On branching process with a threshold and its generalization

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Abstract: This paper studies a special class of population-size-dependent branching processes, in which the offspring distribution is supercritical when the population size does not exceed a given threshold K, and is subcritical or critical when the population size exceeds K. Up to now, the author has found no paper concerning continuous time threshold processes, and study of the discrete case is also limited to the extinction time T and $\mathbb{E}T$. In this paper, both of the two cases were studied more thoroughly: for the discrete case, a limit theorem of the process has been proved; for the continuous case, the existence of Markovian threshold processes has been proved, and an equivalent condition for the process to extinct almost surely was also given. This paper also further generalized discrete time threshold processes by giving a random delay to the threshold's influence of offspring distribution. The properties of these delayed threshold processes were studied, and a limit theorem was obtained.

Keywords: branching process, threshold, extinction, limit theorem

Contents

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1 Introduction

Branching processes, also called Galton-Watson processes, is a type of stochastic process which reflects how population size varies among generations. It was first presented in 1874, in a paper written by Galton and Watson concerning the vanishment of family names. Nowadays, this stochastic process is applied in various research fields in physics, chemistry and biology (for example, in the study of nuclear fission and germ reproduction).

Definition 1.1. A (discrete time) branching process is a sequence $(\zeta_n)_{n\in\mathbb{N}}$ of random variables, with $\zeta_0 \equiv j \in \mathbb{N}^*$, and

$$\zeta_n = \sum_{i=1}^{\zeta_{n-1}} \xi_n^i$$

Where ξ_n^i are i.i.d., non-negative, integer-valued random variables. We call its distribution the **offspring distribution** of the branching process.

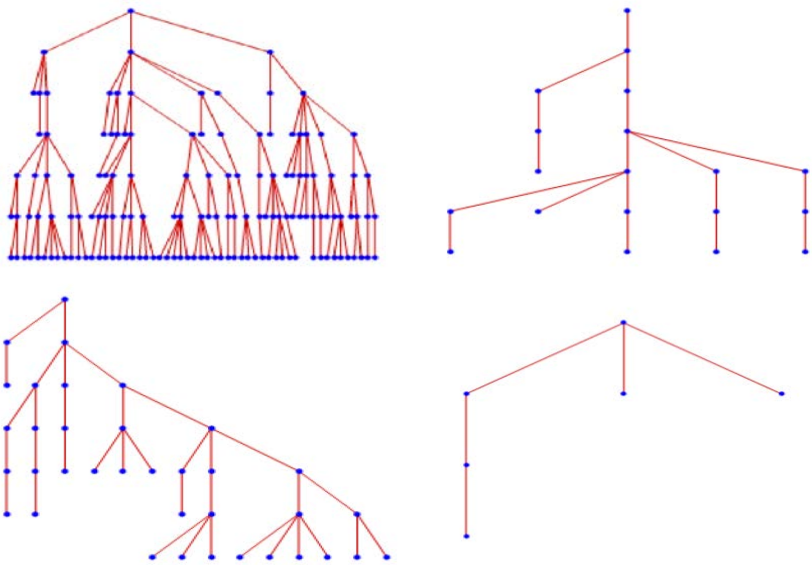

Fig. 1.1. The graphs of randomly generated Galton-Watson trees, with an offspring distribution that has a Poisson distribution with parameter 1.5. Clockwise from the upper right corner: $\zeta_6 = 4$; $\zeta_4 = 0$ (extinction); $\zeta_6 = 9$; $\zeta_6 = 54$.

From Definition 1.1, we can imply that branching processes are Markov chains, and their transition probability can be expressed in terms of their offspring distribution. The properties and limit theorems of traditional branching processes were already studied thoroughly in [2][3]. In Chapter 2 of this paper, several important results about traditional branching processes will be presented. Among them, the conclusion of Lemma 2.3 is especially crucial, classical and attractive: the relationship between m and 1, where m is the expected value of the offspring distribution, plays a dominating role in the stochastic behaviour of a branching process. In the case $m \leq 1$, with the hypothesis that the offspring distribution is non-trivial[1], a branching process will extinct (transfer to 0) with probability 1; while if $m > 1$, there would always be a positive probability that the process could "live forever".

Branching process itself, as a mathematical model, has successfully grasped the essence of many various real-life stochastic processes. Nevertheless, it still has its deficiencies. An relatively obvious one, is that every single member in every generation of a branching process generates its children according to the same offspring distribution. This often does not fit with reality very well. Take animal reproduction as an example: it could be interfered by many other factors, like the temporary influence of some disease, or some potential danger that occurs when a herd gets too large (for example, the break down of ecological balance). The traditional version of branching process lacks the ability to describe these phenomena, thus studying its generalization is of great importance. Moreover, all processes in the natural world are continuous instead of discrete, hence the continuous time generalization of branching processes certainly can describe and reflect the reality better.

In this paper, we mainly study a kind of population-size-dependent branching process (and its further generalizations), which is called branching processes with a threshold (in short, threshold processes). We shall first give the definition of a discrete time threshold process:

Definition 1.2. $\forall K \in \mathbb{N}^*$ (called the threshold), a discrete time threshold process is a sequence $(Z_n)_{n\in\mathbb{N}}$ of random variables, where $Z_0 \equiv j \in \{1,2,\dots,K\}$ and:
(1) If $Z_{n-1} \leq K$, then

$$Z_n = \sum_{i=1}^{Z_{n-1}} \xi_n^i,$$

where ξ_n^i are i.i.d., non-negative, integer-valued random variables, with a distribution $(p_j)_{j\in\mathbb{N}}$ called the inner distribution, sastifying $M \coloneqq \mathbb{E}\xi_n^i > 1$;
(2) If $Z_{n-1} > K$, then

$$Z_n = \sum_{i=1}^{Z_{n-1}} \eta_n^i,$$

where η_n^i are i.i.d., non-negative, integer-valued random variables, with a distribution $(q_j)_{j\in\mathbb{N}}$ called the outer distribution, sastifying $m \coloneqq \mathbb{E}\xi_n^i \leq 1$.

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[1] To avoid trivial cases, in this paper, all offspring distributions ξ with $m \leq 1$ are restricted to sastify $\mathbb{P}(\xi = 0), \mathbb{P}(\xi = 1) < 1$. This restriction also applies to the threshold process and its generalizations in the following text.

Such a generalization of branching processes has its apparent advantages. Still taking animal reproduction as example, the change of offspring distribution on different sides of the threshold K reflects the potential danger caused by a larger population (like it gets easier to be attacked). These threats decrease the survive rate of the descendants, thus the offspring distribution becomes critical or subcritical; but if the size of the herd started to decrease due to this threat, and finally becomes lower than the threshold again, then the potential dangers diminish and the offspring distribution recovers. Hence, threshold process can describe the real-life situations better, and has greater value in practice.

Threshold process is a kind of population-size-dependent branching process. The general properties of population-size-dependent processes were studied in [4], but the limit theorem it obtained does not apply to threshold processes. Athreya, Schuh(2016) (see [1]) specially studied threshold processes. The paper proved that these processes die out with probability 1, and also studied conditions for $\mathbb{E}T < \infty$ to hold. However, other various asymptotic behavious of traditional branching processes, like the rate of convergence of $\mathbb{P}(Z_n > 0)$ (it converges to 0 almost surely in subcritical cases), as far as the author knows, have not been studied by any paper before. In Chapter 3 of this paper, these topics will be studied more thoroughly, and the main result obtained is the following limit theorem:

Theorem 1.1. Let $(Z_n)_{n\in\mathbb{N}}$ be a threshold process, T_n、$D_n (n \in \mathbb{N}^*)$ are random variables defined in section 3 of this paper. When $m < 1$ and the conditions
(1) $\sum_{j=1}^{\infty} j \log j \, q_j < \infty$;
(2) $p_0 = 0$;
(3) There exists $\theta > 0$, such that $\forall 1 \le i \le K, \varphi_i(\theta) := \mathbb{E}\left(exp\left(\theta(T_2 + D_2)\right) \middle| Z_{T_1+D_1} = i\right) < \infty$;
are all true, then for every $j \in \{1,2,\dots,K\}$, under the condition that $Z_0 \equiv j$, there always exists real numbers $0 < b \le a < 1$ such that:

$$0 < \liminf_{n\to\infty} \frac{\mathbb{P}(Z_n > 0)}{b^n}$$

$$\limsup_{n\to\infty} \frac{\mathbb{P}(Z_n > 0)}{a^n} < \infty.$$

In later chapters of this paper, threshold processes will be further generalized. In many real-life cases, there not only exists a threshold that influences the offspring distribution, but also exists a delay (usually random) for this threshold's influence to become effective. In animal reproduction again, since the mutual affect between animals and the environment they live in is a slow process, the offspring distribution would not change immedieately when the population exceeds a threshold, but rather after a ramdon delay. This observation conveys the value of studying the following "delayed threshold process":

Definition 1.3. For a positive integer-valued random variable X, let $X_1, X_2, \ldots$be a sequence of random variables having the distribution of X. For any $K \in \mathbb{N}^*$ (called the threshold), a delayed threshold process is a sequence $(Z_n)_{n\in\mathbb{N}}$ of random variables such that:

$$Z_n = \sum_{i=1}^{Z_{n-1}} \xi_{n,i}^{\mathbf{1}_A(n)}$$

where:
(1) $Z_0 \equiv j \in \{1,2,\dots,K\}$;
(2) $\xi_{n,i}^0$ is i.i.d. non-negative, integer-valued random variables with distribution $(p_j)_{j\in\mathbb{N}}$ such that $M \coloneqq \mathbb{E}\xi_{n,i}^0 > 1$;
(3) $\xi_{n,i}^1$ is i.i.d. non-negative, integer-valued random variables with distribution $(q_j)_{j\in\mathbb{N}}$ such that $m \coloneqq \mathbb{E}\xi_{n,i}^1 \le 1$;
(4) For every $k \in \mathbb{N}$ such that $1 \le k \le n-1$, define $p_n(k)$, a property of k:

$$p_n(k) \Leftrightarrow \big((Z_{k-1} \le K \wedge Z_k \ge K) \vee (Z_{k-1} > K \wedge Z_k \le K)\big) \wedge (\, k + X_k \le n)$$

and assume that $p(0)$ is always true. $A \coloneqq \left\{n: Z_{\max\{0\le k\le n-1:\, p_n(k)\}} > K\right\}$, $\mathbf{1}_A$ is the characteristic function of A.

