(a_1-a_2\right)\left(a_1-a_3\right)\cdots\left(a_1-a_n\right)}+\frac{a_2^{\ r}-a_n^{\ r}}{\left(a_2-a_1\right)\left(a_2-a_3\right)\cdots\left(a_2-a_n\right)}$$

$$+\cdots+\frac{a_n^{\ r}-a_n^{\ r}}{\left(a_n-a_1\right)\left(a_n-a_2\right)\cdots\left(a_n-a_{n-1}\right)}$$

$$=-a_1^{\ r}a_n^{\ r}\frac{a_1^{\ -r-1}+a_1^{\ -r-2}a_n+\cdots+a_n^{\ -r-1}}{\left(a_1-a_2\right)\left(a_1-a_3\right)\cdots\left(a_1-a_{n-1}\right)}-a_2^{\ r}a_n^{\ r}\frac{a_2^{\ -r-1}+a_2^{\ -r-2}a_n+\cdots+a_n^{\ -r-1}}{\left(a_2-a_1\right)\left(a_2-a_3\right)\cdots\left(a_2-a_{n-1}\right)}$$

$$-\cdots-a_{n-1}^{\ r}a_n^{\ r}\frac{a_{n-1}^{\ -r-1}+a_{n-1}^{\ -r-2}a_n+\cdots+a_n^{\ -r-1}}{\left(a_{n-1}-a_1\right)\left(a_{n-1}-a_2\right)\cdots\left(a_{n-1}-a_{n-2}\right)}$$

$$=-a_n^{\ r}f\left(n-1,-1\right)-a_n^{\ r+1}f\left(n-1,-2\right)-\cdots-a_n^{\ -2}f\left(n-1,r+1\right)-a_n^{\ -1}f\left(n-1,r\right)$$

Q.E.D.

2.3.2 The general formula when $r<0$

Obviously,

$$
\begin{aligned}
f(2,r) &= \frac{a_1^{\,r}}{a_1-a_2}+\frac{a_2^{\,r}}{a_2-a_1} \\
&= \frac{a_1^{\,r}-a_2^{\,r}}{a_1-a_2} \\
&= -a_1^{-1}a_2^{-1}\left(a_1^{\,r+1}+a_1^{\,r+2}a_2^{-1}+\cdots+a_2^{\,r+1}\right) \\
&= -\sum_{k_1+k_2=r-1} a_1^{k_1}a_2^{k_2} \qquad (k_1,k_2<0)
\end{aligned}
$$

Using (2.8), we get

$$f(3,r)=\sum_{k_1+k_2+k_3=r-2} a_1^{k_1}a_2^{k_2}a_3^{k_3} \qquad \left(k_1,k_2,k_3\le -1\right)$$

By analogy, the following formula can be deduced.

Theorem (3):

When $r<0$,

$$f(n,r)=h(n,r)=(-1)^{n-1}\sum_{\sum_{i=1}^{n}k_i=r-n+1}\left(\prod_{j=1}^{n}a_j^{\,k_j}\right) \quad \left(k_i \quad 0\right) \tag{2.9}$$

Proof:

Firstly, $f(2,r)=h(2,r)$

Suppose for integer u and any constant integer v, $(u\ge 2, v<0)$, $f(u,v)=h(u,v)$.

Then

$$
\begin{aligned}
f(u+1,v) &= -a_{u+1}{}^{v} f(u,-1) - a_{u+1}{}^{v+1} f(u,-2) - \cdots - a_{n}{}^{-1} f(u+1,v) \\
&= -a_{u+1}{}^{v} h(u,-1) - a_{u+1}{}^{v+1} h(u,-2) - \cdots - a_{n}{}^{-1} h(u,v) \\
&= -a_{u+1}{}^{v} (-1)^{u-1} \sum_{\sum_{i=1}^{u} k_i = -u} \left(\prod_{j=1}^{u} a_j{}^{k_j} \right) \\
&\quad -a_{u+1}{}^{v+1} (-1)^{u-1} \sum_{\sum_{i=1}^{u} k_i = -u-1} \left(\prod_{j=1}^{u} a_j{}^{k_j} \right) \\
&\quad - \cdots - a_{u+1}{}^{-1} (-1)^{u-1} \sum_{\sum_{i=1}^{u} k_i = v-u+1} \left(\prod_{j=1}^{u} a_j{}^{k_j} \right) \\
&= (-1)^{u} \sum_{h=-1}^{v} \left\{ \sum_{\sum_{i=1}^{u} k_i = v-u-h} \left[a_{u+1}{}^{h} \left(\prod_{j=1}^{u} a_j{}^{k_j} \right) \right] \right\} \\
&= (-1)^{u} \sum_{\sum_{i=1}^{u+1} k_i = v-u} \left(\prod_{j=1}^{u+1} a_j{}^{k_j} \right) \\
&= h(u+1,v)
\end{aligned}
$$

Hence, when $r < 0$,

$$
f(n,r) = h(n,r) = (-1)^{n-1} \sum_{\sum_{i=1}^{n} k_i = r-n+1} \left(\prod_{j=1}^{n} a_j{}^{k_j} \right) \qquad (k_i \quad 0)
$$

Q.E.D.

