A Class of Convex Curves

Arising in Capillary Floating Problem

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Abstract

Motivated by Gutkin curves arising in capillary floating and billiard ball problem, we introduce a new projection function for convex plane curves, and study the curves when the defined projection function is constant. These curves can be regarded as a generalization of curves with constant width. We give a necessary and sufficient condition for the existence of such curves, and provide an explicit expression for the radius of curvature. Properties of our curves and their relation to Gutkin curves are also discussed. Finally, we construct and exhibit some curves with constant projection function.

Contents

1. Introduction

1.1 Background

Capillary Floating Problem concerns the equilibrium positions of floating bodies whose positions and orientations are governed by capillary forces. Nanoparticles at fluid interface are becoming a central topic in colloid science (Bresme F and Oettel M, 2007). The study of equilibrium and stability for nanoparticles at fluid interface plays an important role in manufacturing colloidal nanoparticles.

Although it is clear that the round ball floats in equilibrium in any orientation, very little is known for floating bodies of general shape. A case of particular interest appears when the floating particle is a long solid cylinder resting horizontally on the liquid. In this case, the three-dimensional floating problem is reduced to a two-dimensional problem involving the cross-section of the cylinder.

Raphael E., Megelio J.M., Berger M. and Calabi E.[1992] studied the equilibrium positions of a cylinder at a liquid/liquid interface. The convex cross-section of the cylinder is a plane region bounded by a simple closed convex curve. They proved there are at least four neutral equilibrium positions.
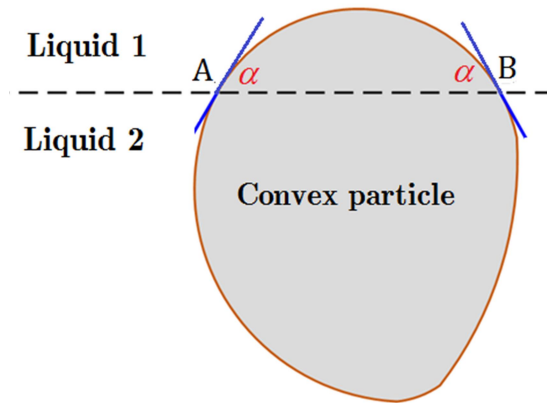

Fig. 1   A convex cross-section in equilibrium position

Under the assumption of gravity-free, the liquid-liquid interface is flat, and it intercepts the convex curve at two points A and B, as shown in Figure 1. If the convex

cylinder is trapped in an equilibrium position, then the contact angles at points A and B are equal to a prescribed angle α, the so-called Young angle (Raphael et al, 1992). The Young angle is determined by the interfacial tensions characterizing the system of solid, liquid, and liquid.

Let C be a smooth simple closed convex plane curve, and $\vec{N} = (\cos\theta, \sin\theta)$ be the outward pointing unit normal, where $\theta \in [0, 2\pi]$ is the oriented angle from x-axis to $\vec{N}$. Since C is smooth and convex. C can be parameterized by θ and written as $C : \vec{r} = \vec{r}(\theta) = (x(\theta), y(\theta))$. Figure 1 shows that the cylinder is in an equilibrium position if and only if there exists an orientation $\theta_0 \in [0, 2\pi]$ such that the chord AB perpendicular to $\vec{N}(\theta_0)$, and the points A and B correspond to parameters $\theta_0 - \alpha$ and $\theta_0 + \alpha$ respectively. Therefore, to locate equilibrium position is equivalent to finding an orientation θ satisfying

$$\left[\vec{r}(\theta_0 + \alpha) - \vec{r}(\theta_0 - \alpha)\right] \cdot \vec{N}(\theta_0) = 0 \qquad (1)$$

Raphael et al [1992] addressed the problem: for a given Young angle α and a given smooth convex curve C, is it always possible to find a θ_0 value which satisfies condition (1)? They found an interesting relationship between this problem and the Four Vertex Theorem in Differential Geometry (see Manfred P. do Carmo, 2004). As a consequence, they proved that for any given α, there exist at least four θ values satisfying (1).

Note that if C is a circle, then given any α, (1) holds for all θ. So one may ask the question: For a given $\alpha \in (0, \pi/2]$, is there a noncircular convex curve $\vec{r} = \vec{r}(\theta)$ such that

$$\left[\vec{r}(\theta + \alpha) - \vec{r}(\theta - \alpha)\right] \cdot \vec{N}(\theta) = 0 \qquad (2)$$

for all $\theta \in [0, 2\pi]$. According to the floating model, if (2) is satisfied for all θ, then

the cylinder with cross-section C floats in neutral equilibrium in any orientation with the contact angle α. A similar question is also raised by Finn R.[2009] in his study of three-dimensional capillary floating problem.

In the study of billiard ball problem, Gutkin E[2012, 2013] answered this question. Consider the general billiard problem which concerns the motion of a ball in a billiard table with boundary C. For each collision, the ball changes the direction according to the reflection rule. The noncircular cross-section satisfying (2) is a billiard table satisfying the following conditions: For any given θ, if we shoot a ball in the direction $\vec{r}(\theta+\alpha)-\vec{r}(\theta-\alpha)$, the refection angles for each collisions are equal to $\pi/2-\alpha$, as shown in Fig. 2.
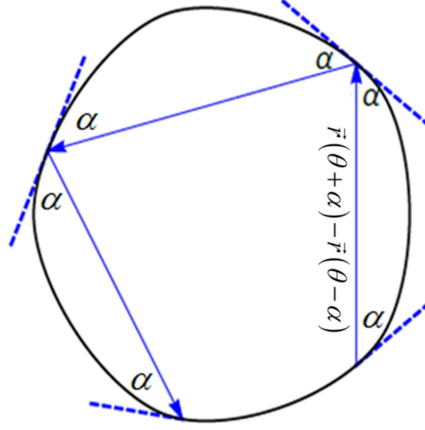

Fig. 2 $\vec{r}(\theta+\alpha)-\vec{r}(\theta-\alpha)$ have equal refection angle $\pi/2-\alpha$ in Gutkin's billiard table

Gutkin obtained a sufficient and necessary condition for a noncircular convex curve satisfying (2).

Gutkin's Theorem (Gutkin, 2012). Let C be a smooth noncircular convex curve, and let $\rho(\theta)=\sum\limits_{k\in\mathbb{Z}}c_k e^{ik\theta}$ be the radius of curvature of C. C satisfies (2) if and only if the following conditions hold.

(i) There exist $n>1$ such that

$$\frac{\sin(n+1)\alpha}{n+1}=\frac{\sin(n-1)\alpha}{n-1} \tag{3}$$

(ii) $c_1 = 0$, $c_k = 0$ for all $k > 1$ such that $\dfrac{\sin(k+1)\alpha}{k+1} \neq \dfrac{\sin(k-1)\alpha}{k-1}$.

(iii) $c_k \neq 0$ for at least one $k > 1$ such that $\dfrac{\sin(k+1)\alpha}{k+1} = \dfrac{\sin(k-1)\alpha}{k-1}$.

Equation (3) can be also be written as

$$n \cot n\alpha = \cot \alpha. \qquad (4)$$

Following Aougab et al [2013], we call a curve satisfying (2) for $\alpha \in (0, \pi/2]$ as an α-Gutkin curve. For an α-Gutkin curve, are the lengths of the chords equal?

