CONSERVATION RELATIONS FOR LOCAL THETA CORRESPONDENCE

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Abstract. We prove Kudla-Rallis's conjecture on first occurrences of orthogonal-symplectic dual pair correspondence, for a local field of characteristic zero.

1. Introduction and main results

Let k be a local field of characteristic zero. Fix a nontrivial unitary character $\psi : \mathrm{k} \to \mathbb{C}^\times$. We shall also fix a parity $\epsilon \in \mathbb{Z}/2\mathbb{Z}$ and a quadratic character $\chi : \mathrm{k}^\times \to \{\pm 1\}$.

Denote by $\mathcal{Q}(\epsilon, \chi)$ the set of isomorphism classes of non-degenerate quadratic spaces V over k such that

- $\dim V$ is finite and has parity ϵ, and
- the discriminant character χ_V of V equals χ.

Recall that the discriminant character χ_V is given by

$$\chi_V(x) := \left(x,\, (-1)^{\frac{m(m-1)}{2}} \det[\langle e_i, e_j \rangle_V]_{1\leq i,j\leq m}\right)_2, \quad x \in \mathrm{k}^\times,$$

where $m := \dim V$, $e_1, e_2, \cdots, e_m$ is a basis of V, $\langle\, ,\, \rangle_V$ is the symmetric bilinear form on V, and $(\, ,\,)_2$ is the Hilbert symbol for k. Also denote by $\mathcal{S}$ the set of isomorphism classes of finite dimensional symplectic spaces over k.

By abuse of notation, we do not distinguish an element of $\mathcal{Q}(\epsilon, \chi)$ with a quadratic space which represents it. Likewise for an element of $\mathcal{S}$ and a symplectic space which represents it. Throughout this article, V always refers to a quadratic space in $\mathcal{Q}(\epsilon, \chi)$ and W a symplectic space in $\mathcal{S}$.

Write

$$1 \to \{\pm 1\} \to \mathrm{Sp}_\epsilon(W) \to \mathrm{Sp}(W) \to 1 \tag{1}$$

for the unique topological central extension of the symplectic group $\mathrm{Sp}(W)$ by $\{\pm 1\}$ such that it splits if ϵ is even, or $W = 0$, or k is isomorphic to $\mathbb{C}$, and it does not split otherwise.

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Put

$$\mathbf{W} := V \otimes W,$$

to be viewed as a symplectic space under the form

$$\langle v \otimes w, v' \otimes w' \rangle_{\mathbf{W}} := \langle v, v' \rangle_V \, \langle w, w' \rangle_W,$$

where $\langle\, ,\, \rangle_V$ and $\langle\, ,\, \rangle_W$ are the non-degenerate symmetric form and the symplectic form on V and W, respectively. Denote by

$$\mathbf{H} := \mathbf{W} \times \mathrm{k}$$

the Heisenberg group associated to $\mathbf{W}$, whose multiplication is given by

$$(u, t)(u', t') := (u + u', t + t' + \langle u, u' \rangle_{\mathbf{W}}).$$

The group $\mathrm{Sp}(\mathbf{W})$ acts on $\mathbf{H}$ as automorphisms by

$$g \cdot (u, t) := (gu, t).$$

It induces an action of $\mathrm{O}(V) \times \mathrm{Sp}_\epsilon(W)$ on $\mathbf{H}$ via the obvious homomorphism

$$\mathrm{O}(V) \times \mathrm{Sp}_\epsilon(W) \to \mathrm{O}(V) \times \mathrm{Sp}(W) \to \mathrm{Sp}(\mathbf{W}).$$

This defines a semidirect product (the Jacobi group)

$$\mathbf{J}_{V,W} := (\mathrm{O}(V) \times \mathrm{Sp}_\epsilon(W)) \ltimes \mathbf{H}. \tag{2}$$

We are concerned with the smooth oscillator representation $\omega_{V,W}$ [Ho1, MVW] of the Jacobi group $\mathbf{J}_{V,W}$. Up to isomorphism, $\omega_{V,W}$ is the unique representation with the following properties: [Ho2, Part II], [MVW, Chapter 2]

- it is a smooth representation if k is non-archimedean, and a smooth Fréchet representation of moderate growth if k is archimedean;
- as a representation of $\mathbf{H}$, it is irreducible with central character ψ;
- for every Lagrangian subspace L of W, denote by $\lambda_{V,L}$ the unique (up to scalar multiplication) nonzero (continuous in the archimedean case) linear functional on $\omega_{V,W}$ which is invariant under $V \otimes L \subset \mathbf{H}$, then $\lambda_{V,L}$ is $\mathrm{O}(V)$-invariant;
- it is genuine as a representation of $\mathrm{Sp}_\epsilon(W)$, namely, the central element $-1 \in \mathrm{Sp}_\epsilon(W)$ acts through the scalar multiplication by $-1 \in \mathbb{C}$.