2.4 Generalization of the cases when $r \in \mathbb{Z}$

$$
f(n,r)=\begin{cases}
\displaystyle\sum_{\sum_{i=1}^{n} k_i = r-n+1} \left(\prod_{j=1}^{n} a_j^{k_j}\right) \quad (k_i \geq 0), \quad r > n-1 \\[2ex]
0, \quad 0 \leq r \leq n-1 \\[2ex]
\displaystyle(-1)^{n-1} \sum_{\sum_{i=1}^{n} k_i = r-n+1} \left(\prod_{j=1}^{n} a_j^{k_j}\right) \quad (k_i < 0), \quad r < 0
\end{cases}
$$

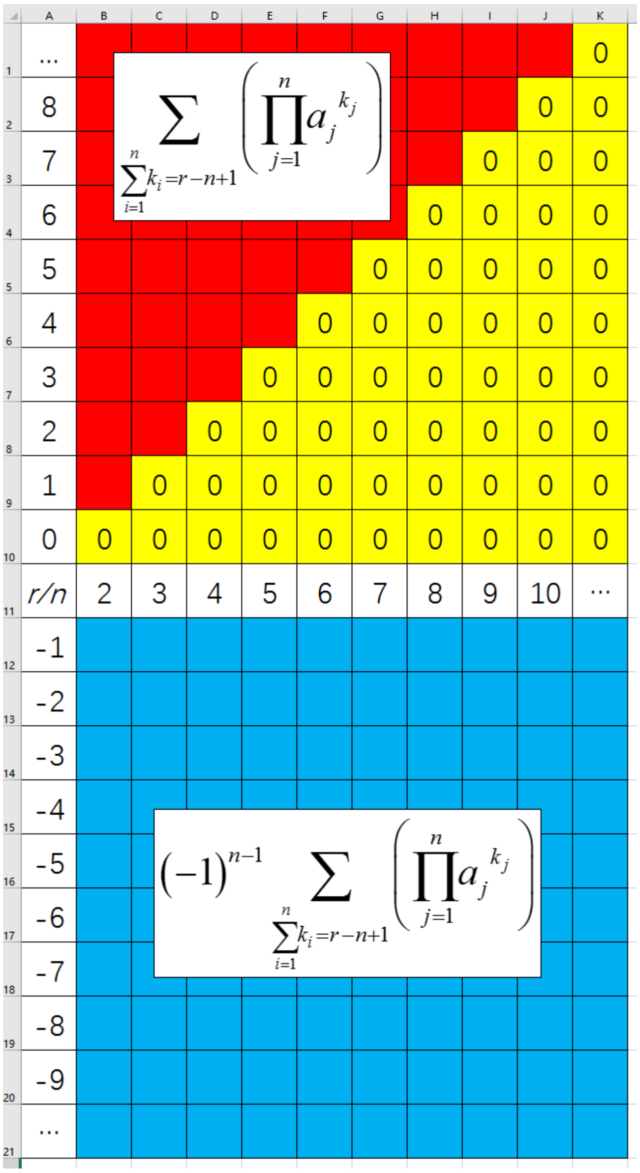

Figure 8

Chapter 3 The connection between Euler's Formula and Lagrange Interpolating Polynomial

Our teacher mentioned that there is some similarities and connections between Euler's Formula and Lagrange Interpolating Polynomial. In the latter,

$$F(x)=\sum_{i=1}^{n} f(x_i)l_i(x) \tag{3.1}$$

where

$$l_i(x)=\frac{(x-x_1)(x-x_2)\cdots(x-x_{i-1})(x-x_{i+1})\cdots(x-x_n)}{(x_i-x_1)(x_i-x_2)\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_n)}$$

the denominator of $l_i(x)$ is the same as the denominator of $f(n,r)$ (of course, the constants changes to x_i). In addition, the coefficient of x^{n-1} is 1, which matches the numerator of $f(n,r)$. Therefore, we found that the polynomial somehow connects with the expanded formula, leading to the proof of some cases of $f(n,r)$.