We have also not found any paper concerning delayed threshold processes. In Chapter 4 of this paper, the extinction time and asymptotic behaviour of delayed threshold processes are both studied, and the main results are the following theorems:

Theorem 1.2. Let $(Z_n)_{n\in\mathbb{N}}$ be a delayed threshold process. Define $T = \min\{n: Z_n = 0\}$. As long as one of the following two conditions
(1) $p_0 > 0$;
(2) $p_0 + p_1 > 0,\ X = C\ a.s.$ (where C is a constant);
holds, for every $j \in \{1,2,\dots,K\}$, we have $\mathbb{P}(T < \infty | Z_0 = j) = 1$.

Theorem 1.3. Let $(Z_n)_{n\in\mathbb{N}}$ be a delayed threshold process, and let $g(s) = \sum_{i=0}^{\infty} q_i s^i$, the PGF (probability generating function) of the outer distribution. When $m < 1$, or when $m = 1$ and

$$\int_0^1 \frac{1-s}{g(s)-s}\, ds < \infty\,,$$

if condition (1) or (2) in Theorem 1.2 holds, and $\sup\{i: p_i > 0\} < \infty$, then for every $j \in \{1,2,\dots,K\}$, we always have $\mathbb{E}(T|Z_0 = j) < \infty$. Specially, when $m = 1$ and $p_0 = 0$, the finiteness of the above integral is equivalent to $\mathbb{E}(T|Z_0 = j) < \infty$.

Theorem 1.4. Let $(Z_n)_{n\in\mathbb{N}}$ be a delayed threshold process, and D_n the stopping time defined in Definition 4.1. If $m < 1$ and the conditions
(1) $\sum_{j=1}^{\infty} j \log j\, q_j < \infty$,
(2) $p_0 = 0$,
(3) $X = C\ a.s.$,
(4) $\exists \theta > 0,\ \forall 1 \le i \le K, \varphi_i(\theta) \coloneqq \mathbb{E}\big(exp\big(\theta(D_2)\big)\,\big|Z_{D_1} = i\big) < \infty$
all holds, then for every $j \in \{1,2,\dots,K\}$, when $Z_0 \equiv j$, there always exists real numbers $0 < b \le a < 1$ such that:

$$0 < \liminf_{n\to\infty} \frac{\mathbb{P}(Z_n > 0)}{b^n},$$

$$\limsup_{n\to\infty} \frac{\mathbb{P}(Z_n > 0)}{a^n} < \infty.$$

In Chapter 5, we will further generalize threshold processes to continuous time Markov chains. This is also a kind of generalized continuous time branching process that, as far as the author is concerned, has not been studied before. The first problem in this continuous time generalization is, how to prove that the corresponding Kolmogorov backward equations has a unique solution (which is equivalent to the existence ane uniqueness of the corresponding continuous time Markov chain)? For common continuous time Markovian branching processes, this problem was already solved very early, see [9]; but in more general cases, since the backward equations is an infinite system of differential equations, it is very hard to solve by simple techniques.

Definition 1.4. Let $0 < a < \infty$ be a constant, $(p_i)_{i\in\mathbb{N}}$ and $(q_i)_{i\in\mathbb{N}}$ be distributions, and $M := \sum_{i=1}^{\infty} ip_i \in (1,\infty), m := \sum_{i=1}^{\infty} iq_i \leq 1$. Also let $\{Z(t): t \geq 0\}$ be a continuous-time, time-homogeneous Markov chain defined on probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and $Z(0) \equiv j \in \{1,2,\dots,K\}$. If when $t \to 0$, we have

$$P_{ij}(t) = \begin{cases} iap_{j-i+1}t + o(t), \text{when } j \geq i-1, j \neq i, i \leq K; \\ iaq_{j-i+1}t + o(t), \text{when } j \geq i-1, j \neq i, i > K; \\ 1 - iat + o(t), \text{when } j = i; \\ o(t), \text{when } j < i-1. \end{cases}$$

then we call $\{Z(t): t \geq 0\}$ a continuous-time Markovian threshold process (with threshold K), in short, a continuous threshold process.

In Chapter 5, we will first prove:

Theorem 1.5. The Kolmogorov backward equations corresponding to Definition 1.4 has a unique solution which does not explode.

Then, we proved the following result concerning the extinction time of continuous threshold processes:

Theorem 1.6. For a continuous threshold process $\{Z(t): t \geq 0\}$, let $T = \inf\{t: Z(t) = 0\}$, Then for every $j \in \{1,2,\dots,K\}$, we have:
(1) When $p_0 = 0$, $\mathbb{P}(T < \infty | Z(0) \equiv j) = 0$;
(2) When $p_0 > 0$, $\mathbb{P}(T < \infty | Z(0) \equiv j) = 1$.

2 Definition and properties of branching processes

Although it is not concerned much in this paper, many researches on branching processes focus on the random tree naturally generated by it (where $\zeta_0 \equiv 1$ is the root vertex, and every vertex is connected with its children), called the Galton-Watson tree. An infinite sequence $(\zeta_n)_{n\in\mathbb{N}}$ can be seen as randomly generated a Galton-Watson tree. Under this comprehension, the sample space Ω of a branching process is defined as the set of all rooted labelled locally finite trees. Details can be found in [3, p.150].

Definition 2.1. For a non-negative, integer-valued random variable X, its probability generating function (PGF) $f(s)$ is defined on $[0,1]$ as:

$$f(s) := \sum_{k=0}^{\infty} \mathbb{P}(X = k)s^k$$

Lemma 2.1. For any random variable X, its PGF $f(s)$ is differentiable in all orders on $[0,1)$, and we have $f(0) = \mathbb{P}(X = 0)$, $f'(1-) = \mathbb{E}X$.
Proof: See [5, p.247]. ■

Lemma 2.2. Let $f_n(s)$ be the PGF of ζ_n. We have $f_n(s) = f_{n-1}(f(s)) = f^{(n)}(s)$ (the n-th iteration of f at s).
Proof: See [3, p.146]. ■

Obviously, if for a branching process there exists n such that $\zeta_n = 0$, then for all integers $n' \geq n$, we have $\zeta_{n'} = 0$, so this branching process "extincted". The following result concerning the extinction probability $\mathbb{P}(\exists 1 \leq n < \infty, \zeta_n = 0)$ is well-known:

Lemma 2.3. For a branching process $(\zeta_n)_{n\in\mathbb{N}}$, assume $\zeta_0 \equiv 1$, and $f(s)$ is the PGF of the offspring distribution,
(1) If $m \leq 1$, then $\mathbb{P}(\exists 1 \leq n < \infty, \zeta_n = 0) = 1$;
(2) If $m > 1$, then $\mathbb{P}(\exists 1 \leq n < \infty, \zeta_n = 0) = q$, where q is the only fixed point of $f(s)$ which is not equal to 1.
Proof: See [3, p.147]. ■

In light of Lemma 2.3, we call a branching process with $m > 1$ supercritical (and also call its offspring distribution a supercritical distribution), $m = 1$ critical, and $m < 1$ subcritical. People have already thoroughly studied the limiting behaviour of branching processes, and limit theorems concerning the growth rate of ζ_n in supercritical cases, the rate of convergence of $\mathbb{P}(\zeta_n > 0)$ in critical and subcritical cases and the conditional distribution of ζ_n on $\{\zeta_n > 0\}$ were all proved. In the following text, we will use a limit theorem of subcritical processes, and here we state it as a lemma:

Lemma 2.4. For a subcritical branching process $(\zeta_n)_{n\in\mathbb{N}}$, assume $\zeta_0 \equiv 1$. The following three statements are equivalent (here we denote $0 * \log 0 = 0$):
(1) $\lim_{n\to\infty} \mathbb{P}(\zeta_n > 0)/m^n > 0$;
(2) $\sup \mathbb{E}(\zeta_n | \zeta_n > 0) < \infty$;
(3) $\mathbb{E}(\xi \log \xi) < \infty$.
Proof: See [3, pp.465-467]. ■

Moreover, for the conditional distribution of a supercritical process on extinction and on non-extinction, we have the following result:

Lemma 2.5. For a supercritical branching process $(\zeta_n)_{n\in\mathbb{N}}$, let its PGF be $f(s) = \sum_{i=0}^{\infty} p_i s^i$, and $q < 1$ be its extinction probability,

(1) When $p_0 > 0$ (which is equivalent to $q > 0$), under the condition $Q \equiv \{\omega: (\zeta_n(\omega))_{n\in\mathbb{N}}$ does not extinct$\}$, $(\zeta_n)_{n\in\mathbb{N}}$ can be decomposed into the sum of a supercritical process $({\zeta_n}^{(A)})_{n\in\mathbb{N}}$ with PGF $\hat{f}(s) = (f\big((1-q)s+q\big)-q)/(1-q) = \sum_{i=0}^{\infty}\widehat{p_i}s^i$ (which implies $\widehat{p_0} = 0$) and a non-negative random sequence $(\widetilde{\zeta_n})_{n\in\mathbb{N}}$. That is, $\zeta_n(\omega) = {\zeta_n}^{(A)}(\omega) + \widetilde{\zeta_n}(\omega)$. Moreover, the offspring distribution of $({\zeta_n}^{(A)})_{n\in\mathbb{N}}$ has the same expected value with the original process. While when $p_0 = 0$, we have $\mathbb{P}(Q) = 1$, and $\zeta_n = {\zeta_n}^{(A)}\ a.s.$.
(2) When $p_0 > 0$, under the condition Q^C, $(\zeta_n)_{n\in\mathbb{N}}$ has the same distribution with a subcritical process with PGF $f^*(s) = f(sq)/q$.
Proof: See [2, pp.47-50]. ■

Finally, we quote a criterion for $\mathbb{E}T < \infty$ to hold:

Lemma 2.6. Let $T = \min\{n: \zeta_n = 0\}$, $f(s) = \sum_{i=0}^{\infty} p_i s^i$. Then when $m < 1$, or when $m = 1$ and

$$\int_0^1 \frac{1-s}{f(s)-s}ds < \infty,$$

we have $\mathbb{E}T < \infty$. Moreover, when $m = 1$, the finiteness of the above integral is equivalent to $\mathbb{E}T < \infty$.
Proof: See [6]. ■

3 A limit theorem of discrete time threshold processes

The definition of a discrete time threshold process is already given in Definition 1.2. In this chapter, we first prove some of its basic properties, and then prove Theorem 1.1, a limit theorem of discrete threshold processes.