The reader is referred to [du, Definition 1.4.1] or [Sun, Section 2] for the notion of "smooth Fréchet representations of moderate growth" in the setting of Jacobi groups.

Denote by $\mathrm{Irr}(\mathrm{O}(V))$ the isomorphism classes of irreducible admissible smooth representations of the orthogonal group $\mathrm{O}(V)$ if k is non-archimedean, and the isomorphism classes of irreducible Casselman-Wallach representations of $\mathrm{O}(V)$ if k is archimedean. The reader may consult [Ca] and [Wal, Chapter 11] for details about Casselman-Wallach representations. Similarly, denote by $\mathrm{Irr}(\mathrm{Sp}_\epsilon(W))$ the isomorphism classes of irreducible admissible genuine smooth representations of $\mathrm{Sp}_\epsilon(W)$ if k is non-archimedean, and the isomorphism classes of irreducible genuine

Casselman-Wallach representations of $\mathrm{Sp}_\epsilon(W)$ if k is archimedean. Throughout this article, π denotes a representation in $\mathrm{Irr}(\mathrm{O}(V))$ and ρ denotes a representation in $\mathrm{Irr}(\mathrm{Sp}_\epsilon(W))$. We are interested in occurrences of π and ρ in the local theta correspondence [Ho3, MVW].

Before we state our main results, we recall two important facts: Kudla's persistence principal [Ku, Propositions 4.1 and 4.5] and non-vanishing of theta liftings in stable range ([Ku, Propositions 4.3 and 4.5], and [PP, Theorem 1] for the archimedean case).

We first consider the case of orthogonal groups. Kudla's persistence principle says that if $W_1, W_2 \in \mathcal{S}$ and $\dim W_1 \leq \dim W_2$, then

$$\mathrm{Hom}_{\mathrm{O}(V)}(\omega_{V,W_1}, \pi) \neq 0 \quad \text{implies} \quad \mathrm{Hom}_{\mathrm{O}(V)}(\omega_{V,W_2}, \pi) \neq 0.$$

Non-vanishing of theta liftings in stable range says that if $\dim W \geq 2 \dim V$, then

$$\mathrm{Hom}_{\mathrm{O}(V)}(\omega_{V,W}, \pi) \neq 0.$$

Define the first occurrence index

$$\mathrm{n}(\pi) := \min\{\frac{1}{2} \dim W \mid W \in \mathcal{S}, \mathrm{Hom}_{\mathrm{O}(V)}(\omega_{V,W}, \pi) \neq 0\}. \tag{3}$$

The conservation relation for orthogonal groups is the following

Theorem A. *For any $V \in \mathcal{Q}(\epsilon, \chi)$ and $\pi \in \mathrm{Irr}(\mathrm{O}(V))$, one has that*

$$\mathrm{n}(\pi) + \mathrm{n}(\pi \otimes \det) = \dim V,$$

where "$\det$*" stands for the determinant character of* $\mathrm{O}(V)$.

Remark: Theorem A was conjectured by Kudla and Rallis [KR3, Conjecture C]. In the non-archimedean case and for π irreducible cuspidal, Theorem A was proved in [Mi, Theorem 2].

Now we consider the case of symplectic groups. For any U in $\mathcal{Q}(\epsilon, \chi)$ (or $\mathcal{S}$), denote by U^- the space U equipped with the form scaled by -1. Two quadratic spaces $V_1, V_2 \in \mathcal{Q}(\epsilon, \chi)$ are said to be in the same Witt tower if the quadratic space $V_1 \oplus V_2^-$ splits. This defines an equivalence relation on $\mathcal{Q}(\epsilon, \chi)$. An equivalence class of this relation is called an (orthogonal) Witt tower.

Denote by $\mathcal{T}(\epsilon, \chi)$ the set of Witt towers in $\mathcal{Q}(\epsilon, \chi)$. By the classification of quadratic spaces over a local field, we know that

$$\sharp(\mathcal{T}(\epsilon, \chi)) = \begin{cases} 2, & \text{if k is non-archimedean;} \\ 1, & \text{if k is isomorphic to } \mathbb{C}; \\ \infty, & \text{if k} = \mathbb{R}. \end{cases} \tag{4}$$

Kudla's persistence principle says that for any given $T \in \mathcal{T}(\epsilon, \chi)$, if $V_1, V_2 \in T$ and $\dim V_1 \leq \dim V_2$, then

$$\mathrm{Hom}_{\mathrm{Sp}_\epsilon(W)}(\omega_{V_1,W}, \rho) \neq 0 \quad \text{implies} \quad \mathrm{Hom}_{\mathrm{Sp}_\epsilon(W)}(\omega_{V_2,W}, \rho) \neq 0.$$