Question 1: <u>Given $\alpha \in (0, \pi/2]$, is there a noncircular Gutkin curve satisfying</u>

$$|\vec{r}(\theta+\alpha) - \vec{r}(\theta-\alpha)| = \text{constant for } \theta \in [0, 2\pi].$$

Since the chord $\vec{r}(\theta+\alpha) - \vec{r}(\theta-\alpha)$ is perpendicular to $\vec{N}(\theta)$ for a Gutkin curve, we have

$$|\vec{r}(\theta+\alpha) - \vec{r}(\theta-\alpha)| = \left| [\vec{r}(\theta+\alpha) - \vec{r}(\theta-\alpha)] \times \vec{N}(\theta) \right|.$$

So a more general question is:

Question 2: <u>Given $\beta \in (0, \pi/2]$, is there a noncircular convex curve satisfying the</u> <u>following condition (5)?</u>

$$\left| [\vec{r}(\theta+\beta) - \vec{r}(\theta-\beta)] \times \vec{N}(\theta) \right| = \text{constant for } \theta \in [0, 2\pi]. \qquad (5)$$

Since

$$\left| [\vec{r}(\theta+\beta) - \vec{r}(\theta-\beta)] \times \vec{N}(\theta) \right| = [\vec{r}(\theta+\beta) - \vec{r}(\theta-\beta)] \cdot \vec{T}(\theta),$$

as shown in Figure 3, $\left| [\vec{r}(\theta+\beta) - \vec{r}(\theta-\beta)] \times \vec{N}(\theta) \right|$ is a projection function. It's the scalar projection of the vector $\vec{r}(\theta+\beta) - \vec{r}(\theta-\beta)$ in the direction $\vec{T}(\theta) = (-\sin\theta, \cos\theta)$. That means a curve satisfying condition (5) has constant scalar projection, regardless of the orientation θ.
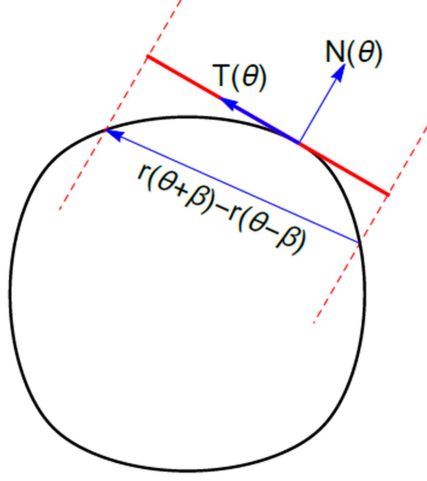
Fig. 3 The scalar projection (in red solid) of $\vec{r}(\theta+\beta)-\vec{r}(\theta-\beta)$ in the direction $\vec{T}(\theta)$

Definition   Let $C:\vec{r}=\vec{r}(\theta)=(x(\theta),y(\theta))$ be a smooth convex plane curve. C is called a curve of constant scalar projection (CCSP) corresponding to β, or β-CCSP curve in short, if

$$\left|\left[\vec{r}(\theta+\beta)-\vec{r}(\theta-\beta)\right]\times\vec{N}(\theta)\right|=\text{constant for }\theta\in[0,2\pi].$$

The circle is a β-CCSP curve for any β. Question 2 is equivalent to the question whether or not a noncircular β-CCSP curve exists. If a noncircular β-CCSP exists, can we construct it? This question is related to a more general inverse problem in geometric tomography: can we reconstruct the shape of a body from its projection function? We'll provide answers for Question 1 and Question 2 in Theorem 1 and Theorem 2.

1.2 Main Theorems

In this paper, we'll give a sufficient and necessary condition for the existence of noncircular β-CCSP curves, and address the geometry of such curves. The following are our main theorems.

At first, we rewrite the scalar projection in terms of the radius of curvature.

Key Lemma. For any $\beta \in (0, \pi/2]$,

$$\left|\left[\vec{r}(\theta+\beta) - \vec{r}(\theta-\beta)\right] \times \vec{N}(\theta)\right| = \int_{-\beta}^{\beta} \rho(\theta-\varphi)\cos\varphi \, d\varphi.$$

where $\rho(\theta)$ be the radius of curvature.

Then, using Fourier expansion technique, we find the existence of noncircular β-CCSP curve is equivalent to the existence of the solutions of trigonometric equations.

Theorem 1. For a given $\beta \in (0, \pi/2]$,

(i) A noncircular β-CCSP curve exists if and only if there exist integer $k>1$ such that

$$\cot k\beta = k \cot \beta. \qquad (6)$$

(ii)　Let $K_\beta = \{k \in \mathbb{N} \mid k > 1, \cot k\beta = k\cot\beta\}$. The radius of curvature of a β-CCSP curve has the form

$$\rho(\theta) = \frac{a_0}{2} + \sum_{k \in K_\beta}\left(a_k \cos k\theta + b_k \sin k\theta\right), \text{ and } \rho(\theta) > 0.$$

Combining Theorem 1 and Gutkin's Theorem, we have the following results on Gutkin curves with constant length of chords.

Theorem 2. A curve C is both a noncircular β-CCSP curve and a β-Gutkin curve, if and only if $\beta = \pi/2$, and thus C is a curve of constant width.

This paper is organized as the following. In Section 2, we provide some notations and preliminaries on curves of constant width. In particular, we give a criterion of curves of constant width in terms of scalar projection. Section 3 is devoted to the proof of the main theorems. We construct some noncircular CCSP curves based Theorem 1. In Section 4, we solve the trigonometric equations (6) and present some properties of the solutions. We also exhibit some noncircular CCSP curves and illustrate the connection between noncircular CCSP curves and Gutkin curves.

2. Preliminaries

2.1 Notations

Throughout this paper, assume that C is a smooth simple closed convex plane curve. Choose a point O inside the curve C as the origin of the coordinate plane. Let θ be the oriented angle from the positive half of x-axis to the outward pointing normal. From the definition of θ, we have $\theta \in [0, 2\pi]$, and the outward pointing unit normal

$$\vec{N}(\theta) = (\cos\theta, \sin\theta)$$.

C can be parameterized by θ and written as $C : \vec{r} = \vec{r}(\theta) = (x(\theta), y(\theta))$. The unit tangential vector

$$\vec{T} = (-\sin\theta, \cos\theta).$$

The support function, denoted by $h(\theta)$, is the distance from the origin to the tangent line at $(x(\theta), y(\theta))$, as shown in Figure 4. Support function $h(\theta)$ is a periodic function of θ with period 2π. It is given by

$$h(\theta) = \vec{r}(\theta) \cdot \vec{N}(\theta) = x(\theta)\cos\theta + y(\theta)\sin\theta \tag{7}$$
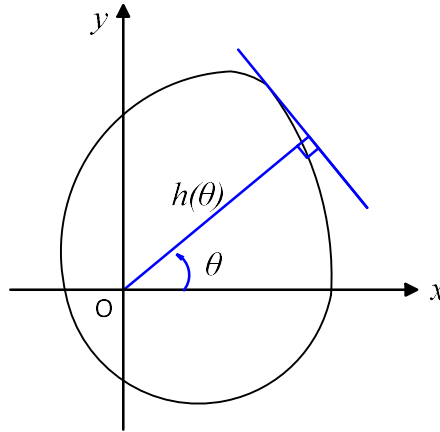
Fig. 4 Support function $h(\theta)$

Differentiation of (7) with respect to θ gives

$$\begin{aligned} h'(\theta) &= x'(\theta)\cos\theta - x(\theta)\sin\theta + y'(\theta)\sin\theta + y(\theta)\cos\theta \\ &= y(\theta)\cos\theta - x(\theta)\sin\theta \end{aligned} \tag{8}$$