In [1], the following result was proved:

Lemma 3.1. Suppose we have the following two stopping times

$$T_1 = min\{n \in \mathbb{N}^*: Z_n \geq K+1 \text{ or } Z_n = 0\},$$
$$D_1 = \begin{cases} \min\{n \in \mathbb{N}^*: n \geq 1, Z_{T_1+n} \leq K\}, \text{when } Z_{T_1} \geq K+1; \\ 0, \text{when } Z_{T_1} = 0. \end{cases}$$

Then for every $j \in \{1,2,\ldots,K\}$, under the condition $Z_0 = j$, we always have $T_1, D_1 < \infty\ a.s.$. Moreover, when $m < 1$, or when $m = 1$ and

$$\int_0^1 \frac{1-s}{g(s)-s}ds < \infty,$$

we have $\mathbb{E}(T_1 + D_1) < \infty$.
Proof: See [1]. ■

Lemma 3.2. $\lambda \coloneqq \min_{1\leq j\leq K} \mathbb{P}(Z_{T1+D1} = 0|Z_0 = j) > 0$.
Proof: For every $j \in \{1,2,\ldots,K\}$, we have

$$\mathbb{P}\left(Z_{T_1+D_1}=0|Z_0=j\right) \geq \mathbb{P}\left(Z_{T_1+D_1}=0, Z_{T_1} \geq K+1|Z_0=j\right)$$
$$\geq \sum_{h=K+1}^{\infty} \mathbb{P}\left(Z_{T_1}=h|Z_0=j\right)\mathbb{P}(Z_{T_1+1}=0 \,|Z_{T_1}=h, Z_0=j\big)$$
$$= \sum_{h=K+1}^{\infty} \mathbb{P}\left(Z_{T_1}=h|Z_0=j\right)(q_o)^h,$$

Since a supercritical process always has an extinction probability stricty less than 1, there always exists some $h \geq K+1$ sastifying $P\left(Z_{T_1}=h|Z_0=j\right)>0$. Besides, no matter the inner distribution is subcritical or critical, we always have $q_0>0$, thus the lemma is proved. ■

The following definitions and derivation were also made in [1]:

At T_1+D_1, we say that the threshold process finished its first cycle. If $Z_{T_1+D_1} \in \{1,2,\dots,K\}$, then it will start its second cycle, which is a new threshold process with initial value $Z_{T_1+D_1}$. Although this value is random, but Lemma 3.1 holds under any condition $Z_0=j$, and the strong Markov property guarantees that the threshold process after time T_1+D_1 is independent with the previous process, hence if we analagously define stopping times T_2 and D_2, then the result of Lemma 3.1 still holds for them in the second cycle. Moreover, as long as the threshold process does not die out in the first two cycles, a third cycle will exist. By analogy, we could successively define the following sequence of stopping times:

$$T_n = \begin{cases} min\{k \in \mathbb{N}^*: Z_{T_1+D_1+\cdots+T_{n-1}+D_{n-1}+k} \geq K+1 \text{ or } Z_{T_1+D_1+\cdots+T_{n-1}+D_{n-1}+k}=0\}, \text{when } Z_{T_1+D_1+\cdots+T_{n-1}+D_{n-1}}>0; \\ 0, \text{when } Z_{T_1+D_1+\cdots+T_{n-1}+D_{n-1}}=0. \end{cases}$$

$$D_n = \begin{cases} \min\{k \in \mathbb{N}^*: Z_{T_1+D_1+\cdots+T_{n-1}+D_{n-1}+T_n+k} \leq K\}, \text{when } Z_{T_1+D_1+\cdots+T_{n-1}+D_{n-1}+T_n} \geq K+1; \\ 0, \text{when } Z_{T_1+D_1+\cdots+T_{n-1}+D_{n-1}+T_n}=0. \end{cases}$$

And then from Lemma 3.1 and 3.2, the result below follows:

Corollary 3.3. For every $n \in \mathbb{N}^*$, we have
$$\mathbb{P}\left(Z_{T_1+D_1}>0,\dots,Z_{T_1+D_1+\cdots+T_n+D_n}>0\right) \leq (1-\lambda)^n.$$

Lemma 3.4. Let the PGF of the inner and outer distribution be $f(s)=\sum_{i=0}^{\infty} p_i s^i$ and $g(s)=\sum_{i=0}^{\infty} q_i s^i$, respectively. Assume $Z_0 \equiv 1$. Then in the corresponding threshold process, the PGF $h_n(s)$ of Z_n sastifies:
$$h_n(s) = \begin{cases} \mathbb{E}(f(s)^{Z_{n-1}}; Z_{n-1} \leq K) + \mathbb{E}(g(s)^{Z_{n-1}}; Z_{n-1}>K), \text{when } n \geq 2; \\ f(s), \text{when } n=1. \end{cases}$$

Proof: By the definition of threshold processes, the distribution of Z_1 must be the inner distribution, thus its PGF is $f(s)$. Using the definition of PGF and the properties of conditional expectation, we have

$$
\begin{aligned}
h_n(s) = \mathbb{E}(s^{Z_n}) &= \mathbb{E}(s^{Z_n}|Z_{n-1} \le K)P(Z_{n-1} \le K) + \mathbb{E}(s^{Z_n}|Z_{n-1} > K)P(Z_{n-1} > K) \\
&= \mathbb{E}\left(\mathbb{E}(s^{\sum_{i=1}^{Z_{n-1}} \xi_n^i}|Z_{n-1}, Z_{n-1} \le K)\right)P(Z_{n-1} \le K) \\
&+ \mathbb{E}\left(\mathbb{E}(s^{\sum_{i=1}^{Z_{n-1}} \eta_n^i}|Z_{n-1}, Z_{n-1} > K)\right)P(Z_{n-1} > K) \\
&= \mathbb{E}(f(s)^{Z_{n-1}}|Z_{n-1} \le K)P(Z_{n-1} \le K) + \mathbb{E}(g(s)^{Z_{n-1}}|Z_{n-1} \le K)P(Z_{n-1} \le K) \\
&= \mathbb{E}(f(s)^{Z_{n-1}}; Z_{n-1} \le K) + \mathbb{E}(g(s)^{Z_{n-1}}; Z_{n-1} > K). \qquad \blacksquare
\end{aligned}
$$

Now we prove Theorem 1.1. This is a limit theorem concerning threshold processes with a subcritical outer distribution, which shows that in such threshold process, the rate that $\mathbb{P}(Z_n > 0) \to 0$ is similar to the rate in a a subcritical branching process: "roughly" exponential.

Proof of Theorem 1.1. We first prove that when $Z_0 = 1$, there exists some $b > 0$ such that

$$0 < \liminf_{n\to\infty} \mathbb{P}(Z_n > 0)/b^n$$

Since $f''(s) > 0$ holds on $[0,1)$, we know $f(s)$ is strictly convex on $[0,1]$. Moreover, $p_0 = 0$, thus $f(0) = 0$, $f(1) = 1$, hence for every $s \in (0,1)$, we have:

$$f(s) = f(0 \cdot (1-s) + 1 \cdot s) < (1-s)f(0) + s \cdot f(1) = s$$

On the other hand, when $q_0 + q_1 = 1$, by $m = q_1 < 1$, we know $g(s)$ is a linear function with a slope less than 1, thus $g(s) > s$ must hold on $[0,1)$. Meanwhile, when $q_0 + q_1 < 1$, by

$$g''(s) = \sum_{i=2}^{\infty} i(i-1)q_i s^{i-2} > 0$$

we know that $g'(s)$ is strictly increasing on $[0,1)$, so by $m = g'(1-) \le 1$, on $[0,1)$ we have:

$$g'(s) - 1 < 0$$

Hence the value of $g(s) - s$ on $[0,1]$ strictly decreases from $q_0 > 0$ at 0 to 0 at 1. Therefore, for every $s \in (0,1)$, $g(s) > s$.

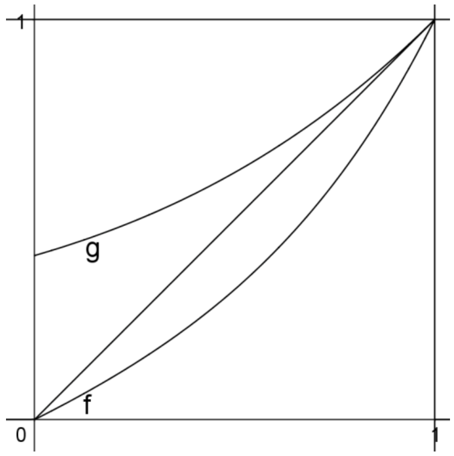

Fig. 3.1. The graph of PGF $f(s)$ and $g(s)$ in the proof of Theorem 1.1. On $[0,1)$ it always holds that $f(s) < s < g(s)$.

In conclusion (noticing $g(0) = q_0 > 0$), on $[0,1)$ we have $f(s) < g(s)$. Hence by Lemma 2.2 and Lemma 3.4, $\forall n \in \mathbb{N}^*$, $\forall s \in [0,1)$, we all have:

$$h_n(s) = \mathbb{E}(f(s)^{Z_{n-1}}; Z_{n-1} \le K) + \mathbb{E}(g(s)^{Z_{n-1}}; Z_{n-1} > K)$$
$$< \mathbb{E}(g(s)^{Z_{n-1}}; Z_{n-1} \le K) + \mathbb{E}(g(s)^{Z_{n-1}}; Z_{n-1} > K) = \mathbb{E}(g(s)^{Z_{n-1}}) = g^{(n)}(s)$$

Thus by Lemma 2.1, we obtain that $\forall n \in \mathbb{N}^*$, $\mathbb{P}(Z_n = 0) = h_n(0) < g^{(n)}(0) = \mathbb{P}(\zeta_n = 0)$, where ζ_n is a simple branching process with offspring distribution $(q_j)_{j\in\mathbb{N}}$. Using the limit theorem of subcritical branching processes in Lemma 2.4, under condition (1), when $m < 1$ we have:

$$\liminf_{n\to\infty} \frac{\mathbb{P}(Z_n > 0)}{m^n} \ge \liminf_{n\to\infty} \frac{\mathbb{P}(\zeta_n > 0)}{m^n} > 0 \qquad (3.1)$$

For any $j \in \{1,2,\dots,K\}$, when $Z_0 = j$, let the corresponding threshold process be $({Z_n}^{(j)})_{n\in\mathbb{N}}$, then obviously $\forall n \in \mathbb{N}^*$, $\mathbb{P}(Z_n > 0) \le \mathbb{P}\left({Z_n}^{(j)} > 0\right)$, so the above conclusion (3.1) holds under any initial value of Z_0.

