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RIGIDITY OF POLYHEDRAL SURFACES, I

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Abstract

We study the rigidity of polyhedral surfaces and the moduli space of polyhedral surfaces using variational principles. Curvature-like quantities for polyhedral surfaces are introduced and are shown to determine the polyhedral metric up to isometry. The action functionals in the variational approaches are derived from the cosine law. They can be considered as 2-dimensional counterparts of the Schlaefli formula.

1. Introduction

We use variational principles to study geometry of polyhedral surfaces in a series of three papers [**19**], [**29**], of which this is the first. Several rigidity and infinitesimal rigidity results are established. As one consequence, for each real number h, we produce a natural parameterization ψ_h of the Teichmüller space of an ideal triangulated surface with boundary. The images of the Teichmüller space under these parameterizations are shown to be convex polytopes by our work (for $h \geq 0$) and the work of Ren Guo [**16**] (for $h < 0$).

A key identity for variational framework for polyhedral metrics in dimension greater than 2 is the Schlaefli formula (see for instance [**37**]). In dimension 3, the Schlaefli formula expresses the volume Vol, the dihedral angles θ_i, and the edge lengths l_i of a tetrahedron in the simple form

$$\frac{\partial Vol}{\partial \theta_i} = \frac{\lambda}{2} l_i \quad \text{and} \quad \frac{\partial \theta_i}{\partial l_j} = \frac{\partial \theta_j}{\partial l_i}$$

where $\lambda = \pm 1$ is the curvature of the ambient hyperbolic space $\mathbf{H}^3$ or the 3-sphere $\mathbf{S}^3$. In 1991, Colin de Verdière [**11**] found a 2-dimensional counterpart of the Schlaefli formula and used it to prove Thurston's circle packing theorem. Our study finds all 2-dimensional counterparts of the Schlaefli formula. They consist of two one-parameter families. All rigidity results in the paper are consequences of the variational principles associated to these 2-dimensional Schlaefli type identities.

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1.1. Polyhedral surfaces, their curvatures, and the main results. Recall that a *closed triangulated surface* $(S, \mathcal{T})$ is the quotient of a finite disjoint union of Euclidean triangles by identifying all edges of triangles in pairs by homeomorphisms. The simplices in the triangulation $\mathcal{T}$ are the quotients of vertices, edges, and triangles in the disjoint union. We use V, E, T to denote the sets of all vertices, edges, and triangles in $\mathcal{T}$. Let $\mathbf{H}^n$, $\mathbf{E}^n$, and $\mathbf{S}^n$ be the n-dimensional space of hyperbolic, Euclidean, and spherical geometries and $\mathbf{R}_{>\mathbf{0}}$ be the set of all positive real numbers.

Definition 1.1. (Polyhedral surfaces) Given a triangulated surface $(S, \mathcal{T})$ and $K^2 = \mathbf{H^2}, \mathbf{S^2}$, or $\mathbf{E^2}$, a K^2*-polyhedral metric* on $(S, \mathcal{T})$ is a map $l : E \to \mathbf{R}_{>0}$ so that if e_1, e_2, e_2 are edges of a triangle, then $l(e_1) + l(e_2) > l(e_3)$ and if $K^2 = \mathbf{S^2}$, $l(e_1) + l(e_2) + l(e_3) < 2\pi$. A *Euclidean (or spherical or hyperbolic) polyhedral surface* is the triple $(S, \mathcal{T}, l)$ where l is an $\mathbf{E^2}$ (or $\mathbf{S^2}$, or $\mathbf{H^2}$) polyhedral metric.

Intuitively, a polyhedral surface is an isometric gluing of geometric triangles along pairs of edges. The boundary of a generic compact convex polytope in the 3-dimensional space $\mathbf{E}^3$, $\mathbf{S}^3$, or $\mathbf{H}^3$ is a polyhedral surface. We emphasize that the triangulation $\mathcal{T}$ is an intrinsic part of a polyhedral surface in our study. Two polyhedral surfaces $(S, \mathcal{T}, l)$ and $(S', \mathcal{T}', l')$ are *triangulation preserving isometric* if there is an isomorphism h between the triangulations $\mathcal{T}$ and $\mathcal{T}'$ so that $l = l' \circ h$. The *(classical) discrete curvature* $k_0 : V \to \mathbf{R}$ of a polyhedral surface $(S, \mathcal{T}, l)$ is the function assigning each vertex 2π less the sum of the inner angles of triangles at the vertex.

A basic problem on polyhedral geometry is to understand the relationship between the metric and its curvature. For a polyhedral surface, the metric is the edge lengths, and discrete curvature comes from the inner angles of triangles. The metric-curvature relation is governed by the cosine law for triangles. Using inner angles, we introduce three families of curvature-like quantities and study the relationships between these curvature-like quantities and the metrics.

Definition 1.2. (Curvatures) Let K^2 be $\mathbf{E}^2$, or $\mathbf{S}^2$ or $\mathbf{H}^2$, and $h \in \mathbf{R}$. Given a K^2 polyhedral metric l on $(S, \mathcal{T})$, the ϕ_h *curvature* of the polyhedral metric l is the function $\phi_h : E \to \mathbf{R}$ sending an edge e to:

$$\phi_h(e) = \int_a^{\pi/2} \sin^h(t)dt + \int_{a'}^{\pi/2} \sin^h(t)dt \tag{1.1}$$

where a, a' are the inner angles facing the edge e. See figure 1.

The ψ_h *curvature* of the metric l is the function $\psi_h : E \to \mathbf{R}$ sending an edge e to

$$\psi_h(e) = \int_0^{\frac{b+c-a}{2}} \cos^h(t)dt + \int_0^{\frac{b'+c'-a'}{2}} \cos^h(t)dt \tag{1.2}$$

where b, b', c, c' are inner angles adjacent to the edge e and a, a' are the angles facing the edge e. See figure 1.

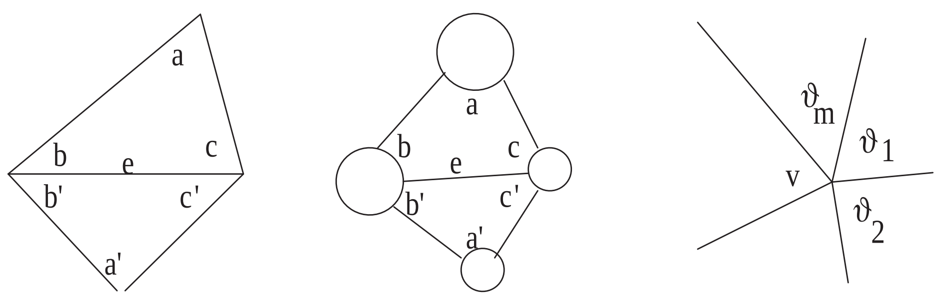

Figure 1

The *h-th discrete curvature* k_h of the metric l is the function $k_h : V \to \mathbf{R}$ sending a vertex v to

$$k_h(v) = (2 - \frac{m}{2})\pi - \sum_{i=1}^{m} \int_{\pi/2}^{\theta_i} \tan^h(t/2)dt \tag{1.3}$$

where the vertex v has degree m and $\theta_1, \ldots, \theta_m$ are all inner angles at v. See figure 1.

REMARK 1.3. The curvatures ϕ_0 and ψ_0 were first introduced by I. Rivin [**36**] and G. Leibon [**25**] respectively. The geometric meaning of them is related to the dihedral angle along edges of a hyperbolic polyhedron associated to the polyhedral surface. In particular, a Euclidean polyhedral surface is Delaunay if and only if $\phi_0(e) \geq 0$ for all $e \in E$, and a hyperbolic polyhedral surface is Delaunay if and only if $\psi_0(e) \geq 0$ for all e.

REMARK 1.4. The positivity of the curvatures ϕ_h and ψ_h is independent of h. To be more precise, due to $(x+y)(\int_0^x \cos^h(t)dt + \int_0^y \cos^h(t)dt) \geq 0$ for $x, y \in [-\pi/2, \pi/2]$, we have $\psi_h(e) \geq 0$ (or $\phi_h(e) \geq 0$) if and only if $\psi_0(e) \geq 0$ (or $\phi_h(e) \geq 0$). Thus the geometric meaning of positive ψ_h curvature is the same Delaunay condition (i.e., $\psi_0 > 0$) for polyhedral metrics.

REMARK 1.5. The curvature $\phi_{-2}(e) = \cot(a) + \cot(a')$ has appeared in the finite element method approximation of the smooth Beltrami-Laplace operator. It is called the discrete cotangent Laplace operator. See for instance Pinkall and Polthier's work [**35**].

Remark 1.6. The 0-th discrete curvature $k_0 = 2\pi - \sum_{i=1}^m \theta_i$ is the (classical) discrete curvature.

Recall that a *circle packing metric* on $(S, \mathcal{T})$ is a polyhedral metric $l : E \to \mathbf{R}_{>0}$ so that there is a map, called the radius assignment, $r : V \to \mathbf{R}_{>0}$ with $l(vv') = r(v) + r(v')$ whenever the edge vv' has end points v and v'.

One of the main results in the series of the papers is:

Theorem 1.7. *Let $h \in \mathbf{R}$ and $(S, \mathcal{T})$ be a closed triangulated surface.*

(a) *A Euclidean circle packing metric on $(S, \mathcal{T})$ is determined up to isometry and scaling by the k_h discrete curvature.*
(b) *A hyperbolic circle packing metric on $(S, \mathcal{T})$ is determined up to isometry by the k_h discrete curvature.*
(c) *A Euclidean polyhedral metric on $(S, \mathcal{T})$ is determined up to isometry and scaling by the ϕ_h curvature.*
(d) *A spherical polyhedral metric on $(S, \mathcal{T})$ is determined up to isometry by the ϕ_h curvature.*
(e) *A hyperbolic polyhedral surface metric on $(S, \mathcal{T})$ is determined up to isometry by the ψ_h curvature.*

Furthermore, the corresponding infinitesimal rigidity results hold.

We remark that Andreev [**1**] and Thurston [**43**] first proved Theorem 1.7(a) for $h = 0$. Theorem 1.7(b) for $h = 0$ was first proved by Thurston [**43**]. Theorem 1.7(c) for $h = 0$ was first proved by Rivin [**36**] and Theorem 1.7(d) for $h = 0$ was first proved by Leibon [**25**]. Theorem 1.7(a),(d) for $h = -2$ shows that the discrete Laplace operator determines the Euclidean and spherical polyhedral metrics.

In this paper, we will prove Theorem 1.7(a),(b) and the corresponding infinitesimal rigidity results (see Theorem 4.1). Theorem 1.7(c),(d),(e) is proved in [**29**] using the machinery built in this paper.

A counterpart of Theorem 1.7(e) for hyperbolic metrics with totally geodesic boundary on an ideal triangulated compact surface is the following. Recall that an *ideal triangulated compact surface* with boundary $(S, \mathcal{T})$ is obtained by removing a small open regular neighborhood of the vertices of a triangulated closed surface. The *edges* of an ideal triangulation $\mathcal{T}$ correspond bijectively to the edges of the triangulation of the closed surface. Given a hyperbolic metric l with geodesic boundary on an ideal triangulated surface $(S, \mathcal{T})$, the triangulation $\mathcal{T}$ is isotopic to a unique geometric ideal triangulation $\mathcal{T}^*$ so that all its edges are geodesics orthogonal to the boundary. The edges in $\mathcal{T}^*$ decompose the surface into hyperbolic right-angled hexagons. The ψ_h *curvature* of the hyperbolic metric l is defined to be the map $\Psi_h : \{$ all edges in $\mathcal{T}\} \to \mathbf{R}$

sending each edge e to

$$\psi_h(e) = \int_0^{\frac{b+c-a}{2}} \cosh^h(t)dt + \int_0^{\frac{b'+c'-a'}{2}} \cosh^h(t)dt \tag{1.4}$$

where a, a' are lengths of arcs in the boundary (in the ideal triangulation $\mathcal{T}^*$) facing the edge and b, b', c, c', are the lengths of arcs in the boundary adjacent to the edge so that a, b, c lie in a hexagon. See figure 1.

Theorem 1.8. *A hyperbolic metric with totally geodesic boundary on an ideal triangulated compact surface is determined by its ψ_h-curvature. Furthermore, if $h \geq 0$, then the set of all ψ_h curvatures on a fixed ideal triangulated surface is an explicit open convex polytope P_h in a Euclidean space so that $P_h = P_0$.*

The case when $h < 0$ has been recently established by Ren Guo [**16**]. He proved that:

Theorem 1.9. *Under the same assumption as in Theorem 1.8, if $h < 0$, the set of all ψ_h curvatures on a fixed ideal triangulated surface is an explicit bounded open convex polytope P_h in a Euclidean space. Furthermore, if $h < h'$, then $P_h \subset P_{h'}$.*

Theorem 1.7 shows that polyhedral metrics are determined by their curvatures. It is natural to investigate the spaces of all curvatures on a given triangulated surface $(S, \mathcal{T})$. A major portion of the paper (§5, §6, §7) studies this problem. The important work of Thurston, Rivin, and Leibon shows that the spaces of k_0, ϕ_0, and ψ_0 curvatures of Delaunay polyhedral metrics and circle packing metrics on $(S, \mathcal{T})$ are convex polytopes. Our results in §6 and §7 (Theorems 6.1, 6.4, 6.7, 7.1) generalize their work to all polyhedral metrics and to curvatures ϕ_h, ψ_h, and k_h for $h \leq -1$ and $h = 0$.

A summary of main results in the subsequent papers [**19**] and [**29**] is as follows. In [**29**], we prove the global rigidity part of Theorem 1.7 and investigate the inversive circle packings introduced by P. L. Bowers and K. Stephenson. We prove a rigidity conjecture of Bowers-Stephenson that the k_0 curvature determines Euclidean and hyperbolic inversive distance circle packing metrics. The case of spherical inversive distance circle packing remains open. Our proof uses a variational principle established by Ren Guo [**18**] for inversive distance circle packing which is valid for Euclidean, spherical, and hyperbolic geometry.

In [**19**], we study various generalized cosine laws for nontriangular regions bounded by three possibly disjoint geodesics in the hyperbolic plane. We establish several variational principles for the corresponding polyhedral surfaces. As a consequence, the work of Penner, Bobenko and Springborn, and Thurston on rigidity of polyhedral surfaces and circle patterns is extended to a very general context.

1.2. The method of proofs and the derivative cosine law. The proofs of the above theorems use variational principles. The use of variational principles on triangulated surfaces in recent time appeared in the seminal work of Colin de Verdière [**11**] in 1991 and I. Rivin [**36**] in 1994. Variational principles on triangulated surfaces have also appeared in [**5**], [**25**], [**9**], [**7**], [**39**], and others. The energy functions used in [**11**], [**36**], [**5**], [**25**], [**9**] are related to the 3-dimensional volume, the Schlaefli formula, or its Legendre transform. Even in the work of [**11**], Colin de Verdière's energy was motivated by the 3-dimensional Schlaefli volume formula [**12**]. In 2004, motivated by the discrete 2-dimensional integrable system, Bobenko and Springborn [**7**] discovered a new collection of energies for triangulated surfaces.

We observe that all energy functions used in [**11**], [**5**], [**36**], [**9**], [**25**], [**7**] are derived from the cosine law and the Legendre transformation. Furthermore, we show that these known energy functions are special cases of two one-parameter families of energy functions derived from the cosine law. These energies functionals can be considered as 2-dimensional analogs of the Schlaefli formula.

Our study is motivated by discretization of 2-dimensional Riemannian geometry. In the discrete setting, the smooth metric is replaced by the polyhedral metric and the Gaussian curvature is replaced by the discrete curvature k_0. The relationship between a polyhedral metric and its curvature is the cosine law for triangles. Thus, the cosine law should be considered as a metric-curvature relation. Just like in Riemannian geometry, it is natural to study the infinitesimal dependence of curvature (inner angles) on the metric (edge lengths). The result is a collection of identities which we call the *derivative cosine law*. Among the most interesting ones are the following. Suppose a triangle in $\mathbf{S}^2$, $\mathbf{E}^2$, or $\mathbf{H}^2$ has inner angles θ_1, θ_2, θ_3 and opposite edge lengths l_1, l_2, l_3. Consider θ_i as a function of (l_1, l_2, l_3); then for i, j, k distinct,

$$\frac{\partial\theta_i/\partial l_j}{\partial\theta_j/\partial l_i} = \frac{\sin\theta_i}{\sin\theta_j} \qquad \text{and} \qquad \frac{\partial\theta_i/\partial l_j}{\partial\theta_i/\partial l_i} = -\cos\theta_k. \tag{1.5}$$

1.3. An example. We illustrate the use of these identities by an example. It shows the main techniques and methods used in our papers. Given a Euclidean triangle of edge lengths l_1, l_2, l_3 and opposite angles $\theta_1, \theta_2, \theta_3$, the cosine law relating them states

$$\cos\theta_i = \frac{l_j^2 + l_k^2 - l_i^2}{2l_j l_k},$$

where $i \neq j \neq k \neq i$. Consider $\theta_i = \theta_i(l_1, l_2, l_3)$ as a smooth function of $l = (l_1, l_2, l_3)$. Then identity (1.5) shows that the differential 1-form

$$\omega = \sum_{i=1}^{3} \ln \tan\left(\frac{\theta_i}{2}\right) dl_i$$

is closed. This closed smooth 1-form is defined on the space $\mathbf{E}^2(3) = \{(l_1, l_2, l_3)|l_i + l_j > l_k\}$ of all Euclidean triangles parameterized by the edge lengths. Since the space $\mathbf{E}^2(3)$ is convex and therefore simply connected, the integral $F(l) = \int_{(1,1,1)}^{l} \omega$ defines a smooth function on the $\mathbf{E}^2(3)$. By definition, this function F satisfies

$$\frac{\partial F}{\partial l_i} = \ln \tan\left(\frac{\theta_i}{2}\right). \tag{1.6}$$

Its Hessian matrix $[\frac{\partial^2 F}{\partial l_r \partial l_s}]_{3\times 3} = [\frac{1}{\sin\theta_r}\frac{\partial \theta_r}{\partial l_s}]_{3\times 3}$ can be shown to be congruent to the Gram matrix of the triangle. Thus the Hessian matrix is positive semi-definite. It follows that the function F is convex on $\mathbf{E}^2(3)$. Property (1.6) says the partial derivative with respect to the i-th edge length (i.e., the metric) of the function F depends only on the opposite angle θ_i (i.e., the curvature). This is similar to the Schlaefli identity. A function with property (1.6) is very useful for variational framework on polyhedral surfaces. We call $F(l_1, l_2, l_3)$ the F-energy of the triangle (l_1, l_2, l_3). Let a Euclidean polyhedral surface $(S, \mathcal{T}, l)$ be given so that E and T are the sets of edges and triangles and $l : E \to \mathbf{R}$ is a metric, also called an *edge length function*. Define an "energy" $W(l)$ of the metric l to be the sum of the F-energies of its triangles, i.e., $W(l) = \sum_{\{e_1,e_2,e_3\}\in T} F(l(e_1), l(e_2), l(e_3))$. Then the function $W(l)$ is convex in l since it is the sum of convex functions. Furthermore, by property (1.6), we have

$$\frac{\partial W(l)}{\partial l_i} = \ln\left(\tan\left(\frac{a}{2}\right)\right) + \ln\left(\tan\left(\frac{a'}{2}\right)\right) = -\phi_{-1}(e_i) \tag{1.7}$$

where a and a' are the two inner angles facing the i-th edge e_i. Identities (1.1) and (1.7) show that the gradient of the convex function W is $-\phi_{-1}$, i.e., $\bigtriangledown W = -\phi_{-1}$.

On the other hand, we have the following well known fact:

Lemma 1.10. *If $U : \Omega \to \mathbf{R}$ is a smooth function of positive definite Hessian matrix defined on an open convex set Ω in $\mathbf{R}^n$, then the gradient map $\bigtriangledown U : \Omega \to \mathbf{R}^n$ is a smooth embedding.*

The function W is not strictly convex. Using the lemma and a little extra work, we prove that the gradient $\bigtriangledown W$ is injective up to scaling, i.e., the ϕ_{-1} curvature determines the metric l up to scaling. This is Theorem 1.7(c) for $h = -1$. Indeed, Theorem 1.7(c) is proved in exactly the same way by using a special collection of closed 1-forms on the space of all Euclidean triangles.

1.4. 2-dimensional Schlaefli formula. The most general form of the cosine law can be stated as follows. Suppose a function $y = y(x)$ where $y = (y_1, y_2, y_3) \in \mathbf{C}^3$ and $x = (x_1, x_2, x_3)$ is in some open connected set in $\mathbf{C}^3$ so that x_i's and y_i's are related by

$$\cos(y_i) = \frac{\cos x_i + \cos x_j \cos x_k}{\sin(x_j)\sin(x_k)} \tag{1.8}$$

where $\{i, j, k\} = \{1, 2, 3\}$. We say $y = y(x)$ is a *cosine law function* and write it as $y = CL(x)$. Let $r_i = \frac{1}{2}(x_j + x_k - x_i)$. Then $r = (r_1, r_2, r_3)$ is a new parameterization so that $x_i = r_j + r_k$.

Theorem 1.11. *For a cosine law function $(y_1, y_2, y_3) = CL(x_1, x_2, x_3)$, all closed 1-forms of the form $w = \sum_{i=1}^3 f(y_i)dg(x_i)$ where f, g are two non-constant smooth functions are up to scaling and complex conjugation,*

$$\omega_h = \sum_{i=1}^3 \int^{y_i} \sin^h(t)dtd(\int^{x_i} \sin^{-h-1}(t)dt) = \sum_{i=1}^3 \frac{\int^{y_i} \sin^h(t)dt}{\sin^{h+1}(x_i)} dx_i$$

for some $h \in \mathbf{C}$, i.e., $f'(t) = \sin^h(t)$ and $g'(t) = \sin^{-h-1}(t)$. In particular, all closed 1-forms of this type are holomorphic or anti-holomorphic.

All closed 1-forms of the form $\sum_{i=1}^3 f(y_i)dg(r_i)$ where f, g are two non-constant smooth functions are up to scaling and complex conjugation,

$$\eta_h = \sum_{i=1}^3 \int^{y_i} \tan^h(t/2)dtd(\int^{r_i} \cos^{-h-1}(t)dt) = \sum_{i=1}^3 \frac{\int^{y_i} \tan^h(t/2)dt}{\cos^{h+1}(r_i)} dr_i$$

for some $h \in \mathbf{C}$, i.e., $f'(t) = \tan^h(\frac{t}{2})$ and $g'(t) = \cos^{-h-1}(t)$. In particular, all closed 1-forms of this type are holomorphic or anti-holomorphic.

This theorem can be considered as finding all 2-dimensional analogs of the Schlaefli identity. Recall that if a spherical or hyperbolic tetrahedron has dihedral angle θ_i and edge length l_i at the i-th edge, then the Schlaefli formula says that the 1-form $\omega = \sum_i l_i d\theta_i$ is closed. Its integration $\int \omega$ is $2\lambda Vol$ where $\lambda = \pm 1$ is the curvature of the space $\mathbf{S}^3$ or $\mathbf{H}^3$ and Vol is the volume of the tetrahedron. By specializing Theorem 1.11 to various cases of $\mathbf{S}^2$, $\mathbf{E}^2$, and $\mathbf{H}^2$ and integrating the 1-forms, we obtain various energy functionals for variational framework on triangulated surfaces. Theorems 3.2 and 3.4 identify all those convex or concave energies constructed in this way.

1.5. Surfaces with boundary. The results obtained in this paper can be generalized without difficulty to compact triangulated surfaces with boundary by doubling across the boundary. The notions of k_h, ϕ_h, and ψ_h curvatures can now be defined for polyhedral metrics on $(S, \mathcal{T})$ by using the corresponding concepts on the closed surface. For simplicity, we will not state the results for triangulated surfaces with boundary.

1.6. The organization of the paper. In section 2, we list some of the properties of the derivatives of the cosine law and prove the part of Theorem 1.11 showing that the listed 1-forms are closed. The proof that the list contains all of the closed 1-form is deferred to Appendix A. In section 3, we deduce various consequences of Theorem 1.11 and identify all convex and concave action functionals derived from the cosine law (Theorems 3.2, 3.4). In §4, we prove the infinitesimal rigidity part of main Theorem 1.7. In §5, we study the shapes of the Teichmüller space in ψ_h-coordinates and prove Theorem 1.8. In §6, we investigate spaces of all ϕ_h and ψ_h curvatures for closed triangulated surfaces, generalizing the main work of Rivin [**36**] and Leibon [**25**]. In §7, we study the space of all k_h discrete curvatures of circle packing metrics. In §8, we discuss some open problems and conjectures. In the appendices, we give a proof of the uniqueness of the energy functions, derive the derivative cosine law of the second kind, and recall known relationships of some energy functions with the Lobachevsky functions.