Combining (7) and (8), we have

$$\begin{cases} x(\theta) = h(\theta)\cos\theta - h'(\theta)\sin\theta \\ y(\theta) = h(\theta)\sin\theta + h'(\theta)\cos\theta \end{cases} \tag{9}$$

Differentiation of (9) gives

$$\begin{cases} x'(\theta) = -[h(\theta) + h''(\theta)]\sin\theta \\ y'(\theta) = [h(\theta) + h''(\theta)]\cos\theta \end{cases} \tag{10}$$

Then the radius of curvature of C is given by

$$\rho(\theta) = \frac{ds}{d\theta} = \sqrt{x'^2(\theta) + y'^2(\theta)} = h(\theta) + h''(\theta),$$

where s is the arc-length parameter of C. Therefore, equation (10) can be also be written as

$$\begin{cases} x'(\theta) = -\rho(\theta)\sin\theta \\ y'(\theta) = \rho(\theta)\cos\theta \end{cases} \tag{11}$$

Since the radius of curvature $\rho(\theta) = h(\theta) + h''(\theta)$ is a periodic function of θ with period 2π. It is natural to consider the Fourier expansion

$$\rho(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\theta + \sum_{n=1}^{\infty} b_n \sin n\theta,$$

where the Fourier coefficients $a_0 = \frac{1}{\pi}\int_0^{2\pi} \rho(\theta)d\theta$, $a_n = \frac{1}{\pi}\int_0^{2\pi} \rho(\theta)\cos(n\theta)d\theta$, and $b_n = \frac{1}{\pi}\int_0^{2\pi} \rho(\theta)\sin(n\theta)d\theta, n = 1, 2, \cdots$.

2.2 Curves of Constant Width

Let $C : \vec{r} = \vec{r}(\theta) = (x(\theta), y(\theta))$ be a closed convex plane curve with the parameter $\theta \in [0, 2\pi]$ defined as before. The width $w(\theta)$ of C in the direction $\vec{N}(\theta) = (\cos\theta, \sin\theta)$ is defined to be the distance between the tangent lines parallel to the given direction $\vec{N}(\theta)$ [Resnikoff 2015], shown in Figure 5. Then

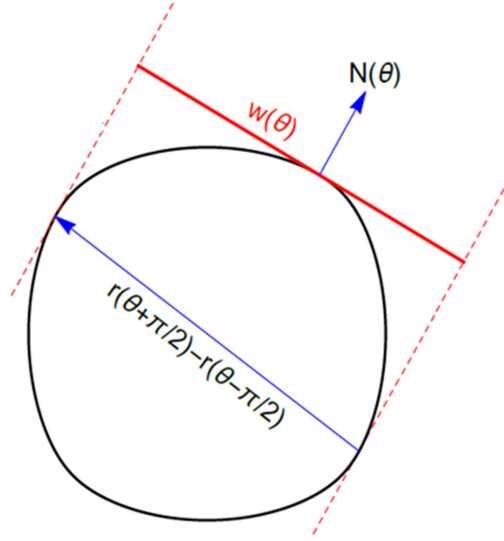$$w(\theta) = h(\theta - \pi/2) + h(\theta + \pi/2)$$

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Fig. 5 The width $w(\theta)$ in the direction $\vec{N}(\theta)$

The closed curve C is said to be of constant width if its width in any direction is a positive constant, i.e. $w(\theta) = \text{constant}$, and the constant is called the width of C. If C is a smooth curve of constant width w, then $\vec{r}(\theta+\pi/2)-\vec{r}(\theta-\pi/2)$ is perpendicular to $\vec{N}(\theta)$, and $\left|\vec{r}(\theta+\pi/2)-\vec{r}(\theta-\pi/2)\right| = w$.

From Figure 5, it's easy to see that $\left|\left[\vec{r}(\theta+\pi/2)-\vec{r}(\theta-\pi/2)\right]\times\vec{N}(\theta)\right|$ is just the width of the curve in the direction $\vec{N}(\theta)$, i.e.

$$\left|\left[\vec{r}(\theta+\pi/2)-\vec{r}(\theta-\pi/2)\right]\times\vec{N}(\theta)\right| = w(\theta)$$

Then a criterion of C to be a curve of constant width can be given in terms of scalar projection. C is a smooth curve of constant width if and only if

$$\left|\left[\vec{r}(\theta+\pi/2)-\vec{r}(\theta-\pi/2)\right]\times\vec{N}(\theta)\right| = \text{constant.} \quad (12)$$

Remark 1: When $\beta=\pi/2$, a noncircular β-CCSP curve exists and it is a smooth noncircular curve of constant width.

Therefore, a β-CCSP curve can be regarded as a generalization of a curve of constant width.

3. Proof of the Main Theorems

From previous section, we know that curves of constant scalar projection (CCSP) exist when $\beta=\pi/2$. They are curves of constant width. One may wonder if there exists noncircular β-CCSP curve for an acute angle β. We'll answer this question for a general angle $\beta \in (0, \pi/2]$ in Theorem 1. Before starting Theorem 1, we need a key lemma for the proof.

We have shown that when $\beta=\pi/2$, the scalar projection $\left|[\vec{r}(\theta+\beta)-\vec{r}(\theta-\beta)]\times\vec{N}(\theta)\right|$ is just width of the curve in the direction $\vec{N}(\theta)$. The following Key Lemma tells us that for a general angle $\beta \in (0, \pi/2]$, the scalar projection can be represented as convolution of cosine function and the radius of curvature of the curve.

Key Lemma. For any $\beta \in (0, \pi/2]$,

$$\left|[\vec{r}(\theta+\beta)-\vec{r}(\theta-\beta)]\times\vec{N}(\theta)\right| = \int_{-\beta}^{\beta} \rho(\theta-\varphi)\cos\varphi\, d\varphi,$$

where $\rho(\theta)$ be the radius of curvature.

Proof: Since

$$\vec{r}(\theta+\beta)-\vec{r}(\theta-\beta)=\left(x(\theta+\beta)-x(\theta-\beta),\, y(\theta+\beta)-y(\theta-\beta)\right),$$

and $\vec{T}(\theta) = (-\sin\theta, \cos\theta)$. A direct calculation gives

$$\left|[\vec{r}(\theta+\beta)-\vec{r}(\theta-\beta)]\times\vec{N}(\theta)\right| = [\vec{r}(\theta+\beta)-\vec{r}(\theta-\beta)]\cdot\vec{T}(\theta)$$
$$= -\sin\theta\left(x(\theta+\beta)-x(\theta-\beta)\right)+\cos\theta\left(y(\theta+\beta)-y(\theta-\beta)\right)$$

Now set

$$g(\varphi) = -\sin\theta\left(x(\theta+\varphi)-x(\theta-\varphi)\right)+\cos\theta\left(y(\theta+\varphi)-y(\theta-\varphi)\right).$$

Then $g(0) = 0$, $g(\beta) = \left|[\vec{r}(\theta+\beta)-\vec{r}(\theta-\beta)]\times\vec{N}(\theta)\right|$. Following (11), we have

$$x'(\varphi) = -\rho(\varphi)\sin\varphi, \quad y'(\varphi) = \rho(\varphi)\cos\varphi$$