We define:

$$C = \max_{1\le i\le K} \mathbb{E}\left(T_n + D_n \middle| Z_{T_1+D_1+\dots+T_{n-1}+D_{n-1}} = i\right)$$
$$S_n = T_1 + D_1 + \dots + T_n + D_n$$
$$\varphi(\theta) = \max_{1\le i\le K} \varphi_i(\theta)$$

Notice that under condition (3), for any positive integer n and integer $1 \le i \le K$, by the strong Markov property of threshold processes, we have:

$$\mathbb{E}\left(exp\left(\theta(T_n + D_n)\right) \middle| Z_{T_1+D_1+\dots+T_{n-1}+D_{n-1}} = i\right) = \mathbb{E}\left(exp\left(\theta(T_2 + D_2)\right) \middle| Z_{T_1+D_1} = i\right) < \infty$$

The next step is using results in Large Deviations Theory, to prove that when $n \to \infty$, $\mathbb{P}(S_n > 2nC)$ converges to 0 exponentially rapid. From condition (3), we can imply that $\exists \theta > 0$, such that $\forall 1 \le i \le K$, the inequality $2C\theta - \log \varphi_i(\theta) > 0$ always holds. The proof of this fact can be found in [5, pp.87-88].

Then by Markov's inequality, for such a θ, $\forall n \in \mathbb{N}^*$, we have

$$e^{2\theta nC}\mathbb{P}(S_n > 2nC) = e^{2\theta nC}\mathbb{P}\left(e^{\theta S_n} > e^{2\theta nC}\right) \le \mathbb{E}e^{\theta S_n}$$

Expanding the right side, we get:

$$\mathbb{E}e^{\theta S_n} = \mathbb{E}e^{\theta S_1}e^{\theta(S_2-S_1)}\dots e^{\theta(S_n-S_{n-1})} = \mathbb{E}\left(\mathbb{E}\left(e^{\theta S_1}e^{\theta(S_2-S_1)}\dots e^{\theta(S_n-S_{n-1})}\middle|\mathcal{F}_{S_1}\right)\right)$$

$$= \mathbb{E}\left(e^{\theta S_1}\mathbb{E}\left(e^{\theta(S_2-S_1)}\dots e^{\theta(S_n-S_{n-1})}\middle|\mathcal{F}_{S_1}\right)\right)$$

$$= \mathbb{E}\left(e^{\theta S_1}\mathbb{E}\left(e^{\theta(S_2-S_1)}\mathbb{E}\left(e^{\theta(S_3-S_2)}\dots e^{\theta(S_n-S_{n-1})}\middle|\mathcal{F}_{S_2}\right)\middle|\mathcal{F}_{S_1}\right)\right) = \cdots$$

$$= \mathbb{E}\left(e^{\theta S_1}\mathbb{E}\left(e^{\theta(S_2-S_1)}\mathbb{E}\left(\dots\mathbb{E}\left(e^{\theta(S_{n-1}-S_{n-2})}\mathbb{E}\left(e^{\theta(S_n-S_{n-1})}\middle|\mathcal{F}_{S_n}\right)\middle|\mathcal{F}_{S_{n-1}}\right)\dots\middle|\mathcal{F}_{S_2}\right)\middle|\mathcal{F}_{S_1}\right)\right)$$

$$\le \mathbb{E}\left(e^{\theta S_1}\mathbb{E}\left(e^{\theta(S_2-S_1)}\mathbb{E}\left(\dots\mathbb{E}\left(e^{\theta(S_{n-1}-S_{n-2})}\varphi(\theta)\middle|\mathcal{F}_{S_{n-1}}\right)\dots\middle|\mathcal{F}_{S_2}\right)\middle|\mathcal{F}_{S_1}\right)\right) \le \dots \le \mathbb{E}\left(e^{\theta S_1}\varphi(\theta)^{n-1}\right)$$

$$\le \varphi(\theta)^n$$

Thus we have $e^{2\theta nC}\mathbb{P}(S_n > 2nC) \le \varphi(\theta)^n$. We take logarithm on both sides and simplify:

$$\mathbb{P}(S_n > 2nC) \le e^{-n(2C\theta - \log\varphi(\theta))}$$

Then by the result of Corollary 3.3, it follows that:

$$\begin{aligned}\mathbb{P}(Z_{2nC}>0)&=\mathbb{P}(Z_{2nC}>0,S_n>2nC)+\mathbb{P}(Z_{2nC}>0,S_n\le 2nC)\\&\le \mathbb{P}(S_n>2nC)+\mathbb{P}\left(Z_{S_n}>0,S_n\le 2nC\right)\le \mathbb{P}(S_n>2nC)+\mathbb{P}\left(Z_{S_n}>0\right)\\&\le \left(e^{\log\varphi(\theta)-2C\theta}\right)^n+(1-\lambda)^n\end{aligned}$$

Where $e^{\log\varphi(\theta)-2C\theta}$ and $1-\lambda$ are both constants less than 1. We may assume that $1-\lambda\ge e^{\log\varphi(\theta)-2C\theta}$. Take an M large enough such that $\sqrt[M]{2}(1-\lambda)<1$. Then for every $n\ge M$ we have:

$$\mathbb{P}(Z_{2nC}>0)\le\left(e^{\log\varphi(\theta)-2C\theta}\right)^n+(1-\lambda)^n\le 2(1-\lambda)^n\le\left(\sqrt[M]{2}(1-\lambda)\right)^n$$

Let $a=\left(\sqrt[M]{2}(1-\lambda)\right)^{\frac{1}{2C}}$. For every $n\in\mathbb{N}^*$, let $n=q_n\cdot 2C+r_n, 0\le r_n<2C$, then:

$$\frac{\mathbb{P}(Z_n>0)}{a^n}<\frac{\mathbb{P}(Z_{2Cq_n}>0)}{a^{2Cq_n}}\cdot\frac{a^{2Cq_n}}{a^{2C(q_n+1)}}$$

$n\to\infty$ is equivalent to $q_n\to\infty$, thus by the result above, when $q\in\mathbb{N}$ we have

$$\limsup_{q\to\infty}\frac{\mathbb{P}(Z_{2Cq}>0)}{a^{2Cq}}\le 1$$

$$\frac{a^{2Cq}}{a^{2C(q+1)}}=\frac{1}{\sqrt[M]{2}(1-\lambda)}$$

And therefore

$$\limsup_{n\to\infty}\frac{\mathbb{P}(Z_n>0)}{a^n}\le\limsup_{q\to\infty}\frac{\mathbb{P}(Z_{2Cq}>0)}{a^{2Cq}}\cdot\frac{a^{2Cq}}{a^{2C(q+1)}}\le\frac{1}{\sqrt[M]{2}(1-\lambda)}<\infty \qquad (3.2)$$

The inequalities (3.1) and (3.2), together, proved the theorem. ■

4 The properties and limit theorem of delayed threshold processes

The main object we study in this chapter is delayed threshold processes. They are already defined in Definition 1.3. A fact worth mentioning is that after introducing such a delay, $(Z_n)_{n\in\mathbb{N}}$ is no longer a Markov chain, so a direct corollary of the Markov property of simple branching processes (that the limit of ζ_n when $n\to\infty$ always exists, and must be either 0 or ∞, see [3]) may not hold.

It should be specifically mentioned that the discussion in this chapter is restricted to the case that the delay X sastifies $\exists N\in\mathbb{N}$, $X\le N\ a.s.$. Under this assumption, we define $N=\inf\{n:\mathbb{P}(X\le n)=1\}<\infty$.

To study the extinction time of delayed threshold processes, we introduce the several random variables below:

Definition 4.1. For a delayed threshold process $(Z_n)_{n\in\mathbb{N}}$, define: (here all variables n and k all take value in positive integers and j is a integer such that $j\ge 2$)

$$R_1=\min(\min\{n:Z_n\le K,\exists k<n,Z_k>K\},\min\{n:Z_n=0\})$$

$$P_{11} = \begin{cases} \min\left(\min\left\{n: Z_{R_1+n} > K\right\}, N\right), \text{when } Z_{R_1} > 0; \\ 0, \text{when } Z_{R_1} = 0. \end{cases}$$

$$T_{11} = \min\left\{n: Z_{R_1+P_{11}+n} \le K\right\};$$

$$P_{1j} = \begin{cases} min\left(min\left\{n: Z_{R_1+P_{11}+T_{11}+\cdots+P_{1(j-1)}+T_{1(j-1)}+n} > K\right\}, N\right), \text{when } Z_{R_1+P_{11}+T_{11}+\cdots+P_{1(j-1)}+T_{1(j-1)}} > 0; \\ 0, \text{when } Z_{R_1+P_{11}+T_{11}+\cdots+P_{1(j-1)}+T_{1(j-1)}} = 0. \end{cases}$$

$$T_{1j} = \min\left\{n: Z_{R_1+P_{11}+T_{11}+\cdots+P_{1(j-1)}+T_{1(j-1)}+P_{1j}+n} \le K\right\};$$

$$N_1 = \begin{cases} \min\left\{n: Z_{R_1+P_{11}+T_{11}+\cdots+P_{1(n-1)}+T_{1(n-1)}+P_{1n}} \le K\right\}, \text{when } Z_{R_1} > 0; \\ 0, \text{when } Z_{R_1} = 0. \end{cases}$$

$$D_1 = R_1 + \sum_{k=1}^{N_1-1} (P_{1k} + T_{1k}) + P_{1N_1}$$

Lemma 4.1. For every $j \in \{1,2,\dots,K\}$, when $Z_0 = j$, we have $R_1 < \infty\ a.s.$, and for every $j \in \mathbb{N}^*$, $T_{1j} < \infty\ a.s.$.

Proof: R_1 can be decomposed into the sum of two random variables:

$$Y_1 = \min\{n: Z_n > K \text{ or } Z_n = 0\}$$

$$Y_2 = \min\left\{n: Z_{X_1+n} \le K\right\}$$

During the time period $0 \le n < Y_1$, the offspring distribution of Z_n must be the inner, supercritical distribution. Thus during this period, the distribution of the delayed threshold process Z_n is identical to a simple threshold process Z_n' which has the same inner and outer distribution with it. Therefore, Y_1 has the same distribution with the stopping time T_1 of simple threshold processes defined in Chapter 3. By Lemma 3.1, we thus know Y_1 is almost surely finite.