3.1 Lagrange Interpolating Polynomial

$$F(x)=\sum_{i=1}^{n} f(x_i)l_i(x) \tag{3.1}$$

where $l_i(x)=\dfrac{(x-x_1)(x-x_2)\cdots(x-x_{i-1})(x-x_{i+1})\cdots(x-x_n)}{(x_i-x_1)(x_i-x_2)\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_n)}$

In order to prove the cases of $f(n,r)$, we have to use a special property of the polynomial, which is that if $f(x)$ is a polynomial of degree $n-1$ or less, then $f(x)=F(x)$.

To prove this property of the polynomial, we need to prove the following lemma.

Lemma (4): A polynomial of degree $n-1$ or less cannot have *n* zeros.

Proof: Suppose a polynomial of degree $n-1$ or less ***does*** have *n* zeros, then this polynomial must have at least $n-1$ stationary points, which means $f'(x)$ has at least $n-1$ zeros. Likewise, $f'(x)$ has at least $n-2$ stationary points, that means $f''(x)$ has at least $n-2$ zeros, and so on. $f(x)$ can only have a constant $n-1$ th order derivative (or does not have one at all). However, the deduction above indicates that it has a $n-1$ th order derivative that has at least one zero, which is contradictory. Thus, such polynomials cannot exist.

Q.E.D.

Lemma (5): If $f(x)$ is a polynomial of degree $n-1$ or less, then $f(x)=F(x)$.

Proof: Suppose $f(x)\neq F(x)$, then assume that $g(x)=f(x)-F(x)$, where $g(x)$ is a polynomial of degree $n-1$ or less. Because on the *n* points ($x_1,x_2,\cdots,x_n$) $f(x)$ and $F(x)$ have the same values, which are $f(x_1),f(x_2),\cdots,f(x_n)$ respectively, $g(x)$ has zeros on those points, making it has *n* zeros. According to **Lemma (4)** this is impossible. Hence $f(x)=F(x)$.

Q.E.D.

3.2 Proof of the extended formula when $r \le n-1$

Proof: Suppose that in Lagrange Interpolating Polynomial, $f(x)=x^r$.
In this case, the coefficient of x^{n-1} in $F(x)$ is

$$=\sum_{i=1}^{n}\frac{x_i^r}{(x_i-x_1)(x_i-x_2)\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_n)}$$
$$=f(n,r)$$

According to **Lemma (5)**, when $n>r$, $F(x)=f(x)=x^r$. Thus all the coefficients of x^k in $F(x)$ is zero except that the coefficient of x^r is 1.

When $n=r+1$, the coefficient of x^{n-1} is $f(n,r)=f(n,n-1)\quad 1$.

When $n>r+1$, the coefficient of x^{n-1} is $f(n,r)=0$.(Because $n-1\neq r$)

3.3 Using the polynomial to proof the extended formula when $n-1<r\le 2n-2$

When $f(x)=x^{n-1}$,

The coefficient of x^{n-1} in $F(x)=f(n,n-1)$ 1

The coefficient of x^{n-2} in $F(x)$

$$
\begin{aligned}
&=0\\
&=\sum_{i=1}^{n}\frac{-\left(x_1+x_2+\cdots+x_{i-1}+x_{i+1}+\cdots+x_n\right)x_i^{n-1}}{\left(x_i-x_1\right)\left(x_i-x_2\right)\cdots\left(x_i-x_{i-1}\right)\left(x_i-x_{i+1}\right)\cdots\left(x_i-x_n\right)}\\
&=\sum_{i=1}^{n}\frac{-\left(x_1+x_2+\cdots+x_n\right)x_i^{n-1}+x_i^{n}}{\left(x_i-x_1\right)\left(x_i-x_2\right)\cdots\left(x_i-x_{i-1}\right)\left(x_i-x_{i+1}\right)\cdots\left(x_i-x_n\right)}\\
&=-\left(x_1+x_2+\cdots+x_n\right)f\left(n,n-1\right)+f\left(n,n\right)
\end{aligned}
$$