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2. The derivative cosine law

We call a smooth function defined in an open set in $\mathbf{R}^n$ *locally convex* (or *locally strictly convex*) if its Hessian matrix is positive semi-definite (or positive definite) at each point. Note that the definition of strictly convex is not the standard one. Let $\{i,j,k\} = \{1,2,3\}$ in this section.

2.1. The derivative cosine law. Given a triangle in $\mathbf{H}^2$, $\mathbf{E}^2$, or $\mathbf{S}^2$ of inner angles $\theta_1, \theta_2, \theta_3$ and edge lengths l_1, l_2, l_3 so that θ_i is facing the l_i-th edge, the cosine law expressing length l_i in terms of the angles θ_r is

$$\cos\left(\sqrt{\lambda} l_i\right) = \frac{\cos\theta_i + \cos\theta_j \cos\theta_k}{\sin\theta_j \sin\theta_k} \tag{2.1}$$

where $\lambda = 1, -1, 0$ is the curvature of the space $\mathbf{S}^2$, $\mathbf{H}^2$, or $\mathbf{E}^2$. Another related cosine law is

$$\cosh(l_i) = \frac{\cosh\theta_i + \cosh\theta_j \cosh\theta_k}{\sinh\theta_j \sinh\theta_k} \tag{2.2}$$

for a right-angle hyperbolic hexagon with three non-adjacent edge lengths l_1, l_2, l_3 and their opposite edge lengths $\theta_1, \theta_2, \theta_3$. Note that (2.2) can be considered the same as (2.1) applied to a triangle in the extension of the hyperbolic plane by the de Sitter plane.

Identities (2.1) and (2.2) show that the cosine laws are specializations of the cosine law function $y = y(x)$ defined by (1.8). The following was proved in [**28**].

Theorem 2.1. *Suppose the cosine law function $y = y(x)$ is defined on an open connected set in $\mathbf{C}^3$ which contains a point (a, a, a) so that $y(a, a, a) = (b, b, b)$. Let $A_{ijk} = \sin y_i \sin x_j \sin x_k$ where $\{i, j, k\} = \{1, 2, 3\}$. Then*

$$A_{ijk} = A_{jki} = A_{123}. \tag{2.3}$$

$$A_{ijk}^2 = 1 - \cos^2 x_i - \cos^2 x_j - \cos^2 x_k - 2 \cos x_i \cos x_j \cos x_k. \tag{2.4}$$

At a point x where $A_{ijk} \neq 0$, then,

$$\frac{\partial y_i}{\partial x_i} = \frac{\sin x_i}{A_{ijk}}, \tag{2.5}$$

$$\frac{\partial y_i}{\partial x_j} = \frac{\partial y_i}{\partial x_i} \cos y_k, \tag{2.6}$$

$$\cos(x_i) = \frac{\cos y_i - \cos y_j \cos y_k}{\sin y_j \sin y_k}. \tag{2.7}$$

Remark 2.2. Formula (2.3) shows that $\frac{\sin y_i}{\sin x_i}$ is independent of the index i. It is the *sine law.*

Remark 2.3. Identity (2.7) can be written in the symmetric form as

$$\cos(\pi - x_i) = \frac{\cos(\pi - y_i) + \cos(\pi - y_j)\cos(\pi - y_k)}{\sin(\pi - y_j)\sin(\pi - y_k)}. \tag{2.8}$$

This reflects the duality of the spherical triangles. Namely, the dual triangle of a spherical triangle has edge lengths $\pi - \theta_i$ and inner angles $\pi - l_i$. In particular, by (2.6) applied to (2.8), we obtain

$$\frac{\partial x_i}{\partial y_j} = -\frac{\partial x_i}{\partial y_i} \cos x_k. \tag{2.9}$$

Remark 2.4. Identity (1.5) follows from (2.6) and (2.9).

Remark 2.5. If we consider (y_i, x_j, x_k) as a function of (y_j, y_k, x_i) in the cosine law, there are similar derivative identities which we call the derivative cosine laws of second kind. See appendix B.

2.2. The tangent law and the radius parameterization. Many geometric problems (circle packing, etc.) prompt us to parameterize a triangle whose i-th edge length (or angle) x_i is by $r_j + r_k$, i.e., one uses (r_1, r_2, r_3) to parameterize (x_1, x_2, x_3) where $r_i = \frac{1}{2}(x_j + x_k - x_i)$. If the x_i are the edge lengths, then the r_i are the radii of the pairwise tangent circles whose centers are the vertices of the triangle. If the x_i are the inner angles, then r_i is $\pi/2$ less the angle between the i-th edge and the circumcircle.

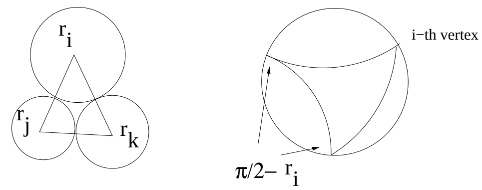

Figure 2

Lemma 2.6. *For the cosine law function $y = y(x)$, write $x_i = r_j + r_k$, or $r_i = \frac{1}{2}(x_j + x_k - x_i)$; under the same assumption as in Theorem 2.1, the following expression is independent of the indices:*

$$\frac{\tan^2(y_i/2)}{\cos^2(r_i)} = -\frac{\cos(r_1 + r_2 + r_3)}{\cos(r_1)\cos(r_2)\cos(r_3)}. \tag{2.10}$$

Furthermore, there is a quantity $A = \sin(y_i)\sin(x_j)\sin(x_k)$ independent of indices so that

$$\frac{1}{\cos(r_i)}\frac{\partial y_i}{\partial r_j} = \frac{2\cos(r_k)}{A\sin(x_k)} \tag{2.11}$$

and

$$\frac{\partial y_i}{\partial r_i} = \frac{2\sin(x_j + x_k)\cos(r_j)\cos(r_k)}{A\sin(x_j)\sin(x_k)}.$$

In particular,

$$\frac{\partial y_i/\partial r_j}{\partial y_j/\partial r_i} = \frac{\cos(r_i)}{\cos(r_j)} = \frac{\tan(y_i/2)}{\tan(y_j/2)}. \tag{2.12}$$

We call identity (2.10) the *tangent law.* In the case of a Euclidean triangle of edge lengths $r_i + r_j$ and opposite angle θ_k, identity (2.10) says that $r_i \tan(\theta_i/2)$ is independent of the index i (the common value is the radius of the inscribed circle).

Proof. To see (2.10), let us calculate $\tan^2(y_i/2)$. It is

$$\tan^2(y_i/2) = \frac{1 - \cos(y_i)}{1 + \cos(y_i)}$$

$$= \frac{\sin(x_j)\sin(x_k) - \cos(x_i) - \cos(x_j)\cos(x_k)}{\sin(x_j)\sin(x_k) + \cos(x_i) + \cos(x_j)\cos(x_k)}$$

$$= -\frac{\cos(x_i) + \cos(x_j + x_k)}{\cos(x_i) + \cos(x_j - x_k)}$$

$$= -\frac{\cos(r_i)\cos(r_1 + r_2 + r_3)}{\cos(r_j)\cos(r_k)}$$

$$= -\frac{\cos^2(r_i)\cos(r_1 + r_2 + r_3)}{\cos(r_1)\cos(r_2)\cos(r_3)}.$$

Thus (2.10) follows.

Next, let us calculate $\frac{\partial y_i}{\partial r_j}$ for $i \neq j$. Note that due to $x_i = r_j + r_k$, we have $\frac{\partial}{\partial r_j} = \frac{\partial}{\partial x_i} + \frac{\partial}{\partial x_k}$. Thus

$$\begin{aligned}\frac{\partial y_i}{\partial r_j} &= \frac{\partial y_i}{\partial x_i} + \frac{\partial y_i}{\partial x_k} \\ &= \frac{\partial y_i}{\partial x_i}(1+\cos(y_j)) \\ &= \frac{\sin(x_i)}{A}\left(\frac{\cos(x_j)+\cos(x_i-x_k)}{\sin(x_i)\sin(x_k)}\right) \\ &= \frac{2\cos(r_i)\cos(r_k)}{A\sin(x_k)}.\end{aligned}$$

This establishes (2.11).

To see $\frac{\partial y_i}{\partial r_i}$, we have

$$\begin{aligned}\frac{\partial y_i}{\partial r_i} &= \frac{\partial y_i}{\partial x_j} + \frac{\partial y_i}{\partial x_k} \\ &= \frac{\partial y_i}{\partial x_i}(\cos(y_k)+\cos(y_j)) \\ &= \frac{\sin(x_i)}{A}\left(\frac{\cos(x_k)+\cos(x_i)\cos(x_j)}{\sin(x_i)\sin(x_j)} + \frac{\cos(x_j)+\cos(x_i)\cos(x_k)}{\sin(x_i)\sin(x_k)}\right) \\ &= \frac{\sin(x_j)\cos(x_j)+\sin(x_k)\cos(x_k)+\cos(x_i)(\cos(x_j)\sin(x_k)+\cos(x_k)\sin(x_j))}{A\sin(x_j)\sin(x_k)} \\ &= \frac{\sin(x_j+x_k)\cos(x_j-x_k)+\cos(x_i)\sin(x_j+x_k)}{A\sin(x_j)\sin(x_k)} \\ &= \frac{2\sin(x_j+x_k)\cos(r_j)\cos(r_k)}{A\sin(x_j)\sin(x_k)}.\end{aligned}$$

q.e.d.

2.3. A proof of Theorem 1.11. We prove that the 1-forms in the list are closed. Indeed, a holomorphic 1-form $\omega_h = \sum_{i=1}^{3} f(y_i)g'(x_i)dx_i$ is closed if and only if $\frac{\partial(f(y_i)g'(x_i))}{\partial x_j} = f'(y_i)g'(x_i)\frac{\partial y_i}{\partial x_j}$ is symmetric in i, j. For ω_h where $f'(t) = \sin^h(t)$ and $g'(t) = \sin^{-h-1}(t)$, we have

$$\begin{aligned}\frac{\partial(f(y_i)g'(x_i))}{\partial x_j} &= f'(y_i)g'(x_i)\frac{\partial y_i}{\partial x_j} = \frac{1}{A}f'(y_i)g'(x_i)\sin(x_i)\cos(y_k) \\ &= \frac{1}{A}\left(\frac{\sin(y_i)}{\sin(x_i)}\right)^h \cos(y_k)\end{aligned}$$

where A is independent of the indices by Theorem 2.1. The above expression is clearly symmetric in i, j due to the sine law. Thus the closedness follows.

To see the holomorphic 1-form $\eta_h = \sum_{i=1}^3 f(y_i)dg(r_i)$ is closed, we check if the quantity $f'(y_i)g'(r_i)\frac{\partial y_i}{\partial r_j}$ is symmetric in i,j for $f'(t) = \tan^h(t/2)$ and $g'(t) = \cos^{-h-1}(t)$. Indeed, it is equal to

$$\frac{\tan^h(y_i/2)}{\cos^{h+1}(r_i)}\frac{\partial y_i}{\partial r_j} = \left(\frac{\tan(y_i/2)}{\cos(r_i)}\right)^h \left(\frac{1}{\cos(r_i)}\frac{\partial y_i}{\partial r_j}\right)$$

where $\frac{1}{\cos(r_i)}\frac{\partial y_i}{\partial r_j}$ is symmetric in i,j by (2.11) and $\frac{\tan(y_i/2)}{\cos(r_i)}$ is independent of i by the tangent law. This shows the closedness.

The proof that these are all closed 1-forms is relatively long and the techniques used will not be used anywhere in the paper. We defer it to Appendix A.

2.4. Hessian matrices of the energy functions. In Theorem 1.11, let $u_i = \int^{x_i} \sin^{-h-1}(t)dt$ in the case of w_h and $u_i = \int^{r_i} \cos^{-h-1}(t)dt$ for η_h; then the closed 1-forms are $w_h = \sum_{i=1}^3 \int^{y_i} \sin^h(t)dtdu_i$ and $\eta_h = \sum_{i=1}^3 \int^{y_i} \tan^h(t/2)dtdu_i$.

Lemma 2.7. *For any two h, h',*

(a) *the Hessian matrices of the two functions $\int^u w_h$ and $\int^u w_{h'}$ are congruent;*
(b) *the Hessian matrices of the two functions $\int^u \eta_h$ and $\int^u \eta_{h'}$ are congruent.*

Proof. In the case of w_h, the Hessian of the function $F(u) = \int^u w_h$ is $[\partial^2 F/\partial u_s \partial u_t]$ where

$$\frac{\partial^2 F}{\partial u_s \partial u_t} = \sin^h(y_t)\frac{\partial y_t}{\partial x_s}\frac{\partial x_s}{\partial u_s}$$
$$= \sin^h(y_t)\sin^{h+1}(x_s)\frac{\partial y_t}{\partial x_s}$$
$$= \left(\frac{\sin(y_t)}{\sin(x_t)}\right)^h (\sin(x_t)\sin(x_s))^h \left(\sin(x_s)\frac{\partial y_s}{\partial x_t}\right). \quad (2.13)$$

Let $q = (\frac{\sin(y_t)}{\sin(x_t)})^h$ be the function which is independent of index t due to the sine law and D be the 3×3 diagonal matrix whose (i,i)-th entry is $\sin^h(x_i)$. Then (2.13) shows that the Hessian matrix is $qD[\sin(x_s)\frac{\partial y_s}{\partial x_t}]D$. It follows that the Hessian matrices for different h's are congruent.

By the same calculation using the tangent law instead of the sine law, for the integration of η_h, we see that the (s,t)-th entry of the Hessian matrix is

$$\left(\frac{\tan(y_t/2)}{\cos(r_t)}\right)^h (\cos(r_s)\cos(r_t))^h \left(\cos(r_t)\frac{\partial y_t}{\partial r_s}\right).$$

This shows that the Hessian matrices for different h's are congruent. q.e.d.

2.5. The Legendre transformation. The integrals $\int w_h$ in Theorem 1.11 are not independent. In fact $\int w_h$ and $\int w_{-h-1}$ are Legendre transformations of each other.

Let us recall briefly the Legendre transforms. Suppose U and V are diffeomorphic connected open sets in $\mathbf{R}^n$ so that the first de Rham cohomology group $H^1_{dR}(U) = 0$. Let $x = (x_1, \ldots, x_n) \in U$ and $y = (y_1, \ldots, y_n) \in V$ and $y = y(x) : U \to V$ be a diffeomorphism so that its Jacobian matrix is symmetric, i.e.,

$$\frac{\partial y_i}{\partial x_j} = \frac{\partial y_j}{\partial x_i}.$$

Then the differential 1-forms $w_U = \sum_{i=1}^n y_i dx_i$ and $w_V = \sum_{i=1}^n x_i dy_i$ are closed in U and V respectively. Their integrations $f(x) = \int_a^x w_U$ and $g(y) = \int_b^y w_V + (a, b)$ where $b = y(a)$ and (a, b) is the dot product are well defined due to $H^1_{dR}(U) = H^1_{dR}(V) = 0$. We call $g(y)$ the *Legendre transformation* of $f(x)$ (and vice versa). By the construction, the Hessian matrices of f and g are inverses of each other. Therefore the Legendre transform of a strictly convex (or concave) function is strictly convex (or concave).

Proposition 2.8. *Let* $f_h(x) = \int_{(\pi/2,\pi/2,\pi/2)}^x \sum_{i=1}^3 \frac{\int_{\pi/2}^{y_i} \sin^h(t)dt}{\sin^{h+1}(x_i)} dx_i$. *Then the Legendre transformation of* $f_h(x)$ *is* $f_{-h-1}(x) - f_{-h-1}(0)$. *In particular, the Legendre transform of* $f_{-1/2}(x)$ *is* $f_{-1/2}(x)$ *up to adding a constant.*

Proof. Let $g_h(y)$ be the Legendre transformation of $f_h(x)$. Then,

$$g_h(y) = \int_{(\pi/2,\pi/2,\pi/2)}^y \sum_{i=1}^3 \left(\int_{\pi/2}^{x_i} \sin^{-h-1}(t)dt \right) d \left(\int_{\pi/2}^{y_i} \sin^h(t)dt \right) + c$$

where $c = g_h(0,0,0) - g_h(\pi/2, \pi/2, \pi/2)$. In the above identity, x and y are related by the cosine law (1.8) which we denote by $y = CL(x)$. Let $w = (w_1, w_2, w_3)$ and $v = (v_1, v_2, v_3)$ so that $w_i = \pi - y_i$ and $v_i = \pi - x_i$. Then by (2.8), we have $v = CL(w)$. Making a change of variables $x_i = \pi - v_i$ and $y_i = \pi - w_i$ in the integral of g_h, we obtain

$$\begin{aligned} g_h(y) &= \int_{(\pi/2,\pi/2,\pi/2)}^{(\pi-w_1,\pi-w_2,\pi-w_3)} \sum_{i=1}^3 \left(\int_{\pi/2}^{\pi-x_i} \sin^{-h-1}(\pi - t) d(\pi - t) \right) \\ &\qquad \sin^h(\pi - w_i) d(\pi - w_i) + c \\ &= \int_{(\pi/2,\pi/2,\pi/2)}^{(\pi-w_1,\pi-w_2,\pi-w_3)} \sum_{i=1}^3 \left(\int_{\pi/2}^{v_i} \sin^{-h-1}(t)dt \right) \sin^h(w_i)dw_i + c \\ &= f_{-h-1}(y) + c \end{aligned}$$

since $v = CL(w)$. Thus $g_h(y) = f_{-h-1}(y) - f_{-h-1}(0)$.

q.e.d.

3. Energy functionals on the moduli spaces of geometric triangles

In this section, we consider 1-forms in Theorem 1.11 defined on the moduli spaces of geometric triangles and determine the convexity of the integrals of the 1-forms.

3.1. Derivative cosine laws for geometric triangles. Take (x_1, x_2, x_3) and (y_1, y_2, y_3) in Theorem 2.1 to be the inner angles $(\theta_1, \theta_2, \theta_3)$ and edge lengths (l_1, l_2, l_3) of a triangle in $\mathbf{E}^2$, $\mathbf{H}^2$, or $\mathbf{S}^2$. Then:

Corollary 3.1. *Let $\{i, j, k\} = \{1, 2, 3\}$. There is a positive quantity A independent of indices so that*

(a) *([**10**])*

$$\frac{\partial \theta_i}{\partial l_j} = -\frac{\partial \theta_i}{\partial l_i} \cos(\theta_k) \quad \textit{and} \quad \frac{\partial \theta_i}{\partial l_i} = \frac{\sin(\theta_i)}{A} > 0.$$

(b) *For spherical triangles,*

$$\frac{\partial l_i}{\partial \theta_j} = \frac{\partial l_i}{\partial \theta_i} \cos(l_k) \quad \textit{and} \quad \frac{\partial l_i}{\partial \theta_i} = \frac{\sin(l_i)}{A} > 0.$$

(c) *For hyperbolic triangles,*

$$\frac{\partial l_i}{\partial \theta_j} = \frac{\partial l_i}{\partial \theta_i} \cosh(l_k) \quad \textit{and} \quad \frac{\partial l_i}{\partial \theta_i} = -\frac{\sinh(l_i)}{A} < 0.$$

(d) *For a hyperbolic right-angled hexagon of three non-pairwise adjacent edge lengths l_1, l_2, l_3 and opposite edge lengths $\theta_1, \theta_2, \theta_3$,*

$$\frac{\partial \theta_i}{\partial l_j} = -\frac{\partial \theta_i}{\partial l_i} \cosh(\theta_k) \quad \textit{and} \quad \frac{\partial \theta_i}{\partial l_i} = \frac{\sinh(\theta_i)}{A} > 0.$$

The proof uses Theorem 2.1 by taking care of the curvature factor $\lambda = \pm 1, 0$ that appeared in (2.1). Note that $\cos(\sqrt{-1}x) = \cosh(x)$, $\sin(\sqrt{-1}x) = \sqrt{-1}\sinh(x)$, and $\sinh(\sqrt{-1}x) = \sqrt{-1}\sin(x)$. Using these relations, part (a) follows from (2.9) where $x_i = \theta_i$ and $y_i = \sqrt{\lambda} l_i$ for $\lambda = \pm 1$. Part (a) for a Euclidean triangle was established in [**10**] and can be checked directly. Parts (b) and (c) follow from Theorem 2.1. To see part (d), note that the cosine law for hexagon can be written as

$$\cos(\pi - \sqrt{-1}\theta_i) = \frac{\cos(\sqrt{-1}l_i) + \cos(\sqrt{-1}l_j)\cos(\sqrt{-1}l_k)}{\sin(\sqrt{-1}l_j)\sin(\sqrt{-1}l_k)}.$$

Thus part (d) follows from Theorem 2.1.

3.2. Closed 1-forms on the space of triangles parameterized by edge lengths. Using Corollary 3.1 and Theorem 1.11, we obtain,

Theorem 3.2. *Let a triangle in* $\mathbf{E}^2$, $\mathbf{H}^2$, *or* $\mathbf{S}^2$ *have inner angles* $\theta_1, \theta_2, \theta_3$ *and opposite edge lengths* l_1, l_2, l_3. *The following is the complete list, up to scaling, of all closed real-valued 1-forms (in variables* l_1, l_2, l_3*) of the form* $\sum_{i=1}^{3} f(\theta_i) dg(l_i)$ *for some non-constant smooth functions* f, g. *Let* $h \in \mathbf{R}$ *and* $u = (u_1, u_2, u_3)$.

(a) *For a Euclidean triangle,*

$$w_h = \sum_{i=1}^{3} \frac{\int^{\theta_i} \sin^h(t)dt}{l_i^{h+1}} dl_i,$$

i.e., $f'(t) = \sin^h(t)$ *and* $g'(t) = t^{-h-1}$. *Furthermore, its integral* $\int^u w_h$ *has a positive semidefinite Hessian matrix in variable* u *where* $u_i = \int_1^{l_i} t^{-h-1} dt$.

(b) *For a spherical triangle,*

$$w_h = \sum_{i=1}^{3} \frac{\int^{\theta_i} \sin^h(t)dt}{\sin^{h+1}(l_i)} dl_i,$$

i.e., $f'(t) = \sin^h(t)$ *and* $g'(t) = \sin^{-h-1}(t)$. *The integral* $\int^u w_h$ *has a positive definite Hessian matrix in* u *where* $u_i = \int_{\pi/2}^{l_i} \sin^{-h-1}(t)dt$.

(c) *For a hyperbolic triangle,*

$$w_h = \sum_{i=1}^{3} \frac{\int^{\theta_i} \sin^h(t)dt}{\sinh^{h+1}(l_i)} dl_i,$$

i.e., $f'(t) = \sin^h(t)$ *and* $g'(t) = \sinh^{-h-1}(t)$.

(d) *For a hyperbolic right-angled hexagon,*

$$w_h = \sum_{i=1}^{3} \frac{\int^{\theta_i} \sinh^h(t)dt}{\sinh^{h+1}(l_i)} dl_i,$$

i.e., $f'(t) = \sinh^h(t)$ *and* $g'(t) = \sinh^{-h-1}(t)$.

In the cases of (b), (c), (d), by taking the Legendre transformation, we also obtain the complete list of all closed 1-forms of the form $\sum_{i=1}^{3} g(l_i) df(\theta_i)$.

Proof. The closedness of these 1-forms is evident due to Theorem 1.11 and Corollary 3.1 except in the case of Euclidean triangles. In the case of $\mathbf{E}^2$, we need to verify that the expression

$$\frac{\partial}{\partial l_j}\left(\frac{\int^{\theta_i} \sin^h(t)dt}{l_i^{h+1}}\right) \tag{3.1}$$

is symmetric in i, j. By Corollary 3.1 that $\frac{\partial \theta_i}{\partial l_j} = -\frac{\partial \theta_i}{\partial l_i} \cos(\theta_k) = -\frac{\sin(\theta_i)}{A} \cos(\theta_k)$, the expression (3.1) is equal to $\frac{1}{l_i^{h+1}} \sin^h(\theta_i) \frac{\partial \theta_i}{\partial l_j} =$

$-(\frac{\sin(\theta_i)}{l_i})^{h+1}\frac{\cos(\theta_k)}{A}$ where A is independent of indices. It is symmetric in i, j due to the sine law.

