Constructing Evans triangle with Pell equation

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Abstract: in this paper, we construct two new primitive Evans triangles by using the Pell equation x^2-Dy^2=1, and give the two forms of the Evans triangle and the corresponding Evans ratio, we also get a new conclusion that the ratio is an integer from a few different forms of the bottom and high.

Key words: Evans triangle; Pell equation; Evans ratio

1 Introduction

In 1977, R. Evans put forward an unsolved problem (problem E2687) in the "American Mathematical Monthly": for all integer triangle, making it a high ratio and the bottom edge of the integer. This problem is included in the famous book named "Unsolved problems in number theory" by R. KGuy. KGuy said that this ratio cannot be 1 and 2, but 3 (such as the length were 4, 13, 15 integer triangle, the first edge of the high of 12), and put forward the problems than can take integer greater than 3?

Ron Evans stated the following problems: on the side of the triangle n integer, integer what can be used as bottom and high ratio? N sign positive or negative, depending on the triangle is acute or obtuse triangle and (for example: 71=-29, X120, y=119 is a solution). He also asked the dual problem: the calculated bottom divides all integer length high triangle. This high ratio and the end of the 1 and 2 is not possible, but the 3 can (for example: the bottom is 4: length 13, 15: as high as 12) if a ratio can appear, whether there is an infinite number of the ratio of the primitive triangle? K.R. S. Sastry gives the angle (3389, 21029, 24360) and (26921, 42041, 6880). In each triangle, and the bottom of the high ratio is 42 (25, 26, 3) and 1/8 (17,113,120) and 8 (ratio in each case, the number of third three of the array is the base of a triangle)

The definition of X, y, z as three integer edge of the Evans triangle, n is a high and is defined as the ratio of Evans. Given a triangle Evans triangle discriminant theorem and construct a class of Evans triangle, its Evans is n three m (m+ 2) integer, third edge is

$$(1.1)\begin{cases} x = 2m^3 + 6m^2 + 5m + 2, \\ y = 2m^3 + 6m^2 + 5m, \\ z = 2m + 2. \end{cases}$$

In this paper [4] and [3], we construct another kind of Evans triangle, which is the Evans ratio of 8m (2m^2-1) - type integer. Its three integers as follow:

$$(1.2)\begin{cases} x = 64m^6 - 48m^4 + 12m^2 - 2, \\ y = 64m^6 - 48m^4 + 8m^2 + 1, \\ z = 4m^2 - 1. \end{cases}$$

The paper [5][6][7] used the same method to construct several other classes of Evans triangle, their Evans ratios of 4m (m+ 1), 2n (M^2-2) and integer type M^2-1. And when m=2k, the results can be obtained by this paper. When m=2k, the result of paper [6] can be the result of paper [3] launch.

In this paper [8] the limitation on the three side of the triangle, the existence of the inquiry in the primitive Evans triangle satisfies the condition of x +y+ 2=2z^3, the two group of solutions, a group of Evans than 4m (m +1) and another group of Evans is 2m (m^2-1) (m^2-2) (m^2 -3). The results of the first group the result is the same. The Evans ratio of 2m (m^2-1) (m^2-2) (m^2 -3) three integer Evans edges respectively as follow:

$$(1.3)\begin{cases} x = (m^2-1)(m^2-3) + m^2(m^2-1)(m^2-2)(m^2-3)(m^4-3m^2+1), \\ y = m^2(m^2-1)(m^2-2)(m^2-3)(m^4-3m^2+1) + m^2 - 2, \\ z = m^2 - 2 + (m^2-1)(m^2-3). \end{cases}$$

The paper[9] presented a Evans triangle Evans than the necessary and sufficient condition for the existence of the conditions thus constructed three types of Etan9 triangle, among which two kinds of paper [6]and paper[7] and four results consistent with previous Evans ratio difference was 2 (m^2-1) (2m^2-1), three of its edge as integer:

$$(1.4)\begin{cases} x = 8m^5 - 8m^3 + m + 1, \\ y = 8m^5 - 8m^3 + m - 1, \\ z = 2m. \end{cases}$$

Paper[10] is also used in this paper[9] to construct a class of Evns triangle, it's Evans ratio of 4m (m^2-1) (2m^4-4m^2 +1), its three integers as folllow:

$$(1.5)\begin{cases} x = m^2 - 2 + 4m^2(m^2-2)(m^2-1)^3, \\ y = 1 + 4m^2(m^2-2)(m^2-1)^3, \\ z = m^2 - 1. \end{cases}$$

In this paper[9], we construct another two Evans triangles according to the properties of the theorem and Pell.