Therefore $f(n,n)=(x_1+x_2+\cdots+x_n)$

The coefficient of x^{n-3} in $F(x)$

$$
\begin{aligned}
&=0\\
&=\sum_{i=1}^{n}\frac{\left(\sum_{j,h=1,j\neq i\neq h\neq j}^{n}x_jx_h\right)x_i^{n-1}}{(x_i-x_1)(x_i-x_2)\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_n)}\\
&=\sum_{i=1}^{n}\frac{\left(\sum_{j,h=1,j\neq h}^{n}x_jx_h\right)x_i^{n-1}-(x_1+x_2+\cdots+x_n)x_i^{n}+x_i^{n+1}}{(x_i-x_1)(x_i-x_2)\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_n)}\\
&=\left(\sum_{j,h=1,j\neq h}^{n}x_jx_h\right)f(n,n-1)-(x_1+x_2+\cdots+x_n)f(n,n)+f(n,n+1)
\end{aligned}
$$

Therefore

$$
\begin{aligned}
f(n,n+1)&=(x_1+x_2+\cdots+x_n)^2-\sum_{j,h=1,j\neq h}^{n}x_jx_h\\
&=\sum_{\sum_{i=1}^{n}k_i=2}\left(\prod_{j=1}^{n}x_j^{k_j}\right)\qquad (k_i\neq 0)
\end{aligned}
$$

And so on,

The coefficient of x^{n-t} in $F(x)$

$$
\begin{aligned}
&= 0 \\
&= \sum_{i=1}^{n} \frac{(-1)^{t-1}\left[\sum\left(\prod_{h=1}^{t-1} x_{j_h}\right)\right] x_i^{n-1}}{(x_i - x_1)(x_i - x_2)\cdots(x_i - x_{i-1})(x_i - x_{i+1})\cdots(x_i - x_n)} \qquad (j_h \text{ are distinct}) \\
&= \sum_{i=1}^{n} \frac{(-1)^{t-1}\left[\sum\left(\prod_{h=1}^{t-1} x_{j_h}\right)\right] x_i^{n-1} + (-1)^{t-2}\left[\sum\left(\prod_{h=1}^{t-2} x_{j_h}\right)\right] x_i^{n} + \cdots (-1)^{0} x_i^{n+t-2}}{(x_i - x_1)(x_i - x_2)\cdots(x_i - x_{i-1})(x_i - x_{i+1})\cdots(x_i - x_n)}
\end{aligned}
$$

$$
\begin{aligned}
&= (-1)^{t-1}\left[\sum\left(\prod_{h=1}^{t-1} x_{j_h}\right)\right] f(n, n-1) \\
&+ (-1)^{t-2}\left[\sum\left(\prod_{h=1}^{t-2} x_{j_h}\right)\right] f(n, n) + \cdots + f(n, n+t-2)
\end{aligned}
\tag{3.2}
$$

Therefore

$$
\begin{aligned}
f(n, n+t-2) = (-1)^{t-2}\left[\sum\left(\prod_{h=1}^{t-1} x_{j_h}\right)\right] f(n, n-1) + (-1)^{t-3}\left[\sum\left(\prod_{h=1}^{t-2} x_{j_h}\right)\right] f(n, n) \\
+ \cdots + (-1)^{0}\left[\sum\left(\prod_{h=1}^{1} x_{j_h}\right)\right] f(n, n+t-3)
\end{aligned}
$$

To prove the right side of the equation above $= \sum_{\sum_{i=1}^{n} k_i = t-1} \left(\prod_{j=1}^{n} x_j^{k_j}\right)$,

Is equivalent to prove that the coefficient of every term in the following polynomial is 1. (Because $\sum_{\sum_{i=1}^{n} k_i = t-1} \left(\prod_{j=1}^{n} x_j^{k_j}\right)$ includes every term of $t-1$ th degree and has the coefficient 1.)

$$
\begin{aligned}
&(-1)^{t-2}\left[\sum\left(\prod_{h=1}^{t-1} x_{j_h}\right)\right] f(n, n-1) + (-1)^{t-3}\left[\sum\left(\prod_{h=1}^{t-2} x_{j_h}\right)\right] f(n, n) \\
&+ \cdots + (-1)^{0}\left[\sum\left(\prod_{h=1}^{1} x_{j_h}\right)\right] f(n, n+t-3)
\end{aligned}
$$