To verify the convexity, note that if $u_i = g(l_i)$ and $w = \sum_{i=1}^{3} f(\theta_i)du_i$ is closed, then the Hessian of the function $F(u) = \int^u w$ is $[\frac{\partial^2 F}{\partial u_r \partial u_s}] = [\frac{f'(\theta_r)}{g'(l_s)}\frac{\partial \theta_r}{\partial l_s}]$.

In the cases (a)–(d), by the sine law and the choice of f, g, $f'(\theta_i)\sin(\theta_i)g'(l_i) = q$ is a positive function independent of the indices. Thus the (r, s)-th entry of the Hessian matrix $[\frac{\partial^2 F}{\partial u_r \partial u_s}]$ can be written as

$$\frac{f'(\theta_r)}{g'(l_s)}\frac{\partial \theta_r}{\partial l_s} = (f'(\theta_r)\sin(\theta_r)g'(l_r))(\frac{1}{g'(l_r)g'(l_s)})\left(\frac{1}{\sin(\theta_r)}\frac{\partial \theta_r}{\partial l_s}\right)$$

$$= \left(\frac{q}{g'(l_r)g'(l_s)})(\frac{1}{\sin(\theta_r)}\frac{\partial \theta_r}{\partial l_s}\right).$$

This shows that the Hessian matrix can be written as $qDLD$ where D is the positive diagonal matrix whose (i, i)-th entry is $\frac{1}{g(l_i)}$ and $L = [\frac{1}{\sin(\theta_r)}\frac{\partial \theta_r}{\partial l_s}]_{3\times 3}$. Recall that given a triangle with inner angles $\theta_1, \theta_2, \theta_3$, its (angle) Gram matrix $[a_{rs}]_{3\times 3}$ satisfies $a_{ii} = 1$ and $a_{ij} = -\cos(\theta_k)$ ($\{i, j, k\} = \{1, 2, 3\}$). On the other hand, by Corollary 3.1(a), the matrix L is equal to the Gram matrix multiplied by the positive function $1/A$. As a consequence, the Hessian of the integral of the 1-forms in (a)-(d) is congruent to the Gram matrix of the triangle. It is well known that the Gram matrix of a Euclidean triangle is positive semi-definite of rank 2 and the Gram matrix of a spherical triangle is positive definite. (See for instance [**28**] for proofs.) Thus the local convexity of the integrations for Euclidean and spherical triangles follows. q.e.d.

Corollary 3.3. *In Theorem 3.2(a), the null space of the Hessian matrix* $Hess(F)$ *of* $F = \int^u w_h$ *at a point* u *is generated by the vector* u *if* $h \neq 0$ *and is generated by* $(1, 1, 1)$ *if* $h = 0$.

Indeed, if $v = (v_1, v_2, v_3)$ is in the kernel of the Hessian, then by definition and the calculation above, we have

$$\sum_{r,s\in\{i,j,k\}} \frac{v_r v_s}{g'(l_r)g'(l_s)} a_{rs} = 0$$

where $[a_{rs}]_{3\times 3}$ is the Gram matrix of the Euclidean triangle. Due to $l_i - l_j\cos(\theta_k) - l_k\cos(\theta_j) = 0$, the null space of the Gram matrix of a Euclidean triangle of edge lengths (l_i, l_j, l_k) is generated by the length vector (l_i, l_j, l_k). It follows that there is a constant $c \in \mathbf{R}$ so that $v_r = cl_r g'(l_r) = cl_r^{-h}$ for $r = i, j, k$. On the other hand, for $h \neq 0$, $u_r = -\frac{1}{h}l_r^{-h}$. Therefore, if $h \neq 0$, $v = -(ch)u$ and if $h = 0$, $v = c(1, 1, 1)$.

3.3. Closed 1-forms on the space of triangles parameterized by radii. In this section, we establish the counterpart of Theorem 3.2 for triangles parameterized by the radii. There are two cases to be discussed: (1) the edge lengths are $l_k = r_i + r_j$ and opposite angles are θ_k and (2) edge lengths are l_i and the opposite angles $\theta_i = r_j + r_k$. Let $\{i, j, k\} = \{1, 2, 3\}$ in this subsection.

Theorem 3.4. *The following are the complete list, up to scaling, of all closed real-valued 1-forms of the form $\sum_{i=1}^{3} f(l_i)dg(r_i)$ (where $\theta_i = r_j + r_k$) and $\sum_{i=1}^{3} f(\theta_i)dg(r_i)$ (where $l_i = r_j + r_k$) for some non-constant smooth functions f, g. Let $h \in \mathbf{R}$ and $u = (u_1, u_2, u_3)$.*

(a) *For a Euclidean triangle of angles θ_i and opposite edge lengths $r_j + r_k$,*

$$\eta_h = \sum_{i=1}^{3} \frac{\int^{\theta_i} \cot^h(t/2)dt}{r_i^{h+1}} dr_i,$$

i.e., $f'(t) = \cot^h(t/2)$ and $g'(t) = t^{-h-1}$. Its integral $\int^u \eta_h$ has a negative semi-definite Hessian matrix in $u = (u_1, u_2, u_3)$ where $u_i = \int_1^{r_i} t^{-h-1}dt$.

(b) *For a hyperbolic triangle of angles θ_i and opposite edge lengths $r_j + r_k$,*

$$\eta_h = \sum_{i=1}^{3} \frac{\int^{\theta_i} \cot^h(t/2)dt}{\sinh^{h+1}(r_i)} dr_i,$$

i.e., $f'(t) = \cot^h(t/2)$ and $g'(t) = \sinh^{-h-1}(t)$. Its integral $\int^u \eta_h$ has a negative semi-definite Hessian matrix in u where $u_i = \int_1^{r_i} \sinh^{-h-1}(t)dt$.

(c) *For a spherical triangle of angles θ_i and opposite edge lengths $r_j + r_k$,*

$$\eta_h = \sum_{i=1}^{3} \frac{\int^{\theta_i} \cot^h(t/2)dt}{\sin^{h+1}(r_i)} dr_i,$$

i.e., $f'(t) = \cot^h(t/2)$ and $g'(t) = \sin^{-h-1}(t)$.

(d) *For a hyperbolic triangle of edge lengths l_i and opposite angles $r_j + r_k$,*

$$\eta_h = \sum_{i=1}^{3} \frac{\int^{l_i} \tanh^h(t/2)dt}{\cos^{h+1}(r_i)} dr_i,$$

i.e., $f'(t) = \tanh^h(t/2)$ and $g'(t) = \cos^{-h-1}(t)$. Its integral $\int^u \eta_h$ has a positive definite Hessian matrix in u where $u_i = \int_1^{r_i} \cos^{-h-1}(t)dt$.

(e) *For a spherical triangle of edge lengths l_i and opposite angles $r_j + r_k$,*

$$\eta_h = \sum_{i=1}^{3} \frac{\int^{l_i} \tan^h(t/2)dt}{\cos^{h+1}(r_i)} dr_i,$$

$f'(t) = \tan^h(t/2)$ *and* $g'(t) = \cos^{-h-1}(t)$.

(f) *For a hyperbolic right-angled hexagon of three non-pairwise adjacent edge lengths l_i and opposite edge lengths $r_j + r_k$,*

$$\eta_h = \sum_{i=1}^{3} \frac{\int^{l_i} \coth^h(t/2)dt}{\cosh^{h+1}(r_i)} dr_i,$$

i.e., $f'(t) = \coth^h(t/2)$ *and* $g'(t) = \cosh^{-h-1}(t)$. *Its integral $\int^u \eta_h$ has a negative definite Hessian matrix in u where $u_i = \int_1^{r_i} \cosh^{-h-1}(t)dt$.*

Proof. The proof of the uniqueness is essentially the same as that of Theorem 1.11 and will be omitted (see appendix A). The proof of closedness of the 1-forms follows from Theorem 1.11 by taking care of the curvature factors. Indeed, for (b) and (c), we take $y_i = \pi - \theta_i$ and $x_i = \pi - \sqrt{\delta} l_i$ in Theorem 1.11 where $\delta = \pm 1$ is the curvature of the space $\mathbf{S}^2$ or $\mathbf{H}^2$. For (d) and (e), we take $y_i = \sqrt{\delta} l_i$ and $x_i = \theta_i$. For (f), we take $y_i = \pi - \sqrt{-1} l_i$ and $x_i = \sqrt{-1}\theta_i$ in Theorem 1.11.

It remains to prove the closedness in case (a). The closedness of the 1-form means $\frac{\partial(f(\theta_i)g'(r_i))}{\partial r_j}$ is symmetric in i, j. Let the R be the radius of the inscribed circle given by $R = r_i \tan(\theta_i/2)$. By Corollary 3.1(a), we have

$$\begin{aligned} \frac{\partial\theta_i}{\partial r_j} = \frac{\partial\theta_i}{\partial l_i} + \frac{\partial\theta_i}{\partial l_k} = \frac{\partial\theta_i}{\partial l_i}(1 - \cos(\theta_j)) &= \frac{2\sin(\theta_i)\sin^2(\theta_j/2)}{A} \\ &= \frac{4r_i \sin^2(\theta_i/2)\sin^2(\theta_j/2)}{AR}. \end{aligned} \tag{3.2}$$

Using (3.2), $f'(t) = \cot^h(t/2)$ and $g'(t) = r^{-h-1}$, we obtain

$$\begin{aligned} \frac{\partial(f(\theta_i)g'(r_i))}{\partial r_j} &= f'(\theta_i)g'(r_i)\frac{\partial\theta_i}{\partial r_j} \\ &= (r_i \tan(\theta_i/2))^{-h}\left(r_i^{-1}\frac{\partial\theta_i}{\partial r_j}\right) = R^{-h}\left(r_i^{-1}\frac{\partial\theta_i}{\partial r_j}\right) \qquad (3.3) \\ &= \frac{4R^{-h-1}}{A}\sin^2(\theta_i/2)\sin^2(\theta_j/2) \qquad (3.4) \end{aligned}$$

is symmetric in i, j, i.e., the 1-form $\eta_h = \sum_i f(\theta_i)dg(r_i)$ in case (a) is closed.

The convexity or concavity of the functions in cases (b), (d), (f) is proved as follows. By Lemma 2.7, for any two h and h', the Hessian matrices of the associated functions $\int^u \eta_h$ and $\int^u \eta_{h'}$ in each case of (b)–(f) are congruent. Thus, to check the convexity or concavity in cases (b), (d), and (f), it suffices to verify it for a specific value of h. These special cases were established by various authors. For cases (a), (b) and $h = 0$, Colin de Verdière [**11**] proved the concavity of the function $\int^u \eta_0$. In case (d), Leibon [**25**] proved the strict convexity for $h = 0$. In case (f), we proved it for $h = 0$ in [**27**]. Finally, to verify the concavity of case (a), we note that (3.3) (which holds even for $i = j$) shows the Hessian matrices of the functions $\int^u \eta_h$ and $\int^u \eta_0$ are congruent. Since $\int^u \eta_0$ is shown in [**11**] to be concave, it follows that $\int^u \eta_h$ is concave in u. q.e.d.

Corollary 3.5. *(See also* [**11**]*). In Theorem 3.4(a), for a Euclidean triangle of edge lengths $l_i = r_j + r_k$, the null space of the Hessian Hess(F) where $F(u) = \int^u \eta_h$ at a point u is generated by the vector u for $h \neq 0$ and by $(1, 1, 1)$ if $h = 0$.*

Indeed, we first note that the null space of the symmetric 3×3 matrix $[a_{st}]$ contains $(1, 1, 1)$ due to the equality $a_{ii} + a_{ij} + a_{ik} = 0$. Next, we note that the rank of $[a_{st}]$ is 2, due to the fact that $|a_{ii}| > |a_{ij}|$, for all $i \neq j$. Thus the null space of $[a_{st}]$ is generated by $(1, 1, 1)$. Now the Hessian matrix H is $R^{-h}[r_s^h r_t^h a_{st}]$ by the calculation above. It follows that the null space of H is generated by $(1, 1, 1)$ if $h = 0$ or by $\frac{1}{h}(r_1^{-h}, r_2^{-h}, r_3^{-h}) = u$ if $h \neq 0$.

4. Infinitesimal and global rigidity of polyhedral surfaces

Suppose $(S, \mathcal{T})$ is a closed triangulated surface so that V, E, and T are the sets of all vertices, edges, and triangles. Let $K^2 = \mathbf{H^2}$, S^2, or $\mathbf{E^2}$. The *moduli space* of all K^2-polyhedral metrics on $(S, \mathcal{T})$, denoted by $P_{K^2}(S, \mathcal{T})$, is the set of all polyhedral metrics $l : E \to \mathbf{R}_{>0}$ so that $l(e_i) + l(e_j) > l(e_k)$ whenever e_i, e_j, e_k are edges of a triangle and if $K^2 = \mathbf{S^2}$, one requires further that $l(e_i) + l(e_j) + l(e_k) < 2\pi$. By definition, the space $P_{K^2}(S, \mathcal{T})$ is a convex polytope in the Euclidean space $\mathbf{R}^E$. Define three maps Φ_h, Ψ_h, and K_h on $P_{K^2}(S, \mathcal{T})$ as follows. The map $\Phi_h : P_{K^2}(S, \mathcal{T}) \to \mathbf{R}^E$ sends a metric to its ϕ_h curvature defined by (1.1). The map $\Psi_h : P_{K^2}(S, \mathcal{T}) \to \mathbf{R}^E$ sends a metric l to its ψ_h curvature defined by (1.2). The map $K_h : P_{K^2}(S, \mathcal{T}) \to \mathbf{R}^V$ sends a polyhedral metric l to its k_h discrete curvature defined by (1.3). Let $\mathbf{R}_{>0}$ act on $\mathbf{R}^E$ by multiplication. Then $P_{E^2}(S, \mathcal{T}) \subset \mathbf{R}^E$ is invariant under the action. The orbit space, denoted by $P_{E^2}(S, \mathcal{T})/\mathbf{R}_{>0}$, is the set of all Euclidean polyhedral metrics on $(S, \mathcal{T})$ modulo scaling. By definition, all maps Φ_h, Ψ_h, and K_h defined on $P_{E^2}(S, \mathcal{T})$ are invariant under the action, i.e., they satisfy the equation $\phi(kx) = \phi(x)$ for all $k \in \mathbf{R}_{>0}$. We use the same notations Φ_h, Ψ_h, and K_h to denote the induced maps from $P_{E^2}(S, \mathcal{T})/\mathbf{R}_{>0}$ to $\mathbf{R}^E$ or $\mathbf{R}^V$. We use $CP_{K^2}(S, \mathcal{T})$ to denote the space of all K^2 circle packing metrics on (S, T) for $K^2 = \mathbf{E}^2, \mathbf{H}^2$, or $\mathbf{S}^2$. In particular, if $K^2 = \mathbf{H^2}, \mathbf{E^2}$, then $CP_{K^2}(S, \mathcal{T}) = \mathbf{R}_{>0}^V$

and $CP_{\mathbf{S}^2}(S,\mathcal{T}) = \{r \in \mathbf{R}^V_{>0} | r(v_1)+r(v_2)+r(v_3) < \pi$ if v_1, v_2, v_3 are vertices of a triangle$\}$. The space of all Euclidean circle packing metrics modulo scaling is denoted by $CP_{E^2}(S,\mathcal{T})/\mathbf{R}_{>0}$.

4.1. The main results. The relationship between the metric l and its various curvatures is encoded by these maps K_h, Φ_h, and Ψ_h. For instance, the rigidity theorem that the ϕ_h curvature determines the metric means that the map Φ_h is injective. The corresponding infinitesimal rigidity theorem says that Φ_h is a local diffeomorphism. Our main result, which implies Theorem 1.7, is

Theorem 4.1. *Suppose $(S,\mathcal{T})$ is a triangulated closed surface and $h \in \mathbf{R}$.*

(a) *The map $\Phi_h : P_{S^2}(S,\mathcal{T}) \to \mathbf{R}^E$ is a smooth embedding.*
(b) *The map $\Phi_h : P_{E^2}(S,\mathcal{T})/\mathbf{R}_{>0} \to \mathbf{R}^E$ is a smooth embedding.*
(c) *The map $K_h : CP_{H^2}(S,\mathcal{T}) \to \mathbf{R}^V$ is a smooth embedding.*
(d) *The map $K_h : CP_{E^2}(S,\mathcal{T})/\mathbf{R}_{>0} \to \mathbf{R}^V$ is a smooth embedding.*
(e) *The map $\Psi_h : P_{H^2}(S,\mathcal{T}) \to \mathbf{R}^E$ is a smooth embedding.*

In particular, in cases (a), (c), (e), the maps are local diffeomorphisms.

In this paper, we will prove parts (c), (d) and the infinitesimal rigidity part of (a), (b), (e) of Theorem 4.1, i.e., the derivatives of the maps in Theorem 4.1 are injective. The injectivity of the maps in parts (a), (b), (e) are proved in [**29**].

The proof uses Lemma 1.10 and the convex or concave energy functions in Theorems 3.2 and 3.4. The following convention will be used. Let $E = \{e_1, \ldots, e_n\}$ be the set of all edges in the triangulation $\mathcal{T}$. If $x : E \to X$, then x_i denotes $x(e_i)$.

4.2. Proof of infinitesimal rigidity part of Theorem 4.1(a) and 4.1(e). Let $f(t) = \int_{\pi/2}^t \sin^{-h-1}(x)dx$ for $t \in (0,\pi)$. Then $f'(t) > 0$ on $(0,\pi)$ and $f(t)$ is strictly increasing. Given $l \in P_{S^2}(S,\mathcal{T})$ with $l_i = l(e_i)$, define $u : E \to \mathbf{R}$ by $u_i = f(l_i)$ and write $u = (u_1, \ldots, u_n)$. Then the map $u = u(l) : P_{\mathbf{S}^2}(S,T) \to \mathbf{R}^E$ is a smooth embedding. Let $\Omega = u(P_{S^2}(S,T))$ which is open in $\mathbf{R}^E$. Recall from Theorem 3.2 that if l_i, l_j, l_k are the edge lengths of a spherical triangle with opposite angles $\theta_i, \theta_j, \theta_k$, then the differential 1-form

$$w_h = \int_{\pi/2}^{\theta_i} \sin^h(t)dtdu_i + \int_{\pi/2}^{\theta_j} \sin^h(t)dtdu_j + \int_{\pi/2}^{\theta_k} \sin^h(t)dtdu_k$$

is closed and its integral $F(u_i,u_j,u_k) = \int_{(\pi/2,\pi/2,\pi/2)}^{(u_i,u_j,u_k)} w_h$ has a positive definite Hessian matrix. Define an energy function $W : \Omega \to \mathbf{R}$ by

$$W(u) = \sum_{\{e_a,e_b,e_c\}\in T} F(u_a,u_b,u_c)$$

where the sum is over all triangles whose edges are $\{e_a, e_b, e_c\}$. By the construction,

$$\frac{\partial W}{\partial u_i} = \int_{\pi/2}^{\alpha} \sin^h(t)dt + \int_{\pi/2}^{\beta} \sin^h(t)dt$$

where α, β are the angles facing the edge e_i, i.e.,

$$\bigtriangledown W = -(\phi_h(e_1), \ldots, \phi_h(e_n)).$$

We claim that the Hessian $Hess(W)$ of W is positive definite. Indeed, take a non-zero vector $v = (v_1, \ldots, v_n) \in \mathbf{R}^n$ and let v^t be the transpose of v. By definition of W,

(4.1)

$$v \cdot Hess(W) \cdot v^t = \sum_{\{e_a, e_b, e_c\} \in T} (v_a, v_b, v_c) \cdot Hess(F)|_{(u_a, u_b, u_c)} \cdot (v_a, v_b, v_c)^t > 0$$

since each summand above is non-negative and one of the summand

$$(v_a, v_b, v_c) \cdot Hess(F) \cdot (v_a, v_b, v_c)^t > 0 \tag{4.2}$$

due to $(v_a, v_b, v_c) \neq 0$ for some (v_a, v_b, v_c).

By Lemma 1.10 applied to W on Ω, we conclude that Φ_h is a local diffeomorphism.

Exactly the same argument shows the infinitesimal rigidity in part 4.1(e). We use the strict convexity of the Legendre transform of the function in Theorem 3.4(b).

4.3. Proofs of infinitesimal rigidity part of Theorem 4.1(b). For an $\mathbf{E}^2$ polyhedral metric $l : E \to \mathbf{R}_{>0}$, let $u_i = f(l_i)$ where $u_i = -\frac{1}{h} l_i^{-h}$ if $h \neq 0$ and $u_i = \ln(l_i)$ if $h = 0$. The space of all $\mathbf{E}^2$ polyhedral metrics on (S, T), parameterized by the edge length function, is the open convex polytope $P_{E^2}(S, T)$ in $\mathbf{R}^E_{>0}$. The map $u = u(l)$ sends $P_{E^2}(S, T)$ onto an open set $\Omega \subset \mathbf{R}^E$. Since $f(\mu l_i) = \mu^{-h} f(l_i)$ for $h \neq 0$ and $f(\mu l_i) = f(l_i) + \ln(\mu)$ for $h = 0$, the space Ω has the following property. If $h \neq 0$, then for any positive number $c \in \mathbf{R}_{>0}$, $c\Omega = \{cx | x \in \Omega\} = \Omega$. If $h = 0$, then for any $c \in \mathbf{R}$, $\Omega + c(1, 1, \ldots, 1) = \{x + c(1, 1, \ldots, 1) | x \in \Omega\} = \Omega$. It follows that the space $P_{E^2}(S, T)/\mathbf{R}_{>0}$ of all Euclidean polyhedral metrics modulo scaling is diffeomorphic, under the map $u = u(l)$, to the set $\Omega \cap P_h$ where $P_h = \{u = (u_1, \ldots, u_n) \in \mathbf{R}^E | \sum_{i=1}^n u_i = -h\}$.

By Theorem 3.2(a), if l_i, l_j, l_k are the edge lengths of a Euclidean triangle with opposite angles $\theta_i, \theta_j, \theta_k$, then the differential 1-form

$$\omega_h = \int_{\pi/2}^{\theta_i} \sin^h(t) dt du_i + \int_{\pi/2}^{\theta_j} \sin^h(t) dt du_j + \int_{\pi/2}^{\theta_k} \sin^h(t) dt du_k$$

is closed and its integral $F(u_i, u_j, u_k) = \int_{(\pi/2, \pi/2, \pi/2)}^{(u_i, u_j, u_k)} \omega_h$ has a positive semidefinite Hessian matrix whose null space is generated by (u_i, u_j, u_k) if $h \neq 0$ and by $(1, 1, 1)$ if $h = 0$.

Now for $u = (u_1, \dots, u_n) \in \Omega$, define the energy function

$$W(u) = \sum_{\{e_a,e_b,e_c\} \in T} F(u_a, u_b, u_c)$$

where the sum is over all triangles $\{e_a, e_b, e_c\}$ in T with edges e_a, e_b, e_c. By the construction, the function $W : \Omega \to \mathbf{R}$ has a positive semi-definite Hessian matrix so that the gradient of W is the $-\phi_h$ curvature. To establish Theorem 4.1(b), it suffices to prove that the gradient map $\bigtriangledown W$ restricted to $\Omega \cap P_h$ is an immersion.