On $\{Y_2 < N\}$ it is obvious that Y_2 is finite, while on $\{Y_2 \ge N\}$, since $Z_{Y_1} \ne 0$, we know that $\forall n \ge N$, $\left(Z_{Y_1-1} \le K \land Z_{Y_1} > K\right)$ and $Y_1 + X_{Y_1} \le Y_1 + n$ must both hold, thus $p_{Y_1+n}(Y_1)$ holds. According to the definition of Y_1 and Y_2, for every k satisfying $Y_1 + 1 \le k \le Y_1 + Y_2 - 1$, it is impossible for $(Z_{k-1} \le K \land Z_k \ge K) \lor (Z_{k-1} > K \land Z_k \le K)$ to be true. Thus for every n satisfying $Y_1 + N \le n \le Y_1 + Y_2$ (since $Y_2 \ge N$, such an n exists), by Definition 4.1 we have

$$\max\{0 \le k \le n-1: p_n(k)\} = Y_1$$

Since on $\{Y_2 \ge N\}$, $Z_{Y_1} > K$, so for such n we must have $\mathbf{1}_A(n) = 1$, that is, the offspring distribution of Z_n must be subcritical. Thus $(Z_{Y_1+N+n})_{n\in\mathbb{N}}$ is in fact identically distributed with a simple branching process with initial value Z_{Y_1+N} and offspring distribution $(q_j)_{j\in\mathbb{N}}$. By Lemma 2.3, we know such a branching process extincts with probability 1, thus we must have $Y_2 < \infty\ a.s.$.

For T_{1j}, when $T_{1j} \ge N$ and $Z_{R_1+P_{11}+\cdots+P_{1j}} > K$ (other cases are all trivial), it can be similarly proved that $(Z_{R_1+P_{11}+\cdots+P_{1j}+N+n})_{n\in\mathbb{N}}$ is identically distributed with a branching process with offspring distribution $(q_j)_{j\in\mathbb{N}}$, and thus also extincts almost surely. Therefore the lemma is proved. ■

Lemma 4.2. For every $j \in \{1,2,\dots,K\}$, when $Z_0 \equiv j$, if $p_0 + p_1 > 0$, then $D_1 < \infty\ a.s.$.

Proof: First, notice that no matter what value Z_{R_1} take, and no matter at what time before R_1 did the process leap over the threshold, and no matter what's the random delay of the affect of these "leap over" on the offspring distribution, as long as we give a set of arbitrary values to these three parameters, the probability $\mathbb{P}(N_1 = 1)$ always have a common lower bound $\lambda = \min((p_0 + p_1)^N, (q_0 + q_1)^N)$. This is because whatever value we take for the three parameters, the event $\{Z_{R_1+1} \le K, \dots, Z_{R_1+N} \le K\}$ is always enough to guarantee that $N_1 = 1$, and when the times of "leaping over" and the corresponding delays are all pinned down, the probability of that event must be the product of several $(p_0 + p_1)$ and $(q_0 + q_1)$ (with N factors in total), thus it is always no less than the smaller one of $(p_0 + p_1)^N$ and $(q_0 + q_1)^N$.

And under the condition $N_1 > 1$, since $Z_{R_1+P_{11}+T_{11}} \le K$, the process between $R_1 + P_{11} + T_{11}$ and $R_1 + P_{11} + T_{11} + P_{12} + T_{12}$ is in fact similar to the process between R_1 and $R_1 + P_{11} + T_{11}$: we still can argue that whatever value $Z_{R_1+P_{11}+T_{11}}$ takes, and whatever the "leaping over" times and corresponding delays are, under these conditions $\mathbb{P}(N_1 = 2)$ still have λ as a common lower bound. Thus $\mathbb{P}(N_1 > 2 | N_1 > 1) \le 1 - \lambda$. Using induction, we can see that for every positive integer n, $\mathbb{P}(N_1 > n+1 | N_1 > 1, \dots, N_1 > n) \le 1 - \lambda$, and thus:

$$\mathbb{P}(N_1 > j) = \mathbb{P}(N_1 > 1)\mathbb{P}(N_1 > 2 | N_1 > 1) \cdots \mathbb{P}(N_1 > j | N_1 > 1, \dots, N_1 > j-1) \le (1 - \lambda)^j$$

Which implies:

$$\mathbb{E}N_1 = \sum_{j=0}^{\infty} \mathbb{P}(N_1 > j) \le \sum_{j=0}^{\infty} (1 - \lambda)^j = \frac{1}{\lambda} < \infty$$

Thus $N_1 < \infty\ a.s.$. Finally, on every $\omega \in \{N_1 < \infty\}$, we have

$$D_1(\omega) = R_1(\omega) + \sum_{k=1}^{N_1(\omega)-1} (P_{1k} + T_{1k}) + P_{1N_1(\omega)} < \infty$$

So by the result of Lemma 4.1, it is now proved that $D_1 < \infty\ a.s.$. ■

After the above preparation, we are able to prove that the extinction time T of delayed threshold processes is almost surely finite under two different conditions, which is Theorem 1.2 in Chapter 1.

Proof of Theorem 1.2.

(1) If $\mathbb{P}(N_1 > 0) = 0$, then $\mathbb{P}(Z_{R_1} = 0) = 1$, so by Lemma 4.1 the conclusion is obvious. Otherwise, we have:

$$\begin{aligned}\mathbb{P}(Z_{D_1} = 0) &\ge \mathbb{P}(Z_{D_1} = 0 | N_1 > 0)\mathbb{P}(N_1 > 0) \\ &= \mathbb{P}(N_1 > 0) \\ &\cdot \sum_{0 \le k \le K: \mathbb{P}(Z_{D_1-1} = k, N_1 > 0) > 0} \mathbb{P}(Z_{D_1-1} = k | N_1 > 0)\mathbb{P}(Z_{D_1} = 0 | Z_{D_1-1} = k, N_1 > 0)\end{aligned}$$

The formula is due to the fact that when $N_1 > 0$, $D_1 = R_1 + P_{11} + T_{11} + \cdots + P_{1N_1}$, so by definition $P_{1N_1} = N$. Again by definition, for every j,

$$P_{1j} = N \iff Z_{R_1+P_{11}+T_{11}+\cdots+T_{1(j-1)}+1}, \dots, Z_{R_1+P_{11}+T_{11}+\cdots+T_{1(j-1)}+P_{1j}-1} \le K$$

Thus Z_{D_1-1} must take value in integers from 0 to K. When $p_0 > 0$, for every k such that $0 \le k \le K$ and $\mathbb{P}(Z_{D_1-1} = k, N_1 > 0) > 0$ (under the condition $\mathbb{P}(N_1 > 0) > 0$, such a k must exist), we have:

$$\begin{aligned}\mathbb{P}(Z_{D_1} = 0|Z_{D_1-1} = k, N_1 > 0) &\\ &= \mathbb{P}((Z_{D_1} = 0|Z_{D_1-1} = k, N_1 > 0)|\mathbf{1}_A(D_1-1) = 1)\mathbb{P}(\mathbf{1}_A(D_1-1) = 1)\\ &+ \mathbb{P}((Z_{D_1} = 0|Z_{D_1-1} = k, N_1 > 0)|\mathbf{1}_A(D_1-1) = 0)\mathbb{P}(\mathbf{1}_A(D_1-1) = 0)\\ &= {q_0}^k\mathbb{P}(\mathbf{1}_A(D_1-1) = 1) + {p_0}^k\mathbb{P}(\mathbf{1}_A(D_1-1) = 0) > 0\end{aligned}$$

And thus

$$\begin{aligned}\mathbb{P}(Z_{D_1} = 0) \ge \mathbb{P}(N_1 > 0) &\\ &\cdot \sum_{0 \le k \le K : \mathbb{P}(Z_{D_1-1} = k, N_1 > 0) > 0} \mathbb{P}(Z_{D_1-1} = k|N_1 > 0)\mathbb{P}(Z_{D_1} = 0|Z_{D_1-1} = k, N_1 > 0)\\ &=: \gamma > 0.\end{aligned}$$

Under the condition $Z_{D_1} > 0$, when $N_1 > 0$, $\forall n \in \mathbb{N}$, by definition we know $p_{D_1+n}(R_1 + P_{11} + T_{11} + \cdots + T_{1(N_1-1)})$ is true (noticing $P_{1N_1} = N$). Meanwhile, from $R_1 + P_{11} + \cdots + T_{1(N_1-1)}$ to D_1 the whole process did not leap over the threshold, thus starting from time D_1, as long as the process does not leap over the threshold, the offspring distribution of Z_n remains the inner supercritical distribution. Also noticing $Z_{D_1} \le K$, we can conclude that $(Z_{D_1+n})_{n\in\mathbb{N}}$ is a new delayed threshold process, having the same offspring distribution with the original process and independent with it. Therefore, on this new process, we can define random variables R_2、P_{2j}、T_{2j}、N_2、D_2 in a manner similar to Definition 4.1 (if $Z_{D_1} = 0$, then all these are defined to be 0). Noticing that in the above argument, γ is in fact the common lower bound of $\mathbb{P}(Z_{D_1} = 0)$ under all conditions $Z_0 = j$, we get:

$$\mathbb{P}(Z_{D_1+D_2} > 0|Z_{D_1} > 0) \le 1 - \gamma < 1$$

And thus $\mathbb{P}(Z_{D_1} > 0, Z_{D_1+D_2} > 0) \le (1-\gamma)^2$. Continuing like this, we define R_n, …, D_n, and let $N^* = \min\{n: Z_{D_1+\cdots+D_n} = 0\}$. Using the property that every period of the process (from 0 to D_1, from D_1 to D_2…) are all mutually independent, we have:

$$\mathbb{E}N^* = \sum_{n=0}^{\infty}\mathbb{P}(N^* > n) = 1 + \sum_{n=1}^{\infty}\mathbb{P}(Z_{D_1} > 0, \dots, Z_{D_1+\cdots+D_n} > 0) \le \sum_{n=0}^{\infty}(1-\gamma)^n = \frac{1}{\gamma} < \infty$$

Thus $N^* < \infty\ a.s.$. Finally, using the conclusion of Lemma 4.2, on every $\omega \in \{N^* < \infty\}$ we have:

$$T(\omega) \le \sum_{n=0}^{N^*(\omega)} D_n(\omega) < \infty.$$

(2) The procedure of the proof does not differs much from the one above, though for $0 \le k \le K$ such that $\mathbb{P}(Z_{D_1-1} = k, N_1 > 0) > 0$, since $p_0 > 0$ no longer necessarily holds, we need to use other methods to prove that $\mathbb{P}(Z_{D_1} = 0|Z_{D_1-1} = k, N_1 > 0) > 0$.