Consider a term in $\prod_{j=1}^{n} x_j^{k_j}$ ($\sum_{i=1}^{n} k_i = t-1$). Its coefficient (the number of time it appears) in $\left[\sum\left(\prod_{h=1}^{1} x_{j_h}\right)\right] f(n,n+t-3)$ equals to the number of terms it includes in $\sum\left(\prod_{h=1}^{1} x_{j_h}\right)$. In brief, the coefficient is the number of distinct letters in $\prod_{j=1}^{n} x_j^{k_j}$ (suppose it is m). Similarly, its coefficient in $\left[\sum\left(\prod_{h=1}^{2} x_{j_h}\right)\right] f(n,n+t-4)$ equals to the number of terms it includes in $\sum\left(\prod_{h=1}^{2} x_{j_h}\right)$. This number is the number of terms of two distinct letters in $\prod_{j=1}^{n} x_j^{k_j}$, which is $\binom{m}{2}$. Hence, the coefficient of this term in $\prod_{j=1}^{n} x_j^{k_j}$ (the total number of times it appears) is

$$
\begin{aligned}
\sum_{i=1}^{m} (-1)^{i-1}\binom{m}{i} &= \binom{m}{0} - \sum_{j=0}^{m} (-1)^{j}\binom{m}{j} \\
&= 1
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
f(n,n+t-2) = {} & (-1)^{t-2}\left[\sum\left(\prod_{h=1}^{t-1} x_{j_h}\right)\right] f(n,n-1) + (-1)^{t-3}\left[\sum\left(\prod_{h=1}^{t-2} x_{j_h}\right)\right] f(n,n) \\
& + \cdots + (-1)^{0}\left[\sum\left(\prod_{h=1}^{1} x_{j_h}\right)\right] f(n,n+t-3) \\
= {} & \sum_{\sum_{i=1}^{n} k_i = t-1} \left(\prod_{j=1}^{n} x_j^{k_j}\right)
\end{aligned}
$$

Q.E.D.

Once again, using another method, we obtained and proved part of **theorem (2)**,

When $n-1<r\leq 2n-2$,

$$f\left(n,r\right)=\sum_{\sum_{i=1}^{n}k_i=r-n+1}\left(\prod_{j=1}^{n}x_j^{\,k_j}\right) \tag{2.6}$$

3.4 A new equation derived from the application of the extended formula in Lagrange Interpolating Polynomial

We noticed that the coefficient of each term of the polynomial can be expressed by the extended formula. Also, the coefficient of in x^{n-i} in $F(x)$ is

$$\begin{cases} 1, & i=1 \\ 0, & i>1 \end{cases} \qquad \left(\text{Supposing that } f(x)=x^{n-1}\right)$$

Thus, we can obtain a new equation using this relation.

According to (3.2)

$$=(-1)^{t-1}\left[\sum\left(\prod_{h=1}^{t-1} x_{j_h}\right)\right] f(n,n-1)$$

$$+(-1)^{t-2}\left[\sum\left(\prod_{h=1}^{t-2} x_{j_h}\right)\right] f(n,n)+\cdots+f(n,n+t-2)$$

$$=\sum_{i=1}^{t}\left\{(-1)^{t-i}\left[\sum\left(\prod_{h=1}^{t-i} x_{j_h}\right)\right] f(n,n+i-2)\right\} \tag{3.3}$$

Hence, the constant term in $F(x)$

$$=0$$

$$=\sum_{i=1}^{n}\left\{(-1)^{n-i}\left[\sum\left(\prod_{h=1}^{n-i} x_{j_h}\right)\right] f(n,n+i\quad 2)\right\} \tag{3.4}$$

Define

$$g(n,r)=\sum_{j_h=1}^{n}\left(\prod_{h=1}^{r} x_{j_h}\right) \qquad \left(g(n,0)=1\right) \tag{3.5}$$

Then we have

$$\begin{aligned} &f(n,2n-2)g(n,0)-f(n,2n-3)g(n,1) \\ &+\cdots+(-1)^{n-1} f(n,n-1)g(n,n-1)=0 \end{aligned} \tag{3.6}$$

Using "Σ" to express the sum, we have

Theorem (4):

$$\sum_{i=1}^{n}(-1)^{n-i} f(n,n+i-2)g(n,n-i) \quad 0 \tag{3.7}$$

Chapter 4 Other applications of the formula, conjectures and predictions

We noticed that the constants a_i in $f(n,r)$ do not have to be random and meaningless constants. Rather, they could be some fixed constants or functions that are distinct. Thus, we believe that some useful equations can be derived from this. The following equations are the ones we found by substituting a_i with certain constants and functions.