2 Lemma

Lemma 2.1.[9] Set n as positive integer, then n is a Evans ratio of a Evans triangle, if there is only a positive integer a, b, c,d, ac >bd, then

$$\frac{1}{n}=\frac{a^2-b^2}{2ab}+\frac{c^2-d^2}{2cd}.$$

Note 2.1: the above theorem shows that: to solve the Evans ratio, equivalent to determine the positive integer a, b, c, d , then

$$\frac{2abcd}{(ad+bc)(ac-bd)}$$

As positive integer. This is similar to the three sides were: $x=ab(c^2+d^2)$, $y=cd(a^2+b^2)$, $z=(ad+bc)(ac-bd)$, the high side of the triangle h_z and z is a positive integer $\frac{2abcd}{(ad+bc)(ac-bd)}$ bviously we can assume:

$$\gcd(a, b) = \gcd(c,d) = 1.$$

Note 2.2 : from Note 1 can be seen: a, b, c, d were not odd and gcd(a, b) =gcd (C, d) =1. Can be drawn from the above discussion: solving the Evans ratio, we can assume that $a=\gcd(a, c)\gcd(a, d)\, a_1$, $b=\gcd(b, c)\gcd(b,d)\, b_1$, $c=\gcd(a, c)\gcd(b, c)\, c_1$, $d=\gcd(a, d)\gcd(b,d)\, d_1$. When a_1, b_1, c_1, d_1 both were positive integers, and

$$n=\frac{2abcd}{(ad+bc)(ac-bd)}=\frac{\gcd(a,c)\gcd(b,c)\gcd(a,d)\gcd(b,d)a_1b_1c_1d_1}{(\gcd(a,d)^2a_1d_1+\gcd(b,c)^2b_1c_1)(\gcd(a,c)^2a_1c_1-\gcd(b,d)^2b_1d_1)}.$$

Easy to prove that

$\gcd(a_1b_1c_1d_1(\gcd(a,d)^2a_1d_1+\gcd(b,c)^2b_1c_1)(\gcd(a,c)^2a_1c_1-\gcd(b,d)^2b_1d_1))=1$. So that

$\frac{2abcd}{(ad+bc)(ac-bd)}$ is an integer when and only if there are both positive integer a_1, b_1, c_1, d_1, then

$$\frac{\gcd(a,c)\gcd(b,c)\gcd(a,d)\gcd(b,d)}{(\gcd(a,d)^2a_1d_1+\gcd(b,c)^2b_1c_1)(\gcd(a,c)^2a_1c_1-\gcd(b,d)^2b_1d_1)}$$

Is positive integer.

Lemma 2.2.[11] Set D is a positive integer and is not a perfect square, then the equation is

$$x^2 - Dy^2 = 1$$

(1)There are infinitely many solutions of x, y. Set $x_1^2-Dy_1^2=1, x_1>0, y_1>0,$ in all the x>0,y>0, make the smallest solution in the group of $x+y\sqrt{D}$ as((x_1, y_1) as (1) the basic solution), then all (1)solutions by

$$x+y\sqrt{D}=\pm(x_1+y_1\sqrt{D})^k$$

K is an arbitrary integer.

3 Two theorems and proofs that high and bottom ratio are integers

This section by using Pell equation x^2-Dy^2=1 constructed two new kinds of primitive Evans triangle, given the two kinds of Evans triangle shape of three sides of the form and the corresponding Evans.