Lemma 4.2. *The restriction map $W| : \Omega \cap P_h \to \mathbf{R}$ has a positive definite Hessian matrix. In particular, the associated gradient map $\bigtriangledown(W|) : \Omega \cap P_h \to P_0 = \mathbf{R}^{n-1}$ is a local diffeomorphism.*

Proof. Let $A : \mathbf{R}^n \to P_0$ be the orthogonal projection. Then by definition, $\bigtriangledown(W|) = A(\bigtriangledown(W)|)$ on the subspace $\Omega \cap P_h$. Furthermore, the restriction of the Hessian $Hess(W)$ to $P_0 \times P_0$ is the Hessian matrix $Hess(W|)$ of $W|$. Thus it suffices to show that the null space of the Hessian $Hess(W)$ at a point u is transverse to the plane P_0. To see this, take a vector $v = (v_1, \dots, v_n) \in \mathbf{R}^n$ so that

$$vHess(W)|_u v^t = 0$$

where v^t is the transpose of the row vector v. By the definition of the function W, the above identity is equivalent to

$$\sum_{\{e_a,e_b,e_c\} \in T} (v_a, v_b, v_c) Hess(F)|_{(u_a,u_b,u_c)} (v_a, v_b, v_c)^t = 0.$$

Each term in the summation is non-negative due to the convexity of F. It follows that all terms are vanishing, i.e.,

$$(v_a, v_b, v_c) Hess(F)|_{(u_a,u_b,u_c)} (v_a, v_b, v_c)^t = 0.$$

By Corollary 3.3, there is a constant $C_{\{a,b,c\}}$ depending only on the triangle $\{e_a, e_b, e_c\}$ so that if $h \neq 0$,

$$(v_a, v_b, v_c) = C_{\{a,b,c\}}(u_a, u_b, u_c) \tag{4.3}$$

and if $h = 0$,

$$(v_a, v_b, v_c) = C_{\{a,b,c\}}(1, 1, 1). \tag{4.4}$$

Take two triangles sharing the same edge e_a; say these triangles are $\{e_a, e_b, e_c\}$ and $\{e_a, e_i, e_j\}$. Then (4.3) and (4.4) hold for both triangles. In the case $h \neq 0$, by (4.3), we obtain $C_{\{a,b,c\}} u_a = C_{\{a,i,j\}} u_a = v_a$. Since $u_a \neq 0$ for $h \neq 0$, this implies that $C_{\{a,b,c\}} = C_{\{a,i,j\}}$. On the other hand, the surface is assumed to be connected. Thus there is a constant C so that $C_{\{a,b,c\}} = C$ for all triangles $\{e_a, e_b, e_c\}$. By the same argument using (4.4), we conclude that for $h = 0$, $C_{\{a,b,c\}}$ is again a constant. In summary, we have $v = Cu$ if $h \neq 0$ and $v = C(1, 1, \dots, 1)$ if $h = 0$. Since the vector v is not in the subspace P_0 unless $v = 0$, this shows that $W| : \Omega \cap P_h \to \mathbf{R}$ has a positive definite Hessian matrix. By Lemma 1.10, it follows that $\bigtriangledown(W)| : \Omega \cap P_h \to \mathbf{R}^n$ is an immersion. q.e.d.

4.4. A proof of Theorem 4.1(c). The proof is straightforward due to the strict convexity of the energy functional in Theorem 3.4(b) (by replacing h by $-h$). Namely, given a hyperbolic triangle of edge lengths $l_k = r_i + r_j$, $\{i, j, k\} = \{1, 2, 3\}$, and opposite angles θ_k, the differential 1-form

$$\eta_h = \sum_{i=1}^{3} \int_{\pi/2}^{\theta_i} \tan^h(t/2)dtdu_i$$

is closed where $u_i = \int_1^{r_i} \sinh^{h-1}(t)dt$. Furthermore, the Hessian matrix of the function $F(u_1, u_2, u_3) = \int^u \eta_h$ is positive definite. By definition, we have

$$\frac{\partial F}{\partial u_i} = \int_{\pi/2}^{\theta_i} \tan^h(t)dt.$$

Let $V = \{v_1, \ldots, v_m\}$ be the set of all vertices in the triangulation. For a hyperbolic circle packing metric $r : V \to \mathbf{R}_{>0}$, define $u = u(r) : V \to \mathbf{R}$ by $u_i = f(r(v_i))$ where $f(x) = \int_1^x \sinh^{h-1}(t)dt$. The image of $\mathbf{R}^V_{>0}$ under the map $u = u(r)$ is the open cube J^V where $J = f(\mathbf{R}_{>0})$. Define a smooth function W on J^V by

$$W(u_1, \ldots, u_m) = \sum_{\{v_a, v_b, v_c\} \in T} F(u_a, u_b, u_c) \tag{4.5}$$

where the sum is over all triangles with vertices v_a, v_b, v_c.

By the construction, W has a positive definite Hessian matrix and its gradient is the h-th discrete curvature k_h.

Thus Theorem 4.1(d) follows from Lemma 1.10 applied to the energy function W.

Remark 4.3. For $h = 0$, the above proof was first given by Colin de Verdière [**11**].

4.5. A proof of Theorem 4.1(d). The proof of part (d) is essentially the same as that of parts (b), (c). We sketch the main steps. First, by Theorem 3.4(a) (by replacing h by $-h$), the integral $F(u) = \int^u \eta$ of the closed 1-form

$$\eta = \sum_{i=1}^{3} \int^{\theta_i} \tan^h(t/2)dtdu_i$$

is locally concave where $u_i = \frac{1}{h} r_i^h$ for $h \neq 0$ and $u_i = \ln r_i$ for $h = 0$. Furthermore, by Corollary 3.5, the null space of the Hessian of $F(u)$ at a point u is generated by u if $h \neq 0$ and by $(1, 1, 1)$ if $h = 0$. Now for a Euclidean circle packing metric $r : V \to \mathbf{R}_{>0}$, define a new function $u : V \to \mathbf{R}$ by $u_i = f(r_i)$ where $f(t) = \frac{1}{h} t^h$ for $h \neq 0$ and $f(t) = \ln(t)$ for $h = 0$. We write $u = (u_1(r), \ldots, u_m(r)) \in \mathbf{R}^V$. The image of $\mathbf{R}^V_{>0}$

under $u = u(r)$ is an open convex cube I^V where $I = f(\mathbf{R}_{>0})$. Define a function W on I^V by the same formula (4.5). Then this function W is concave with gradient $\bigtriangledown W$ equal to the h-th discrete curvature. The space $CP_{E^2}(S, \mathcal{T})/\mathbf{R}_{>0}$ of all circle packing metrics modulo scaling is homeomorphic to $I^V \cap P_h$ where $P_h = \{(u_1, \ldots, u_m) \in \mathbf{R} | \sum_{i=1}^k u_i = h\}$ under $u = u(l)$. Now due to Corollary 3.5 and by the same argument as in §4.3, the function $W| : I^V \cap P_h \to \mathbf{R}$ is strictly concave. Thus by the convexity of $I^V \cap P_h$, the map $\bigtriangledown W| : I^V \cap P_h \to \mathbf{R}^V$ is an embedding.

5. Parameterizations of the Teichmüller Space of a Surface with Boundary

5.1. Ideal triangulated surfaces. An *ideal triangulation* of a compact surface is defined as follows. Take a closed triangulated surface $(X, \mathcal{T}^*)$ with vertex set V. Let $N(V)$ be a small open regular neighborhood of V. Then $S = X - N(V)$ is a compact surface with an ideal triangulation $\mathcal{T} = \{\sigma \cap S | \sigma \in \mathcal{T}^*\}$. It is well known that each compact surface S with $\partial S \neq \emptyset$ and negative Euler characteristic admits an ideal triangulation. We use E to denote the set of all edges in $\mathcal{T}$, i.e., $E = \{e \cap S | e \text{ an edge in } T^*\}$. If σ is a triangle in $\mathcal{T}^*$, we call $\sigma \cap S$ a *hexagon* (a 2-cell) in $\mathcal{T}$ and we use T to denote the set of all hexagons in $\mathcal{T}$.

The geometric realization of a hexagon is the right-angled hyperbolic hexagon. The following is a well known result. See [**6**], [**22**] for a proof.

Lemma 5.1. *For any $l_1, l_2, l_3 \in \mathbf{R}_{>0}$, there exists a hyperbolic right-angled hexagon, unique up to isometry, whose three pairwise non-adjacent edges have lengths l_1, l_2, l_3.*

5.2. Coordinates for Teichmüller spaces and the main results. Suppose $(S, \mathcal{T})$ is an ideal triangulated compact surface with boundary. For each $h \in \mathbf{R}$, we produce a parameterization ψ_h of the Teichmüller space of the surface S in this section. For $h = 0$, this parameterization was first found in [**27**].

Let $E = \{e_1, \ldots, e_m\}$ be the set of all edges in $\mathcal{T}$. As a convention, if $x : E \to X$ is a function, we use x_i to denote $x(e_i)$. Given $\mathcal{T}$ and any $l : E \to \mathbf{R}_{>0}$, we produce a hyperbolic metric with geodesic boundary on S by making each hexagon in $\mathcal{T}$ a right-angled hexagon of given length $l(e)$'s (using Lemma 5.1) and gluing them isometrically along edges. Thus each vector $l \in \mathbf{R}^E_{>0}$ produces a hyperbolic metric with geodesic boundary, still denoted by l, on S. It is known that each hyperbolic metric with geodesic boundary on S is Teichmüller equivalent to exactly one such metric l. (See for instance [**44**], [**27**] for a proof.) This gives a parameterization of the Teichmüller space $Teich(S)$ of S by $\mathbf{R}^E_{>0}$.

Let $g_h(x) = \int_0^x \cosh^h(t)dt$. For a hyperbolic metric with geodesic boundary l on S, recall that the ψ_h *curvature* of the metric is defined to be

$$\psi_h(e) = g_h\left(\frac{b+c-a}{2}\right) + g_h\left(\frac{b'+c'-a'}{2}\right)$$

where b, c, b', c' are the lengths of the edges (in ∂S) adjacent to the edge e and a, a' are the lengths of the edges (in ∂S) facing the edge e. See figure 1. Denote $\Psi_h : \mathbf{R}^E_{>0} \to \mathbf{R}^E$ the map sending a metric l to its ψ_h curvature.

Theorem 5.2. *For any $h \in \mathbf{R}$ and an ideal triangulated surface $(S, \mathcal{T})$, the map $\Psi_h : Teich(S) \to \mathbf{R}^E$ is a smooth embedding.*

An *edge cycle* in $(S, \mathcal{T})$ is an edge loop in the 1-skeleton of the dual cellular decomposition of (S, T). To be more precise, an edge cycle consists of edges $\{e_{i_1}, \ldots, e_{i_r}\}$ in E and hexagons $\{H_1, \ldots, H_r\}$ in $\mathcal{T}$ so that for all indices s, e_{i_s} and $e_{i_{s+1}}$ are adjacent to a hexagon H_i where $e_{i_{r+1}} = e_{i_1}$.

Theorem 5.3. *For any $h \in \mathbf{R}$ and an ideal triangulated surface $(S, \mathcal{T})$, $\Psi_h(Teich(S)) = \Psi_0(Teich(S)) = \{z \in \mathbf{R}^E |$ for each edge cycle $\{e_{i_1}, \ldots, e_{i_m}\}$, $\sum_{s=1}^m z(e_{i_s}) > 0\}$. Furthermore, the image $\Psi_h(Teich(S))$ is an open convex polytope independent of the parameter $h \geq 0$.*

For $h < 0$, we conjecture that a similar result holds, i.e., if $h < h' < 0$, then $\Psi_h(Teich(S)) \subset \Psi_{h'}(Teich(S))$ and each $\Psi_h(Teich(S))$ is an open convex polytope. This has been established by Ren Guo [**16**].

Remark 5.4. (a) Theorems 5.2 and 5.3 were proved for $h = 0$ in [**27**].

(b) Whether these new coordinates are related to the quantum Teichmüller space ([**14**], [**23**], [**3**], [**42**]) is an interesting question.

(c) An edge cycle $\{e_{i_1}, \ldots, e_{i_m}\}$ is called *fundamental* if each edge appears at most twice. It is proved in [**27**] that the convex set $\{z \in \mathbf{R}^E |$ for each edge cycle $\{e_{i_1}, \ldots, e_{i_m}\}$, $\sum_{s=1}^m z(e_{i_s}) > 0\}$ is defined by a finite set of linear inequalities $\sum_{s=1}^m z(e_{i_s}) > 0$ where $\{e_{i_1}, \ldots, e_{i_m}\}$ is a fundamental edge cycle. Thus, $\Psi_h(Teich(S))$ is an open convex polytope in $\mathbf{R}^E$.

5.3. A proof of Theorem 5.2. The proof of Theorem 5.2 is very similar to that of Theorem 4.1(c) and is a simple application of the strict convexity of the energy functions introduced in §3. By Theorem 3.4(f) (by replacing h by $-h-1$) and Legendre transformation, for a hyperbolic right-angled hexagon of three non-adjacent edge lengths l_i, l_j, l_k and opposite edge lengths $\theta_i, \theta_j, \theta_k$ where $\theta_i = r_j + r_k$, $i \neq j \neq k \neq i$, the following 1-form

$$\omega = \sum_{i=1}^{3} \int_0^{r_i} \cosh^h(t)dtd(\int_1^{l_i} \tanh^{h+1}(t/2)dt)$$

is closed. Let $u_i = \int_1^{l_i} \tanh^{h+1}(t/2)dt$ and $u = (u_1, u_2, u_3)$. Then

$$w = \sum_{i=1}^{3} \int_0^{r_i} \cosh^h(t)dtdu_i,$$

and the integral $F(u) = \int_{(1,1,1)}^{u} w$ has a negative definite Hessian matrix in J^3 where $J = f(\mathbf{R}_{>0})$ and $f(x) = \int_1^x \tanh^{h+1}(t/2)dt$. Furthermore, $\frac{\partial F}{\partial u_i} = \int_0^{r_i} \cosh^h(t)dt$.

For a hyperbolic metric $l : E \to \mathbf{R}_{>0}$ on $(S, \mathcal{T})$, let $u : E \to \mathbf{R}$ be $u_i = \int_1^{l(e_i)} \tanh^{h+1}(t/2)dt$. Then the set of all possible values of u forms the open convex cube J^E. Define an energy function $W : J^E \to \mathbf{R}$ by

$$W(u) = \sum_{\{e_a,e_b,e_c\}\in T} F(u_a, u_b, u_c)$$

where the sum is over all hexagons with edges e_a, e_b, e_c. By definition and exactly the same argument as in §4.2, W has a negative definite Hessian matrix. Furthermore, by the construction of F,

$$\frac{\partial W}{\partial u_i} = \psi_h(e_i)$$

i.e., $\bigtriangledown W = \Psi_h$. By Lemma 1.10, we conclude that the map $\bigtriangledown W : J^E \to \mathbf{R}^E$ is a smooth embedding. This proves Theorem 5.2.

5.4. Degenerations of hyperbolic hexagons. Suppose a hyperbolic right-angled hexagon has three non-pairwise adjacent edge lengths l_1, l_2, l_3 and opposite edge lengths $\theta_1, \theta_2, \theta_3$ so that the edge of length θ_i is opposite to the edge of length l_i. Let r_i be $\frac{1}{2}(\theta_j + \theta_k - \theta_i)$ $(i, j, k$ distinct) and call r_i the *r-coordinate* at the edge of length θ_i. Note that $\theta_i = r_j + r_k$.

Lemma 5.5. (a) *If $i \neq j$, then $l_j \geq \cosh^{-1}(\coth(\theta_i))$. In particular, $\lim_{\theta_i \to 0} l_j(\theta_1, \theta_2, \theta_3) = \infty$ so that the convergence is uniform in $\theta = (\theta_1, \theta_2, \theta_3)$.*

(b) *$|r_i| \geq \cosh^{-1}(\frac{1}{2}\coth(\frac{l_i}{2}))$. In particular, $\lim_{l_i \to 0} |r_i(l_1, l_2, l_3)| = \infty$ so that the convergence is uniform in l.*

(c) *Suppose a sequence of hexagons satisfies that $|r_1|, |r_2|, |r_3|$ are uniformly bounded. Then $\lim_{l_i \to \infty} \theta_j(l)\theta_k(l) = 0$ so that the convergence is uniform in l.*

Proof. For (a), we use the cosine law that

$$\begin{aligned}\cosh(l_j) &= \frac{\cosh(\theta_j) + \cosh(\theta_i)\cosh(\theta_k)}{\sinh(\theta_i)\sinh(\theta_k)} \\ &\geq \frac{\cosh(\theta_i)\cosh(\theta_k)}{\sinh(\theta_i)\sinh(\theta_k)} \\ &\geq \coth(\theta_i).\end{aligned}$$

Since $\lim_{\theta_i \to 0} \coth(\theta_i) = \infty$, it follows that $\lim_{\theta_i \to 0} l_j = \infty$ and the convergence is uniform in θ_i.

For (b), we use the tangent law (2.10) for hexagons that

$$\tanh^2(l_i/2) = \frac{\cosh(r_j)\cosh(r_k)}{\cosh(r_i)\cosh(r_i+r_j+r_k)}$$

$$= \frac{1}{\cosh(r_i)[\cosh(r_i)(1+\tanh(r_j)\tanh(r_k)) + \sinh(r_i)(\tanh(r_j)+\tanh(r_k))]}$$

$$\geq \frac{1}{\cosh(r_i)[\cosh(r_i)(1+1) + |\sinh(r_i)|(1+1)]}$$

$$\geq \frac{1}{4\cosh^2(r_i)}.$$

It follows that $\cosh^2(r_i) \geq \frac{1}{4\tanh^2(l_i/2)}$. Thus part (b) follows and the convergence is uniform in l.

For part (c), by the assumption that $|r_i|$'s are uniformly bounded, it follows that $\theta_i = r_j + r_k$ are uniformly bounded from above. Now the cosine law says that

$$\cosh(l_i) = \frac{\cosh(\theta_i) + \cosh(\theta_j)\cosh(\theta_k)}{\sinh(\theta_j)\sinh(\theta_k)}$$

$$\leq \frac{C}{\sinh(\theta_j)\sinh(\theta_k)}$$

for some constant C. Thus $\sinh(\theta_j)\sinh(\theta_k) \leq \frac{C}{\cosh(l_i)}$. Since $\sinh(t) \geq t$ for $t \geq 0$, it follows that $\lim_{l_i \to \infty} \theta_j \theta_k = 0$ and the convergence is uniform in l. q.e.d.

5.5. A proof of Theorem 5.3. Recall that $\Psi_h : \mathbf{R}^E_{>0} \to \mathbf{R}^E$ is the map sending a hyperbolic metric $l \in \mathbf{R}^E_{>0}$ to its ψ_h curvature. Let Ω be the convex set $\{z \in \mathbf{R}^E |$ whenever $e_{i_1}, \ldots, e_{i_k}$ form an edge cycle, $\sum_{j=1}^k z(e_{i_j}) > 0\}$. We will prove the theorem in two steps. First, we show $\Psi_h(\mathbf{R}^E_{>0}) \subset \Omega$. Then, we show $\Psi_h(\mathbf{R}^E_{>0})$ is a closed subset of Ω. Since Theorem 5.2 shows that $\Psi_h(\mathbf{R}^E_{>0})$ is open in Ω, it follows from the connectivity of Ω that $\Psi_h(\mathbf{R}^E_{>0}) = \Omega$.

To see $\Psi_h(\mathbf{R}^E_{>0}) \subset \Omega$, take a hyperbolic metric $l \in \mathbf{R}^E_{>0}$ and an edge cycle $\{e_{n_1}, \ldots, e_{n_k}\}$ with associated hexagons $\{H_1, \ldots, H_k\}$. Let $z = \Psi_h(l)$ and a_i be the length of the edge in the hexagon H_i adjacent to both e_{n_i} and $e_{n_{i+1}}$. Denote the lengths of the edges in H_i opposite to e_{n_i} and $e_{n_{i+1}}$ by b_i and c_i. Then by definition, $\sum_{i=1}^m z(e_{n_i}) = \sum_{i=1}^k (\int_0^{\frac{a_i+b_i-c_i}{2}} \cosh^h(t)dt + \int_0^{\frac{a_i+c_i-b_i}{2}} \cosh^h(t)dt)$ where we have rearranged the sum according to the hexagon H_i. Now each term in the above summation is positive due to $\int_0^{t+c} \cosh^h(s)ds + \int_0^{t-c} \cosh^h(s)ds > 0$ for any $t > 0$. It follows that the sum $\sum_{i=1}^k z(e_{n_i}) > 0$.

To see that $\Psi_h(\mathbf{R}^E_{>0})$ is closed in Ω, take a sequence $l^{(m)} \in \mathbf{R}^E_{>0}$ so that $\Psi_h(l^{(m)})$ converges to a point $b \in \Omega$. We claim that $l^{(m)}$ contains a subsequence converging to a point $a \in \mathbf{R}^E_{>0}$. By taking a subsequence, we may assume that $\lim_{m\to\infty} l^{(m)} = a \in [0,\infty]^E$. The goal is to use $b \in \Omega$ to show that for each edge $e \in E$, $a(e) \in (0,\infty)$.

Suppose otherwise that there is an edge $e \in E$ so that $a(e) \in \{0,\infty\}$. We will derive a contradiction for the two cases $a(e) = 0$ and $a(e) = \infty$.

Call an edge of a hexagon in $\mathcal{T}$ which appears in ∂S a *boundary arc.* The *r-coordinate* of a boundary arc s is the r-coordinate (introduced in §5.4) of s in the unique hexagon containing it.

Case 1. $a(e) = 0$ for some $e \in E$, i.e., $\lim_{m\to\infty} l^{(m)}(e) = 0$. Let H, H' be the hexagons containing e and r_m, r'_m be the r-coordinates of boundary arcs in H, H' (in metric $l^{(m)}$) opposite to e. By Lemma 5.5(b), we have $\lim_{m\to\infty} |r_m| = \lim_{m\to\infty} |r'_m| = \infty$. Due to the assumption that $b \in \Omega$,

$$z_m(e) = \int_0^{r_m} \cosh^h(t)dt + \int_0^{r'_m} \cosh^h(t)dt$$

is bounded in m. On the other hand, $h \geq 0$ implies that $\int_0^\infty \cosh^h(t)dt = \infty$. It follows that one of the limits $\lim_{m\to\infty} r_m$ or $\lim_{m\to\infty} r'_m$ is ∞ and other one is $-\infty$. Say $\lim_{m\to\infty} r_m = -\infty$. Let r''_m and r'''_m be the r-coordinates of the other two boundary arcs of H (in metric $l^{(m)}$). Since $r_m + r''_m$ and $r_m + r'''_m$ are both non-negative (being the length of an edge), we obtain $\lim_{m\to\infty} r''_m = \infty$ and $\lim_{m\to\infty} r'''_m = \infty$.

We summarize the above discussion to two rules governing the r-coordinates of boundary arcs in hexagons in the metrics $l^{(m)}$.

Rule I. If the r-coordinate of a boundary arc converges to $-\infty$, then the r-coordinates of the other two boundary arcs in the same hexagon converge to ∞.

Rule II. If x and y are two boundary arcs opposite to an edge so that the r-coordinate of x converges to $\pm\infty$, then the r-coordinate of y converges to $\mp\infty$.

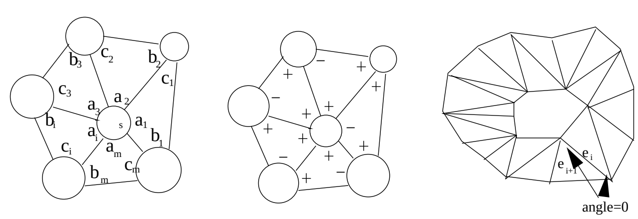

Figure 3

We claim that these two rules are contradicting to each other on $(S, \mathcal{T})$. Indeed, since $a(e) = 0$, there exists a boundary arc whose r-coordinates converge to $\pm\infty$. Now use Rule II to find a boundary arc, denoted by a_1, whose r-coordinates converge to $-\infty$. Say this boundary arc lies in the boundary component s of ∂S. See figure 3. Label all boundary arcs in s by $a_1, a_2, \ldots, a_n$ in a cyclic order so that a_i is in the hexagon H_i. Let the other two boundary arcs in H_i be b_i and c_i so that c_i and b_{i-1} are opposite to the same edge in $\mathcal{T}$. Assign a boundary arc $+$ (or $-$) if its r-coordinates converge to ∞ (or $-\infty$) as $m \to \infty$. In particular, a_1 is assigned $-$. Rule I says both b_1 and c_1 are assigned $+$. Now applying Rule II to c_2, b_1, we see that c_2 is assigned $-$. Now applying Rule I to c_2 and H_2, we see that b_2 and a_2 are assigned $+$. Repeating this, we see that b_i are assigned $+$ and c_i are assigned $-$. Therefore a_i are assigned $+$. However, $a_{n+1} = a_1$ is assigned $-$ by the choice of a_1. This is a contradiction.

The above argument shows that if the r-coordinates of a boundary arc in the metrics $l^{(m)}$ are not bounded, then we obtain a contradiction. Therefore, we may assume r-coordinates of all boundary arcs are bounded.

Case 2. There exists an edge e so that $a(e) = \infty$, i.e., $\lim_m l^{(m)}(e) = 0$, and all r-coordinates of boundary arcs in the metrics $l^{(m)}$ are bounded. Let H be a hexagon containing the edge e. By the assumption $\lim_m l^{(m)}(e) = \infty$ and Lemma 5.5(a), (c), there exists an edge e' in H and a boundary arc x in H adjacent to e and e' so that

1) The length of x in $l^{(m)}$ tends to 0 as m tends to ∞ (by Lemma 5.5(c) applied to $l^{(m)}(e) \to \infty$), and
2) $\lim_m l^{(m)}(e') = \infty$ (by Lemma 5.5(a) applied to $l^{(m)}(x) \to 0$).