Non-vanishing of stable range theta liftings says that if $V \in T$ and $\dim V \geq \min\{\dim V' \mid V' \in T\} + 2\dim W$, then

$$\mathrm{Hom}_{\mathrm{Sp}_\epsilon(W)}(\omega_{V,W}, \rho) \neq 0.$$

Define the first occurrence index

$$\mathrm{m}_T(\rho) := \min\{\dim V \mid V \in T,\ \mathrm{Hom}_{\mathrm{Sp}_\epsilon(W)}(\omega_{V,W}, \rho) \neq 0\}. \tag{5}$$

The conservation relation for non-archimedean symplectic groups is the following

Theorem B. *Assume that* k *is non-archimedean. For any* $W \in \mathcal{S}$ *and* $\rho \in \mathrm{Irr}(\mathrm{Sp}_\epsilon(W))$*, one has that*

$$\sum_{T \in \mathcal{T}(\epsilon,\chi)} \mathrm{m}_T(\rho) = 2\dim W + 4.$$

Remark: Theorem B was conjectured by Kudla and Rallis [KR3, Conjecture A]. They also proved the result for ρ irreducible cuspidal [KR3, Corollary 3].

The situation is more complicated in the case of real symplectic groups due to the abundance of real orthogonal Witt towers. We observe that if $T_1, T_2 \in \mathcal{T}(\epsilon,\chi)$ are two different Witt towers and $V_i \in T_i$ $(i = 1, 2)$, then $V_1 \oplus V_2^-$ has even dimension (from the parity assumption), trivial discriminant character (from the same discriminate character assumption), and does not split. Therefore we must have

$$\text{the split rank of } (V_1 \oplus V_2^-) \leq \frac{\dim V_1 + \dim V_2 - 4}{2}. \tag{6}$$

We say that T_1 and T_2 are adjacent if the equality holds in (6). When k is non-archimedean, the two Witt towers in $\mathcal{T}(\epsilon,\chi)$ are adjacent. When $\mathrm{k} = \mathbb{R}$, every Witt tower in $\mathcal{T}(\epsilon,\chi)$ has exactly two adjacent Witt towers (in $\mathcal{T}(\epsilon,\chi)$).

We put

$$\mathrm{m}(\rho) := \min\{\mathrm{m}_T(\rho) \mid T \in \mathcal{T}(\epsilon,\chi)\}. \tag{7}$$

We have the following conservation relation for real symplectic groups.

Theorem C. *Assume that* $\mathrm{k} = \mathbb{R}$*. For any* $W \in \mathcal{S}$ *and* $\rho \in \mathrm{Irr}(\mathrm{Sp}_\epsilon(W))$*, one has that*

$$\min\{\mathrm{m}_{T_1}(\rho) + \mathrm{m}_{T_2}(\rho) \mid T_1, T_2 \in \mathcal{T}(\epsilon,\chi), T_1 \neq T_2\} = 4n + 4,$$

where $2n := \dim W$*. In fact we have the following more precise assertions.*

- **(a):** *We have* $\mathrm{m}(\rho) \leq 2n + 2$*.*
- **(b):** *If* $\mathrm{m}(\rho) = 2n + 2$*, then*

  $$\sharp\{T \in \mathcal{T}(\epsilon,\chi) \mid \mathrm{m}_T(\rho) = 2n + 2\} = 2,$$

  and the two Witt towers in the above set are adjacent.
- **(c):** *If* $\mathrm{m}(\rho) \leq 2n + 1$*, then*
  - *there is a unique Witt tower* $T_\rho \in \mathcal{T}(\epsilon,\chi)$ *such that* $\mathrm{m}_{T_\rho}(\rho) = \mathrm{m}(\rho)$*;*

- *there exists a Witt tower* $T \in \mathcal{T}(\epsilon, \chi)$ *adjacent to* T_ρ *such that*

$$\mathrm{m}_T(\rho) + \mathrm{m}(\rho) = 4n + 4;$$

- *for all Witt towers* $T \in \mathcal{T}(\epsilon, \chi)$ *different from* T_ρ*, one has that*

$$\mathrm{m}_T(\rho) + \mathrm{m}(\rho) \geq 4n + 4,$$

and the inequality is strict if T *is not adjacent to* T_ρ*.*

Remarks: (a) A. Paul proved an analog of Kudla-Rallis's conjecture for unitary-unitary dual pair correspondence for $\mathrm{k} = \mathbb{R}$ [Pa, Conjecture 1.2], for a discrete series representation, or a representation irreducibly induced from a discrete series representation. Method of this article applies to general dual pairs, and in particular establishes the validity of [Pa, Conjecture 1.2] without any restrictions. (b) For complex symplectic groups, there is no conservation relation to formulate for the simple reason that there is only one Witt tower in $\mathcal{Q}(\epsilon, \chi)$.