Theorem 3.1.Evans ratio is 2m (M^2-2) (4m^4-8m^2 +1) (4m^4-8m^2 +3) Evans triangle, and three integer sides respectively x=D (a^2 +d^2),y=a^2 +D2d^2, z=D +1), and: $a=(m^2-1)\left[(m^2-1)^2+3m^2(m^2-2)\right]$, $d=m\left[3(m^2-1)^2+m^2(m^2-2)\right]$, $D=m^2-2$, $m\in Z$.

The proof is proved by **lemma 2.1** that the positive integer a=c, d|b, ac-bd=1 are from the **lemma 2.1**.

$$n=\frac{2abcd}{(ad+bc)(ac-bd)}=\frac{2a^2Dd^2}{a(d+Dd)}=\frac{2adD}{1+D}$$

besides D=b/d

Then the solution of the ac - bd = a^2 -Dd^2 = 1, and the solution of the Pell equation x2 - Dy2 = 1.

When $D=m^2-2(m\geq 3)$, the answer of x^2-(m^2-2)y^2=1 is x_1=m^2-1,y_1=m, then Evans Ratio is $n=\frac{2adD}{1+D}=2m(m^2-2)$,this the result of paper(6). By lemma 2.2 $x_k+y_k=(x_1+y_1\sqrt{D})^k$, then $x_2=2m^4-4m^2+1$, $y_2=2m(2m^2-1)$, to Evans ratio of $4m(m^2-1)(2m^4-4m^2+1)$,that is the result of paper(10) .

Because $x_3=(m^2-1)\left[(m^2-1)^2+3m^2(m^2-2)\right]$, $y_3=m\left[3(m^2-1)^2+m^2(m^2-2)\right]$, the Evans ratio as follow :

$n=\frac{2adD}{1+D}=2m(m^2-2)$ (4m^4-8m^2 +1) (4m^4-8m^2 +3),and three integer are

$$\begin{cases} x=ab(c^2+d^2), \\ y=cd(a^2+b^2), \\ z=(ad+bc). \end{cases}$$

To common factor ad

$$(1.8)\begin{cases} x=(m^2-2)[(m^2-1)^2[(m^2-1)^2+3m^2(m^2-2)]^2+m^2[3(m^2-1)^2+m^2(m^2-2)]^2], \\ y=[(m^2-1)[(m^2-1)^2+3m^2(m^2-2)]]^2+(m^2-2)^2[m[3(m^2-1)^2+m^2(m^2-2)]]^2, \\ z=m^2-1. \end{cases}$$

Theorem 3.2. Evans ratio is $\frac{m(m^2-1)(m^2-4)}{2}$ Etans triangle, three integer edge of $x=\frac{m^8-9m^6+27m^4-27m^2-4}{4}$, $y=\frac{m^8-9m^6+27m^4-31m^2+16}{4}$, $z=m^2-3$, while $m\in Z$, $2\nmid m.$ 。

The proof, when $D=m^2-4(m>3,2?m)$, Pell equation $x^2-(m^2-4)y^2=1$, initial solution $x_1=\frac{m^3-3m}{2}$, $y_1=\frac{m^2-1}{2}$, Evans ratio is $n=\frac{2adD}{1+D}=\frac{m(m^2-1)(m^2-4)}{2}$.

the three integers as follow:

$$\begin{cases} x=ab(c^2+d^2), \\ y=cd(a^2+b^2), \\ z=(ad+bc). \end{cases}$$

To common factor ad

$$\begin{cases} x=\frac{m^8-9m^6+27m^4-27m^2-4}{4}, \\ y=\frac{m^8-9m^6+27m^4-31m^2+16}{4}, \\ z=m^2-3. \end{cases}$$

Through the proof of the theorem we can know that this method can be used to calculate the (x_i, y_i) can be constructed from a number of groups of positive integers (a, b, c, d) can be obtained in a myriad of new Evans triangles.

4 The ratio of bottom and the high is an integer

In this section we discuss the case of a base with a high ratio of integers. Similar to **lemma 1**, we have the following conclusions.