By repeatedly using these two properties, we obtain a cycle of edges, say $\{e_{n_1}, \ldots, e_{n_k}\}$ so that

1) $\lim_m l^{(m)}(e_{n_i}) = \infty$,
2) e_{n_i} and $e_{n_{i+1}}$ lie in a hexagon H_i so that $e_{n_{k+1}} = e_{n_1}$ for $i = 1, 2, \ldots, k$,
3) the length $a_i^{(m)}$ of the boundary arc in H_i adjacent to e_{n_i} and $e_{n_{i+1}}$ converges to 0 (in the metrics $l^{(m)}$).

By definition, the sum of the ψ_h curvature at $e_{n_1}, \ldots, e_{n_k}$ in metric $l^{(m)}$ is

$$\sum_{i=1}^{k} z^{(m)}(e_{n_i}) = \sum_{i=1}^{k} \int_0^{\frac{a_i^{(m)}+b_i^{(m)}-c_i^{(m)}}{2}} \cosh^h(t)dt + \int_0^{\frac{a_i^{(m)}-b_i^{(m)}+c_i^{(m)}}{2}} \cosh^h(t)dt \tag{5.1}$$

where $b_i^{(m)}$ and $c_i^{(m)}$ are the lengths of the boundary arcs in H_i facing e_{n_i} and $e_{n_{i+1}}$ and $z^{(m)}(e) = \psi_h(e)$ in the metric $l^{(m)}$. Due to the assumption that all r-coordinates are bounded in the metrics $l^{(m)}$, the lengths of all boundary arcs are bounded. In particular, $b_i^{(m)}$ and $c_i^{(m)}$ are bounded. Thus, as m tends to infinity, due to $\lim_m a_i^{(m)} = 0$, the limit of the summation $\lim_m \sum_{i=1}^k z^{(m)}(e_{n_i}) = 0$. But by definition $\lim_m \sum_{i=1}^k z^{(m)}(e_{n_i}) = \sum_{i=1}^k b(e_{n_i}) > 0$. This is a contradiction.

6. Moduli spaces of polyhedral metrics, I

In this section, we describe the spaces of all ϕ_h and ψ_h curvatures on a triangulated surface for $h = 0$ and $h \leq -1$. The main theorems are the counterparts of Theorem 5.3 for closed triangulated surfaces.

In §6.1 we analyze how triangles degenerate in $\mathbf{E}^2$, $\mathbf{H}^2$, and $\mathbf{S}^2$. The main theorems for Euclidean, hyperbolic and spherical polyhedral surfaces are proved in §6.2, §6.3, and §6.4 respectively.

6.1. Degenerations of geometric triangles. For $K^2 = \mathbf{H}^2, \mathbf{S}^2$ or $\mathbf{E}^3$, let $K^2(3) \subset [0,\infty]^3$ denote the space of all triangles in K^2 parameterized by the edge lengths $l = (l_1, l_2, l_3)$. A point $l \in \overline{K^2(3)} - K^2(3) \subset [0,\infty]^3$ is called a *degenerated triangle.* The inner angles $\theta_1, \theta_2, \theta_3$ of a degenerated triangle l are defined as follows. Take a sequence $l^{(n)} = (l_1^{(n)}, l_2^{(n)}, l_3^{(n)})$ in $K^2(3)$ converging to l so that their inner angles $(\theta_1^{(n)}, \theta_2^{(n)}, \theta_3^{(n)})$ converge to $\theta = (\theta_1, \theta_2, \theta_3) \in [0,\pi]^3$. Then we call $\theta_1, \theta_2, \theta_3$ inner angles of the degenerated triangle l. Note that θ_i's depend on the choice of the converging sequences $l^{(n)}$. For instance, a degenerated triangle of edge lengths $1, 1, 0$ can have inner angles $\alpha, \pi - \alpha, 0$ for any $\alpha \in [0,\pi]$. We use $\{i, j, k\} = \{1, 2, 3\}$ in this section.

6.1.1. Degenerated Euclidean triangles. For degenerated Euclidean triangles, we normalize the length by $\sum_{i=1}^3 l_i^{(n)} = 1$. Then $l_1 + l_2 + l_3 = 1$. There are two possibilities:

(E1) One of $l_i = 0$, and $l_j = l_k > 0$. In this case, $\theta_i = 0$.

(E2) $l_1, l_2, l_3 > 0$ and $l_i = l_j + l_k$. In this case, $\theta_i = \pi$, $\theta_j = \theta_k = 0$.

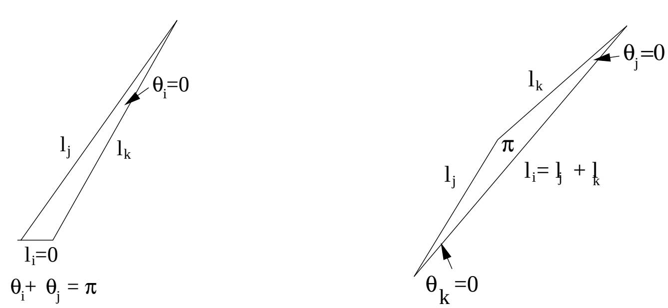

Figure 4

6.1.2. Degenerated hyperbolic triangles. There are four types of degenerated hyperbolic triangles of lengths $l \in [0, \infty]^3$ and angles $\theta \in [0, \pi]^3$:

(H1) One of the edge lengths is ∞. In this case, there are two lengths $l_i = l_j = \infty$ so that $\theta_k = 0$.

(H2) All edge lengths l_k's are finite so that some $l_i = 0$ and some $l_j > 0$. In this case, $l_k = l_j > 0$ and $\theta_i = 0$.

(H3) All l_i's are 0. In this case, $\sum_{i=1}^{3} \theta_i = \pi$.

(H4) All edge lengths are in $\mathbf{R}_{>0}$ and $l_i = l_j + l_k$. In this case, $\theta_i = \pi$, $\theta_j = \theta_k = 0$.

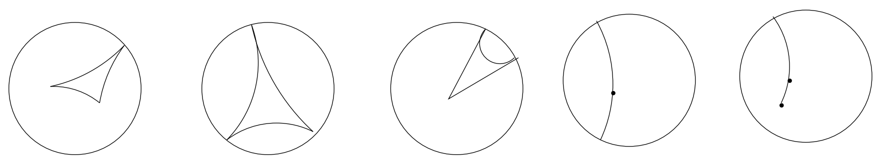

Figure 5

6.1.3. Degenerated spherical triangles. The space of all spherical triangles parameterized by the edge length is $\mathbf{S}^2(3) = \{(l_1, l_2, l_3) \in \mathbf{R}^3 | l_i + l_j > l_k, \text{ and } l_1 + l_2 + l_3 < 2\pi \text{ where } \{i, j, k\} = \{1, 2, 3\}\}$. Take a degenerated spherical triangle of edge lengths $l = (l_1, l_2, l_3)$ and inner angles $\theta_1, \theta_2, \theta_3$. Since the closure $\overline{\mathbf{S}^2(3)}$ is defined by the inequalities $l_i + l_j \geq l_k$ and $l_1 + l_2 + l_3 \leq 2\pi$, it follows that if $l_i = 0$ then $l_j = l_k$, and if $l_i = \pi$ then $l_j + l_k = \pi$. We classify degenerated spherical triangles into six types:

(S1) $l = (0, 0, 0)$. In this case, $\theta_i + \theta_j + \theta_k = \pi$.

(S2) $l_i = 0$ and $l_j = l_k = \pi$. In this case, $\theta_i = \theta_j + \theta_k - \pi$.

(S3) $l_i = 0$ and $l_j = l_k \in (0, \pi)$. In this case, $\theta_i = 0$ and $\theta_j + \theta_k = \pi$.

(S4) $l_i = \pi$ and $l_j + l_k = \pi$ so that $l_j, l_k \in (0, \pi)$. In this case, $\theta_i = \pi$ and $\theta_j = \theta_k$.

(S5) $(l_1, l_2, l_3) \in (0, \pi)^3$ and $l_i = l_j + l_k$ for some i, j, k. In this case, $\theta_i = \pi$ and $\theta_j = \theta_k = 0$.

(S6) $(l_1, l_2, l_3) \in (0, \pi)^3$ and $l_1 + l_2 + l_3 = 2\pi$. In this case, all $\theta_i = \pi$.

Note that in the last two cases (S5) and (S6), each inner angle is well defined.

6.2. Moduli spaces of Euclidean polyhedral surfaces. We will prove:

Theorem 6.1. *Suppose $(S, \mathcal{T})$ is a closed triangulated surface so that $E = \{e_1, \ldots, e_n\}$ is the set of all edges.*

(a) *The space $\Phi_0(P_{E^2}(S, \mathcal{T})) \subset \mathbf{R}^E$ is in the affine plane*

$$\mathbf{A} = \left\{ z \in \mathbf{R}^E | \sum_{e \in E} z(e) = \pi(|E| - |T|) \right\} \tag{6.1}$$

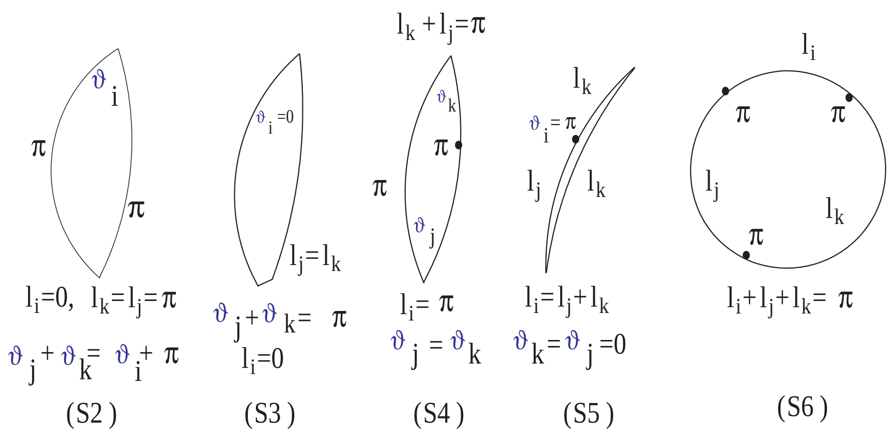

Figure 6

so that it is a connected component of the open set defined by the inequalities in (i) and bounded by $\mathbf{W}_{i;jk}$ *in (ii):*

(i) *for any proper subset* $I \subset E$ *so that no triangle has exactly two edges in* I,

$$\sum_{e \in I} z(e) < \pi|I| - \pi|F_I| \tag{6.2}$$

with $F_I = \{\sigma \in F|$ *all edges of* σ *are in* $I\}$;

(ii) *the hypersurfaces* $\mathbf{W}_{i;jk}$ *which are the* Φ_0 *image of the codimension-1 submanifold* $\{z \in \mathbf{R}^E_{>0} | z(e_i) = z(e_j) + z(e_k)$ *where* e_i, e_j, e_k *form the edges of a triangle*$\} \cap \overline{P_{E^2}(S, \mathcal{T})}$.

(b) *If* $h \le -1$, *then the space* $\Phi_h(P_{E^2}(S, \mathcal{T}))$ *is a proper smooth codimension-1 submanifold in* $\mathbf{R}^E$.

Numerical calculation shows that $\Phi_0(P_{E^2}(S, \mathcal{T}))$ is not convex in general. Polyhedral metrics $\{z \in \mathbf{R}^E_{>0} | z(e_i) = z(e_j) + z(e_k)$ where e_i, e_j, e_k form the edges of a triangle$\} \cap \overline{P_{E^2}(S, \mathcal{T})}$ are non-degenerated with respect to a different triangulation obtained by the diagonal switch surgery operation on $\mathcal{T}$.

Proof. To see part (a), first note that (6.1) is the Gauss-Bonnet theorem for Euclidean polyhedral surfaces. It follows that $\Phi_0(P_{E^2}(S, \mathcal{T})) \subset \mathbf{A}$. Let X be the space of all normalized polyhedral metrics defined by $P_{E^2}(S, \mathcal{T}) \cap \{z \in \mathbf{R}^E | \sum_{e \in E} z(e) = 1\}$. We have $\Phi_0(P_{E^2}(S, \mathcal{T})) = \Phi_0(X)$ by definition. It is a theorem of Rivin [**36**] that the map $\Phi_0 : X \to \mathbf{A}$ is an embedding. It follows that $\Phi_0(X)$ is open and connected in $\mathbf{A}$ by dimension counting. To finish the proof, we need to analyze the boundary of $\Phi_0(X)$ in $\mathbf{A}$. We will show that if $l^{(m)}$ is a sequence of polyhedral metrics in X converging to a boundary point $p \in \overline{X} - X$, then $\Phi_0(l^{(m)})$

contains a subsequence converging to a point either in $\mathbf{W}_{i;jk}$ or in an affine plane where one of the inequalities in (6.2) becomes an equality. Furthermore, we will prove that (6.2) holds for all non-degenerated polyhedral metrics.

To this end, let us assume, after taking a subsequence, that angles of each triangle (in $\mathcal{T}$) in metrics $l^{(m)}$ converge and $\Phi_0(l^{(m)})$ converges to a point $w \in \mathbf{R}^E$. There are two cases which could occur for the degenerated metric p : (1) $p(e) > 0$ for all $e \in E$ and there is a triangle with edges e_i, e_j, e_k so that $p(e_i) = p(e_j) + p(e_k)$; (2) the set $I = \{e \in E | p(e) = 0\} \neq \emptyset$ and $I \neq E$. In case (1), by definition, we have $w \in \mathbf{W}_{i;jk}$. In case (2), by the triangular inequality, there is no triangle σ having exactly two edges in I. Let F_I be the set of all triangles with all edges in I and G_I be the set of all triangles with exactly one edge in I. For angles measured in the metric p, by definition,

$$\sum_{e \in I} \phi_0(e) = \pi|I| - \sum_{\sigma \in F_I} (a + b + c) - \sum_{\sigma \in G_I} a \tag{6.3}$$

where the first sum is over triangles σ in F_I with inner angles a, b, c and the second sum is over all triangles σ in G_I with an inner angle a facing an edge in I. But the angle $a = 0$ for triangles in G_I by (E2). It follows from (6.3) that $\sum_{e \in I} \phi_0(e) = -\pi(|F_I| - |I|)$. This shows that the point w is in an affine surface defined by an equality from condition (6.3). The above argument also shows that condition (6.2) holds for non-degenerated polyhedral metrics due to $G_I \neq \emptyset$, $a > 0$, and $\sum_{\sigma \in G_I} a > 0$ for non-degenerated metrics. This establishes part (a).

To see part (b) for $h \leq -1$, by Theorem 4.1, the restriction map $\Phi_h|_X : X \to \mathbf{R}^E$ is an embedding and its image is a smooth codimension-1 submanifold. Thus it suffices to show that $\Phi_h(X)$ is a closed subset of $\mathbf{R}^E$. To this end, take a sequence $\{l^{(m)}\}$ of points in X converging to a point $p \in \overline{X} - X \subset [0,1]^E$ so that angles of each triangle (in $\mathcal{T}$) in metrics $l^{(m)}$ converge. In particular, we may assume that $\Phi_h(l^{(m)})$ converges to a point w in $[-\infty, \infty]^E$. We will show that one of the coordinates of w is infinite. Suppose otherwise that $w \in \mathbf{R}^E$. We will derive a contradiction as follows.

By the same argument as above, we see there are two cases: (1) $p(e) > 0$ for all edges e and there is a triangle σ with edges e_i, e_j, e_k so that $p(e_i) = p(e_j) + p(e_k)$, or (2) there is an edge e so that $p(e) = 0$.

In the argument below, all angles and lengths are measured in the metric p. In case (1), let the inner angles of the triangle σ be α, β, γ where $(\alpha, \beta, \gamma) = (0, \pi, 0)$ so that α faces e_j. Now let σ' be the triangle adjacent to σ along e_j and α' be the angle in σ' facing e_j. By definition,

$$\phi_h(e_j) = \int_\alpha^{\pi/2} \sin^h(t)dt + \int_{\alpha'}^{\pi/2} \sin^h(t)dt$$

is finite. Due to the divergence of $\int_{\pi/2}^{0} \sin^h(t)dt = -\infty$ for $h \leq -1$ and $\alpha = 0$, it follows that $\alpha' = \pi$. Thus, the inner angles of σ' must be $0, 0, \pi$. In summary, we obtain the following rule: if α, α' are two angles facing an edge so that $\alpha = 0$, then $\alpha' = \pi$. Now applying this rule to triangle σ', we obtain a third $(0, 0, \pi)$-angled triangle σ'' adjacent to σ'. Since there are only a finite number of triangles in T, by repeatedly applying this rule, we obtain an edge cycle $\{e_{n_1}, \ldots, e_{n_k}\}$ so that $e_{n_i}, e_{n_{i+1}}$ are adjacent to a triangle σ_i ($e_{n_{k+1}} = e_{n_1}$) and the angle of σ_i facing e_{n_i} is π. The inner angles of σ_i are $\pi, 0, 0$. We call such an edge cycle a $(\pi, 0, 0)$-*angled edge cycle*.

Lemma 6.2. *There are no* $(\pi, 0, 0)$*-angled edge cycles in a degenerated* K^2 *polyhedral metric for* $K^2 = \mathbf{E}^2$*, or* $\mathbf{H}^2$*, or* $\mathbf{S}^2$*.*

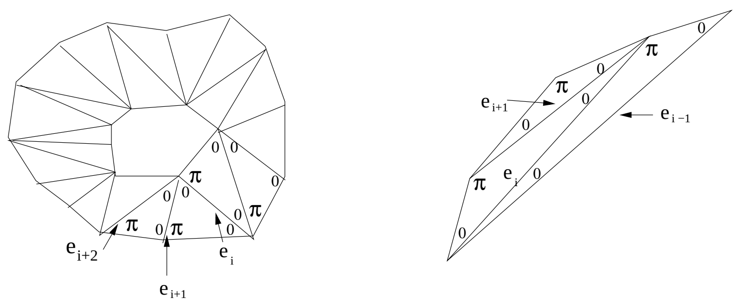

Figure 7

Proof. Suppose otherwise that such an edge cycle exists. Take a sequence of non-degenerated polyhedral metrics converging to the degenerated metric. We obtain a (non-degenerated) polyhedral metric l on $(S, \mathcal{T})$ so that the inner angle of σ_i facing e_{n_i} is larger than the other two angles in σ_i. Using the fact that in a Euclidean (or hyperbolic or spherical) triangle, the larger angle faces the edge of longer length, we see that the length $l(e_{n_i})$ of e_{n_i} is strictly larger than the length of $l(e_{n_{i+1}})$ of $e_{n_{i+1}}$. Thus, we obtain

$$l(e_{n_1}) > l(e_{n_2}) > \cdots > l(e_{n_k}) > l(e_{n_{k+1}}) = l(e_{n_1}).$$

This is absurd. q.e.d.

By this lemma, we conclude that case (1) does not occur.

In case (2) where some edge $e \in E$ has length $p(e) = 0$, there must be some $e' \in E$ so that $p(e') > 0$ due to the normalization assumption $\sum_{x \in E} p(x) = 1$. It follows that there is a triangle σ having two edges e, e' so that $p(e) = 0$ and $p(e') > 0$. Thus the inner angle α of σ facing e must be 0. By the same argument as above, if α' is the other angle facing e, then $\alpha' = \pi$. This implies that the triangle σ' containing α'

must have inner angles $(\pi, 0, 0)$. By the same argument as above, we produce a $(\pi, 0, 0)$-angled edge cycle. By Lemma 6.2, this is impossible. This ends the proof of part (b). q.e.d.

We remark that Theorem 6.1(a) is an improvement of the main result of Rivin [**36**].

Corollary 6.3. *(Rivin) The space* $\Phi_0(P_{E^2}(S, \mathcal{T})) \cap [0, \infty)^E \subset \mathbf{A}$ *is the convex polytope defined by condition (*6.2*) in Theorem 6.1 and inequalities* $0 \leq z(e) \leq \pi$ *for all* $e \in E$.

Proof. First, we have the obvious inequality that $\phi_0(e) \in [-\pi, \pi]$. We will use the same notations as above. It suffices to show that the condition $\mathbf{W}_{i;jk}$ does not arise in the limits of ϕ_0 curvatures of polyhedral metrics $\Phi_0(P_{E^2}(S, \mathcal{T})) \cap [0, \infty)^E$. Suppose otherwise that there is a sequence of metrics $\{l^{(m)}\}$ in $P_{E^2}(S, \mathcal{T}) \cap \Phi_0^{-1}([0, \infty)^E)$ so that the sequence $l^{(m)}$ converges to a degenerated polyhedral metric $p \in \mathbf{W}_{i;jk}$. By definition, $p(e) \in (0, \infty)$ for all $e \in E$ and there is a triangle σ with edges e_i, e_j, e_k so that $p(e_i) = p(e_j) + p(e_k)$. Let the two inner angles facing the edge e_i be a and a' so that a is in the triangle σ. Then $a = \pi$ and the inner angles of σ are $\pi, 0, 0$. By definition $\phi_0(e_i) = \pi - a - a'$ and $\phi_0(e) \geq 0$ for all $e \in E$. It follows that $a' = 0$. Since the only degenerated triangles in p are $(0, 0, \pi)$-angled triangles, this implies the triangle σ' adjacent to σ along e_i must have inner angles $0, 0, \pi$. To summarize, we see that the Delaunay condition that $\phi_0(e) \in [0, \infty)$ forces the propagation of $(0, 0, \pi)$-angled triangles. By repeatedly using this propagation rule, we obtain a $(0, 0, \pi)$-angled edge cycle in the degenerated metric p. But by Lemma 6.2, this is impossible. q.e.d.

6.3. Moduli space of hyperbolic polyhedral metrics.

We will prove:

Theorem 6.4. (a) *The space* $\Psi_0(P_{H^2}(S, \mathcal{T}))$ *is a connected component of the open set in* $\mathbf{R}^E$ *defined by the following inequalities in (i), (ii) and bounded by hypersurfaces* $\mathbf{W}_{i;jk}$. *Letting* $z \in \mathbf{R}^E$*:*

(i) *For each edge cycle* $\{e_{n_1}, \ldots, e_{n_k}\}$, $\sum_{i=1}^k z(e_{n_i}) > 0$.

(ii) *For any subset* I *of* E *with the property that no triangle has exactly two edges in* I, *let* F'_I *be the set of all triangles having at least one edge in* I, *then*

$$\sum_{e \in I} z(e) < \frac{\pi |F'_I|}{2}.$$

(iii) *The hypersurface* $\mathbf{W}_{i;jk}$ *which is the image under* Ψ_0 *of the codimension-1 submanifold* $\{z \in \mathbf{R}^E_{>0} | z(e_i) = z(e_j) + z(e_k)$ *where* e_i, e_j, e_k *are the edges of a triangle*$\} \cap \overline{P_{H^2}(S, \mathcal{T})}$.

(b) *For $h \le -1$, the space $\Psi_h(P_{H^2}(S,T))$ is a connected component of the open set in $\mathbf{R}^E$ bounded by $\Phi_h(P_{E^2}(S,T))$ and inequalities (i) in part (a).*

Proof. The proof of part (a) follows the same argument used in the proof of Theorem 6.1(a). We will use the same notations as in subsection 6.2. First by Leibon's rigidity theorem, Ψ_0 is a smooth embedding. It follows that $\Psi_0(P_{H^2}(S,\mathcal{T}))$ is an open connected set in $\mathbf{R}^E$. We need to determine its boundary. Take a sequence of points $\{l^{(m)}\}$ converging to a boundary point p of $P_{H^2}(S,\mathcal{T})$ in $[0,\infty]^E$ so that the angles of each triangle (in T) in the metrics $l^{(m)}$ converge and $\Psi_0(l^{(m)})$ converges to w in $\mathbf{R}^E$. We will show that w lies either in a codimension-1 hyperplane defined by an equality in (i) or (ii), or w is in the hypersurface $\mathbf{W}_{i;jk}$. Furthermore, we prove that (i) and (ii) hold for points in $P_{H^2}(S,\mathcal{T})$.

There are four types of degenerated hyperbolic triangles $(H1),\ldots,(H4)$ that appear in the degenerated metric p as discussed in §6.1 and figure 5. In the proof below, all edge lengths and angles are measured in the degenerated metric p.