To conclude this introduction, the authors would like to acknowledge the deep influence of the ideas of Kudla and Rallis [KR1, KR2, KR3] on this article. The proof of our results follows their approach closely. Our main contribution (Lemmas 3.2 and 5.4) is to pinpoint and to recognize the role of certain structure results about degenerate principal series representations, which fortunately can be read off from results in the existing literature. (Method of the current article, together with analogous results on degenerate principal series for other classical groups, will imply similar conservation relations for other dual pairs, which we will leave to the interested reader. The appropriate statements to be made are in [Mi], for example.) As pointed out by Kudla and Rallis [KR3], the conservation relations imply theta dichotomy phenomenon in the non-archimedean case ([KR3, Conjecture B]. Note that the latter was recently established by Zorn [Zo], and by Gan, Gross and Prasad [GGP]. Note also Harris, Kudla and Sweet [HKS] proved some important cases of theta dichotomy for unitary-unitary dual pair correspondence earlier. For $\mathrm{k} = \mathbb{R}$, the corresponding (though more complicated) result was established by Adams and Babasch [AB, Corollary 5.3], using Vogan's version of the Langlands classification. The conservation relations proved in this article thus yield a shorter proof for [AB, Corollary 5.3], as one of the by-products.

2. Doubling method

Recall that W is a symplectic space over k, of dimension $2n \geq 0$. We form the symplectic space $\mathbb{W} := W \oplus W^-$ and note that $\Delta := \{(w, w) \in W \oplus W^-\}$ is a Lagrangian subspace of $\mathbb{W}$. Denote by $\mathrm{P}(\Delta)$ the parabolic subgroup of $\mathrm{Sp}(\mathbb{W})$ stabilizing Δ. Write

$$1 \to \{\pm 1\} \to \mathrm{P}_\epsilon(\Delta) \to \mathrm{P}(\Delta) \to 1 \tag{8}$$

for the topological central extension of $\mathrm{P}(\Delta)$ which is induced by the extension (1) (for $\mathbb{W}$). Denote by $|\cdot|_{\mathrm{k}}$ the normalized absolute value on k. For ease of notation, we use $|\cdot|$ to denote the following positive character on $\mathrm{P}_\epsilon(\Delta)$:

$$\mathrm{P}_\epsilon(\Delta) \rightarrow \mathrm{P}(\Delta) \xrightarrow{\text{restriction on } \Delta} \mathrm{GL}(\Delta) \xrightarrow{\det} \mathrm{k}^\times \xrightarrow{|\cdot|_{\mathrm{k}}} \mathbb{R}_+^\times.$$

For every $V \in \mathcal{Q}(\epsilon, \chi)$, recall that ([Ho2, Theorem 5.1]) there is a unique (up to scalar multiplication) nonzero (continuous in the archimedean case) linear functional $\lambda_{V,\Delta}$ on $\omega_{V,\mathbb{W}}$ which is invariant under

$$V \otimes \Delta \subset \mathbf{J}_{V,\mathbb{W}} = (\mathrm{O}(V) \times \mathrm{Sp}_\epsilon(\mathbb{W})) \ltimes ((V \otimes \mathbb{W}) \times \mathrm{k}).$$

It is invariant under $\mathrm{O}(V)$ by the definition of $\omega_{V,\mathbb{W}}$ in the Introduction.

By using the Schrodinger model of $\omega_{V,\mathbb{W}}$ ([Ho2, Part II], [MVW, Chapter 2]), one immediately has the following

Lemma 2.1. *There is a unique character χ_Δ on $\mathrm{P}_\epsilon(\Delta)$, which depends on χ (and ψ), such that*

$$\lambda_{V,\Delta}(p \cdot v) = \chi_\Delta(p)|p|^{\frac{\dim V}{2}} \lambda_{V,\Delta}(v), \quad p \in \mathrm{P}_\epsilon(\Delta),\, v \in \omega_{V,\mathbb{W}}.$$

for every $V \in \mathcal{Q}(\epsilon, \chi)$.

For $s \in \mathbb{C}$, define the following normalized degenerate principal series representation of $\mathrm{Sp}_\epsilon(\mathbb{W})$:

$$\mathrm{I}_\chi(s) := \{f \in C^\infty(\mathrm{Sp}_\epsilon(\mathbb{W})) \mid f(px) = \chi_\Delta(p)|p|^{s+\frac{2n+1}{2}} f(x),\, p \in \mathrm{P}_\epsilon(\Delta),\, x \in \mathrm{Sp}_\epsilon(\mathbb{W})\}.$$

Under right translations, this is a smooth genuine representation of $\mathrm{Sp}_\epsilon(\mathbb{W})$.

The functional $\lambda_{V,\Delta}$ induces a $\mathrm{Sp}_\epsilon(\mathbb{W})$-intertwining map

$$\begin{array}{rccl} \Phi: & \omega_{V,\mathbb{W}} & \rightarrow & \mathrm{I}_\chi(\frac{\dim V}{2} - \frac{2n+1}{2}), \\ & v & \mapsto & (g \mapsto \lambda_{V,\Delta}(g \cdot v)). \end{array}$$

Denote by $\mathrm{R}_{\mathbb{W}}(V)$ the image of Φ (equipped with the quotient topology in the archimedean case):

$$\mathrm{R}_{\mathbb{W}}(V) := \Phi(\omega_{V,\mathbb{W}}) \subseteq \mathrm{I}_\chi(\frac{\dim V}{2} - \frac{2n+1}{2}). \tag{9}$$

Rallis and, Kudla and Rallis, prove that $\mathrm{R}_{\mathbb{W}}(V)$ is the maximal (Hausdorff in the archimedean case) quotient of $\omega_{V,\mathbb{W}}$ on which $\mathrm{O}(V)$ acts trivially. See [Ra] and [KR1].