1. According to **Theorem (2)**,

$$\begin{aligned} f(n,n-1) &= \frac{a_1^{\,n-1}}{(a_1-a_2)(a_1-a_3)\cdots(a_1-a_n)} + \frac{a_2^{\,n-1}}{(a_2-a_1)(a_2-a_3)\cdots(a_2-a_n)} \\ &\quad + \cdots + \frac{a_n^{\,n-1}}{(a_n-a_1)(a_n-a_2)\cdots(a_n-a_{n-1})} \\ &= 1 \end{aligned}$$

Substitute a_i with i, we get

$$\begin{aligned} & \frac{a_1^{\,n-1}}{(a_1-a_2)(a_1-a_3)\cdots(a_1-a_n)} + \frac{a_2^{\,n-1}}{(a_2-a_1)(a_2-a_3)\cdots(a_2-a_n)} \\ & + \cdots + \frac{a_n^{\,n-1}}{(a_n-a_1)(a_n-a_2)\cdots(a_n-a_{n-1})} \\ & = 1 \\ & = \frac{1^{n-1}}{(1-2)(1-3)\cdots(1-n)} + \frac{2^{n-1}}{(2-1)(2-3)\cdots(2-n)} \\ & + \cdots + \frac{n^{n-1}}{(n-1)(n-2)\cdots(n-n+1)} \\ & = \sum_{i=1}^{n} (-1)^{n-i} \frac{i^{n-1}}{(i-1)!(n-i)!} \end{aligned}$$

That is

Theorem (5):

$$\sum_{i=1}^{n} (-1)^{n-i} \frac{i^{n}}{i!(n-i)!} = 1 \tag{4.1}$$

Noticing that $\left(\begin{array}{c} n \\ i \end{array}\right) = \frac{n!}{i!(n-i)!}$, (4.1) can be written as

$$\sum_{i=1}^{n} (-1)^{n-i} i^{n} \left(\begin{array}{c} n \\ i \end{array}\right) = n! \tag{4.2}$$

2. According to **Theorem (2)**,

$$f(4,4) = a_1 + a_2 + a_3 + a_4 \tag{4.3}$$

Substitute

$a_1 = \sin\alpha\cos\beta\cos\gamma$

$a_2 = \cos\alpha\sin\beta\cos\gamma$

$a_3 = \cos\alpha\cos\beta\sin\gamma$

$a_4 = -\sin\alpha\sin\beta\sin\gamma$

Because that the trigonometric three-angle formula (expansion of sin(a+b+c)) states that

$$\begin{aligned} \sin(\alpha+\beta+\gamma) &= \sin\alpha\cos\beta\cos\gamma + \cos\alpha\sin\beta\cos\gamma \\ &\quad + \cos\alpha\cos\beta\sin\gamma - \sin\alpha\sin\beta\sin\gamma \end{aligned} \tag{4.4}$$

We get

$$\begin{aligned} &\frac{(\sin\alpha\cos\beta\cos\gamma)^4}{(\sin\alpha\cos\beta\cos\gamma - \cos\alpha\sin\beta\cos\gamma)(\sin\alpha\cos\beta\cos\gamma - \cos\alpha\cos\beta\sin\gamma)(\sin\alpha\cos\beta\cos\gamma + \sin\alpha\sin\beta\sin\gamma)} + \\ &\frac{(\cos\alpha\sin\beta\cos\gamma)^4}{(\cos\alpha\sin\beta\cos\gamma - \sin\alpha\cos\beta\cos\gamma)(\cos\alpha\sin\beta\cos\gamma - \cos\alpha\cos\beta\sin\gamma)(\cos\alpha\sin\beta\cos\gamma + \sin\alpha\sin\beta\sin\gamma)} + \\ &\frac{(\cos\alpha\cos\beta\sin\gamma)^4}{(\cos\alpha\cos\beta\sin\gamma - \sin\alpha\cos\beta\cos\gamma)(\cos\alpha\cos\beta\sin\gamma - \cos\alpha\sin\beta\cos\gamma)(\cos\alpha\cos\beta\sin\gamma + \sin\alpha\sin\beta\sin\gamma)} + \\ &\frac{(-\sin\alpha\sin\beta\sin\gamma)^4}{(-\sin\alpha\sin\beta\sin\gamma - \sin\alpha\cos\beta\cos\gamma)(-\sin\alpha\sin\beta\sin\gamma - \cos\alpha\sin\beta\cos\gamma)(-\sin\alpha\sin\beta\sin\gamma - \cos\alpha\cos\beta\sin\gamma)} \\ &= \sin\alpha\cos\beta\cos\gamma + \cos\alpha\sin\beta\cos\gamma + \cos\alpha\cos\beta\sin\gamma - \sin\alpha\sin\beta\sin\gamma \\ &= \sin(\alpha+\beta+\gamma) \end{aligned}$$