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$$g'(\varphi) = -\sin\theta\left[-\rho(\theta+\varphi)\sin(\theta+\varphi) - \rho(\theta-\varphi)\sin(\theta-\varphi)\right]$$
$$+\cos\theta\left[\rho(\theta+\varphi)\cos(\theta+\varphi) + \rho(\theta-\varphi)\cos(\theta-\varphi)\right]$$
$$= \rho(\theta+\varphi)\left[\sin\theta\sin(\theta+\varphi) + \cos\theta\cos(\theta+\varphi)\right]$$
$$+\rho(\theta-\varphi)\left[\sin\theta\sin(\theta-\varphi) + \cos\theta\cos(\theta-\varphi)\right]$$
$$= \rho(\theta+\varphi)\cos\varphi + \rho(\theta-\varphi)\cos\varphi$$
$$= \cos\varphi\left[\rho(\theta+\varphi) + \rho(\theta-\varphi)\right]$$

Consequently, we have

$$g(\beta) = \left|\left[\vec{r}(\theta+\beta) - \vec{r}(\theta-\beta)\right] \times \vec{N}(\theta)\right|$$
$$= \int_0^\beta g'(\varphi)d\varphi$$
$$= \int_0^\beta \rho(\theta+\varphi)\cos\varphi d\varphi + \int_0^\beta \rho(\theta-\varphi)\cos\varphi d\varphi$$
$$= \int_{-\beta}^\beta \rho(\theta-\varphi)\cos\varphi d\varphi$$

□

Since the radius of curvature $\rho(\theta)$ is a periodic function of θ with period 2π. The Fourier expansion of $\rho(\theta)$ is used in the proof of the main theorem. Recall the Fourier expansion $\rho(\theta) = \dfrac{a_0}{2} + \sum_{n=1}^\infty a_n\cos n\theta + \sum_{n=1}^\infty b_n\sin n\theta$. We are now in a position to prove the main theorem.

Theorem 1. For a given $\beta \in (0, \pi/2]$,

(i) A noncircular β-CCSP curve exists if and only if there exist integer $k>1$ such that

$$\cot k\beta = k\cot\beta.$$

(ii) Let $K_\beta = \{k \in \mathbb{N} \mid k > 1, \cot k\beta = k\cot\beta\}$. The radius of curvature of a β-CCSP curve has the form

$$\rho(\theta) = \frac{a_0}{2} + \sum_{k \in K_\beta}\left(a_k\cos k\theta + b_k\sin k\theta\right), \text{ and } \rho(\theta) > 0.$$

Proof: In terms of the Fourier series,

$$\rho(\theta-\varphi)\cos\varphi$$

$$=\cos\varphi\left(\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos(n\theta-n\varphi)+\sum_{n=1}^{\infty}b_n\sin(n\theta-n\varphi)\right)$$

$$=\cos\varphi\left(\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos n\theta\cos n\varphi+\sum_{n=1}^{\infty}a_n\sin n\theta\sin n\varphi\right)$$

$$+\cos\varphi\left(\sum_{n=1}^{\infty}b_n\sin n\theta\cos n\varphi-\sum_{n=1}^{\infty}b_n\cos n\theta\sin n\varphi\right)$$

Applying the Key Lemma, and the following facts

$$\int_{-\beta}^{\beta}\cos\varphi\cos n\varphi\,d\varphi=\frac{1}{2}\int_{-\beta}^{\beta}\cos(n+1)\varphi\,d\varphi+\frac{1}{2}\int_{-\beta}^{\beta}\cos(n-1)\varphi\,d\varphi$$

$$=\begin{cases}\dfrac{\sin 2\beta}{2}+\beta, & n=1\\[2mm]\dfrac{\sin(n+1)\beta}{n+1}+\dfrac{\sin(n-1)\beta}{n-1}, & n\neq 1\end{cases}$$

and

$$\int_{-\beta}^{\beta}\cos\varphi\sin n\varphi\,d\varphi=\frac{1}{2}\int_{-\beta}^{\beta}\sin(n+1)\varphi\,d\varphi+\frac{1}{2}\int_{-\beta}^{\beta}\sin(n-1)\varphi\,d\varphi=0,$$

we obtain

$$\left|[\vec{r}(\theta+\beta)-\vec{r}(\theta-\beta)]\times\vec{N}(\theta)\right|=\int_{-\beta}^{\beta}\rho(\theta-\varphi)\cos\varphi\,d\varphi$$

$$=\frac{a_0}{2}\int_{-\beta}^{\beta}\cos\varphi\,d\varphi+\sum_{n=1}^{\infty}\left[a_n\cos n\theta\int_{-\beta}^{\beta}\cos\varphi\cos n\varphi\,d\varphi\right]+\sum_{n=1}^{\infty}\left[b_n\sin n\theta\int_{-\beta}^{\beta}\cos\varphi\cos n\varphi\,d\varphi\right]$$

$$=a_0\sin\beta+(a_1\cos\theta+b_1\sin\theta)\left(\frac{1}{2}\sin 2\beta+\beta\right)$$

$$+\sum_{n=2}^{\infty}(a_n\cos n\theta+b_n\sin n\theta)\left(\frac{\sin(n+1)\beta}{n+1}+\frac{\sin(n-1)\beta}{n-1}\right).$$

Note that for $\beta\in(0,\pi/2]$, $0<\frac{1}{2}\sin 2\beta=\sin\beta\cos\beta<\sin\beta<\beta$ which implies

$$\frac{1}{2}\sin 2\beta+\beta\neq 0.$$

Thus $\left|[\vec{r}(\theta+\beta)-\vec{r}(\theta-\beta)]\times\vec{N}(\theta)\right|$ is constant for all θ if and only if $a_1=b_1=0$, and

$$a_n\left(\frac{\sin(n+1)\beta}{n+1}+\frac{\sin(n-1)\beta}{n-1}\right)=b_n\left(\frac{\sin(n+1)\beta}{n+1}+\frac{\sin(n-1)\beta}{n-1}\right)=0 \qquad (13)$$

for $n>1$. Next, we'll show that equation

$$\frac{\sin(n+1)\beta}{n+1}+\frac{\sin(n-1)\beta}{n-1}=0 \qquad (14)$$

is equivalent to $\cot n\beta = n\cot \beta$ for $\beta \in (0, \pi/2]$. Since

$$\sin(n \pm 1)\beta = \sin n\beta \cos \beta \pm \cos n\beta \sin \beta,$$

$$\frac{\sin(n+1)\beta}{n+1} + \frac{\sin(n-1)\beta}{n-1} = 0 \quad \Leftrightarrow \quad \sin n\beta \cos \beta = n\cos n\beta \sin \beta \quad (15)$$

For $\beta \in (0, \pi/2]$ and $n>1$, if $\sin n\beta = 0$, then we have $\sin \beta \neq 0$ and $\cos n\beta \neq 0$,

which contradicts (15). Thus $\sin n\beta \neq 0$, and therefore (15) implies $\cot n\beta = n\cot \beta$.

Let

$$K_\beta = \{k \in \mathbb{N} \mid k > 1, \cot k\beta = k\cot \beta\}.$$

Following (13) we know that for any $n \notin K_\beta$, the Fourier coefficients $a_n = b_n = 0$. As a

consequence, the radius of curvature of a β-CCSP curve has the form

$$\rho(\theta) = \frac{a_0}{2} + \sum_{k \in K_\beta} (a_k \cos k\theta + b_k \sin k\theta).$$

In order to ensure the convexity, we also need $\rho(\theta) > 0$.

$\square$

If $a_k = b_k = 0$ for any $k \in K_\beta$, then the radius of curvature of a β-CCSP curve is

a positive constant, and thus it is a circular. In order for the curve to be noncircular,

we need at least one $k \in K_\beta$ such that the Fourier coefficient $a_k \neq 0$ or $b_k \neq 0$.