Lemma 6.5. *In the type (H1) degeneration where $p(e)=\infty$ for some edge e, there exists an edge cycle $\{e_{n_1},\ldots,e_{n_k};\sigma_1,\ldots,\sigma_k\}$ so that the lengths of e_{n_i} are infinite and the angle between $e_{n_i},e_{n_{i+1}}$ in the triangle σ_i is 0.*

Proof. Take a triangle σ adjacent to e. Then by triangle inequality for edge lengths of a triangle, there is another edge e' in σ so that $p(e')=\infty$ and the angle between e,e' in σ is 0. Now considering the triangle σ' adjacent to σ along e' and repeating this process, we obtain an edge cycle $\{e_{n_1},\ldots,e_{n_k}\}$ so that the lengths of e_{n_i} are infinite and the angle between $e_{n_i},e_{n_{i+1}}$ in the triangle σ_i is 0. q.e.d.

We call the edge cycle in the lemma a $(\infty,\infty,0)$ *edge cycle.* For such an edge cycle, by the definition of ψ_0, the summation $\sum_{i=1}^m \psi_0(e_{n_i})$ is equal to the summation $\sum_{i=1}^m a_i$ where a_i is the angle between e_{n_i} and $e_{n_{i+1}}$ in the triangle adjacent to both edges. Since $a_i=0$, therefore $\sum_{i=1}^m \psi_0(e_{n_i})=0$. This shows that p is in the plane defined by the equality case of condition (i) for some edge cycle. It also shows that condition (i) holds for all hyperbolic polyhedral metrics in $P_{H^2}(S,\mathcal{T})$ since $a_i>0$ for non-degenerated triangles.

In the type (H2) and (H3) cases that $p(e)<\infty$ for all $e\in E$ and some $p(e')=0$, let $I=\{e\in E|p(e)=0\}$. By the triangular inequalities, there is no triangle with exactly two edges in I. Take a triangle σ with inner angles $\theta_i,\theta_j,\theta_k$ so that one of the edges of σ is in I. If all edges of the triangle are in I, then the sum $\theta_i+\theta_j+\theta_k=\pi$. In this case the sum

$$(6.4)\qquad (\theta_j+\theta_k-\theta_i)/2+(\theta_i+\theta_k-\theta_j)/2+(\theta_i+\theta_j-\theta_k)/2=\pi/2.$$

If only one edge of σ is in I, say the edge e_i facing θ_i is in I, then by the assumption $p(e_i)=0$ and $p(e_j)=p(e_k)>0$. The assumption implies that $\theta_i=0$ and $\theta_j+\theta_k=\pi$. Thus

$$(6.5)\qquad (\theta_j+\theta_k-\theta_i)/2=\pi/2.$$

Now the summation $\sum_{e\in I}\psi_0(e)$ can be expressed as

$$\sum_{\sigma;e_i,e_j,e_k\in I}((\theta_j+\theta_k-\theta_i)/2+(\theta_i+\theta_k-\theta_j)/2+(\theta_i+\theta_j-\theta_k)/2)$$
$$+\sum_{\sigma;e_i\in I,e_j\notin I}(\theta_j+\theta_k-\theta_i)/2,$$

where the first sum is over all triangles σ whose three edges e_i,e_j,e_k are in I and the second sum is over all triangles σ with exactly one edge e_i in I. Equalities (6.4) and (6.5) show that in both cases the contribution to $z(e)$'s from each triangle is $\pi/2$. It follows that $\sum_{e\in I}\psi_0(e)=\pi|F_I'|/2$, i.e., p lies in the plane defined by the equality case of condition (ii). This also shows that the inequality in condition (ii) holds for all metrics in $P_{H^2}(S,\mathcal{T})$ since (6.4) and (6.5) become strictly less than $\pi/2$ for non-degenerated hyperbolic triangles.

In the type (H4) degeneration, by definition, $w\in\mathbf{W}_{i;jk}$.

To prove part (b), by Theorem 4.1, $\Psi_h:P_{H^2}(S,\mathcal{T})\to\mathbf{R}^E$ is a smooth embedding and its image is an open subset of $\mathbf{R}^E$. It remains to find the boundary points of $\Psi_h(P_{H^2}(S,\mathcal{T}))$ in $\mathbf{R}^E$. Take a sequence of metrics $\{l^{(m)}\}$ converging to a boundary point p of $P_{H^2}(S,\mathcal{T})$ in $[0,\infty]^E$ so that angles of the triangle (in $\mathcal{T}$) in metrics $l^{(m)}$ converge and $\Psi_h(l^{(m)})$ converges to w in $\mathbf{R}^E$. We will show that w is either in $\Phi_h(P_{E^2}(S,\mathcal{T}))$ or in an affine surface defined by the equality case of (i) for some edge cycle. Furthermore, we will prove that (i) holds. There are four types of degenerations of hyperbolic triangles in p according to §6.1. If p contains a triangle of type (H1) where there is an edge e so that $p(e)=\infty$, then by Lemma 6.5 there exists an edge cycle of type $(\infty,\infty,0)$. Then by the same argument as in the proof of Theorem 5.3 and identity (5.1) (§5.5 case 2) where $\cosh(t)$ is replaced by $\cos(t)$ and $\lim_{m\to\infty}a_i^{(m)}=0$, we conclude that $\sum_{i=1}^m\psi_h(e_{n_i})=0$ along the edge cycle. Thus the point w is in the plane defined by an equality in (i). The proof also shows that (i) holds for all edge cycles due to $\int_0^{\frac{b+c-a}{2}}\cos^h(t)dt+\int_0^{\frac{c+a-b}{2}}\cos^h(t)dt>0$ for $a,b,c\in(0,\pi)$ and $a+b+c<\pi$.

If p satisfies $p(e)<\infty$ for all $e\in E$ and contains a triangle of type (H2), i.e., there are two edges e',e'' with $p(e')=0$ and $p(e'')>0$, we find a triangle σ with three edges e_i,e_j,e_k so that $p(e_i)=0$ and $p(b_j)=p(e_k)>0$. Let the inner angles of σ be a,b,c so that a is facing

e_i. Let σ' be the triangle adjacent to σ along e_i so that the inner angles are a', b', c' with a' facing e_i. Then by the choice of σ, $a = 0$ and $b+c = \pi$. On the other hand,

$$\psi_h(e_i) = \int_0^{\frac{b+c-a}{2}} \cos^h(t)dt + \int_0^{\frac{b'+c'-a'}{2}} \cos^h(t)dt.$$

By the assumption $\frac{b+c-a}{2}$ is $\pi/2$. Due to the divergence of $\int_0^{\pi/2} \cos^h(t)dt$ for $h \leq -1$ and the assumption that $\psi_h(e_i)$ is finite, we must have $b'+c'-a' = -\pi$. By the assumption that $a', b', c' \geq 0$ and $a'+b'+c' \leq \pi$, we must have $(a', b', c') = (\pi, 0, 0)$. Now by the same argument applied to $b' = 0$, we produce a new $(0, 0, \pi)$-angled triangle adjacent to σ'. In this way, we obtain a $(0, 0, \pi)$-angled edge cycle in the triangulation. By Lemma 6.2, this is impossible, i.e., type (H2) degenerated metric p does not occur.

If all edges in the p metric have zero length, then each triangle degenerates to a Euclidean triangle. Evidently if $a + b + c = \pi$, then $(a + b - c)/2 = \pi/2 - c$. Thus $\int_0^{(a+b-c)/2} \cos^h(t)dt = \int_0^{\pi/2-c} \cos^h(t)dt = \int_c^{\pi/2} \sin^h(t)dt$. Thus $\psi_h(e) = \phi_h(e)$. It follows that w is in the image $\Phi_h(P_{E^2}(S, \mathcal{T}))$.

In the last case all edge lengths are in $(0, \infty)$ and p contains a triangle of type (H4), i.e., there is a triangle σ with edges e_i, e_j, e_k so that $p(e_i) = p(e_j) + p(e_k)$. Then the inner angles of σ are $\pi, 0, 0$. By the same argument as in the type (H2) degeneration, due to $h \leq -1$, we see that the triangle adjacent to σ along e_i (also, e_j, e_k) has inner angles $0, 0, \pi$. It follows that there must be a $(0, 0, \pi)$-edge cycle in p. This again contradicts Lemma 6.2. q.e.d.

Corollary 6.6. *(Leibon) The space $\Psi_0(P_{H^2}(S, \mathcal{T})) \cap (0, \pi]^E$ is a convex polytope defined by condition (ii) in Theorem 6.4(a).*

Proof. It suffices to show that for the Delaunay condition where $\phi_0(e) \in (0, \pi]$, both constraints (i) and (iii) in Theorem 6.4 are not necessary.

First of all, we show that condition (iii) $\mathbf{W}_{i;jk}$ does not arise in the limits of Delaunay polyhedral metrics. Suppose otherwise that there is a sequence of metrics $\{l^{(m)}\}$ converging to p in $P_{H^2}(S, \mathcal{T}) \cap \Psi_0^{-1}((0, \pi]^E)$ so that the angles of each triangle in metrics $l^{(m)}$ converge and the sequence $\Psi_0(l^{(m)})$ converges to a point $w \in \mathbf{W}_{i;jk}$. In the degenerated metric p, let a, b, c be the inner angles in the triangle σ facing the edges e_i, e_j, e_k and let a', b', c' be angles of the triangle σ' adjacent to σ along e_i so that a, a' are facing e_i. Then $(a, b, c) = (\pi, 0, 0)$ and $(b + c - a)/2 = -\pi/2$. Since $\psi_0(e_i) = \frac{1}{2}(b+c-a+b'+c'-a') \geq 0$ and $(b'+c'-a') \leq \pi$, it follows that $(b' + c' - a')/2 = \pi/2$. This in turn implies that $\{a', b', c'\} = \{0, 0, \pi\}$ with $a' = 0$. In summary, the Delaunay condition that $\psi_0(e) \in [0, \pi]$ forces the propagation of $(0, 0, \pi)$ angled triangles. By repeatedly using this propagation rule, we construct a $(0,0,\pi)$-angled edge cycle in the degenerated metric p. But by Lemma 6.2, this is impossible. Finally, it is clear that condition (i) follows from the Delaunay condition that $\psi_0(e) > 0$. q.e.d.

6.4. The moduli space of spherical polyhedral surfaces. In this section we investigate the space of all spherical polyhedral metrics on $(S, \mathcal{T})$ in terms of the ϕ_h curvature for $h = 0$ and $h \leq -1$.

A *degenerated* spherical polyhedral metric l on a triangulated surface $(S, \mathcal{T})$ is a point in the boundary of $P_{S^2}(S, \mathcal{T}) \subset \mathbf{R}^E$, i.e., one of the triangles is degenerated. A degenerated spherical polyhedral metric is called a *bubble* if all triangles in the metric are of types (S1) and (S2) introduced in §6.1. Since a type (S2) triangle is represented by a region in the 2-sphere bounded by two geodesics of length π, i.e., a *secant*, geometrically a bubble polyhedral surface is obtained by taking a finite set of secants and identifying edges in pairs. Let $\mathbf{W}_{i,jk} = \Phi_0(\mathrm{Y})$ where $Y = \{z \in (0,\pi)^E | z(e_i) = z(e_j) + z(e_k)$ where e_i, e_j, e_k form the edges of a triangle$\} \cap \overline{\mathbf{P}_{S^2}(S, \mathcal{T})}$. Similarly, let $\mathbf{U}_{ijk} = \Phi_0(\mathrm{Z})$ where $Z = \{z \in (0,\pi)^E | z(e_i) + z(e_j) + z(e_k) = 2\pi$ where e_i, e_j, e_k form the edges of a triangle$\} \cap \overline{\mathbf{P}_{S^2}(S, \mathcal{T})}$. Both of them are codimension-1 hypersurfaces in $\mathbf{R}^E$.

Theorem 6.7. *Suppose $(S, \mathcal{T})$ is a closed triangulated connected surface so that E is the set of all edges in the triangulation.*

(a) *The space $\Phi_0(P_{S^2}(S, \mathcal{T}))$ of all ϕ_0 curvatures of spherical polyhedral metrics on $(S, \mathcal{T})$ is a connected component of the open set in $\mathbf{R}^E$ bounded by the hypersurfaces $\mathbf{W}_{i,jk}, \mathbf{U}_{ijk}$ and defined by the following set of linear inequalities: for any disjoint sets $I, J \subset E$ so that no triangle $\sigma \in T$ has exactly three edges in J, or exactly two edges in $I \cup J$,*

$$\sum_{e \in I} z(e) - \sum_{e \in J} z(e) < \pi |G(I,J)| - \pi |F(I)| + \pi(|I| - |J|), \tag{6.6}$$

where $F(I)$ consists of all triangles with all three edges in I and $G(I,J)$ consists of all triangles with either (1) two edges in J and one edge in I or (2) exactly one edge in J and the other two edges not in I.

(b) *Let $h \leq -1$. The space $\Phi_h(P_{S^2}(S, \mathcal{T}))$ of all ϕ_h curvatures of spherical polyhedral metrics on $(S, \mathcal{T})$ is a connected component of the open set in $\mathbf{R}^E$ bounded by the Φ_h images of the bubble degenerated spherical surfaces.*

6.4.1. Proof of Theorem 6.7(a). By Theorem 4.1(a), the map $\Phi_0 : P_{S^2}(S, \mathcal{T}) \to \mathbf{R}^E$ is a smooth embedding. It remains to analyze the boundary of the open set $\Omega = \Phi_0(P_{S^2}(S, \mathcal{T}))$ in $\mathbf{R}^E$. To this end, take a sequence $\{l^{(m)}\}$ in $P_{S^2}(S, \mathcal{T})$ converging to a point $p \in \overline{P_{S^2}(S, \mathcal{T})} -$

$P_{S^2}(S,\mathcal{T})$ so that the angles of each triangle in metrics $l^{(m)}$ in $\mathcal{T}$ converge and $\Phi_0(l^{(m)})$ converges to a point $w \in \partial\Omega$. If all edge lengths in the degenerated metric p are in the open interval $(0,\pi)$, then all degenerated triangles in the metric p are of types (S5) or (S6) due to the classification in §6.1. Thus by definition $w \in \mathbf{W}_{i;jk}$ or $w \in \mathbf{U}_{ijk}$ for some e_i, e_j, e_k forming edges of a triangle in T. Now if some edge lengths in the metric p are 0 or π, let

$$I = \{e \in E | p(e) = 0\}$$

and

$$J = \{e \in E | p(e) = \pi\}.$$

We have $I \cap J = \emptyset$ and $I \cup J \neq \emptyset$. Furthermore, the triangle inequality and $l_1 + l_2 + l_3 \leq 2\pi$ for edge lengths imply that no triangle $\sigma \in T$ has all edges in J, or exactly two edges in $I \cup J$. We claim that (6.6) becomes an equality for this choice of I, J. Furthermore, we shall prove that (6.6) holds for all metrics in $P_{S^2}(S,\mathcal{T})$.

In the proof below, all edge lengths and angles are measured in the degenerated metric p. Consider a triangle σ with an edge in $I \cup J$. Let $\theta_i, \theta_j, \theta_k$ be the inner angles and e_i, e_j, e_k be the edges in σ so that θ_r faces e_r. There are four possibilities: (I) all edges of σ are in I; (II) one edge of σ is in I and the other two edges are in J; (III) one edge of σ is in I and the other two are not in $I \cup J$; and (IV) one edge of σ is in J and the other two are not in $I \cup J$. We will analyze the angles θ_r in each of these four cases.

Case I: All e_r's are in I, and thus the triangle σ is of type (S1). We obtain

$$\theta_i + \theta_j + \theta_k = \pi. \tag{6.7}$$

Furthermore, the left-hand side of (6.7) is strictly greater than π for non-degenerated spherical triangles.

Case II: $e_i \in I$ and $e_j, e_k \in J$. Thus the triangle σ is of type (S2). Then, by the classification,

$$\theta_i - \theta_j - \theta_k = -\pi. \tag{6.8}$$

Furthermore, the left-hand side of (6.8) is strictly greater than $-\pi$ for non-degenerated spherical triangles.

Case III: $e_i \in I$ and $e_j, e_k \notin I \cup J$. Then σ is of type (S3) so that

$$\theta_i = 0. \tag{6.9}$$

Furthermore, the left-hand side of (6.9) is strictly greater than 0 for non-degenerated triangles.

Case IV, $e_i \in J$ and $e_j, e_k \notin I \cup J$. Then σ is of type (S4) and

$$-\theta_i = -\pi. \tag{6.10}$$

Furthermore, the left-hand side of (6.10) is strictly greater than $-\pi$ for non-degenerated triangles.

Now the left-hand side of (6.6) can be expressed as

$$\sum_{e\in I}\phi_0(e)-\sum_{e\in J}\phi_0(e)=-\sum_{e\in I}(\alpha+\beta)+\sum_{e\in J}(\alpha+\beta)+\pi(|I|-|J|) \tag{6.11}$$

where α,β are angles facing the edge e.

Break the first two summations in the right-hand side of (6.11) into groups according to the triangles of cases I, II, III, and IV. Then

$$\begin{aligned}&-\sum_{e\in I}(\alpha+\beta)+\sum_{e\in J}(\alpha+\beta)\\ &=-\sum_{\sigma\in \text{ case I}}(\theta_i+\theta_j+\theta_k)-\sum_{\sigma\in \text{ case II}}(\theta_i-\theta_j-\theta_k)\\ &\quad-\sum_{\sigma\in \text{ case III}}\theta_i-\sum_{\sigma\in \text{ case IV}}(-\theta_i).\end{aligned} \tag{6.12}$$

By equalities (6.7)–(6.10), the expression (6.12) is $-\pi|F(I)|+\pi|G(I\cup J)|$. This verifies that the condition (6.6) becomes an equality for the degenerated metric p. On the other hand, for a non-degenerated spherical triangle, the left-hand sides of (6.7)–(6.10) become strictly greater than the right-hand side. Thus the above argument shows (6.11) is strictly less than $\pi|G(I\cup J)|-\pi|F(I)|$ for any metric in $P_{S^2}(S,\mathcal{T})$, i.e., (6.6) holds for non-degenerated metrics.

6.4.2. *A proof of Theorem 6.7(b).* By Theorem 4.1(a), the map $\Phi_h : P_{S^2}(S,\mathcal{T})\to \mathbf{R}^E$ is a smooth embedding. To prove Theorem 6.7(b), we need to show that boundary points of $\Phi_h(P_{S^2}(S,\mathcal{T}))$ in $\mathbf{R}^E$ come from the images of the bubbled metrics under Φ_h. To this end, take a sequence of points $\{l^{(m)}\}$ in $P_{S^2}(S,\mathcal{T})$ converging to a point $p\in \overline{P_{S^2}(S,\mathcal{T})}-P_{S^2}(S,T)\subset[0,\infty]^E$ so that inner angles of each triangle in metrics $l^{(m)}$ in $\mathcal{T}$ converge and $\Phi_h(l^{(m)})$ converge to $w\in\mathbf{R}^E$. The goal is to show that all triangles in the metric p are of types (S1) or (S2). All angles are measured in the metric p below.

Lemma 6.8. *If α and β are two angles facing an edge so that $\alpha\in\{0,\pi\}$, then $\beta=\pi-\alpha\in\{0,\pi\}$.*

Indeed, this is due to the assumption that $\int_\alpha^{\pi/2}\sin^h(t)dt+\int_\beta^{\pi/2}\sin^h(t)dt\in\mathbf{R}$ and both integrals $\int_0^{\pi/2}\sin^h(t)dt$ and $\int_\pi^{\pi/2}\sin^h(t)dt$ diverge.

Proposition 6.9. *No triangle in p has inner angles equal to 0 or π.*

Proof. We begin by introducing some terminologies. If $\theta\in\{0,\pi\}$ and $l\in[0,\pi]$, by a *$[\theta,l]$-triangle* we mean a degenerated spherical triangle with one inner angle θ so that the length of the opposite edge is l. Thus, it suffices to show that there are no $[\theta,l]$-triangles in p. Suppose otherwise, by Lemma 6.8, p contains both $[0,l]$- and $[\pi,l]$-triangles. Let

σ be such a triangle of angles $\theta_1, \theta_2, \theta_3$ and opposite edge lengths l_1, l_2, l_3 where $\theta_1 = 0$ or π. We will discuss three cases according to $l_1 = \pi, 0$ or is in $(0, \pi)$.

Case 1: $l_1 = \pi$. By Lemma 6.8, we may assume that $\theta_1 = 0$, i.e., σ is a $[0, \pi]$-triangle. According to the classification in subsection 6.1, the type of a $[0, \pi]$-triangle is (S2) so that the inner angles $(\theta_1, \theta_2, \theta_3) = (0, 0, \pi)$ and the opposite edge lengths $(l_1, l_2, l_3) = (\pi, 0, \pi)$. Let τ be the triangle adjacent to the l_3-th edge of σ and let β be the angle in τ facing the l_3-th edge. By Lemma 6.8, $\beta = \pi - \theta_3 = 0$. Thus τ is a $[0, \pi]$-triangle. In summary, we see that $[0, \pi]$-triangles propagate through one of its edges. In particular, there exists an edge cycle so that each triangle in the cycle is a $[0, \pi]$-triangle. Since the inner angles of a $[0, \pi]$-triangle are $0, 0, \pi$ and by Lemma 6.8, this edge cycle is a $\{0, 0, \pi\}$-angled edge cycle. According to Lemma 6.2, this is impossible. Therefore, there are no $[0, \pi]$-triangle in p. By Lemma 6.8, there are no $[\pi, \pi]$-triangles in the metric p.

Case 2: $l_1 = 0$. By Lemma 6.8, we may assume that σ is a $[\pi, 0]$-triangle where $\theta_1 = \pi$ and $l_1 = 0$. According to the classification in subsection 6.1, the type of σ must be either (S1) or (S2). If σ is of type (S2), then its angles and lengths are: $(\theta_1, \theta_2, \theta_3) = (\pi, \pi, \pi)$ and $(l_1, l_2, l_3) = (0, \pi, \pi)$. Thus σ is a $[\pi, \pi]$-triangle. This is impossible by case 1. Thus σ must be of type (S1). In this case, the angles of σ must be $(\theta_1, \theta_2, \theta_3) = (\pi, 0, 0)$ and lengths $(l_1, l_2, l_3) = (0, 0, 0)$. Let τ be the triangle adjacent to σ along the l_2-th edge. Then due to $\theta_2 = 0$ and Lemma 6.8, the angle of τ facing the l_2-th edge is π. It follows that τ is a $[\pi, 0]$-triangle. Thus we see that a $[\pi, 0]$-triangle propagates through one of its edges. Therefore, we obtain an edge cycle so that all triangles in the cycle are of type $[\pi, 0]$. By the analysis above, each such $[\pi, 0]$-triangle of type (S1) has inner angles $0, 0, \pi$. Therefore, we obtain a $\{0, 0, \pi\}$-angled edge cycle. This contradicts Lemma 6.2. Using Lemma 6.8, we see that there are no $[0, 0]$-triangles in p.

Case 3: $l_1 \in (0, \pi)$. By Lemma 6.8, we may assume that σ is a $[0, l_1]$-triangle. According to the classification of degenerated triangles, the triangle σ must be of types (S3), (S4), or (S5).

If σ is of type (S3), then $(\theta_1, \theta_2, \theta_3) = (0, \pi, 0)$ and $(l_1, l_2, l_3) = (l_1, l_1, 0)$. Thus σ is also a $[0, 0]$-triangle. According to case 2, this cannot occur.

If σ is of type (S4), then $(\theta_1, \theta_2, \theta_3) = (0, 0, \pi)$ and $(l_1, l_2, l_3) = (l_1, \pi - l_1, \pi)$. This implies that σ is a $[\pi, \pi]$-triangle which is impossible by case 1.

Thus the type of σ must be (S5) so that $(\theta_1, \theta_2, \theta_3) = (0, 0, \pi)$ and $(l_1, l_2, l_3) = (l_1, l_2, l_1 + l_2) \in (0, \pi)^3$. Let τ be a triangle adjacent to σ along the l_3-th edge. Due to $\theta_3 = \pi$ and Lemma 6.8, the inner angle of τ facing the l_3-th edge must be 0. Thus τ is a $[0, l_3]$-triangle where

$l_3 \in (0, \pi)$. Thus a type (S5) $[0, l_1]$-triangle propagates through one of its edges. By the discussion above, a type (S5), $[0, l_3]$-triangle has inner angles $0, 0, \pi$, and it follows that there exists a $\{0, 0, \pi\}$-angled edge cycle in the metric p. This contradicts Lemma 6.2. q.e.d.