Theorem 4.1. Set n is a positive integer, then n is the bottom of an integer edge triangle with a high ratio, when and only when the a,b,c,d, ac>bd, lead to

$$n=\frac{(ad+bc)(ac-bd)}{2abcd}.$$

Note 4.1: the above theorem shows that the solution to the bottom with a high integer ratio, equivalent to determine the positive integer a, b, c, d , then

$$\frac{(ad+bc)(ac-bd)}{2abcd}$$

As positive integer. This is similar to the three sides were: $x = ab\left(c^2 + d^2\right)$, $y = cd\left(a^2 + b^2\right)$, z= (ad+bc)(ac-bd) , bottom of the triangle on the edge of the high h_z ratio is a positive integer ,such as $\frac{(ad+bc)(ac-bd)}{2abcd}$. We could assume

gcd(a, b) = gcd(c, d) = 1.

Note 4.2: Could be seen from the **notes 4.1**:a, b, c, d, cannot just have a number of other odd , and numbers are gcd(a, b) = gcd(c, d) = 1.

It can be concluded from the above discussion: for the bottom and high ratios, we can assume that: $a = \gcd(a,c)\gcd(a,d)a_1$, $b = \gcd(b,c)\gcd(b,d)b_1$, $c = \gcd(a,c)\gcd(b,c)c_1$, $d = \gcd(a,d)\gcd(b,d)d_1$. This moment a_1, b_1, c_1,d_1, both are positive integers. and

$$n = \frac{(ad+bc)(ac-bd)}{2abcd} = \frac{(\gcd(a,d)^2 a_1 d_1 + \gcd(b,c)^2 b_1 c_1)(\gcd(a,c)^2 a_1 c_1 - \gcd(b,d)^2 b_1 d_1)}{\gcd(a,c)\gcd(b,c)\gcd(a,d)\gcd(b,d) a_1 b_1 c_1 d_1}.$$

Easy to prove that $\gcd\left(a_1b_1c_1d_1, \left(\gcd(a,d)^2 a_1d_1 + \gcd(b,c)^2 b_1c_1\right)\left(\gcd(a,c)^2 a_1, c_1 - \gcd(b,d)^2 b_1d_1\right)\right) = 1$. So we should make $\frac{(ad+bc)(ac-bd)}{2abcd}$ as integers, then $a_1=b_1=c_1=d_1=1$and

$$\frac{(\gcd(a,d)^2 + \gcd(b,c)^2)(\gcd(a,c)^2 - \gcd(b,d)^2)}{2\gcd(a,c)\gcd(b,c)\gcd(a,d)\gcd(b,d)}$$

is positive integer.

Note 4.3: can be seen from the above discussion, the bottom with a high integer ratio. If we assume that gcd (a, c) =s, gcd(a, d)= t, gcd (B, c) =u, gcd (b, d) =v, there are s, t, u, v both are coprime , and triangle similar length respectively.

$$x = s^2u^2 + t^2v^2,\ y = s^2t^2 + u^2v^2, z = (s^2 - v^2)(u^2 + t^2),\ h_z = 2stuv.$$

First of all, we consider the following equation:

$$\frac{(s^2 - v^2)(u^2 + t^2)}{2stuv} = k,$$

K is integer, s,t,u,v are both positive integer. So we could

$$2 \not| sv, \quad \gcd(2tu, u^2+t^2)|2, \quad \gcd(sv, s^2-v^2)=1.$$

Firstly, we consider when 2/u, t=1, we could $2u\left|s^2-v^2, sv\right|u^2+1$, then

$$\begin{cases} u^2+1=k_1sv, \\ s^2-v^2=2k_2u. \end{cases}$$

By the second equation of the above equation, we have

$$\begin{cases} s+v=2k_3u_1, \\ s-v=2k_4u_2, \ u=2u_1u_2, \ k_2=k_3k_4. \end{cases}$$