Remark 2. A β-CCSP curve has constant scalar projection

$$\left| [\vec{r}(\theta + \beta) - \vec{r}(\theta - \beta)] \times \vec{N}(\theta) \right| = a_0 \sin \beta$$

Theorem 1 reduces the existence of noncircular β-CCSP curve to the study the

trigonometric equation $\cot n\beta = n\cot \beta$. In the following Section 4, we will

completely characterize $\beta \in (0, \pi/2]$ such that $\cot n\beta = n\cot \beta$.

Here we only present two simple examples. Our constructions of the curves are

based on Theorem 1. Recall from (11) that a simple closed convex plane curve

satisfies the differential equations

$$x'(\theta) = -\rho(\theta)\sin\theta, \quad y'(\theta) = \rho(\theta)\cos\theta.$$

For a curve with $\rho(\theta) = 2 + \cos n\theta$, we get the parametric equations of the curve

$$\begin{cases} x(\theta) = c_1 + 2\cos\theta - \dfrac{1}{2(n-1)}\cos(n-1)\theta + \dfrac{1}{2(n+1)}\cos(n+1)\theta, \\ y(\theta) = c_2 + 2\sin\theta + \dfrac{1}{2(n-1)}\sin(n-1)\theta + \dfrac{1}{2(n+1)}\sin(n+1)\theta, \end{cases}$$

where c_1, c_2 are constants. For convenience, we take $c_1 = c_2 = 0$ in the examples.

Example 1. Let $n=4$, and $\rho(\theta) = 2 + \cos n\theta$. We have

$$\begin{cases} x(\theta) = 2\cos\theta - \dfrac{1}{6}\cos 3\theta + \dfrac{1}{10}\cos 5\theta \\ y(\theta) = 2\sin\theta + \dfrac{1}{6}\sin 3\theta + \dfrac{1}{10}\sin 5\theta \end{cases}$$

For $n=4$, there is only one acute angle $\beta \approx 0.855$ such that $\cot n\beta = n\cot\beta$, as shown in the left panel of Figure 6.
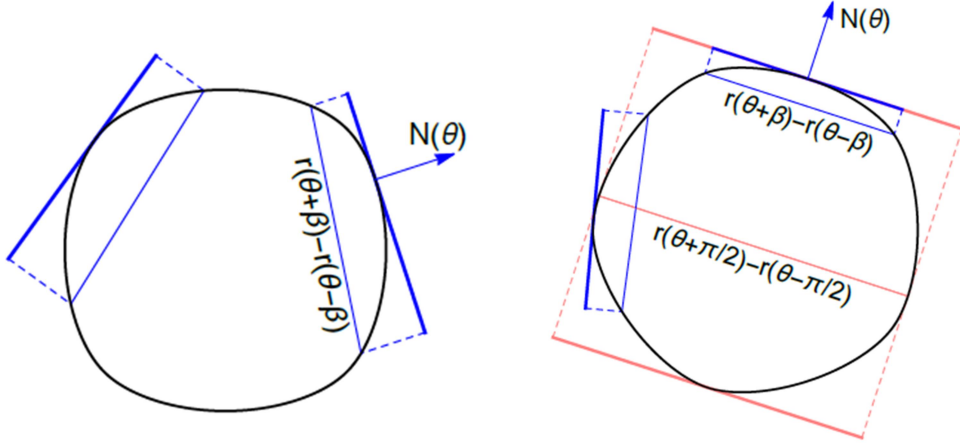

Fig. 6 β-CCSP curve with $\rho(\theta) = 2 + \cos 4\theta$ and $\beta \approx 0.855$(left); a curve of constant width with $\rho(\theta) = 2 + \cos 5\theta$ corresponding to $\beta \approx 0.659$ and $\pi/2$ (right)

Example 2. Let $n=5$, and $\rho(\theta) = 2 + \cos 5\theta$. Then

$$\begin{cases} x(\theta) = 2\cos\theta - \dfrac{1}{8}\cos 4\theta + \dfrac{1}{12}\cos 6\theta \\ y(\theta) = 2\sin\theta + \dfrac{1}{8}\sin 4\theta + \dfrac{1}{12}\sin 6\theta \end{cases}$$

It is a curve of constant width. Besides $\beta = \pi/2$, there is an acute angle $\beta \approx 0.659$ such that $\cot n\beta = n\cot\beta$, as shown in the right panel of Figure 6.

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Combining Theorem 1 and Gutkin's Theorem, we have the following results on Gutkin curves with constant length of chords.

Theorem 2. A curve C is both a noncircular β-CCSP curve and a β-Gutkin curve, if and only if $\beta = \pi / 2$, and thus C is a curve of constant width.

Proof: The sufficiency is obvious. Next, we'll prove the necessity. Let

$$K_\beta = \left\{ k \in \mathbb{N} \mid k > 1, \cot k\beta = k \cot \beta \right\},$$

$$K_\beta^* = \left\{ k \in \mathbb{N} \mid k > 1, k \cot k\beta = \cot \beta \right\}.$$

Since C is both a noncircular β-CCSP curve and a β-Gutkin curve, following Gutkin's Theorem and Theorem 1, the Fourier expansion of the radius of curvature has the form

$$\rho(\theta) = \frac{a_0}{2} + \sum_{k \in K_\beta \cap K_\beta^*} \left(a_k \cos k\theta + b_k \sin k\theta \right).$$

$\forall k \in K_\beta \cap K_\beta^*$, i.e k satisfies

$$\cot k\beta = k \cot \beta, \quad k \cot k\beta = \cot \beta$$

Then we obtain $\cot k\beta = \cot \beta = 0$ and thus $\beta = \dfrac{\pi}{2}$ and $k > 1$ is odd, which implies C is a curve of constant width. The Fourier series of the radius of curvature has the form

$$\rho(\theta) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_{2k+1} \cos(2k+1)\theta + \sum_{k=1}^{\infty} b_{2k+1} \sin(2k+1)\theta.$$

$$\square$$

For any α-Gutkin curve,

$$\left| \vec{r}(\theta + \alpha) - \vec{r}(\theta - \alpha) \right| = \left\| \left[\vec{r}(\theta + \alpha) - \vec{r}(\theta - \alpha) \right] \times \vec{N}(\theta) \right\|$$

Following Theorem 2, we know that the lengths of chords $\left| \vec{r}(\theta + \alpha) - \vec{r}(\theta - \alpha) \right| = \left\| \left[\vec{r}(\theta + \alpha) - \vec{r}(\theta - \alpha) \right] \times \vec{N}(\theta) \right\| =$ constant is not possible for an α-Gutkin curve when $\alpha \in (0, \pi / 2)$.

4. Properties of the Trigonometric Equations

By Theorem 1, the angle β plays an important role for the existence of noncircular curves of constant scalar projection. In particular, if a noncircular curve satisfying

$$\left|[\vec{r}(\theta+\beta)-\vec{r}(\theta-\beta)]\times\vec{N}(\theta)\right|=\text{constant for } \theta,$$

then β must satisfy $\cot n\beta = n\cot\beta$ for some integer $n>1$. There is an obvious solution $\beta = \pi/2$ when n is odd. However, $\beta = \pi/2$ is not a solution when n is even.