Now to finish the proof, we see that all degenerated triangles in p are of types (S1) or (S2). We claim all triangles in p are of types (S1) or (S2). This is due to the fact that any triangle adjacent to a triangle of types (S1) or (S2) along an edge must be degenerated (since all edge lengths of types (S1) and (S2) are 0 or π). By assumption, these adjacent triangles must be of types (S1) or (S2). Since the surface is connected and the metric p is degenerated, it follows that all triangles in p are of types (S1) or (S2).

7. Moduli spaces of polyhedral surfaces, II: circle packing metrics

In Thurston's notes [**43**], he showed that the space of all discrete curvatures k_0 of Euclidean or hyperbolic circle packing metrics on a triangulated surface is a convex polytope. (Also see [**15**], [**32**], [**30**], [**10**] for different proofs.) The goal of this section is to give a description of the space of all k_h curvatures when $h \leq -1$.

Let $(S, \mathcal{T})$ be a triangulated closed surface so that E and V are sets of all edges and vertices. Let $CP_{K^2}(S, \mathcal{T})$ be the space of all circle packing metrics on $(S, \mathcal{T})$ in K^2 geometry where $K^2 = \mathbf{S}^2, \mathbf{E}^2$, or $\mathbf{H}^2$. Recall that $K_h : CP_{K^2}(S, \mathcal{T}) \to \mathbf{R}^V$ sends a circle packing metric to its k_h-th discrete curvature.

Theorem 7.1. *Suppose $h \leq -1$ and $(S, \mathcal{T})$ is a closed triangulated surface.*

(a) *The space $K_h(CP_{E^2}(S, \mathcal{T}))$ is a proper codimension-1 hypersurface in $\mathbf{R}^V$.*
(b) *The space $K_h(CP_{H^2}(S, \mathcal{T}))$ is an open set whose boundary is contained in $K_h(CP_{E^2}(S, \mathcal{T}))$ in $\mathbf{R}^V$.*

7.1. Degeneration of Euclidean and hyperbolic triangles.

The following result on degeneration of triangles will be used to analyze the singularities that appear in the variational framework. Part of the lemma was proved already in [**30**] and [**43**].

Lemma 7.2. *([**30**], [**43**]). Suppose a Euclidean or hyperbolic triangle has edge lengths l_1, l_2, l_3 and opposite angles $\theta_1, \theta_2, \theta_3$. Let $\{i, j, k\} = \{1, 2, 3\}$ and $l_i = r_j + r_k$.*

(a) *If the triangle is hyperbolic, then $\theta_i \leq 2\tan^{-1}(\frac{1}{\sqrt{2\sinh(r_i)}})$. In particular, $\lim_{r_i \to \infty} \theta_i(r_1, r_2, r_3) = 0$ so that the convergence is uniform in $r = (r_1, r_2, r_3)$.*

(b) *If the triangle is hyperbolic and* $l_i \to \infty$*, then after taking a subsequence, one of* l_j *or* l_k*, say* l_j*, tends to* ∞*, so that the angle* θ_k *between* l_i*-th and* l_j*-th edges tends to zero.*

(c) *Suppose* $r_i \geq c$ *for a fixed constant* $c > 0$*. Then* $\lim_{r_j \to 0} \theta_i(r_1, r_2, r_3) = 0$ *and the convergence is uniform in* (r_1, r_2, r_3)*.*

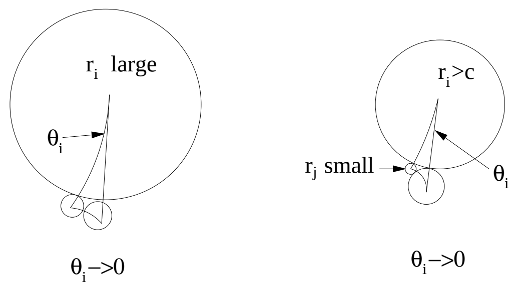

Figure 8

Proof. To see (a), recall that the tangent law for hyperbolic triangle (2.10) says,

$$\tan^2(\theta_i/2) = \frac{\sinh(r_j)\sinh(r_k)}{\sinh(r_i)\sinh(r_1+r_2+r_3)}.$$

Due to $\sinh(x+y) \geq 2\sinh(x)\sinh(y)$ for $x, y > 0$, it follows that

$$\tan^2(\theta_i/2) \leq \frac{\sinh(r_j)\sinh(r_k)}{\sinh(r_i)\sinh(r_j+r_k)} \leq \frac{1}{2\sinh(r_i)}. \tag{7.1}$$

Thus part (a) holds.

Part (b) follows from part (a). Indeed, since $l_i = r_j + r_k$ and l_i tends to infinity, one of r_j or r_k must tend to infinity after taking a subsequence. Say r_k tends to infinity. Then due to $l_j \geq r_k$, l_j converges to infinity. By part (a), θ_k tends to 0.

To see part (c) for hyperbolic triangles, using (7.1), we obtain

$$\begin{aligned}\tan^2(\theta_i/2) &\leq \frac{\sinh(r_j)}{\sinh(r_i)}\frac{\sinh(r_k)}{\sinh(r_j+r_k)}\\ &\leq \frac{\sinh(r_j)}{\sinh(c)}\frac{\sinh(r_k)}{\sinh(r_j+r_k)}\\ &\leq \frac{\sinh(r_j)}{\sinh(c)}.\end{aligned}$$

Thus part (c) follows.

To see part (c) for Euclidean triangles, recall that the radius of the inscribed circle of a Euclidean triangle is $R = \sqrt{\frac{r_1 r_2 r_3}{r_1+r_2+r_3}}$. Thus by

$\tan(\theta_i/2) = \frac{R}{r_i}$, we obtain

$$\tan^2(\theta_i/2) = \frac{r_j r_k}{r_i(r_1+r_2+r_3)} \le \frac{r_j}{c}.$$

Thus we obtain the uniform convergence of θ_i to 0.

We remark that a geometric way to see part (a) is as follows. Make the i-th vertex the Euclidean center of the Poincare disk model. For large radius r_i, the Euclidean diameter of the hyperbolic disk C of radius r_i centered at the origin is almost 1. This forces the Euclidean diameter of any hyperbolic disk tangent to C to be very small. Thus the angle θ_i is very small no matter how one chooses the radii r_j and r_k. q.e.d.

7.2. A proof of Theorem 7.1 (a). We identify the space $CP_{E^2}(S, \mathcal{T})$ of all Euclidean circle packing metrics with $\mathbf{R}^V_{>0}$ by the radius parameter. Let $X = \{r \in \mathbf{R}^V_{>0} | \sum_{v \in V} r(v) = 1\}$ be the space of all normalized circle packing metrics. By definition and Theorem 4.1, $K_h(X) = K_h(CP_{E^2}(S, \mathcal{T}))$ where $K_h| : X \to \mathbf{R}^V$ is an embedding and its image is a codimension-1 smooth submanifold. It remains to show that when $h \le -1$, $K_h(X)$ is a closed subset of $\mathbf{R}^V$. To this end, take a sequence of points $\{r^{(m)}\}$ in X so that $K_h(r^{(m)})$ converges to a point in $\mathbf{R}^V$. We will prove that $\{r^{(m)}\}$ contains a convergent subsequence in X.

Since the space X is bounded, by taking a subsequence if necessary, we may assume that $r^{(m)}$ converges to a point p in the closure $\bar{X}$ of X in $[0, \infty)^V$ and the inner angles of each triangle in metrics $r^{(m)}$ converge. If $p \in X$, we are done. If otherwise, the set $I = \{v \in V | p(v) = 0\}$ is non-empty and $I \ne V$. Since the surface S is connected, there exists a triangle $\sigma \in T$ with vertices, say v_1, v_2, v_3, so that $p(v_2) = 0$ and $p(v_1) > 0$.

We claim that $\lim_{m \to \infty} k_h^{(m)}(v_1) = \infty$ where $k_h^{(m)}$ is the k_h curvature in the metrics $r^{(m)}$. This will contradict the assumption that $\lim_m K_h(r^{(m)})$ is in $\mathbf{R}^V$.

To see the claim, consider those triangles τ having v_1 as a vertex. Let θ be the inner angle in τ at the vertex v_1. By definition,

$$k_h(v_1) = (2 - m/2)\pi - \sum_\tau \int_{\pi/2}^{\theta} \tan^h(t/2)\, dt \tag{7.2}$$

where the sum is over all such triangles τ and m is the degree of v_1.

We now analyze the angle θ. If v_j and v_k are the other two vertices of τ, then there are two cases: (1) both $p(v_j)$ and $p(r_k)$ are positive or (2) one of $p(v_j), p(v_k)$ is zero. In case (1), the triangle τ is non-degenerated since $p(v_1) > 0$ and thus $\theta \in (0, \pi)$. The contribution of $\int_{\pi/2}^{\theta} \tan^h(t/2) dt$ to the sum (7.2) is finite. In case (2), say $p(r_j) = 0$, by Lemma 7.2(c), the angle θ in the metrics $r^{(m)}$ converges to 0 as m tends to infinity. Thus the contribution of the term from the triangle τ to (7.2) is negative infinity (i.e., $\int_{\pi/2}^{0} \tan^h(t)dt = -\infty$, due to $h \le -1$). By the choice of v_1, v_2, v_3, $p(v_2) = 0$ and $p(v_1) > 0$, it follows that case (2) exists. This establishes the claim and hence the proof of Theorem 7.1(a).

7.3. A proof of Theorem 7.1(b). We again identify $CP_{H^2}(S, \mathcal{T})$ with $\mathbf{R}^V_{>0}$ by the radius parameter. By Theorem 4.1, the map $K_h : \mathbf{R}^V_{>0} \to \mathbf{R}^V$ is an embedding. The goal is to prove the image $K_h(\mathbf{R}^V_{>0})$ is a domain bounded by $K_h(CP_{E^2}(S, \mathcal{T}))$, i.e., boundary points of $K_h(\mathbf{R}^V_{>0})$ are in $K_h(CP_{E^2}(S, \mathcal{T}))$. To this end, take a sequence $\{r^{(m)}\}$ converging to a boundary point $p \in [0, \infty]^V$ of $\mathbf{R}^V_{>0}$ so that $K_h(r^{(m)})$ converges to a point $w \in \mathbf{R}^V$. We may assume, after taking a subsequence, that the inner angles of each triangle in metrics $r^{(m)}$ converge. We will show that $w \in K_h(CP_{E^2}(S, \mathcal{T}))$.

Since the point p is in the boundary of $\mathbf{R}^V_{>0}$ in $[0, \infty]^V$, there are three possibilities: (1) there is a vertex v so that $p(v) = \infty$, (2) $p(v) < \infty$ for all $v \in V$ and there are $v_1, v_2 \in V$ so that $p(v_2) = 0$ and $p(v_1) > 0$, or (3) $p(v) = 0$ for all $v \in V$.

In the first case, say $p(v_1) = \infty$. Then by Lemma 7.2(a), all angles θ at vertex v_1 converge to 0 uniformly. It follows that the h-th discrete curvature at v_1, $k_h(v_1) = (2 - m/2)\pi - \sum_\theta \int_{\pi/2}^{\theta} \tan^h(t/2)dt$ diverges to ∞ due to $h \le -1$. This contradicts $\lim_m K_h(r^{(m)}) \in \mathbf{R}^V$.

In case (2), exactly the same argument used in §7.2 works due to the fact that Lemma 7.2(c) holds for Euclidean and hyperbolic triangles. This again contradicts $\lim_m K_h(r^{(m)}) \in \mathbf{R}^V$.

The only case left is that $p(v) = 0$ for all $v \in V$. In this case, the metrics $r^{(m)}$ are degenerating to Euclidean circle packing metrics after a scaling. By Theorem 7.1(a) where $K_h(CP_{E^2}(S, \mathcal{T}))$ is closed in $\mathbf{R}^V$, it follows that $\lim_m K_h(r^{(m)})$ is in $K_h(CP_{E^2}(S, \mathcal{T}))$.

8. Open Problems

8.1. The space of all geometric triangulations with prescribed curvature. The investigation in the previous sections leads us to propose the following.

Conjecture 8.1. *Suppose $(S, \mathcal{T})$ is a closed triangulated surface. Let $\Pi : P_{K^2}(S, \mathcal{T}) \to \mathbf{R}^V$ be the curvature map sending a metric l to its k_0 curvature and $p \in \mathbf{R}^V$.*

(a) *For $K^2 = \mathbf{E}^2$ or $\mathbf{H}^2$, the space $\Pi^{-1}(p)$ is either the empty set or a smooth manifold diffeomorphic to $\mathbf{R}^{|E|-|V|}$.*

(b) *For $K^2 = \mathbf{S}^2$, the space $\Pi^{-1}(p)$ is either the empty set or a smooth manifold diffeomorphic to $\mathbf{R}^{|E|-|V|+\mu}$ where μ is the dimension of the group of conformal automorphisms of a spherical polyhedral metric $l \in \Pi^{-1}(p)$.*

One supporting evidence comes from Teichmüller spaces on surfaces with boundary so that the boundary lengths are prescribed. This was discussed in subsection 5.3. The conjecture for $S = \mathbf{S}^2$ was first investigated by S. S. Cairns in [**8**]. It was also related to the work of E. Steinitz [**40**] on the moduli space of all convex polytopes in the 3-space of the same combinatorial type. Cairns was trying to show that for spherical polyhedral metrics on $(\mathbf{S}^2, \mathcal{T})$, $\Pi^{-1}(0)$ is either homeomorphic to a Euclidean space or is the empty set. His first proof in 1941 contained a gap and later in [**8**] he proved that the set $\Pi^{-1}(0)$ is connected. The question whether $\Pi^{-1}(0)$ is a cell for spherical polyhedral metrics on the 2-sphere became Cairns conjecture ([**2**]). In [**2**], E. Bloch, R. Connelly, and D. Henderson proved that for Euclidean polyhedral metrics on a simplicially triangulated disk, the space $\Pi^{-1}(0)$ is homeomorphic to a Euclidean space. Another evidence for the conjecture comes from the work of [**36**] and [**25**]. They show that the space of all Delaunay $\mathbf{E}^2$ or $\mathbf{H}^2$ polyhedral metrics (i.e., $\psi_0(e) \geq 0$) with prescribed discrete curvature is a cell.

The following result implies that the spaces $\Pi^{-1}(p)$ are smooth manifolds in the Euclidean and hyperbolic cases.

Proposition 8.2. *Suppose $(S, \mathcal{T})$ is a closed triangulated surface. Then*

(a) *The curvature map $\Pi : P_{H^2}(S, \mathcal{T}) \to \mathbf{R}^V$ is a submersion.*
(b) *The curvature map Π defined on $P_{E^2}(S, \mathcal{T})$ is a submersion to the affine space*

$$\mathbf{A} = \left\{ z \in \mathbf{R}^V | \sum_{v \in v} z(v) = 2\pi\chi(S) \right\}$$

of $\mathbf{R}^V$ defined by the Gauss-Bonnet identity.

Proof. We will use the following notation. If v is a vertex and e is an edge having v as a vertex, we denote it by $e > v$. Given v, the set of elements in $\{e \in E | e > v\}$ will be counted with multiplicity, i.e., if the two end points of e are v, then e will be counted twice. The following simple lemma was proved in [**36**] and [**25**].

Lemma 8.3. *Suppose v is a vertex.*

(a) *(Rivin) For a Euclidean polyhedral metric, $\sum_{e>v} \phi_0(e) = 2\pi - k(v)$;*
(b) *(Leibon) $\sum_{e>v} \psi_0(e) = 2\pi - k(v)$.*

Lemma 8.4. *The linear map $L : \mathbf{R}^E \to \mathbf{R}^V$ sending a vector $z \in \mathbf{R}^E$ to $\sum_{e>v} z(e)$ is an epimorphism.*

Proof. For a finite set Z, we identify the dual space of $\mathbf{R}^Z$ with $\mathbf{R}^Z$ using the standard basis. Then the dual map $L^* : \mathbf{R}^V \to \mathbf{R}^E$ is $L^*(f)(e) =$

$\sum_{v<e} f(v)$ for $e \in E$. It suffices to show that L^* is injective. To see this, suppose $f \in \mathbf{R}^V$ so that $L^*(f) = 0$, i.e., $f(v) = -f(v')$ whenever v, v' are end points of an edge. Then $f = 0$ follows by considering a triangle with vertices v, v', v''. Indeed, we have $f(v) = -f(v') = f(v'') = -f(v)$. Thus $f(v) = 0$. q.e.d.

Now to prove Proposition 8.2(a), consider the affine map $A : \mathbf{R}^E \to \mathbf{R}^V$ so that $A(z)(v) = 2\pi - \sum_{e>v} z(e)$. Then Lemma 8.3 shows that $\Pi = A \circ \Psi_0$. It follows that $D(\Pi) = -LD(\Psi_0)$. Now by Lemma 8.4, the derivative of A is $-L$ which is surjective. By Leibon's theorem, $D(\Psi_0)$ is onto. Therefore, $D(\Pi)$ is onto. To prove Proposition 8.2(b), by Lemma 8.3, we have $\Pi = A \circ \Phi_0$. By Rivin's rigidity theorem, the rank of $D(\Phi_0)$ is $|E| - 1$. By Lemma 8.4, it follows that the rank of $D(\Pi)$ is at least $|V| - 1$. But on the other hand, by the Gauss-Bonnet formula, $\Pi(P_{E^2}(S, \mathcal{T}))$ lies in the affine space $\{z \in \mathbf{R}^V \,|\, \sum_{v \in V} z(v) = 2\pi\chi(S)\}$. Thus, the rank of $D(\Pi)$ is $|V| - 1$ and Π is a submersion to the affine space. q.e.d.

The special case of Conjecture 8.1 addresses the space $\Pi^{-1}(0)$, i.e., the space of all geometric triangulations of constant curvature metrics on a surface. There exists the obvious map $\phi : \Pi^{-1}(0) \to Teich(S)$ from $\Pi^{-1}(0)$ to the Teichmüller space by forgetting the triangulation. The fiber $\phi^{-1}(0)$ can be interpreted as the space of all geodesic triangulations isotopic to $\mathcal{T}$ in a fixed constant curvature metric. There are two related questions for ϕ. Namely, when is $\Pi^{-1}(0)$ non-empty and when is ϕ surjective? Both of these questions have been solved by a combination of the works of various authors, especially [**13**]. Call a triangulation *geometric* if there exists a constant curvature metric on the surface so that each cell in the triangulation is geodesic, i.e., the triangulation is isotopic to a geodesic triangulation in some constant curvature metric. The question whether $\Pi^{-1}(0)$ is non-empty is the same as asking if the triangulation is geometric. This was solved in the work of Koebe, Thurston [**43**], Colin de Verdière [**11**], [**13**], Marden-Rodin [**30**], and others.

Recall that a triangulation of a space is called *simplicial* if the triangulation is isomorphic to a simplicial complex. We summarize the above discussion into the following.

Proposition 8.5. *Suppose $\mathcal{T}$ is a triangulation of a closed surface S.*

(a) *([**43**], [**11**], [**24**], [**30**]) A triangulation $\mathcal{T}$ is the geometric triangulation in some constant curvature metric on S if and only if the lift of the triangulation to the universal cover is simplicial.*
(b) *([**13**]) If $\mathcal{T}$ is a geometric triangulation in some constant curvature metric, then $\mathcal{T}$ is isotopic to a geodesic triangulation in any constant curvature metric.*

A triangulation $\mathcal{T}$ of a closed surface S is said to support an *angle structure* if one can assign each vertex in each triangle a positive number, called angle, so that the sum of the angles at each vertex is 2π, and each triangle with these angle assignments becomes a K^2 geometric triangle where $K^2 = \mathbf{H}^2$ if $\chi(S) < 0$, $K^2 = \mathbf{E}^2$ if $\chi(S) = 0$, and $K^2 = \mathbf{S}^2$ if $\chi(S) > 0$. It can be shown ([**11**], [**10**]) that for closed triangulated surfaces of non-positive Euler characteristic, the existence of an angle structure is equivalent to the fact that the triangulation is geometric. However, R. Stong [**41**] has constructed a non-geometric triangulation of the 2-sphere which supports an angle structure. See also the related work of [**17**].

8.2. Global rigidity of polyhedral metrics in various curvatures. Those non-convex or concave energy functions in Theorems 3.2 and 3.4 have the corresponding variational principles on triangulated surfaces.

Problem *For any* $h \in \mathbf{R}$*,*

(a) *show that a hyperbolic polyhedral surface is determined by its* ϕ_h *curvature;*
(b) *show that a spherical polyhedral surface is determined by its* ψ_h *curvature.*

8.3. Cellular decompositions of the Teichmüller spaces. *One interesting consequence of Theorem 7.1 and Lemma 7.2 concerns the cell decompositions of the Teichmüller space, first observed in* [**33**] *for* ψ_0*-curvature.*

Recall that the arc-complex of a compact surface S with boundary is the following simplicial complex, denoted by $A(S)$. The vertices of $A(S)$ are isotopy classes $[a]$ of proper arcs a in S which are homotopically non-trivial relative to the boundary of S. A simplex in $A(S)$ is a collection of distinct vertices $[a_1], \ldots, [a_k]$ so that $a_i \cap a_j = \emptyset$ for all $i \neq j$. For instance, the isotopy class of an ideal triangulation corresponds to a simplex of maximal dimension in $A(S)$. The *non-fillable subcomplex* $A_\infty(S)$ of $A(S)$ consists of those simplexes $([a_1], \ldots, [a_k])$ so that one component of $S - \cup_{i=1}^k a_i$ is not simply connected. The simplexes in $A(S) - A_\infty(S)$ are called *fillable*.

As a convention in this subsection, if X is a simplicial complex, then $|X|$ denotes the geometric realization of X. If σ is a simplex with vertices $v_1, \ldots, v_k$, then the product space $int(\sigma) \times \mathbf{R}_{>0} = \{\sum_{i=1}^k h_i v_i | h_i > 0, \sum_{i=1}^k h_i = 1\} \times \mathbf{R}_{>0}$ can be identified naturally with the open cone over $int(\sigma)$, denoted by $c(\sigma) = \{\sum_{i=1}^k c_i v_i | c_i > 0\}$ by sending $(\sum_i h_i v_i, \lambda)$ to $\lambda \sum_i h_i v_i$. Using this convention, the product space $(|A(S)| - |A_\infty(S)|) \times \mathbf{R}_{>0}$ can be naturally identified with the union of all open cones over interior points of fillable simplexes in $|A(S)|$, i.e., each point in $(|A(S)| - |A_\infty(S)|) \times \mathbf{R}_{>0}$ is of the form $x = \sum_{i=1}^k c_i [a_i]$ where $c_i > 0$

for exactly one fillable simplex $([a_1], \ldots, [a_k])$. For $x = \sum_{i=1}^{k} c_i[a_i]$ in $(|A(S)| - |A_\infty(S)|) \times \mathbf{R}_{>0}$, let $([a_1], \ldots, [a_n])$ be an ideal triangulation containing the fillable simplex $([a_1], \ldots, [a_k])$. Assign each edge $[a_i]$ the positive number $z_i = c_i$ if $i \leq k$ and zero otherwise. Then this assignment z satisfies the positive edge cycle condition in Theorem 5.3 for the ideal triangulation $([a_1], \ldots, [a_n])$. By Theorem 5.3 for $h \geq 0$, there exists a hyperbolic metric on S whose ψ_h-coordinate in the ideal triangulation $([a_1], \ldots, [a_n])$ is z. For the ψ_0-coordinate, this fact has also been established by Hazel [**21**].

On the other hand, the following results of Ushijima [**44**] and Kojima [**24**] show that:

Theorem 8.6. *For a compact hyperbolic surface S with totally geodesic boundary, there is an ideal triangulation so that the ψ_0-coordinate of the metric in the ideal triangulation is non-negative. Furthermore, the set of all edges in the ideal triangulation with positive ψ_0-coordinate form a fillable simplex in $A(S)$ and the fillable simplex is unique.*

Combining with the observation that $\psi_0(e) > 0$ if and only if $\psi_h(e) > 0$, we can replace positivity of the ψ_0-coordinate in Theorem 8.6 by ψ_h. As a consequence, one can define an injective map

$$\Pi_h : Teich(S) \to (|A(S)| - |A_\infty(S)|) \times \mathbf{R}_{>0}$$

by sending the equivalence class of a hyperbolic metric to the point $\sum_{i=1}^{n} z_i[a_i]$ where $(a_1, \ldots, a_n)$ is the ideal triangulation produced in Theorem 8.6 and z_i is the ψ_h-coordinate of the metric at the i-th edge in the ideal triangulation. The discussion above shows that Π_h is onto. Thus we obtain (see also [**20**]):

Corollary 8.7. *For any compact surface with boundary and of negative Euler characteristic and $h \geq 0$, the map*

$$\Pi_h : Teich(S) \to (|A(S)| - |A_\infty(S)|) \times \mathbf{R}_{>0}$$

is a homeomorphism equivariant under the action of the mapping class group. In particular, for each h, the map Π_h produces a natural cell-decomposition of the moduli space of surfaces with boundary.