After the simplification, we get

Theorem (6):

$$\begin{aligned}&\frac{(\sin\alpha\cos\beta\cos\gamma)^3}{\sin(\alpha-\beta)\sin(\alpha-\gamma)\cos(\beta-\gamma)}+\frac{(\cos\alpha\sin\beta\cos\gamma)^3}{\sin(\beta-\alpha)\sin(\beta-\gamma)\cos(\alpha-\gamma)}\\&+\frac{(\cos\alpha\cos\beta\sin\gamma)^3}{\sin(\gamma-\alpha)\sin(\gamma-\beta)\cos(\alpha-\beta)}-\frac{(\sin\alpha\sin\beta\sin\gamma)^3}{\cos(\alpha-\beta)\cos(\beta-\gamma)\cos(\alpha-\gamma)}\\&=\sin(\alpha+\beta+\gamma)\end{aligned} \tag{4.5}$$

Of course, $\alpha\neq\beta\neq\gamma\neq\alpha;\quad |\alpha-\beta|,|\beta-\gamma|,|\alpha-\gamma|\neq\frac{\pi}{2}$

3. Similar to **Theorem (6)**, according to the trigonometric three-angle formula (expansion of cos(a+b+c))

$$\begin{aligned}&\cos(\alpha+\beta+\gamma)=\cos\alpha\cos\beta\cos\gamma\quad\cos\alpha\sin\beta\sin\gamma\\&-\sin\alpha\cos\beta\sin\gamma-\sin\alpha\sin\beta\cos\gamma\end{aligned} \tag{4.6}$$

And **Theorem (2)** (4.3)

We get **Theorem (7):**

$$\begin{aligned}&\frac{(\cos\alpha\cos\beta\cos\gamma)^3}{\cos(\alpha-\beta)\cos(\beta-\gamma)\cos(\alpha-\gamma)}-\frac{(\cos\alpha\sin\beta\sin\gamma)^3}{\sin(\alpha-\beta)\sin(\alpha-\gamma)\cos(\beta-\gamma)}\\&-\frac{(\sin\alpha\cos\beta\sin\gamma)^3}{\sin(\beta-\alpha)\sin(\beta-\gamma)\cos(\alpha-\gamma)}-\frac{(\sin\alpha\sin\beta\cos\gamma)^3}{\sin(\gamma-\alpha)\sin(\gamma-\beta)\cos(\alpha-\beta)}\\&=\cos(\alpha+\beta+\gamma)\end{aligned} \tag{4.7}$$

Also, $\alpha \neq \beta \neq \gamma \neq \alpha$; $|\alpha-\beta|,|\beta-\gamma|,|\alpha-\gamma| \neq \frac{\pi}{2}$

Due to the limitation of time, we only obtained those equations. However, we believe that by applying the extended Euler's Formula, a lot of more useful formulas can be acquired.

Chapter 5 Conclusion

In this paper, we successfully expanded the simple fractional Euler's Formula (1.1) to the cases when the number of terms is an integer equal to or greater than two, and the exponent is any integer. We proved our theory step by step. Then, we connected the extended formula to Lagrange interpolating polynomial and proved the former using this relation. Also, a new equation was obtained through this connection. At last, we explored and applied the extended formula to derive a set of new equations, as well as propose some conjectures.

Index

Index

References

[1]*Fundamentals of Numerical Analysis*, Zhi Guan&Jinfu Lu, Higher Education Press, 2010

[2]*An Extension of Euler’s Formula*, Wenqian Huang, Fujian Middle School Mathematics, 5(2017):6-7

Appendix

```
#include <iostream>
#include <ctime>
#include <cstdlib>
using namespace std;
int a[22]={0};
long long int power(int a,int b)
{
   int i;
   long long int x=1;
   for(i=1;i<=b;i++)
            x*=a;
   return x;
}
long double f(int n,int r)
{
   int i,j,numerator=1;long double denominator=1,x=0;
   for(i=1;i<=n;i++)
   {
            numerator=power(a[i],r);
            cout<<" "<<numerator<<"/";
            for(j=1;j<=n;j++)
            {
                     if(i==j)continue;
                     denominator*=(a[i]-a[j]);
                     cout<<"("<<a[i]<<"-"<<a[j]<<")";
            }
            cout<<" = "<<numerator<<"/"<<denominator<<" =
"<<numerator/denominator<<"\n";
            x+=numerator/denominator;
            cout<<" sum="<<x<<"\n";
```

```
            denominator=1;
    }
    return x;
}
int random(int start,int end)
{
    int x=start+(end-start)*rand()/(RAND_MAX+1.0);
    return x;
}
int main() {
    srand(unsigned(time(0)));
    int i;int n,r;bool b;
    while(1)
    {
    cout<<"\n Please input n(n<20),r:  ";
    cin>>n;
    if(n<=0||n>=20)
    return 0;
    cin>>r;
    for(i=1;i<=n;i++)
    {
            b=false;
            while(!b)
            {
                    a[i]=random(-10,10);
            b=true;
                    for(int j=1;j<i;j++)
            {
                    if(a[j]==a[i])b=false;
                    }
            }
            cout<<"   a["<<i<<"] = "<<a[i]<<"\n";
    }
    long double result=f(n,r);
```

```
    cout<<"  f(n,r)= "<<result<<" \n\n";
    }
    return 0;
}
```