Next, we only consider the solutions of $\cot n\beta = n\cot\beta$ in $(0,\pi/2)$. When β is restricted to $(0, \pi/2)$, we have $\cot\beta > 0$, and

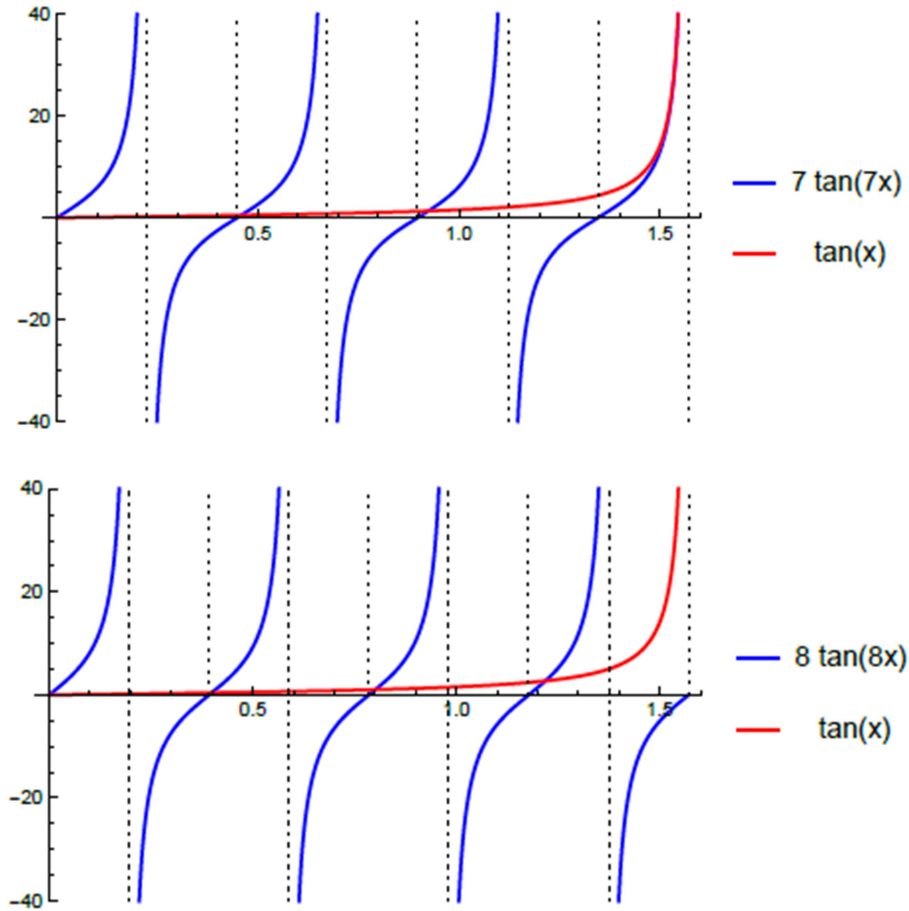$$\cot n\beta = n\cot\beta \quad \Leftrightarrow \quad n\tan n\beta = \tan\beta.$$

Fig. 7 Graphs of functions $y = n\tan nx$ and $y = \tan x$ for n=7 and *n*=8

To study the solutions of the equation $n \tan nx = \tan x$ in $(0, \pi/2)$, we cut the interval $(0, \pi/2)$ into n disjoint sub-intervals $I_m^{(n)} = \left(\dfrac{m-1}{2n}\pi, \dfrac{m}{2n}\pi \right)$, $m = 1, 2, \cdots, n$.

The endpoints $\dfrac{m}{2n}\pi$, $m = 1, 2, \cdots, n-1$ don't satisfy $n \tan nx = \tan x$, so we can ignore them in the analysis. The graphs of the functions $y = n \tan nx$ and $y = \tan x$ for $n=7$ and $n=8$ are shown in Figure 7.

Lemma 1. Let $f_n(x) = n \tan nx - \tan x$ and $I_m^{(n)} = \left(\dfrac{m-1}{2n}\pi, \dfrac{m}{2n}\pi \right)$, $m = 1, 2, \cdots, n$. Then

(i) For any odd $n>1$, $\displaystyle\lim_{x \to \frac{\pi}{2}^-} f_n(x) = 0$, and

(ii) For any integer $n > 1$, odd $m \le n$, and $x \in I_m^{(n)}$, $f_n'(x) > 0$.

Proof: (i) For any odd $n>1$,

$$f_n(x) = n \tan nx - \tan x$$
$$= \frac{n \sin nx \cos x - \cos nx \sin x}{\cos nx \cos x}$$
$$= \frac{(n-1)\sin(n+1)x + (n+1)\sin(n-1)x}{\cos(n+1)x + \cos(n-1)x}$$

For any odd $n>1$, the numerator and denominator converge to 0 when $\theta \to \pi/2 -$. According to L'Hôpital's rule,

$$\lim_{\theta \to \frac{\pi}{2}^-} f_n(x) = \lim_{x \to \frac{\pi}{2}^-} \frac{(1-n^2)\big(\cos(n+1)x + \cos(n-1)x\big)}{(n+1)\sin(n+1)x + (n-1)\sin(n-1)x}$$
$$= \lim_{x \to \frac{\pi}{2}^-} \frac{(n^2-1)\big((n+1)\sin(n+1)x + (n-1)\sin(n-1)x\big)}{(n+1)^2 \cos(n+1)x + (n-1)^2 \cos(n-1)x}$$
$$= 0$$

(ii) For any integer $n > 1$, $m < n$, and $x \in I_m^{(n)}$,

$$x \in I_m^{(n)} \subset \left(0, \frac{\pi}{2} - \frac{\pi}{2n} \right) \quad \Rightarrow \quad 0 < \tan x < \tan\left(\frac{\pi}{2} - \frac{\pi}{2n} \right) = \cot\left(\frac{\pi}{2n} \right) \le \frac{2n}{\pi}.$$

Then the differentiation of $f_n(x)$

$$f_n'(x) = n^2 \sec^2 nx - \sec^2 x \geq n^2 - 1 - \left(\frac{2n}{\pi}\right)^2 = \left(1 - \frac{4}{\pi^2}\right)n^2 - 1 > 0 \text{ for } n > 1.$$

When $n>1$ is odd, and $m = n$, for any $x \in I_m^{(n)} = I_n^{(n)} = \left(\frac{n-1}{2n}\pi, \frac{\pi}{2}\right)$, let

$x^* = x - \frac{n-1}{2n}\pi.$ Then $nx^* \in (0, \pi/2)$, and

$$\tan nx = \tan\left(nx - \frac{n-1}{2}\pi\right) = \tan nx^* > 0.$$

For any $n>1$, we have

$$0 < nx^* = n\left(x - \frac{n-1}{2n}\pi\right) = n\left(x - \frac{\pi}{2}\right) + \frac{\pi}{2} < x < \frac{\pi}{2}.$$

And then $\sec x > \sec nx^* > 0$, and

$$f_n'(x) = n^2 \sec^2 nx - \sec^2 x = n^2 \sec^2 nx^* - \sec^2 x > \left(n^2 - 1\right)\sec^2 nx^* > 0 \text{ for } n > 1.$$

Therefore, for any integer $n > 1$, odd $m \leq n$, and $x \in I_m^{(n)}$, $f_n'(x) > 0$.

$\square$

Proposition 1. For a given integer $n>1$, let $B_n = \left\{\beta \in (0, \pi/2) \middle| \cot n\beta = n\cot\beta\right\}$. Then

(i) $\#B_n = \begin{cases} \dfrac{n-1}{2} - 1, & \text{if } n>1 \text{ is odd,} \\[3mm] \dfrac{n}{2} - 1, & \text{if } n \text{ is even,} \end{cases}$

where $\#B_n$ denotes the number of elements of the set B_n.

(ii) For any $\beta \in \bigcup\limits_{n=2}^{\infty} B_n$, β/π is irrational.