Notice that under the condition of (2) we have $N = C$. When $N_1 > 0$, it can be first implied that $p_{D_1-1}(R_1 + P_{11} + T_{11} + \cdots + P_{1(N_1-1)})$ holds, because from $P_{1N_1} = C$ and $T_{1(N_1-1)} \ge 1$ we know:

$$R_1 + P_{11} + T_{11} + \cdots + P_{1(N_1-1)} + C \le R_1 + P_{11} + T_{11} + \cdots + P_{1(N_1-1)} + T_{1(N_1-1)} + C - 1$$
$$= D_1 - 1.$$

Next, for every $k > R_1 + P_{11} + T_{11} + \cdots + P_{1(N_1-1)}$, the only value for $p_{D_1-1}(k)$ to possibly hold is $R_1 + P_{11} + T_{11} + \cdots + P_{1(N_1-1)} + T_{1(N_1-1)}$, but this is also impossible, because

$$R_1 + P_{11} + T_{11} + \cdots + P_{1(N_1-1)} + T_{1(N_1-1)} + C = D_1 > D_1 - 1.$$

And thus $\max\{0 \le k \le (D_1 - 1) - 1: p_{D_1-1}(k)\} = R_1 + P_{11} + T_{11} + \cdots + P_{1(N_1-1)}$, so

$$\mathbf{1}_A(D_1 - 1) = 1\ a.s.$$

Hence $\mathbb{P}(Z_{D_1} = 0 | Z_{D_1-1} = k, N_1 > 0) = {q_0}^k > 0$. The remaining proof is same with(1).■

At the beginning of this chapter, we have already mentioned that delayed threshold processes do not have the Markov property, so we are not able to prove that $(Z_n)_{n\in\mathbb{N}}$ must converge either to 0 or to ∞ by the Markov property. Therefore, when the conditions of Theorem 1.2 do not hold, we could not deny the possibility that under some circumstances, a delayed threshold process may infinitely leap over the threshold, neither extinct nor explode.

Theorem 1.3 further gives a set of conditions for $\mathbb{E}T$ to be finite. Now we prove it.

Proof of Theorem 1.3. First we assume that

$$\int_0^1 \frac{1-s}{g(s)-s} ds < \infty$$

Consider Y_1 and Y_2 defined in the proof of Lemma 4.1. Y_1 has the same distribution with T_1 in Lemma 3.1, thus $\mathbb{E}(Y_1 | Z_0 = j) < \infty$. For Y_2, let τ be the extinction time of a branching process with offspring distribution $(q_j)_{j\in\mathbb{N}}$, then by Lemma 2.6 $\mathbb{E}\tau < \infty$. Meanwhile, on $\{Y_2 > N\}$, by the deduction in Lemma 4.1 we know $(Z_{Y_1+N+n})_{n\in\mathbb{N}}$ is a branching process with offspring distribution $(q_j)_{j\in\mathbb{N}}$, thus on any $\{Z_0 = j\}$ we all have:

$$\mathbb{E}(Y_2 | Z_{Y_1+N} = k) \le \mathbb{E}(\max(\tau_1, \dots, \tau_k) | Z_{Y_1+N} = k) \le \mathbb{E}(\tau_1 + \cdots + \tau_k | Z_{Y_1+N} = k) = k\mathbb{E}\tau < \infty$$

And therefore on $\{Y_2 > N\}$ we have: (let $S = \sup\{i: p_i > 0\}$)

$$\mathbb{E}Y_2 = \sum_{k=K+1}^{\infty} \mathbb{P}(Z_{Y_1+N} = k)\, \mathbb{E}(Y_2 | Z_{Y_1+N} = k)$$

$$\le \max_{0\le i\le K*S^N:\, \mathbb{P}(Z_{Y_1+N-1}=i)>0} \sum_{k=K+1}^{\infty} \mathbb{P}(Z_{Y_1+N} = k | Z_{Y_1+N-1} = i)\, k\, \mathbb{E}\tau$$

$$\le KS^N M \mathbb{E}\tau < \infty$$

In the last inequality above, $M < \infty$ can be implied from $S < \infty$. Thus $\mathbb{E}R_1 = \mathbb{E}Y_1 + \mathbb{E}Y_2 < \infty$. By the same method through which we expanded $\mathbb{E}Y_2$, we can prove that $\mathbb{E}T_{1j} < \infty$. Then by noticing $\mathbb{E}P_{1j} \le N$, we know that there is a common upper bound C for every $\mathbb{E}(P_{1j} + T_{1j})$. Since $\mathbb{E}N_1 < \infty$, by applying Wald's equation, we obtain:

$$\mathbb{E}D_1 = \mathbb{E}R_1 + \mathbb{E}\sum_{k=1}^{N_1-1} (P_{1k} + T_{1k}) + \mathbb{E}P_{1N_1} \le \mathbb{E}R_1 + \mathbb{E}(N_1 - 1)C + \mathbb{E}P_{1N_1} < \infty$$

Thus $\mathbb{E}D_1 < \infty$ holds under any condition $\{Z_0 = j\}$. So by the property that $(Z_{D_1+\cdots+D_k+n})_{n\in\mathbb{N}}$ is independent with $(Z_n)_{n\in\mathbb{N}}$, we know there is a common upper bound for all $\mathbb{E}D_n$:

$$C' \coloneqq \max_{1\le j\le K} \mathbb{E}(D_1\,|Z_0 = j) < \infty$$

Noticing $\mathbb{E}N^* < \infty$, applying Wald's equation again will yield the first result of this theorem:

$$\mathbb{E}T \le \mathbb{E}\sum_{n=0}^{N^*} D_n = \mathbb{E}\left(\sum_{n=0}^{N^*} \mathbb{E}D_n\right) \le \mathbb{E}N^*C' < \infty$$

Next, we assume that

$$\int_0^1 \frac{1-s}{g(s)-s}\,ds = \infty$$

Due to the restriction that $p_0 = 0$, the whole process will only be able to die out when the offspring distribution is the outer one. We use the method of embedding a critical branching process $(\zeta_n)_{n\in\mathbb{N}}$ into $(Z_n)_{n\in\mathbb{N}}$ to prove that under any condition $\{Z_0 = j\}$ we all have $\mathbb{E}T = \infty$.

Consider the time ϱ_1 when the offspring distribution turns into the outer distribution for the first time. Among the particles in Z_{ϱ_1}, we randomly choose a particle to be the root ζ_0 of $(\zeta_n)_{n\in\mathbb{N}}$; starting from this moment to the moment ϱ_2 that the offspring distribution returns to the inner distribution (or the whole process extincts), we let the value of $(\zeta_n)_{n\in\mathbb{N}}$ be euqal to the corresponding generations of the sub-Galton-Watson tree evolved from ζ_0. If $Z_{\varrho_2} > 0$, then we wait until the moment ϱ_3 when the offspring distribution changes again to the outer distribution. Let $\zeta_{\varrho_2-\varrho_1} = j$. In Z_{ϱ_3} we randomly choose j particles as the particles of $\zeta_{\varrho_2-\varrho_1}$, and until the moment ϱ_4 when the offspring distribution returns back again to the inner distribution (or the process extincts), we define $(\zeta_n)_{n\in\mathbb{N}}$ as the sum of the particle number in the corresponding generations of the sub-Galton-Watson tree generated by these j particles.

Continuing like this, we enbedded a critical process with offspring distribution $(q_j)_{j\in\mathbb{N}}$ into the original process. If we let the extinction time of $(\zeta_n)_{n\in\mathbb{N}}$ be T', then obviously $T \ge T'$. By Lemma 2.6, under the conditions of this theorem, $\mathbb{E}T' = \infty$, and thus $\mathbb{E}T = \infty$. ■

At the end of this chapter, we generalize Theorem 1.1, which is proved in Chapter 3, to a limit theorem for delayed threshold processes. This result (Theorem 1.4) shows that, although influenced by both the threshold and the delay (which we assume to be constant in this theorem), the rate of convergence of $\mathbb{P}(Z_n > 0)$ in a delayed threshold process is still similar to a subcritical branching process: exponentially rapid.

Proof of Theorem 1.4. The idea and many technical details of this proof is similar to Theorem 1.1, so we omit some explanation of the details.

In the proof of Theorem 1.1 we proved that under condition (2), on $[0,1)$ we have $f(s) < g(s)$. When $X = C\ a.s.$ and $Z_0 = 1$, $n \ge C + 1$, we can expand the PGF $h_n(s)$ of Z_n as below:

$$h_n(s) = \mathbb{E}(f(s)^{Z_{n-1}}; Z_{n-C-1} \le K) + \mathbb{E}(g(s)^{Z_{n-1}}; Z_{n-C-1} > K)$$

$$< \mathbb{E}(g(s)^{Z_{n-1}}; Z_{n-1} \leq K) + \mathbb{E}(g(s)^{Z_{n-1}}; Z_{n-1} > K) = \mathbb{E}(g(s)^{Z_{n-1}}) = g^{(n)}(s)$$

Where the first equation can be proved similar with Theorem 1.1. Also notice that when $n \leq C$, $h_n(s) \equiv f^{(n)}(s) < g^{(n)}(s)$, we thus know that for every positive integer n, we have $\mathbb{P}(Z_n > 0) = h_n(0) < g^{(n)}(0) = \mathbb{P}(\zeta_n > 0)$, where $(\zeta_n)_{n\in\mathbb{N}}$ is a simple branching process with Z_n's outer distribution as its offspring distribution. Hence similar to Theorem 1.1, under condition (1) we can get:

$$\liminf_{n\to\infty} \frac{\mathbb{P}(Z_n > 0)}{m^n} > 0$$

Meanwhile, condition (4) guarantees that for every integer $n \geq 2$,

$$C \coloneqq \max_{1\leq i\leq K} \mathbb{E}\left(D_n | Z_{D_1+\cdots D_{n-1}} = i\right) < \infty$$

This is because there always exists an A large enough such that on $\{D_n > A\}$, $\exp\left(\theta(D_2)\right) > D_2$ is always true. If we further define $S_n = T_1 + D_1 + \cdots + T_n + D_n$ and $\varphi(\theta) = \max_{1\leq i\leq K} \varphi_i(\theta)$, then similar with Theorem 1.1, we obtain:

$$e^{2\theta nC}\mathbb{P}(S_n > 2nC) \leq \varphi(\theta)^n$$
$$\mathbb{P}(S_n > 2nC) \leq e^{-n(2C\theta - \log\varphi(\theta))}$$

The remaining proof is almost identical with Theorem 1.1, so we omit it. The final conclusion is that, there exists a positive constant $a = (\sqrt[M]{2}(1-\gamma))^{1/2C}$ (where M is large enough and γ is the constant defined in the proof of Theorem 1.2), such that

$$\limsup_{n\to\infty} \frac{\mathbb{P}(Z_n > 0)}{a^n} < \infty$$

This proves the theorem. ■

5 Properties of continuous time Markovian threshold processes

In the two chapters above, we studied simple and delayed discrete time threshold processes. In this chapter we generalize threshold processes from another perspective, which is changing the process from discrete time into continuous time, and discuss problems like whether the extinction time is finite. We only study continuous threshold processes which remain their Markov property, which are called coutinuous time Markovian threshold processes (in short, continuous threshold processes). The charasteristic of these processes is that the time period between two transitions is exponentially distributed.