Note that there is a unique continuous homomorphism

$$\mathrm{Sp}_\epsilon(W) \times \mathrm{Sp}_\epsilon(W^-) \rightarrow \mathrm{Sp}_\epsilon(\mathbb{W})$$

which makes the diagrams in

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \{\pm 1\}\times\{\pm 1\} & \longrightarrow & \mathrm{Sp}_\epsilon(W)\times\mathrm{Sp}_\epsilon(W^-) & \longrightarrow & \mathrm{Sp}(W)\times\mathrm{Sp}(W^-) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \{\pm 1\} & \longrightarrow & \mathrm{Sp}_\epsilon(\mathbb{W}) & \longrightarrow & \mathrm{Sp}(\mathbb{W}) & \longrightarrow & 1 \end{array}$$

commutative, where the first vertical arrow is the multiplication map. Therefore every representation of $\mathrm{Sp}_\epsilon(\mathbb{W})$ is a representation of $\mathrm{Sp}_\epsilon(W)\times\mathrm{Sp}_\epsilon(W^-)$ through the restriction.

Let $\rho\in\mathrm{Irr}(\mathrm{Sp}_\epsilon(W))$ be as in the Introduction. Identify $\mathrm{Sp}_\epsilon(W^-)$ with $\mathrm{Sp}_\epsilon(W)$ in the obvious way and write $\rho^\vee\in\mathrm{Irr}(\mathrm{Sp}_\epsilon(W^-))$ for the contragredient of ρ.

The following criterion for non-vanishing of theta lifting is, by now, quite standard [Ho1, Ra].

Lemma 2.2. *For any $V\in\mathcal{Q}(\epsilon,\chi)$, we have*

$$\mathrm{Hom}_{\mathrm{Sp}_\epsilon(W)}(\omega_{V,W},\rho)\neq 0$$

if and only if

$$\mathrm{Hom}_{\mathrm{Sp}_\epsilon(W)\times\mathrm{Sp}_\epsilon(W^-)}(\mathrm{R}_{\mathbb{W}}(V),\rho\widehat{\otimes}\rho^\vee)\neq 0.$$

Here and henceforth, "$\widehat{\otimes}$" stands for the completed projective tensor product if k is archimedean, and the algebraic tensor product if k is non-archimedean.

On the other hand, the theory of local Zeta integrals [PSR, LR] implies

Lemma 2.3. *For any $s\in\mathbb{C}$, we have*

$$\mathrm{Hom}_{\mathrm{Sp}_\epsilon(W)\times\mathrm{Sp}_\epsilon(W^-)}(\mathrm{I}_\chi(s),\rho\widehat{\otimes}\rho^\vee)\neq 0.$$

3. Two results on degenerate principal series representations

Lemma 3.1. *Let $m\geq 2n+1$ be an integer with parity ϵ, then*

$$\sum_{V\in\mathcal{Q}(\epsilon,\chi),\,\dim V=m}\mathrm{R}_{\mathbb{W}}(V)=\mathrm{I}_\chi(\frac{m}{2}-\frac{2n+1}{2}).$$

Consequently for any $\rho\in\mathrm{Irr}(\mathrm{Sp}_\epsilon(W))$, we have

$$\mathrm{m}(\rho)\leq\begin{cases}2n+1, & \epsilon=1,\\ 2n+2, & \epsilon=0.\end{cases}$$

Proof. The first assertion is in [KR2] (non-archimedean) and [LZ1, LZ2] (archimedean). The rest follows immediately from the first assertion and Lemma 2.2. □

For symplectic groups, the key observation of this article is the following lemma, which can be read off from [KR2, Introduction] (non-archimedean) and [LZ1, Section 4] ($\mathrm{k}=\mathbb{R}$).