Proof: (i) $\forall x \in I_m^{(n)} = \left(\frac{m-1}{2n}\pi, \frac{m}{2n}\pi\right), m = 1, 2, \cdots, n.$ Let $x_m^* = x - \frac{m-1}{2n}\pi$, then

$nx_m^* \in (0, \pi/2)$, and

$$n\tan nx = n\tan\left(nx_m^* + \frac{m-1}{2}\pi\right) = \begin{cases} n\tan nx_m^* > 0, & m \text{ is odd,} \\ -n\cot nx_m^* < 0, & m \text{ is even.} \end{cases}$$

However, the function $y = \tan x$ is positive within each interval $I_m^{(n)}$. Therefore, for the equation $n \tan nx = \tan x$ in $(0, \pi/2)$, $n>1$. There is no solution in $I_m^{(n)}$ with even m. Only solutions in $I_m^{(n)}$ with odd m are possible. Based on Lemma 1, the difference $f_n(x) = n \tan nx - \tan x$ is monotone in $I_m^{(n)}$ when $m<n$. And the values of $f_n(x)$ at the endpoints satisfy

$$f_n\left(\frac{m-1}{2n}\pi\right) = 0 - \tan\left(\frac{m-1}{2n}\pi\right) = \begin{cases} 0, & m=1 \\ <0, & \text{odd } m>1 \end{cases}$$

$$\lim_{x \to \frac{m}{2n}\pi-} f_n(x) = \begin{cases} \infty, & \text{odd } m<n, \\ 0, & \text{odd } m=n. \end{cases}$$

That means $f_n(x) > 0$ in $I_1^{(n)}$, and $f_n(x) < 0$ in $I_n^{(n)}$ if n is odd. Then there is no solution in $I_1^{(n)}$ or $I_n^{(n)}$. The values of $f_n(x)$ at the two endpoints have opposite signs in $I_m^{(n)}$ for odd $m \in (1, n)$. Therefore, there is one and only one solution in each $I_m^{(n)}$ for odd $m \in (1, n)$. If n is odd there are $(n-1)/2-1$ solutions located in $I_{2k+1}^{(n)} = \left(\frac{k}{n}\pi, \frac{2k+1}{2n}\pi\right)$, $k = 1, 2, \cdots, \frac{n-1}{2}-1$. If n is even, there are $n/2-1$ solutions located in $I_{2k+1}^{(n)}$, $k = 1, 2, \cdots, \frac{n}{2}-1$.

(ii) Cyr V. [2011] showed that if $\delta \in (0, 1/2) \cap \mathbb{Q}$ and $n, m \in \mathbb{Z}$ such that $\sin(m\pi\delta) \neq 0$, then $\dfrac{\sin(n\pi\delta)}{\sin(m\pi\delta)}$ is either $-1, 0, 1$ or irrational, where $\mathbb{Q}$ is the set of all rational numbers. For any $\beta \in \bigcup_{n=2}^{\infty} B_n$, we have

$$\cot n\beta = n \cot \beta \Longleftrightarrow \frac{\sin(n+1)\beta}{\sin(n-1)\beta} = -\frac{n+1}{n-1} \notin \{-1, 0, 1\} \cup \mathbb{Q}.$$

According to Cyr's results, β/π is irrational.

□

Proposition2. If the radius of curvature of C has the form

$$\rho(\theta) = \frac{a_0}{2} + a_n \cos n\theta + b_n \sin n\theta, \quad n > 3,$$

and $\rho(\theta) > 0$, then there exist two distinct acute angles such that C is both an α-Gutkin curve and a β-CCSP curve. In particular, if $n > 3$ is odd, α and β can be chosen such that $\alpha + \beta = \pi/2$.

Proof: When $n>3$, Gutkin showed that there exist at least one solution of $\tan nx = n \tan x$ in $(0, \pi/2)$. In addition, by Proposition 1, there is at least one solution of $\cot nx = n \cot x$ in $(0, \pi/2)$. The equations $\tan nx = n \tan x$ and $\cot nx = n \cot x$ have no common solutions in $(0, \pi/2)$. Therefore, there exist two distinct $\alpha, \beta \in (0, \pi/2)$ such that $\tan n\alpha = n \tan \alpha$ and $\cot n\beta = n \cot \beta$. In the special case when $n>3$ is odd,

$$\tan nx = n \tan x \Leftrightarrow \cot n\left(\frac{\pi}{2} - x\right) = n \cot\left(\frac{\pi}{2} - x\right)$$

That means if α is a solution of $\tan nx = n \tan x$, then $\beta = \pi/2 - \alpha$ is a solution of $\cot nx = n \cot x$. Therefore, when $n>3$ is odd, we can choose complementary α and β.

□

Example 1 (Continued). Let $n=4$, and $\rho(\theta) = 2 + \cos n\theta$.

For $n=4$, there is only one acute angle $\alpha \approx 1.150$ such that $\tan n\alpha = n \tan \alpha$. The chords $\vec{r}(\theta + \alpha) - \vec{r}(\theta - \alpha), \theta \in [0, 2\pi]$, of the curve have constant contact angle α., shown in left panel in Figure 8. In addition, there is only one acute angle $\beta \approx 0.855$ such that $\cot n\beta = n \cot \beta$. The chords $\vec{r}(\theta + \beta) - \vec{r}(\theta - \beta), \theta \in [0, 2\pi]$, have constant scalar projection $4\sin\beta$, shown in right panel in Figure 8.
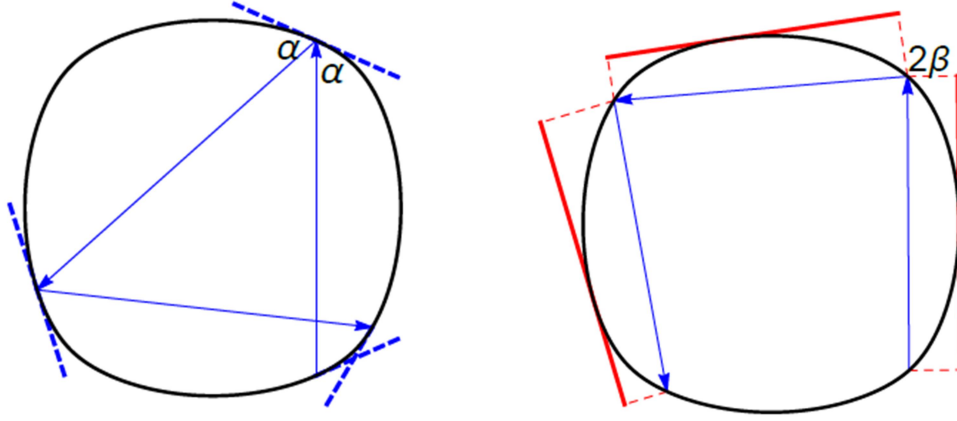
Fig. 8 A curve with $\rho(\theta) = 2 + \cos 4\theta$ with contact angle $\alpha \approx 1.150$ (left) and constant scalar projection $4\sin\beta$ with $\beta \approx 0.855$(right)

Example 2 (Continued). Let $n=5$, and $\rho(\theta) = 2 + \cos 5\theta$.