Some of the results:

```
Please input n(n<20),r:  7 0
  a[1] = 6
  a[2] = -2
  a[3] = 1
  a[4] = -8
  a[5] = -4
  a[6] = 0
  a[7] = 2
1/(6--2)(6-1)(6--8)(6--4)(6-0)(6-2) = 1/134400 = 7.44048e-006
sum=7.44048e-006
1/(-2-6)(-2-1)(-2--8)(-2--4)(-2-0)(-2-2) = 1/2304 = 0.000434028
sum=0.000441468
1/(1-6)(1--2)(1--8)(1--4)(1-0)(1-2) = 1/675 = 0.00148148
sum=0.00192295
1/(-8-6)(-8--2)(-8-1)(-8--4)(-8-0)(-8-2) = 1/241920 = 4.1336e-006
sum=0.00192708
1/(-4-6)(-4--2)(-4-1)(-4--8)(-4-0)(-4-2) = 1/-9600 = -0.000104167
sum=0.00182292
1/(0-6)(0--2)(0-1)(0--8)(0--4)(0-2) = 1/-768 = -0.00130208
sum=0.000520833
1/(2-6)(2--2)(2-1)(2--8)(2--4)(2-0) = 1/-1920 = -0.000520833
sum=-1.58819e-022
 f(n,r)= -1.58819e-022
```

```
Please input n(n<20),r:  11 0
  a[1] = 6
  a[2] = 5
  a[3] = 9
  a[4] = 1
  a[5] = -2
  a[6] = -1
  a[7] = -4
  a[8] = 0
  a[9] = 2
  a[10] = 7
  a[11] = -3
1/(6-5)(6-9)(6-1)(6--2)(6--1)(6--4)(6-0)(6-2)(6-7)(6--3) = 1/1.8144e+006 = 5.51146e-007
sum=5.51146e-007
1/(5-6)(5-9)(5-1)(5--2)(5--1)(5--4)(5-0)(5-2)(5-7)(5--3) = 1/-1.45152e+006 = -6.88933e-007
sum=-1.37787e-007
1/(9-6)(9-5)(9-1)(9--2)(9--1)(9--4)(9-0)(9-2)(9-7)(9--3) = 1/2.07567e+008 = 4.81771e-009
sum=-1.32969e-007
1/(1-6)(1-5)(1-9)(1--2)(1--1)(1--4)(1-0)(1-2)(1-7)(1--3) = 1/-115200 = -8.68056e-006
sum=-8.81352e-006
1/(-2-6)(-2-5)(-2-9)(-2-1)(-2--1)(-2--4)(-2-0)(-2-2)(-2-7)(-2--3) = 1/266112 = 3.75782e-006
sum=-5.05571e-006
1/(-1-6)(-1-5)(-1-9)(-1-1)(-1--2)(-1--4)(-1-0)(-1-2)(-1-7)(-1--3) = 1/-120960 = -8.2672e-006
sum=-1.33229e-005
1/(-4-6)(-4-5)(-4-9)(-4-1)(-4--2)(-4--1)(-4-0)(-4-2)(-4-7)(-4--3) = 1/9.2664e+006 = 1.07917e-007
sum=-1.3215e-005
1/(0-6)(0-5)(0-9)(0-1)(0--2)(0--1)(0--4)(0-2)(0-7)(0--3) = 1/90720 = 1.10229e-005
sum=-2.19206e-006
1/(2-6)(2-5)(2-9)(2-1)(2--2)(2--1)(2--4)(2-0)(2-7)(2--3) = 1/302400 = 3.30688e-006
sum=1.11482e-006
1/(7-6)(7-5)(7-9)(7-1)(7--2)(7--1)(7--4)(7-0)(7-2)(7--3) = 1/-6.6528e+006 = -1.50313e-007
sum=9.64506e-007
1/(-3-6)(-3-5)(-3-9)(-3-1)(-3--2)(-3--1)(-3--4)(-3-0)(-3-2)(-3-7) = 1/-1.0368e+006 = -9.64506e-007
sum=-3.10193e-025
 f(n,r)= -3.10193e-025
```

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