First, we use the infinitesimal method to give the definition of continuous time Markovian branching processes (in short, continuous branching processes):

Definition 5.1. Let $0 < a < \infty$ be a constant, $(p_i)_{i\in\mathbb{N}}$ a distribution, and $M = \sum_{i=1}^{\infty} ip_i \in (0,\infty)$. Also let $\{\zeta(t): t \geq 0\}$ be a continuous time, time-homogeneous Markov chain on probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and state space $\mathbb{N}$. $P_{ij}(t) \coloneqq \mathbb{P}(\zeta(t) = j | \zeta(0) = i)$, $\zeta(0) \equiv j \in \mathbb{N}^*$. If when $t \to 0$ we have:

$$P_{ij}(t) = \begin{cases} iap_{j-i+1}t + o(t), \text{when } j \geq i-1, j \neq i; \\ 1 - iat + o(t), \text{when } j = i; \\ o(t), \text{when } j < i-1. \end{cases}$$

Then we call $\{\zeta(t): t \geq 0\}$ a continuous branching process, and $(p_i)_{i\in\mathbb{N}}$ its offspring distribution.

The definition above actually only defined the Q-matrix (transition rate matrix) $\mathbb{Q}$. This matrix determines the transition probability function $P_{ij}(t)$ (transition probability matrix $\mathbb{P}(t)$) by the following two equivalent equations, known as the Kolmogorov backward and forward equation:

$$\begin{cases} \frac{d}{dt}\mathbb{P}(t) = \mathbb{Q}\mathbb{P}(t) \quad \text{(backward equation)} \\ \frac{d}{dt}\mathbb{P}(t) = \mathbb{P}(t)\mathbb{Q} \quad \text{(forward equation)} \end{cases}$$

The boundary condition is $\mathbb{P}_{ij}(0+) = E$, where E is the unit matrix having the same size with $\mathbb{Q}$, and if $\mathbb{Q}$ has countably infinite rows, then E is also a countably infinite unit matrix.

The solution of the two equations exists in most cases, but if not bounded by any other conditions, there may exist solutions such that $\sum_{j\in\mathbb{N}} P_{ij}(t) < 1$, which means that the Markov chain transferred infinitely many times in a finite time period t, a phenomenon known as explosion. Thus from such solutions we could not really construct a correspoding Markov chain. On the other hand, if the backward (or forward) equation has a unique solution such that $\sum_{j\in\mathbb{N}} P_{ij}(t) = 1$ for every $t \geq 0$ and $i,j \in \mathbb{N}$, then we can imply that there exists a unique continuous time Markov chain with $P_{ij}(t)$ as its transition probability function. The proof can be found in [7]. However, for continuous branching processes, the restriction $\sum_{i=1}^{\infty} ip_i < \infty$ in Definition 5.1 is a sufficient condition for the backward equation to have a unique, non-exploding solution. This is proved in [9]. Hence, the continuous time Markov chain described in Definition 5.1 always exists.

Lemma 5.1. For a continuous branching process $\{\zeta(t): t \geq 0\}$ and any $\delta > 0$, the sequence $(\zeta_n)_{n\in\mathbb{N}} \coloneqq (\zeta(n\delta))_{n\in\mathbb{N}}$ is always a discrete branching process (called the embedded branching process), and has $F(s) = \sum_{k=0}^{\infty} P_{1k}(\delta)\, s^k$ as its offspring distribution's PGF. This offspring distribution has an expectation $m' = e^{a(M-1)\delta}$ (where M is the expectation of the continuous time offspring distribution), so m' and M has the same order relation with 1.
Proof: See [2, pp.106-111]. ∎

Lemma 5.2. Let $\{\zeta(t): t \geq 0\}$ be a continuous branching process with offspring distribution $(p_i)_{i\in\mathbb{N}}$, and $f(s)$ is the PGF of the offspring distribution. Define

$$A = \{\omega \in \Omega: t \to \infty\text{时}\zeta(t,\omega) \to \infty\}$$
$$B = \{\omega \in \Omega: t \to \infty\text{时}\zeta(t,\omega) \to 0\}$$

Then $\mathbb{P}(A \cup B) = 1$, $\mathbb{P}(B) = q$, where q is the only fixed point of $f(s)$ which is not equal to 1.
Proof: See [2, pp.107-108]. ∎

Definition 1.4 in Chapter 1 used an infinitesimal method similar with Definition 5.1 to define continuous time threshold processes. Our first task here is to prove that the corresponding Kolmogorov backward equations of Definition 1.4 has a unique non-exploding solution, just like Definition 5.1 does. In fact, this is not easier than proving the properties of continuous threshold processes. First, through the following Lemma 5.3, we give an equivalent condition for the corresponding backward equations of a Q-matrix to uniquely exist:

Lemma 5.3. Define the embedded discrete Markov chain $(X_n)_{n\in\mathbb{N}}$ of a Q-matrix (q_{ij}) as a chain having

$$p_{ij}=\begin{cases}-q_{ij}/q_{ii}, \text{when } i\neq j, q_{ii}\neq 0;\\ 0, \text{when } i=j, q_{ii}\neq 0 \text{ or } i\neq j, q_{ii}=0;\\ 1, \text{when } i=j, q_{ii}=0\,.\end{cases}$$

as its transition probability from state i to j, and having a probability distribution $\pi(x)$ strictly greater than 0 on the whole state space $\mathcal{S}$ as its initial distribution. Also define a sequence $(\tau_n)_{n\in\mathbb{N}}$ of r.v.'s (called the waiting time) that, under the condition of sequence $(X_n)_{n\in\mathbb{N}}$, are exponentially distributed and mutually independent, with τ_k having expectation $-\frac{1}{q_{X_kX_k}}$. Also define the jump times $N(t)$ at time t:

$$N(t)=\begin{cases}\min\{n\geq 0: \tau_0+\cdots+\tau_n>t\}, \text{when } \sum_{i=0}^{\infty}\tau_i>t;\\ \infty, \text{otherwise}\end{cases}$$

Then: (1) The corresponding Kolmogorov backward equation of Q-matrix (q_{ij}) has a solution $P^*{}_{ij}(t)$, such that for every $t\geq 0$, $i,j\in\mathbb{N}$, there holds $\sum_{j\in\mathbb{N}}P^*{}_{ij}(t)\leq 1$, and if the equality sign always holds, then this solution is the unique solution of the backward equation; (2) The equivalent condition for $\sum_{j\in\mathbb{N}}P^*{}_{ij}(t)=1$ to hold for every $t\geq 0, i,j\in\mathbb{N}$ is that: $\forall t\geq 0$, $N(t)<\infty\ a.s.$.

Remark. The enbedded discrete Markov chain of a continuous time chain is defined as the embedded discrete Markov chain of its Q-matrix.

Proof: See [7, pp.71-75]. ■

By Definition 1.4, the transition probability of the embedded discrete Markov chain of a continuous threshold process is:

$$p_{ij}=\begin{cases}-\dfrac{iap_{j-i+1}}{-ia}=p_{j-i+1}, \text{when } j\geq i-1, j\neq i, i\leq K;\\ -\dfrac{iaq_{j-i+1}}{-ia}=q_{j-i+1}, \text{when } j\geq i-1, j\neq i, i>K;\\ 0, \text{when } j<i-1 \text{ or } j=i.\end{cases}$$

We can see that the transition probability of $(X_n)_{n\in\mathbb{N}}$ when $i\leq K$ is in fact identical with the embedded chain of a continuous branching process with offspring distribution $(p_i)_{i\in\mathbb{N}}$, and when $i>K$, identical with the embedded chain of a continuous branching process with offspring distribution $(q_i)_{i\in\mathbb{N}}$. Thus, the properties of these two simple embedded chains may help us to prove that $N(t)<\infty\ a.s.$.

Lemma 5.4. Let $\{\zeta(t): t\geq 0\}$ be a continuous branching process with offspring distribution $(p_i)_{i\in\mathbb{N}}$. Let $T_1=\inf\{\text{t}: \zeta(t)>K \text{ or } \zeta(t)=0\}$, then for every $j\in\{1,2,\dots,K\}$, we have $\mathbb{P}(T_1<\infty|\zeta(0)\equiv j)=1$.

Proof: We first prove this lemma when $\zeta(0) \equiv 1$. Using the definition in Lemma 5.2 of events A and B, we pick an arbitrary $\delta > 0$ and construct an embedded discrete branching process $\zeta_n = \zeta(n\delta)$ according to Lemma 5.1.

The extinction of $\{\zeta(t): t \geq 0\}$ (event B) is equivalent to the extinction of $(\zeta_n)_{n\in\mathbb{N}}$. Meanwhile, the result of Lemma 5.1 shows that $(\zeta_n)_{n\in\mathbb{N}}$ is still a supercritical branching process, so by Lemma 2.5, under the condition A, if we let $F(s) = \sum_{k=0}^{\infty} P_{1k}(\delta)\, s^k$ (where $P_{1k}(\delta)$ is the transition probability of $\zeta(t)$), and $\mathbb{P}(B) = q$, then $(\zeta_n)_{n\in\mathbb{N}}$ can be decomposed into the sum of a supercritical process $({\zeta_n}^{(A)})_{n\in\mathbb{N}}$ whose offspring distribution's PGF is $\hat{F}(s) = (F((1-q)s+q)-q)/(1-q) \coloneqq \sum_{i=0}^{\infty} \widehat{p_i} s^i$ and a non-negative stochastic sequence $(\widetilde{\zeta_n})_{n\in\mathbb{N}}$, where $\sum_{i=0}^{\infty} i\widehat{p_i} = \sum_{i=0}^{\infty} ip_i = M > 1$. Thus $\widehat{p_1} < 1$.

Since $\widehat{p_0} = 0$, $X_1 \coloneqq \min\{\text{n}: {\zeta_n}^{(A)} > 0\}$ is geometrically distributed with parameter $1 - \widehat{p_1} > 0$, thus $X_1 < \infty\ a.s.$. Let ${\zeta_{X_1}}^{(A)} = k$, if $k \leq K$, then further define $X_2 = \min\{\text{n}: {\zeta_{n+X_1}}^{(A)} > k\}$, which is geometrically distributed with parameter $1 - \widehat{p_1}^k > 0$. We also have $X_2 < \infty\ a.s.$. Continuing like this, if ${\zeta_{X_1+X_2}}^{(A)} \leq K$, then further define $X_3 < \infty\ a.s.$... This process will end with no more than K iterations before the stopping time $T_1' = \min\{n: {\zeta_n}^{(A)} > K\}$, because from the moment $X_1 + \cdots + X_n$ to $X_1 + \cdots + X_{n+1}$, the value ${\zeta_n}^{(A)}$ increased at least 1. Meanwhile, the value of ζ_n is a sequence of values intercepted from the continuous process $\{\zeta(t): t \geq 0\}$, thus $T_1 \leq T_1'$, and therefore we have $\mathbb{P}(T_1 < \infty | \zeta(0) \equiv 1, A) = 1$.