Lemma 3.2. *Assume that* k *is not isomorphic to* $\mathbb{C}$. *Let* $V_1 \in \mathcal{Q}(\epsilon, \chi)$ *with* $m_1 := \dim V_1 \geq 2n+1$. *Then as* $\mathrm{Sp}_\epsilon(\mathbb{W})$*-representations,*

$$\frac{\mathrm{I}_\chi(\frac{m_1}{2} - \frac{2n+1}{2})}{\sum_{V \in \mathcal{Q}(\epsilon,\chi),\, \dim V = m_1,\, V \ncong V_1} \mathrm{R}_{\mathbb{W}}(V)} \cong \begin{cases} \mathrm{R}_{\mathbb{W}}(V_1'), & \text{if there exists a quadratic space } V_1' \text{ of dimension } 4n+2-m_1 \\ & \text{which belongs to the same Witt tower as } V_1; \\ 0, & \text{otherwise.} \end{cases}$$

4. Proof of conservation relation for symplectic groups

We start with the following result of Kudla and Rallis [KR3, Lemma 4.2] (non-archimedean) and of Loke [LL, Theorem 1.2.1] ($\mathrm{k} = \mathbb{R}$). Recall that a quadratic space $V \in \mathcal{Q}(\epsilon, \chi)$ is called quasi-split if its split rank $\geq \frac{\dim V - 2}{2}$.

Lemma 4.1. *Assume that* ϵ *is even. If* $V \in \mathcal{Q}(\epsilon, \chi)$ *is not quasi-split, then*

$$\mathrm{Hom}_{\mathrm{Sp}_\epsilon(W)}(\omega_{V,W}, \mathbb{C}) \neq 0$$

implies that V *has split rank* $\geq 2n$, *in particular* $\dim V \geq 4n+4$. *Here* $\mathbb{C}$ *stands for the unique one-dimensional genuine representation of* $\mathrm{Sp}_\epsilon(W)$.

The following result is also known, at least in the non-archimedean case ([KR3, Theorem 3.8]). We include a proof for the sake of completeness.

Lemma 4.2. *Let* $T_1, T_2 \in \mathcal{T}(\epsilon, \chi)$ *be two different Witt towers. Then*

$$\mathrm{m}_{T_1}(\rho) + \mathrm{m}_{T_2}(\rho) \geq 4n+4.$$

Proof. For $i = 1, 2$, let $V_i \in \mathcal{Q}(\epsilon, \chi)$ be such that $m_{T_i}(\rho) = \dim V_i$ and

$$\mathrm{Hom}_{\mathrm{Sp}_\epsilon(W)}(\omega_{V_i,W}, \rho) \neq 0. \tag{10}$$

Then $V_1 \oplus V_2^-$ has even dimension, trivial discriminant character, and does not split. By (6), it is not quasi-split.

Recall [MVW] that (10) for $i = 2$ is equivalent to

$$\mathrm{Hom}_{\mathrm{Sp}_\epsilon(W)}(\omega_{V_2^-,W}, \rho^\vee) \neq 0. \tag{11}$$

Combining (10) for $i = 1$ and (11), we get

$$\mathrm{Hom}_{\mathrm{Sp}_{\epsilon_0}(W)}(\omega_{V_1 \oplus V_2^-,W}, \mathbb{C}) \neq 0.$$

Here $\epsilon_0 := 0 \in \mathbb{Z}/2\mathbb{Z}$. Since $V_1 \oplus V_2^-$ is not quasi-split, we conclude from Lemma 4.1 that $\dim V_1 + \dim V_2 \geq 4n+4$. The result follows. □

Lemma 4.3. *Assume that* k *is not isomorphic to* $\mathbb{C}$. *Then there are two different* $T_1, T_2 \in \mathcal{T}(\epsilon, \chi)$ *such that*

$$\mathrm{m}_{T_1}(\rho) + \mathrm{m}_{T_2}(\rho) \leq 4n+4.$$

Proof. Pick a quadratic space $V_0 \in \mathcal{Q}(\epsilon, \chi)$ so that

$$\dim V_0 = \mathrm{m}(\rho) \quad (\leq 2n+2) \quad \text{and} \quad \mathrm{Hom}_{\mathrm{Sp}_\epsilon(W)}(\omega_{V_0,W}, \rho) \neq 0.$$

From Lemma 2.3, we may pick a nonzero element μ in

$$\mathrm{Hom}_{\mathrm{Sp}_\epsilon(W)\times \mathrm{Sp}_\epsilon(W^-)}(\mathrm{I}_\chi(\frac{4n+4-\mathrm{m}(\rho)}{2} - \frac{2n+1}{2}), \rho\widehat{\otimes}\rho^\vee).$$

Denote by V_1 the quadratic space of dimension $4n+4-\mathrm{m}(\rho)$ which belongs to the same Witt tower as V_0. It suffices to show that there is a quadratic space $V \in \mathcal{Q}(\epsilon, \chi)$ such that $\dim V = \dim V_1$, $V \not\cong V_1$ and μ does not vanish on $\mathrm{R}_{\mathbb{W}}(V)$. Suppose this is not the case, then μ factors to a nonzero linear map on

$$\frac{\mathrm{I}_\chi(\frac{4n+4-\mathrm{m}(\rho)}{2} - \frac{2n+1}{2})}{\sum_{V\in\mathcal{Q}(\epsilon,\chi),\, \dim V = 4n+4-\mathrm{m}(\rho),\, V\not\cong V_1} \mathrm{R}_{\mathbb{W}}(V)}.$$

This is impossible by Lemma 3.2 and the minimality in the definition of $\mathrm{m}(\rho)$. □

Lemma 4.4. *If $T_1, T_2 \in \mathcal{T}(\epsilon, \chi)$ are two different Witt towers so that*

$$\mathrm{m}_{T_1}(\rho) + \mathrm{m}_{T_2}(\rho) = 4n+4,$$

then T_1 and T_2 are adjacent.