We remark that the underlying cells for various h's are the same. The attaching maps for the cells are different. In particular, if $h \neq h'$ and $h, h' \geq 0$, then $\Pi_{h'}^{-1}\Pi_h$ is a self-homeomorphism of the Teichmüller space preserving the cell-structure derived from Π_0 and commuting with the action of the mapping class group, i.e., it induces a self-homeomorphism of the moduli space of the surface. These self-homeomorphisms of $Teich(S)$ deserve a further study. For instance, we do not know if these maps are smooth. Finally, Guo's result [**16**] for ψ_h with $h < 0$ also produces cellular structures on the Teichmüller space $Teich(S)$.

8.4. Miscellaneous remarks. It will be interesting to know if these edge invariants ϕ_h, ψ_h, and k_h correspond to some curvatures in Riemannian geometry as triangulations become finer and converge to a Riemannian metric.

Appendix A. Proof of uniqueness of the 1-forms

The goal of this appendix is to prove the uniqueness part of Theorem 1.11.

Theorem 1.11. *For the cosine law function $y = y(x)$, all closed 1-forms of the form $w = \sum_{i=1}^{3} f(y_i)dg(x_i)$ where f, g are two non-constant smooth functions, are up to scaling and complex conjugation*

$$\omega_h = \sum_{i=1}^{3} \int^{y_i} \sin^h(t)dtd(\int^{x_i} \sin^{-h-1}(t)dt) = \sum_{i=1}^{3} \frac{\int^{y_i} \sin^h(t)dt}{\sin^{h+1}(x_i)} dx_i$$

for some $h \in \mathbf{C}$.

All closed 1-forms of the form $\sum_{i=1}^{3} f(y_i)dg(r_i)$ where f, g are two non-constant smooth functions, are up to scaling and complex conjugation

$$\eta_h = \sum_{i=1}^{3} \int^{y_i} \tan^h(t/2)dtd(\int^{r_i} \cos^{-h-1}(t)dt) = \sum_{i=1}^{3} \frac{\int^{y_i} \tan^h(t/2)dt}{\cos^{h+1}(r_i)} dr_i$$

for some $h \in \mathbf{C}$. In particular, all closed 1-forms are holomorphic or anti-holomorphic.

Proof. Let $\{i, j, k\} = \{1, 2, 3\}$. q.e.d.

Lemma A1. *Suppose $y = y(x)$ is the cosine law function and f, g are two smooth non-constant functions.*

(a) *If $f(y_i)/g(x_i)$ is independent of the indices for all x, then there are constants h, μ, c_1, c_2 so that $f(t) = c_1 \sin^h(t) \sin^\mu(\bar{t})$ and $g(t) = c_2 \sin^h(t) \sin^\mu(\bar{t})$.*
(b) *If $r_i = 1/2(x_j + x_k - x_i)$, and $f(y_i)/g(r_i)$ is independent of the indices for all r, then there are constants h, μ, c_1, c_2 so that $f(t) = c_1 \tan^h(t) \tan^\mu(\bar{t})$ and $g(t) = c_2 \cos^h(t) \cos^\mu(\bar{t})$.*

Proof. We use f_z and $f_{\bar{z}}$ to denote the partial derivatives $\frac{\partial f}{\partial z}$ and $\frac{\partial f}{\partial \bar{z}}$ respectively. Note that $\partial y_i/\partial \bar{x}_j = 0$. Taking $\frac{\partial}{\partial x_k}$ to the identity $\frac{f(y_i)}{g(x_i)} = \frac{f(y_j)}{g(x_j)}$, we obtain

$$\frac{f_z(y_i)}{g(x_i)} \frac{\partial y_i}{\partial x_k} = \frac{f_z(y_j)}{g(x_j)} \frac{\partial y_j}{\partial x_k}$$

By the derivative cosine law that $\frac{\partial y_i/\partial x_k}{\partial y_j/\partial x_k} = \frac{\sin(y_i)\cos(y_j)}{\sin(y_j)\cos(y_i)}$ and $\frac{f(y_i)}{g(x_i)} = \frac{f(y_j)}{g(x_j)}$, we obtain

$$\frac{f_z(y_i)\sin(y_i)}{f(y_i)\cos(y_i)} = \frac{f_z(y_j)\sin(y_j)}{f(y_j)\cos(y_j)}.$$

The variables y_i, y_j are independent. This shows that there is a constant $h \in \mathbf{C}$ so that

$$\frac{f_z(t)}{f(t)} = h\cot(t),$$

i.e.,

$$\frac{\partial \ln(f(z))}{\partial z} = \frac{\partial(h \ln \sin(z))}{\partial z}.$$

If we take $\frac{\partial}{\partial \bar{x}_k}$ to the equation $\frac{f(y_i)}{g(x_i)} = \frac{f(y_j)}{g(x_j)}$ and use $\partial y_i/\partial \bar{x}_k = 0$, we obtain, by the same argument as above,

$$\frac{\partial(\ln f(z))}{\partial \bar{z}} = \frac{\partial(\mu \ln \sin(\bar{z}))}{\partial \bar{z}}$$

for some constant $\mu \in \mathbf{C}$. This implies that $f(z) = c_1 \sin^h(z)\sin^\mu(\bar{z})$. Now substituting it back to $f(y_i)/g(x_i)$ and using the sine law, we obtain that $\frac{g(x_i)}{\sin^h(x_i)\sin^\mu(\bar{x}_i)}$ is independent of the indices i. Thus it must be a constant. This shows that $g(z) = c_2 \sin^h(z)\sin^\mu(\bar{z})$ for some constant c_2.

The proof of the second part (b) is exactly the same as part (a) where we use the tangent law that $\tan(y_i/2)/\cos(r_i)$ is independent of i instead of the sine law. QED.

To prove the uniqueness part of Theorem 1.11, we write the closed 1-form $w = \sum_{i=1}^3 f(y_i)dg(x_i)$ as

$$w = \sum_{i=1}^{3} f(y_i)g_z(x_i)dx_i + f(y_i)g_{\bar{z}}(x_i)d\bar{x}_i.$$

The 1-form w is closed if and only if for $i \neq j$, the expressions $\frac{\partial(f(y_i)g_z(x_i))}{\partial x_j}$ and $\frac{\partial(f(y_i)g_{\bar{z}}(x_i))}{\partial \bar{x}_j}$ are symmetric in i, j and

(A.1) $$\frac{\partial(f(y_i)g_z(x_i))}{\partial \bar{x}_j} = \frac{\partial(f(y_j)g_{\bar{z}}(x_j))}{\partial x_i}.$$

The symmetry of i, j in $\frac{\partial(f(y_i)g_z(x_i))}{\partial x_j}$ and Theorem 2.1 show that

$$f_z(y_i)g_z(x_i)\sin(x_i) = f_z(y_j)g_z(x_j)\sin(x_j).$$

By Lemma A1, there are constants c_1, c_2, α, β so that

(A.2) $$f_z(t) = c_1 \sin^\alpha(t)\sin^\beta(\bar{t})$$

and

(A.3) $$g_z(t) = c_2 \sin^{-\alpha-1}(t)\sin^{-\beta}(\bar{t}).$$

By the same argument using the symmetry of i, j in $\frac{\partial(f(y_i)g_{\bar z}(x_i))}{\partial \bar x_j}$, we obtain

$$f_{\bar z}(t) = c_3 \sin^h(t) \sin^\mu(\bar t) \tag{A.4}$$

and

$$g_{\bar z}(t) = c_4 \sin^{-h}(t) \sin^{-\mu-1}(\bar t) \tag{A.5}$$

for some constants c_3, c_4, h, μ. Substituting (A.2)–(A.5) into (A.1), we obtain

$$\begin{aligned} &c_2 c_3 \sin^h(y_i) \sin^\mu(\bar y_i) \sin^{-\alpha-1}(x_i) \sin^{-\beta+1}(\bar x_i) \cos(\bar y_k)\bar B \\ &= c_1 c_4 \sin^\alpha(y_j) \sin^\beta(\bar y_j) \sin^{-h+1}(x_j) \sin^{-\mu-1}(\bar x_j) \cos(y_k) B, \end{aligned} \tag{A.6}$$

where $B = \frac{\sin(x_i)}{\sin(y_i)}$ is a function symmetric in i, j, k. We claim (A.6) implies that $c_1c_2c_3c_4 = 0$. Indeed, suppose otherwise that $c_1c_2c_3c_4 \neq 0$. We will derive a contradiction as follows. Identity (A.6) can be written as

$$\begin{aligned} &c_2 c_3 \sin^{h-\alpha-1}(y_i) \sin^{\mu+1-\beta}(\bar y_i) \cos(\bar y_k) B^{-\alpha-1} (\bar B)^{-\beta+2} \\ &= c_1 c_4 \sin^{-h+\alpha+1}(y_j) \sin^{-\mu-1+\beta}(\bar y_j) \cos(y_k) B^{-h+2} (\bar B)^{-\mu-1}. \end{aligned}$$

As a consequence, we conclude that

$$(\sin^{h-\alpha-1}(y_i) \sin^{\mu+1-\beta}(\bar y_i))(\sin^{h-\alpha-1}(y_j) \sin^{\mu+1-\beta}(\bar y_j)) \frac{\cos(\bar y_k)}{\cos(y_k)} \tag{A.7}$$

is independent of the indices i, j, k. In particular, identity (A.7) is equal to

$$(\sin^{h-\alpha-1}(y_i) \sin^{\mu+1-\beta}(\bar y_i))(\sin^{h-\alpha-1}(y_k) \sin^{\mu+1-\beta}(\bar y_k)) \frac{\cos(\bar y_j)}{\cos(y_j)}.$$

This shows that

$$\begin{aligned} &(\sin^{h-\alpha-1}(y_k) \sin^{\mu+1-\beta}(\bar y_k)) \frac{\cos(y_k)}{\cos(\bar y_k)} \\ &\qquad = (\sin^{h-\alpha-1}(y_j) \sin^{\mu+1-\beta}(\bar y_j)) \frac{\cos(y_j)}{\cos(\bar y_j)}. \end{aligned}$$

Since y_j, y_k are independent variables, both sides must be constant. But that is impossible.

As a consequence, we see that $c_1c_2c_3c_4 = 0$. Since we assume that f and g are non-constant functions, we have $|c_1|+|c_3| \neq 0$ and $|c_2|+|c_4| \neq 0$. Now if $c_3 = 0$, then $c_1 \neq 0$ due to $|c_1| + |c_3| > 0$. But (A.6) shows that $c_1c_4 = 0$. Since $c_1 \neq 0$, we must have $c_4 = 0$. This shows, by (A.4) and (A.5), that $f_{\bar z} = g_{\bar z} = 0$, i.e., f and g are holomorphic. By (A.2) and (A.3), due to the holomorphic property of f, g, it follows that $\beta = -0$, i.e., $f(z) = c_1 \int^z \sin^\alpha(t)dt$ and $g(z) = c_2 \int^z \sin^{-\alpha-1}(t)dt$. The same argument shows that if $c_1 = 0$, then f, g are anti-holomorphic, given by (A.4) and (A.5) with $h = 0$. This establishes Theorem 1.11 for the w_h family.

The proof for the forms $w = \sum_{i=1}^{3} f(y_i)dg(r_i)$ is exactly the same, using the tangent law (that $\tan(y_i/2)/\cos(r_i)$ is independent of the indices) and Lemma A1(b). q.e.d.

Appendix B. Derivative Cosine Law of the Second Kind

Suppose that $y = y(x)$ is the cosine law function so that

$$\cos(y_i) = \frac{\cos(x_i) + \cos(x_j)\cos(x_k)}{\sin(x_j)\sin(x_k)} \tag{B.1}$$

where $\{i, j, k\} = \{1, 2, 3\}$. This convention of $\{i, j, k\} = \{1, 2, 3\}$ is assumed in this appendix.

Then we know that

$$\cos(x_i) = \frac{\cos(y_i) - \cos(y_j)\cos(y_k)}{\sin(y_j)\sin(y_k)}. \tag{B.2}$$

Identity (B.2) shows that

$$\cos(y_i) = \cos(y_j)\cos(y_k) + \sin(y_j)\sin(y_k)\cos(x_i). \tag{B.3}$$

We consider $y_i = y_i(y_j, y_k, x_i)$ and $x_j = x_j(y_j, y_k, x_i)$ as functions of y_j, y_k and x_i. Let $A^*_{ijk} = \sin(y_i)\sin(y_j)\sin(x_k)$ and $A_{ijk} = \sin(x_i)\sin(x_j)\sin(y_k)$. Both A^*_{ijk} and A_{ijk} are independent of the indices due to the sine law.

Derivative cosine law II. *The derivatives of functions $y_i = y_i(y_j, y_k, x_i)$ and $x_j = x_j(y_j, y_k, x_i)$ satisfy*

$$\frac{\partial y_i}{\partial y_j} = \cos(x_k) \tag{B.4}$$

$$\frac{\partial y_i}{\partial x_i} = \frac{A^*_{ijk}}{\sin(y_i)} = \frac{A_{ijk}}{\sin(x_i)} \tag{B.5}$$

$$\frac{\partial x_j}{\partial y_k} = -\sin(x_j)\cot(y_i) \tag{B.6}$$

$$\frac{\partial x_j}{\partial y_j} = \frac{\sin(x_k)}{\sin(y_i)} \tag{B.7}$$

$$\frac{\partial x_j}{\partial x_i} = -\frac{\sin(x_j)\cos(x_k)}{\sin(x_i)} \tag{B.8}$$

Proof. Taking derivative $\partial/\partial x_i$ to (B.3), we have

$$-\sin(y_i)\frac{\partial y_i}{\partial x_i} = -\sin(y_j)\sin(y_k)\sin(x_i).$$

Dividing it by $-\sin(y_i)$, we obtain (B.5).

To see (B.4), take $\partial/\partial y_j$ to (B.3). We obtain

$$\text{(B.9)} \qquad -\sin(y_i)\frac{\partial y_i}{\partial y_j} = -\sin(y_j)\cos(y_k) + \cos(y_j)\sin(y_k)\cos(x_i).$$

Let $c_i = \cos(x_i)$ and $s_i = \sin(x_i)$. By the sine law, then, (B.9) can be written as

$$\begin{aligned}\frac{\partial y_i}{\partial y_j} &= \frac{\sin(y_j)}{\sin(y_i)}\cos(y_k) - \cos(y_j)\cos(x_i)\frac{\sin(y_k)}{\sin(y_i)} \\ &= \frac{s_j\cos(y_k)}{s_i} - \frac{\cos(y_j)\cos(x_i)s_k}{s_i} \\ &= \frac{1}{s_i}(s_j\frac{c_k + c_ic_j}{s_is_j} - s_kc_i\frac{c_j + c_ic_k}{s_is_k}) \\ &= \frac{1}{s_i^2}(c_k + c_ic_j - c_ic_j - c_i^2c_k) \\ &= \frac{1}{s_i^2}(c_ks_i^2) \\ &= c_k.\end{aligned}$$

This verifies (B.4).

To see the partial derivatives of $x_j = x_j(y_j, y_k, x_i)$, we use the sine law

$$\text{(B.10)} \qquad \sin(x_j) = \sin(x_i)\sin(y_j)/\sin(y_i).$$

Take the partial derivative of (B.10) with respect to y_k. We obtain

$$\cos(x_j)\frac{\partial x_j}{\partial y_k} = -\sin(x_i)\sin(y_j)\frac{\cos(y_i)}{\sin^2(y_i)}\frac{\partial y_i}{\partial y_k}.$$

By (B.4), we obtain that

$$\begin{aligned}\frac{\partial x_j}{\partial y_k} &= -\frac{\sin(x_i)\sin(y_j)\cos(y_i)}{\sin^2(y_i)} \\ &= -\sin(x_j)\cot(y_i)\end{aligned}$$

where the last equation is due to the sine law. This establishes (B.6).

To see (B.7), we take the partial derivative with respect to x_i of (B.10). It becomes

$$\cos(x_j)\frac{\partial x_j}{\partial x_i} = \frac{\sin(y_j)}{\sin^2(y_i)}(\cos(x_i)\sin(y_i) - \sin(x_i)\cos(y_i)\partial y_i/\partial x_i).$$

Using identity (B.5) and the sine law, the above is

$$\frac{\cos(x_i)\sin(y_i) - \cos(y_i)A_{ijk}}{\sin^2(y_i)}\sin(y_j)$$

$$= \frac{\cos(x_i)\sin(y_i) - \cos(y_i)\sin(y_i)\sin(x_j)\sin(x_k)}{\sin^2(y_i)}\sin(y_j)$$

$$= \frac{\cos(x_i) - (\cos(x_i) + \cos(x_j)\cos(x_k))}{\sin(y_i)}\sin(y_j)$$

$$= -\frac{\cos(x_j)\cos(x_k)}{\sin(y_i)}\sin(y_j)$$

$$= -\frac{\cos(x_j)\cos(x_k)}{\sin(x_i)}\sin(x_j).$$

Now dividing both sides by $\cos(x_j)$, we obtain identity (B.8).

Finally, take the partial derivative with respect to y_j to (B.10). Using (B.4) and the sine law, we obtain

$$\cos(x_j)\frac{\partial x_j}{\partial y_j} = \sin(x_i)\frac{\cos(y_j)\sin(y_i) - \sin(y_j)\cos(y_i)\partial y_i/\partial y_j}{\sin^2(y_i)}$$

$$\text{(B.11)} \qquad = \frac{\sin(x_i)}{\sin^2(y_i)}(\cos(y_j)\sin(y_i) - \sin(y_j)\cos(y_i)\cos(x_k)).$$

Let $C_r = \cos(y_r)$ and $S_r = \sin(y_r)$. Then by (B.2), equation (B.11) becomes

$$= \frac{\sin(x_i)}{S_i^2}\left(C_j S_i - S_j C_i \frac{C_k - C_i C_j}{S_i S_j}\right)$$

$$= \frac{\sin(x_i)}{S_i^3}(C_j S_i^2 - C_i C_k + C_i^2 C_j)$$

$$= \frac{\sin(x_i) S_k}{S_i^2}\left(\frac{C_j - C_i C_k}{S_i S_k}\right)$$

$$= \frac{S_k \sin(x_i)\cos(x_j)}{S_i^2}$$

$$= \frac{\sin(x_k)\cos(x_j)}{\sin(y_i)}.$$

Dividing both sides by $\cos(x_j)$, we obtain (B.7). q.e.d.

We remark that identity (B.6) for Euclidean triangles was in [**10**], Lemma A1(d).

Corollary B2. *Let $h \in \mathbf{C}$.*

(a) *Consider y_j, y_k, x_i as variables and x_i fixed. Then the differential 1-form*

$$\frac{\int^{x_j} \sin^h(t)dt}{\sin^{h+1}(y_j)}dy_j + \frac{\int^{x_k} \sin^h(t)dt}{\sin^{h+1}(y_k)}dy_k$$

is closed.

(b) *([**19**]) Consider y_j, y_k, x_i as variables and x_i fixed. Then the differential 1-form*

$$\left(\int^{x_k} \sin^h(t)dt\right) \sin^{h-1}(y_j)dy_j + \left(\int^{x_j} \sin^h(t)dt\right) \sin^{h-1}(y_k)dy_k$$

is closed.

(c) *([**19**]) Consider x_i, x_j, x_k as variables and x_i fixed. Then the differential 1-form*

$$\left(\int^{y_k} \sin^h(t)dt\right) \sin^{h+1}(x_j)dx_j + \left(\int^{y_j} \sin^h(t)dt\right) \sin^{h+1}(x_k)dx_k$$

is closed.

The proof is a simple application of identities (B.6) and (B.7) in the above theorem. We omit the detail. The integrations of the 1-forms for $h = 0, -1$ in part (b) for geometric triangles were first discovered by Bobenko-Springborn [**7**]. Bobenko-Springborn showed the integral of the 1-form for $h = 0$ can be identified with the dilogarithmic function. In the work of [**19**], a further study of the applications of the derivative cosine law of the second kind are carried out.

Appendix C. Relationship to the Lobachevsky Function

In the special cases of $h = \pm 1$ or 0, some of integrations $\int^u w_h$ and $\int^u \eta_h$ in Theorem 1.11 and Corollary B2 or their Legendre transformations have been found explicitly by various authors. We give a brief summary in this appendix.

Following Milnor [**31**], let $\Lambda(z) = \int_0^z -\ln(2\sin(t))dt$ be the (complex valued) Lobachevsky function defined as a multi-valued complex analytic function (depending on the choice of the branch of $\ln(t)$ and the path). This function is related to the dilogarithm function (see [**31**]).

Let $y = y(x)$ be the cosine function defined by (1.8), $x_i = r_j + r_k$ for $\{i, j, k\} = \{1, 2, 3\}$ and $r = (r_1, r_2, r_3)$. Then we have:

Proposition C1. *The following identities hold up to addition of a constant.*

(a) *([**26**])*

(C.1)

$$\int^x \sum_{i=1}^3 \ln\tan(y_i/2)dx_i = -\sum_{i=1}^3 \Lambda(\pi/2 - r_i) + \Lambda(\pi/2 - r_1 - r_2 - r_3) + \sqrt{-1}\pi\left(\sum_{i=1}^3 r_i\right)$$

(b) *(Leibon [**25**])*

$$\int^r 2\ln\sin(y_i/2)dr_i = \sum_{i=1}^3 [\Lambda(\pi/2 - r_i) + \Lambda(r_i + r_{i+1}) + \sqrt{-1}\pi r_i] + \Lambda(\pi/2 - r_1 - r_2 - r_3) \tag{C.2}$$

where $r_4 = r_1$.

(c) *(Bobenko-Springborn [**7**]) Consider x_1, x_2, y_3 as variables and fixing y_3. The integral*

$$\int^{(x_1,x_2)} \ln\tan(y_1/2)dx_2 + \ln\tan(y_2/2)dx_1 \tag{C.3}$$

$$= \Lambda(\pi/2 - r_1) + \Lambda(\pi/2 - r_2) - \Lambda(\pi/2 - r_3) + \Lambda(\pi/2 - r_1 - r_2 - r_3) + \sqrt{-1}\pi/2(x_1 + x_2) + c,$$

where the constant c depends only on y_3.

Proof. The proof is straightforward by checking the derivatives of both sides. In part (a), the partial derivative with respect to x_i of the left-hand side is $\ln(\tan(y_i/2))$ by definition. By the tangent law (2.10), we have

$$2\ln(\tan(y_i/2)) = \ln\cos(r_i) + \ln\cos(r_1 + r_2 + r_3) - \ln\cos(r_j) - \ln\cos(r_k) + \sqrt{-1}\pi.$$

The right-hand side of the above equation is the x_i-th partial derivative of the right-hand side of (C.1) by the definition of the Lobachevsky function.

In part (b), we use the following identity:

$$\sin^2(y_i/2) = \frac{1 - \cos(y_i)}{2} = \frac{\sin(x_j)\sin(x_k) - \cos(x_i) - \cos(x_j)\cos(x_k)}{2\sin(x_j)\sin(x_k)} = -\frac{\cos(r_i)\cos(r_1 + r_2 + r_3)}{\sin(r_i + r_k)\sin(r_i + r_j)}.$$

Now take the logarithm of this function and compare with the partial derivatives of the right-hand side of (C.2).

The proof of (C.3) is the same as above. We omit the details. q.e.d.

These integrations in the cases of spherical or hyperbolic triangles have geometric interpretations. To be more precise, for x, y to be the inner angles and edge lengths of a spherical triangle, the integral in Proposition C1(a) is the volume of the ideal hyperbolic octahedron which is the convex hull of the six intersection points of the three circles at the sphere at infinity forming a triangle of inner angles x_1, x_2, x_3 (see [**26**]). If x, y are the inner angles and edge lengths of a hyperbolic triangle, Leibon [**25**] showed that the integral in Proposition C1(b) is the volume of the ideal prism which is the convex hull of the six intersection points at the sphere at infinity of the three circles forming a triangle of inner angles x_1, x_2, x_3. For a spherical triangle of inner angles x_1, x_2, x_3, the integral in Proposition C1(b) was shown by P. Doyle [**25**] to be the volume of the hyperbolic tetrahedron with exactly three vertices at infinity and a finite vertex v so that the dihedral angles at the edges from v are x_1, x_2, x_3.

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