The curve with $\rho(\theta) = 2 + \cos 5\theta$ is a curve of constant width. Besides $\alpha = \beta = \pi/2$, there is only one acute angle $\alpha \approx 0.912$ and $\beta = \pi/2 - \alpha \approx 0.659$ satisfying $\tan n\alpha = n \tan \alpha$ and $\cot n\beta = n \cot \beta$ respectively. In this case, α and β are complementary. The chords $\vec{r}(\theta + \alpha) - \vec{r}(\theta - \alpha), \theta \in [0, 2\pi]$, have contact angle α, as shown in the left panel of Figure 9. The chords $\vec{r}(\theta + \beta) - \vec{r}(\theta - \beta), \theta \in [0, 2\pi]$, have constant scalar projection, as shown in right panel.
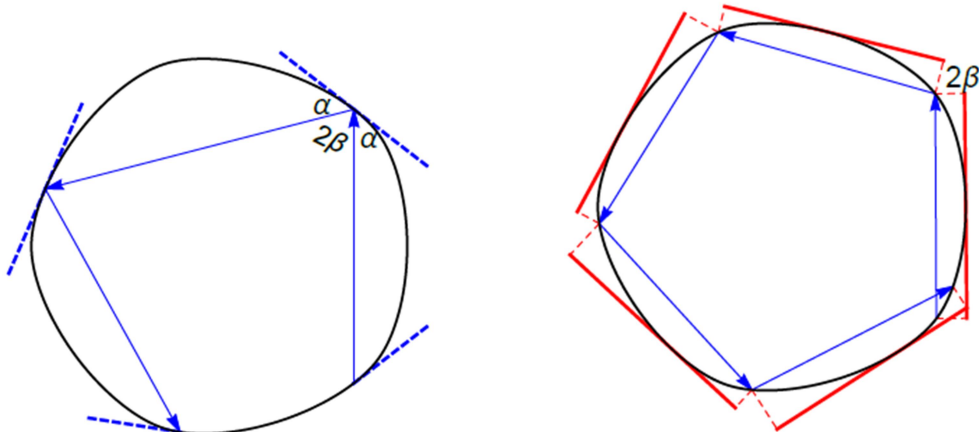

Fig. 9 A curve of constant width with $\rho(\theta) = 2 + \cos 5\theta$ with contact angle $\alpha \approx 0.912$ (left) and constant scalar projection $4\sin\beta$ with complementary $\beta \approx 0.659$ (right)

5. Discussion

We have shown that the radius of curvature of a noncircular β-CCSP curve has the form

$$\rho(\theta) = \frac{a_0}{2} + \sum_{k \in K_\beta} \left(a_k \cos k\theta + b_k \sin k\theta \right). ,$$

where $a_0 > 0$, and $a_k \neq 0$ or $b_k \neq 0$ for at least one

$$k \in K_\beta = \{ k \in \mathbb{N} \mid k > 1, \cot k\beta = k \cot \beta \}.$$

When $\beta=\pi/2$, a β-CCSP curve is a curve of constant width, with radius of curvature

$$\rho(\theta) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_{2k+1} \cos(2k+1)\theta + \sum_{n=1}^{\infty} b_{2k+1} \sin(2k+1)\theta.$$

When β is restricted to $(0, \pi/2)$, for any given $n>1$, only finitely many values of $\beta \in B_n = \{ \beta \in (0, \pi/2) \mid \cot n\beta = n \cot \beta \}$ such that $\cot n\beta = n \cot \beta$ holds. For any $\beta \in B_n$, then

$$n \in K_\beta = \{ k \in \mathbb{N} \mid k > 1, \cot k\beta = k \cot \beta \}.$$

Besides n, is there any other integer $m \in K_\beta$? This is equivalent to the following open problem.

Open problem. Let $m>1$ and $n>1$ be distinct integers. Are there common solutions in $(0, \pi/2)$ for the trigonometric equations $\cot mx = m \cot x$ and $\cot nx = n \cot x$?

If the answer of the problem is "no", then the radius of curvature of any noncircular β-CCSP curve with $\beta \in (0, \pi/2)$ has the simple form

$$\rho(\theta) = \frac{a_0}{2} + a \cos n\theta + b \sin n\theta,$$

where $n>1$ such that $\cot n\beta = n \cot \beta$.

References

1. Aougab T., Sun X.,Tabachnikov S.,et al. On curves and polygons with the equiangular chord property. Pacific Journal of Mathematics. 2013. 274: 305-324.

2. Bresme F, Oettel M. Nanoparticles at fluid interfaces. Journal of Physics Condensed Matter, 2007, 19(41):3385-3391.

3. Cyr V. A number theoretic question arising in the geometry of plane curves and in billiard dynamics. Proceedings of the American Mathematical Society, 2011, 140:3035-3040.

4. Finn, R. Floating bodies subject to capillary attractions. J. Math. Fluid Mech. 2009. 11: 443-458.

5. Gardner R.J. Geometric Tomography, Second Edition. Cambridge University Press. 2006

6. Gutkin, E. Capillary floating and the billiard ball problem. J. Math. Fluid Mech. 2012.14: 363-382.

7. Gutkin, E. Addendum to: Capillary floating and the billiard ball problem. J. Math. Fluid Mech. 2013.15, 425-430.

8. Manfred P. do Carmo. Differential Geometry of Curves and Surfaces (English reprint edition). Pearson Education Asia Limited and China Machine Press. 2004.

9. Peng J.G., Chen Q. Differential Geometry (in Chinese). Higher Education Press, Beijing, 2002.

10. Raphël E., di Meglio J.M., Berger M.,Calabi E. Convex particles at interfaces. J. Phys. I France, 1992. 2:571–579.

11. Resnikoff H L. On curves and surfaces of constant width. Mathematics, 2015.

12. Stein E.M. and Shakarchi R. Fourier Analysis: An Introduction (English reprint edition). Princeton University Press, New Jersey. 2003.

Appendix . Mathematica Code for Figure 6

```
Clear[t];
x1[t_] := 2 Cos[t] - 1/6 Cos[3 t] + 1/10 Cos[5 t];
y1[t_] := 2 Sin[t] + 1/6 Sin[3 t] + 1/10 Sin[5 t];
pt1 = ParametricPlot[{x1[t], y1[t]}, {t, 0, 2 Pi}, PlotStyle → Black];


f2[t_, n_] := Sin[(n + 1) t] / (n + 1) + Sin[(n - 1) t] / (n - 1);
Plot[f2[t, 4], {t, 0, Pi / 2}];
solb = FindRoot[f2[t, 4] == 0, {t, 0.85}];
b1 = solb[[1, 2]]

pl = {};
t2 = {Pi / 10, 4 Pi / 5};
For[i = 1, i < 3, i++,
 t = t2[[i]];
 sol4 = NSolve[(y - y1[t]) == -1 / Tan[t] * (x - x1[t]) &&
     (y - y1[t - b1]) == Tan[t] * (x - x1[t - b1]), {x, y}];
 x2 = sol4[[1, 1, 2]];
 y2 = sol4[[1, 2, 2]];
 sol5 = NSolve[(y - y1[t]) == -1 / Tan[t] * (x - x1[t]) &&
     (y - y1[t + b1]) == Tan[t] * (x - x1[t + b1]), {x, y}];
 x3 = sol5[[1, 1, 2]];
 y3 = sol5[[1, 2, 2]];
 g1 = Graphics[{Blue, Dashed, Line[{{x3, y3}, {x1[t + b1], y1[t + b1]}}]}];
 g2 = Graphics[{Blue, Dashed, Line[{{x2, y2}, {x1[t - b1], y1[t - b1]}}]}];
 g3 = Graphics[{Blue, Thick, Line[{{x2, y2}, {x3, y3}}]}];
 g4 = Graphics[{Blue, Line[{{x1[t - b1], y1[t - b1]}, {x1[t + b1], y1[t + b1]}}]}];
 AppendTo[pl, g1];
 AppendTo[pl, g2];
 AppendTo[pl, g3];
 AppendTo[pl, g4];
]
```