Proof. Let $V_1 \in T_1$ and $V_2 \in T_2$ be quadratic spaces such that

$$\dim V_1 + \dim V_2 = 4n+4,$$

and

$$\mathrm{Hom}_{\mathrm{Sp}_\epsilon(W)}(\omega_{V_i,W}, \rho) \neq 0, \quad i = 1, 2.$$

As in the proof of Lemma 4.2, the quadratic space $V_1 \oplus V_2^-$ must have split rank $\geq 2n$. Together with $\dim V_1 + \dim V_2 = 4n+4$, this implies that T_1 and T_2 are adjacent. □

Theorem B and Theorem C now follow by combining Lemmas 3.1, 4.2, 4.3 and 4.4.

5. The case of orthogonal groups

The proof of conservation relation for orthogonal groups is similar to that for symplectic groups, though technically less complicated. We will be contented to sketch a proof in this section.

Let $V \in \mathcal{Q}(\epsilon, \chi)$ be a quadratic space over k of dimension $m \geq 0$. As in the symplectic case, we form the quadratic space $\mathbb{V} := V \oplus V^-$ and note that $\nabla := \{(v, v) \in V \oplus V^-\}$ is a maximal isotropic subspace of $\mathbb{V}$. Denote by $\mathrm{P}(\nabla)$ the parabolic subgroup of $\mathrm{O}(\mathbb{V})$ stabilizing ∇. Again, we simply use $|\cdot|$ to denote the following positive character on $\mathrm{P}(\nabla)$:

$$\mathrm{P}(\nabla) \xrightarrow{\text{restriction on } \nabla} \mathrm{GL}(\nabla) \xrightarrow{\det} \mathrm{k}^\times \xrightarrow{|\cdot|_{\mathrm{k}}} \mathbb{R}_+^\times.$$

Denote by $\lambda_{\nabla,W}$ the unique (up to scalar multiplication) nonzero (continuous in the archimedean case) linear functional on $\omega_{\mathbb{V},W}$ which is invariant under

$$\nabla\otimes W\subset \mathbf{J}_{\mathbb{V},W}=(\mathrm{O}(\mathbb{V})\times \mathrm{Sp}_{\epsilon_0}(W))\ltimes((\mathbb{V}\otimes W)\times \mathrm{k}).$$

Here $\epsilon_0:=0\in\mathbb{Z}/2\mathbb{Z}$.

By using the Schrodinger model of $\omega_{\mathbb{V},W}$ ([Ho2, Part II], [MVW, Chapter 2]), one immediately has the following

Lemma 5.1. *The functional $\lambda_{\nabla,W}$ transforms through the unique genuine character under the action of $\mathrm{Sp}_{\epsilon_0}(W)$, and satisfies*

$$\lambda_{\nabla,W}(p\cdot v)=|p|^{\frac{\dim W}{2}}\lambda_{\nabla,W}(v),\quad p\in \mathrm{P}(\nabla),\ v\in\omega_{\mathbb{V},W}.$$

For $s\in\mathbb{C}$, define the normalized degenerate principal series representation of $\mathrm{O}(\mathbb{V})$:

$$\mathrm{J}(s):=\{f\in C^\infty(\mathrm{O}(\mathbb{V}))\mid f(px)=|p|^{s+\frac{m-1}{2}}f(x),\ p\in\mathrm{P}(\nabla),\ x\in\mathrm{O}(\mathbb{V})\}.$$

Under right translations, this is a smooth representation of $\mathrm{O}(\mathbb{V})$.

The functional $\lambda_{\nabla,W}$ induces a $\mathrm{O}(\mathbb{V})$-intertwining map

$$\begin{array}{rccl}\Psi: & \omega_{\mathbb{V},W} & \to & \mathrm{J}(\frac{\dim W}{2}-\frac{m-1}{2}),\\ & v & \mapsto & (g\mapsto \lambda_{\nabla,W}(g.v)).\end{array}$$

Denote by $\mathrm{R}_{\mathbb{V}}(W)$ the image of Ψ (equipped with the quotient topology in the archimedean case):

$$\mathrm{R}_{\mathbb{V}}(W)=:\Psi(\omega_{\mathbb{V},W})\subseteq \mathrm{J}(\frac{\dim W}{2}-\frac{m-1}{2}). \tag{12}$$

Rallis [Ra] proves that $\mathrm{R}_{\mathbb{V}}(W)$ is the maximal (Hausdorff in the archimedean case) quotient of $\omega_{\mathbb{V},W}$ on which $\mathrm{Sp}_{\epsilon_0}(W)$ acts through the genuine character. See [Zhu] for the archimedean case.