Then get the results: $s=k_3u_1+k_4u_2$, $v=k_3u_1-k_4u_2$, set it into $u^2+1=k_1sv$,

$$4u_1^2u_2^2+1=k_1k_3^2u_1^2-k_1k_4^2u_2^2.$$

As

$$(k_1k_3^2-4u_2^2)u_1^2=1+k_1k_4^2u_2^2.$$

Make $k_1=4u_2^2+1, k_3=1, k_4=2$, so $u_1=2u_2^2+1$. lead $u_2=m$, we should get it

$$u=2m(2m^2+1), \ s==2m^2+1-2m, \ v=2m^2+1-2m.$$

$$\begin{cases} x=4m^2(2m^2+1)^2(2m^2+1+2m)^2+(2m^2+1-2m)^2, \\ y=4m^2(2m^2+1)^2(2m^2+1-2m)^2+(2m^2+1+2m)^2, \\ z=4m(2m^2+1)(4m^2(2m^2+1)^2+1). \end{cases}$$

Here $k=4(4m^2+1)$. If we take $k_1=1$, $u_1=1$, $k_4=2m$, then there is

$$k_3^2-(4m^2+4)u_2^2=1.$$

With the relevant the conclusion of Pell, we take $u_2=m$, $k_3=2m^2+1$, $k=2m(2m^2+1)$. $U_2=2m(2m^2+1)$, $k_3=8m^4+8m^2+1$, then $k=2m(8m^4+8m^2+1)$.

Secondly, we consider another case of $2|u, u=2u_1$. At this point we have $2ut\left|s^2-v^2, sv\right|u^2+t^2$, then

$$\begin{cases} 4u_1^2+t^2=k_1sv, \\ s^2-v^2=4k_2u_1t. \end{cases}$$

We take the second equation

$$\begin{cases} s = k_3 u_1 + k_4 t, \\ v = k_3 u_1 - k_4 t, \quad k_2 = k_3 k_4. \end{cases}$$

Put into $4u_1^2 + t^2 = k_1 sv$

$$4u_1^2 + t^2 = k_1 k_3^2 u_1^2 - k_1 k_4^2 t^2.$$

As

$$(k_1 k_3^2 - 4) = (k_1 k_4^2 + 1) t^2.$$

If we take

$$\begin{cases} t^2 - k_1 k_3^2 = -4, \\ u_1^2 - k_1 k_4^2 = 1. \end{cases}$$

Then we could select $t = m, k_3 = 1, k_1 = m^2 - 4, 2 \nmid m, u_1 = \frac{m \ \ m^2 - m}{2}, k_4 = \frac{m \ \ m^2 - 1}{2}$

$$k == \frac{m(m^2-1)(m^2-4)}{2}.$$

From the discussion above, we can see that we can get a large class of different forms of Evans than the Pell equation, and we can get the conclusion that a large class of different forms of the bottom and the high ratio is an integer.

Reference:

1．Ronald J.Evans.Problem E2685（J）Amer.Math.Monthly，1977.（84）；820

2．Rinchad K.Guy.Unsolved Problem in number Theory（M）.New York：Spring-verlag，1991：104.

3．YoungMing Leo, XiaoChun Lee. About evans problem of a number of conclusion. Wuhan University of science and technology (natural science edition) 2000 (12): 421-423.

4．YoungMing Leo, XiaoChun Lee .About evans Triangle of a family new solution [j]. Air force radar Institute of technology, 2001, 15 (2): 63-64.

5．XinRong Gan, ZhaoDong Yao, building, about evans problem [j]. Jianghan University, 2001, 12 (2): 78-79.

6. Xin Bian, a class of the original evans Triangle [j] mathematics research, 2007: 52-53.

7. Wynn Lee, YanHong Wang， About a class of evans Triangle [j]. Mathematics communications, 2008 (17): 34-35.

8. Wynn Lee， For a class of the original evans Triangle inquiry [j] mathematics

research, 2010 (13): 31-32.
9. Xin Bian, Evans Triangle sufficient conditions and its application [j]. Teaching mathematics, 2010 (17): 16-18.
10.Xin Bian, ZhongMin Lee. Evans problem of a class of new solution [j]. Teaching mathematics, 2011 (2): 68-69.
11. Zhao Ke, SunKay. Talk about the indefinite equation. Harbin University Press, 2011. 15-17.