If $p_0 = 0$, then by Lemma 5.2 the event A occurs almost surely, so the argument above has already proved the lemma. While if $p_0 > 0$, under the condition of B, by Lemma 2.5, $(\zeta_n)_{n\in\mathbb{N}}$ has the same distribution with a subcritical process with $F^*(s) = F(sq)/q$ as its offspring distribution's PGF, so if we define $T_1'' = \min\{n: \zeta_n = 0\}$, then on B by Lemma 2.3 we have $T_1'' < \infty\ a.s.$. Similar to the case on A, on B we also have $T_1 \leq T_1''$. Summarizing, we get $\mathbb{P}(T_1 < \infty | \zeta(0) \equiv 1) = 1$.

Finally, under the condition $\zeta(0) \equiv k > 1$, we define the sub-process evolved from the j-th particle of $\zeta(0)$ be $\{\zeta^{(j)}(t): t \geq 0\}$, and $T_1^j \coloneqq \inf\{\text{t}: \zeta^{(j)}(t) > K \text{ or } \zeta^{(j)}(t) = 0\}$. Then by the argument above, $T_1^j < \infty\ a.s.$ holds for every j. If there exists some $1 \leq j \leq k$ such that $\zeta^{(j)}(T_1^j) > K$, then we certainly have $T_1 \leq T_1^j < \infty$; otherwise, we have $\max(T_1^1, \dots, T_1^k) < \infty$ and $\zeta(\max(T_1^1, \dots, T_1^k)) = 0$, which also implies $T_1 \leq \max(T_1^1, \dots, T_1^k) < \infty$. ■

Proof of Theorem 1.5. Consider the embedded discrete Markov chain $(X_n)_{n\in\mathbb{N}}$ of a continuous threshold process. We will prove that under any initial value k (a positive integer), we always have $\forall t \geq 0$, $N(t) < \infty\ a.s.$, and then by Lemma 5.3 we can obtain the conclusion of this theorem. First we define a set of stopping times: ($N \geq 2$ is a positive integer)

$$D_0 \coloneqq \min\{\text{n}: X_n \leq K\}$$
$$T_1 \coloneqq \min\{n \geq R: X_n > K \text{ or } X_n = 0\}$$
$$D_N \coloneqq \min\{n \geq T_N: X_n \leq K\}$$
$$T_N \coloneqq \min\{n \geq D_{N-1}: X_n > K \text{ or } X_n = 0\}$$

From the moment 0 to D_0, X_n and the corresponding waiting times τ_0、...、τ_{D_0-1} are respectively identically distributed with the embedded chain and its waiting times of a continuous branching process with initial value k and offspring distribution $(q_i)_{i\in\mathbb{N}}$. Because the Kolmogorov backward equation of this branching process has a unique non-exploding solution, by Lemma 5.2 and 5.3 we know $D_0 < \infty\, a.s.$, and thus for any $0 \le t \le \tau_0 + \cdots + \tau_{D_0 - 1}$, we all have $N(t) < \infty\, a.s.$.

Noticing that the $(X_n)_{n\in\mathbb{N}}$ will decrease at most 1 on every jump, we must have $X_{D_0} > 0$. Then, using the strong Markov property of $(X_n)_{n\in\mathbb{N}}$, it can be seen that from the moment D_0 to T_1, the chain X_n and the waiting times in the corresponding period are again identically distributed with the embedded chain and waiting times of a continuous branching process with X_{D_0} as its initial value and $(p_i)_{i\in\mathbb{N}}$ as its offspring distribution. Thus its Kolmogorov backward equation also has a unique non-exploding solution. So using the result of Lemma 5.3 and Lemma 5.4, we know $T_1 - D_0 < \infty\, a.s.$, so for $0 \le t \le \tau_0 + \cdots + \tau_{T_1 - 1}$, there always holds $N(t) < \infty\, a.s.$.

The two parts of argument above can be repeated: Assume that for some N we already know $N(t) < \infty\, a.s.$ for all t such that $0 \le t \le \tau_0 + \cdots + \tau_{(T_N)-1}$. If $X_{T_N} = 0$, then the embedded chain $(X_n)_{n\in\mathbb{N}}$ will not jump anymore after T_N, so it immediately follows that for any $t \ge 0$, $N(t) < \infty\, a.s.$. If $X_{T_N} > 0$, then by Lemma 5.2, 5.3 and the strong Markov property, we can extend the range of t sastifying our condition to $0 \le t \le \tau_0 + \cdots + \tau_{(D_N)-1}$. Then since $(X_n)_{n\in\mathbb{N}}$ decreases at most 1 on every jump, we still have $X_{D_N} > 0$ (in fact, $X_{D_N} = K\, a.s.$), so by Lemma 5.3, 5.4 and the strong Markov property, for every $0 \le t \le \tau_0 + \cdots + \tau_{(T_{n+1})-1}$ we have $N(t) < \infty\, a.s.$. By induction, we can thus conclude that for every N, $N(t) < \infty\, a.s.$ holds for $0 \le t \le \tau_0 + \cdots + \tau_{(T_N)-1}$.

If $\mathbb{P}(\forall n, X_{T_N} > 0) = 0$, then the argument above has already proved the theorem. Otherwise, on $\{\forall n, X_{T_N} > 0\}$ we still need to prove:

$$\lim_{N\to\infty} \sum_{i=0}^{(T_N)-1} \tau_i = \infty\, a.s.$$

To see this, notice two facts: first, for any natural number N, τ_{D_N} is exponentially distributed with a parameter no larger than K, and the waiting times are mutually independent ; second, on $\{\forall n, X_{T_N} > 0\}$ for every positive integer N we have $T_N - D_{N-1} \ge 1$. From the former fact we know:

$$\mathbb{P}\left(\sum_{N=0}^{\infty} \tau_{D_N} < \infty\right) \le \mathbb{P}\left(\lim_{N\to\infty} \tau_{D_N} = 0\right) \le \mathbb{P}\left(\exists M > 0, \forall N \ge M, \tau_{D_N} \le 1\right) = 0$$

Thus the left-hand probability is equal to zero. Then by the second fact:

$$\lim_{N\to\infty} \sum_{i=0}^{(T_N)-1} \tau_i \ge \sum_{N=0}^{\infty} \tau_{D_N} = \infty\, a.s.$$

So by applying Lemma 5.3, we can get the conclusion of this theorem. ■

After proving Theorem 1.5, we can study the properties of continuous threshold processes. We now give the proof of Theorem 1.6, which uses the tool of embedded chains developed above.

Proof of Theorem 1.6.

(1) By Lemma 5.4 we know, when $p_0 = 0$, $T_1 \coloneqq \inf\{t\colon Z(t) > K \text{ or } Z(t) = 0\}$ is almost surely finite (before the moment T_1, $Z(t)$ is identically distributed with the continuous

branching process in Lemma 5.4), and $Z(T_1) > K\ a.s.$ From the Markov property, we further know that after T_1, $Z(t)$ is identically distributed with a continuous branching process independent from the past process, with initial value $Z(T_1)$ and offspring distribution $(q_i)_{i\in\mathbb{N}}$. From Lemma 5.2 we know that such a process extincts with probability 1, thus $D_1 \coloneqq \inf\{t > T_1: Z(t) \le K\} < \infty\ a.s.$. Noticing that the transition probability of the embedded chain $(X_n)_{n\in\mathbb{N}}$ of $Z(t)$ determines that it decreases at most 1 on every jump, thus we must have $Z(D_1) = K\ a.s.$. Applying Lemma 5.4 again on the process starting from D_1, it is not hard to find that $T_2 \coloneqq \inf\{t > D_1: Z(t) > K \text{ or } Z(t) = 0\} < \infty\ a.s.$, and $Z(T_2) > K\ a.s.$; then it again follows that $D_2 \coloneqq \inf\{t > T_2: Z(t) \le K\} < \infty\ a.s.$ and $Z(D_2) = K\ a.s.$, et cetera.

The argument above shows that the state K is recurrent, and any state $j \in \{1,2,\dots,K\}$ will definitely transfer to K in finite time. So it immediately follows that $\mathbb{P}(T < \infty | Z(0) \equiv j) = 0$ (because state 0 is absorbing).

(2) First by Lemma 5.2, 5.4, we still have $T_1, D_1 < \infty\ a.s.$, and if $Z(T_1) \ne 0$ then there must holds $Z(D_1) = K.$ Consider the embedded chain $(X_n)_{n\in\mathbb{N}}$, using the definition of $N(t)$ in Lemma 5.3, we have:

$$\mathbb{P}(Z(T_2) = 0 | Z(T_1) \ne 0) \ge \mathbb{P}\left(X_{N(D_1)+1} = 0 \middle| X_{N(T_1)} \ne 0\right) = {p_0}^k \coloneqq \lambda > 0$$

If we continue like this to define stopping times T_n and D_n, then by the Markov property of $Z(t)$ we have:

$$\mathbb{P}(Z(T_n) = 0 | Z(T_1) \ne 0, \dots, Z(T_{n-1}) \ne 0) \ge \mathbb{P}\left(X_{N(D_n)+1} = 0 \middle| X_{N(T_1)} \ne 0, \dots, X_{N(T_{n-1})} \ne 0\right) = \lambda$$

And thus:

$$\mathbb{P}(T = \infty | Z(0) \equiv j) \le \lim_{n\to\infty} \mathbb{P}(Z(T_1) \ne 0, \dots, Z(T_n) \ne 0) \le \lim_{n\to\infty} (1-\lambda)^n = 0.$$ ■

Theorem 1.6 reflects the difference of properties between continuous and discrete threshold processes: In discrete processes, all particles in the same generation reproduces their children simultaneously, thus the threshold does not influence its stochastic behaviour very subtly; while in continuous processes, every particle reproduces at a different time, so it is more sensitive to the threshold, and the threshold influences it more subtly. We can see this in the proof clearly: the main reason that state K is recurrent when $p_0 = 0$, is that continuous processes only decrease at most 1 on every jump. This difference between continuous and discrete processes, suits with our intuitive comprehension of natural processes like animal reproduction.

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