Again we have the following criterion for non-vanishing of theta lifting [Ho1, Ra].

Lemma 5.2. *For any $W\in\mathcal{S}$, we have*

$$\mathrm{Hom}_{\mathrm{O}(V)}(\omega_{V,W},\pi)\neq 0$$

if and only if

$$\mathrm{Hom}_{\mathrm{O}(V)\times\mathrm{O}(V^-)}(\mathrm{R}_{\mathbb{V}}(W),\pi\widehat{\otimes}\pi^\vee)\neq 0.$$

Again the theory of local Zeta integrals [PSR, LR] implies that

Lemma 5.3. *For any $s\in\mathbb{C}$, we have*

$$\mathrm{Hom}_{\mathrm{O}(V)\times\mathrm{O}(V^-)}(\mathrm{J}(s),\pi\widehat{\otimes}\pi^\vee)\neq 0.$$

From the non-vanishing of stable range theta liftings, we clearly have

$$0 \leq \mathrm{n}(\pi),\ \mathrm{n}(\pi \otimes \det) \leq m.$$

We also recall the well-known fact that the first occurrence index of the determinant character of $\mathrm{O}(V)$ is m. See for example, [Ra, Appendix] and [PP, Appendix C]. Similar to the proof of Lemma 4.2, this implies that

$$\mathrm{n}(\pi) + \mathrm{n}(\pi \otimes \det) \geq m.$$

For orthogonal groups, the key observation of this article is the following

Lemma 5.4. *Assume that* $\dim W \geq m-1$*, then as* $\mathrm{O}(\mathbb{V})$*-representations,*

$$\begin{aligned} & \mathrm{J}(\frac{\dim W}{2} - \frac{m-1}{2}) / \mathrm{R}_{\mathbb{V}}(W) \\ \cong & \begin{cases} \mathrm{R}_{\mathbb{V}}(W') \otimes \det_{\mathbb{V}}, & \text{if there exists a symplectic space } W' \\ & \text{of dimension } 2m-2-\dim W; \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Here "$\det_{\mathbb{V}}$*" stands for the determinant character of* $\mathrm{O}(\mathbb{V})$*.*

Of course the condition that there exists a symplectic space W' of dimension $2m-2-\dim W$ is simply $\dim W \leq 2m-2$. We phrase it in this way with the sole purpose that the statements for orthogonal and symplectic groups will look parallel.

Proof. The assertion is clear from the results of Yamana [Ya, Corollary 8.8] (non-archimedean), Lee [LL, Appendix] ($\mathrm{k} = \mathbb{R}$), and Lee and Zhu [LZ2, Theorem 1] ($\mathrm{k} = \mathbb{C}$). □

We are now ready to prove Theorem A. The case of $m = 0$ is trivial. So assume that $m \geq 1$. Without loss of generality, assume that $\mathrm{n}(\pi) \geq \mathrm{n}(\pi \otimes \det)$. If $\mathrm{n}(\pi) \leq m/2$, then

$$\mathrm{n}(\pi) + \mathrm{n}(\pi \otimes \det) \leq m$$

and we are done. So assume that $\mathrm{n}(\pi) \geq (m+1)/2$ and let $W_0 \in \mathcal{S}$ be a symplectic space of dimension $2\,\mathrm{n}(\pi) - 2 \geq m-1 \geq 0$. From Lemma 5.3, we may pick a nonzero element μ in

$$\mathrm{Hom}_{\mathrm{O}(V)\times \mathrm{O}(V^-)}(\mathrm{J}(\frac{\dim W_0}{2} - \frac{m-1}{2}), \pi \widehat{\otimes} \pi^\vee).$$

Note that μ vanishes on $\mathrm{R}_{\mathbb{V}}(W_0)$, by Lemma 5.2 and the minimality in the definition of $\mathrm{n}(\pi)$. Thus it follows from Lemma 5.4 that μ factors to a nonzero element of

$$\begin{aligned} & \mathrm{Hom}_{\mathrm{O}(V)\times \mathrm{O}(V^-)}(\mathrm{R}_{\mathbb{V}}(W_0') \otimes \det_{\mathbb{V}}, \pi \widehat{\otimes} \pi^\vee) \\ = \ & \mathrm{Hom}_{\mathrm{O}(V)\times \mathrm{O}(V^-)}(\mathrm{R}_{\mathbb{V}}(W_0'), (\pi \otimes \det) \widehat{\otimes} (\pi \otimes \det)^\vee). \end{aligned}$$

Here $W_0' \in \mathcal{S}$ is the symplectic space of dimension $2m-2-\dim W_0 \geq 0$. Therefore

$$\mathrm{n}(\pi \otimes \det) \leq m - 1 - \frac{\dim W_0}{2} = m - \mathrm{n}(\pi)$$

and we conclude the proof.

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