A family of sequences generated by eliminating some prime factors from the sequence of the naturals

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Table of the Paper

ABSTRACT

This paper starts with an interesting property of the sequence of the naturals $\{A_n\}$, i.e. $\{A_n\}:=\{1,2,3,...\}$. Considering the generated sequence $\{L_n\}$ by eliminating all the factor 2 in A_n if it has, we show that the partial sum of the new sequence $\{L_n\}$ from 2^{n-1} th term to (2^n-1) th term is square number 2^{2n-2}.

This property is further extended to the more general case: $\{H(n)\}$ is the sequence with each term $H(n)$ generated according to the following transforming rule:

ER $(\{p_1, p_2,..., p_m\})$: Given m primes $p_i, i=1,2,...,m$, for each $i \in \{1,2,...,m\}$, if A_n has factor p_i, then substitute A_n with A_n / p_i until it does not have factor p_i.

We focus on the partial sum of $\{H(n)\}$. In **section one**, we research on Question 1: Given one prime p, we eliminate all the factor p in $\{A_n\}$ according to the transforming rule ER$(\{p\})$. Then the new sequence $\{L_n\}$ obtained has the property as follows:

$$\sum_{i=kp^{n-1}}^{kp^n-1} L_i = \frac{(p-1)}{2} k^2 p^{2n-1}.$$

The main results in this section are Theorem 1, 2 and the relative corollaries.

In **section two,** we mainly study Question 2: Given two prime p,q, we eliminate all factors p and q in $\{A_n\}$ according to the transforming rule ER$(\{p,q\})$. Then the new sequence $\{H(n)\}$ obtained has the property as follows:

$$\sum_{i=1}^{kp^r q^s} H(i) = \frac{k^2 pq(p^{2r}-1)(q^{2s}-1)}{2(p+1)(q+1)} + \sum_{i=1}^{kp^r} H(i) + \sum_{i=1}^{kq^s} H(i) - \sum_{i=1}^{k} H(i)$$

(or $\displaystyle\sum_{i=kp^r}^{kp^r q^s} H(i) = \frac{k^2 pq(p^{2r}-1)(q^{2s}-1)}{2(p+1)(q+1)} + \sum_{i=k}^{kq^s} H(i)$).

We also focus on the number of valued "1" point of $\{H(n)\}$ from this section, where the term in $\{A_n\}$ which becomes the number 1 in $\{H(n)\}$ is called a valued "1" point of

$\{H(n)\}$.Let $Y(n)=\{i\,|\,H(i)=1, i\in N, i\le n\}$, we have

$$|Y(kp^r q^s)|=|Y(kp^r)|+|Y(kq^s)|-|Y(k)|+rs,$$

The main results in this section are Theorem 3, 4, 5, 6 and the relative corollaries.

In **section three,** we extend to the case of given m primes $p_i, i=1,2,...,m$, i.e. Question 3, and obtain similar results in Theorem 7,8,9,10 and the relative corollaries.

In **section four,** we propose Question 4 and obtain several concise inequalities to estimate the partial sum of the first n terms and the number of valued "1" point in the first n terms. Furthermore, we derive a new proof for the noted fact that the set of primes is infinite.

For given two primes case, we have (Theorem 11,12)

$$\frac{pq}{2(p+1)(q+1)}x^2-2x<\sum_{i\le x}H(i)<\frac{pq}{2(p+1)(q+1)}x^2+2x$$

$$\frac{1}{2}\frac{\log x}{\log p}\left(\frac{\log x}{\log q}+1\right)<|Y(x)|<\frac{1}{2}\left(\frac{\log x}{\log p}+1\right)\left(\frac{\log x}{\log q}+2\right) \qquad \text{(where } p<q\text{)}$$

For given m primes case, we have (Theorem 13,14)

$$\frac{1}{2}\prod_{i=1}^{m}\frac{p_i}{p_i+1}x^2-2^{m-1}x<\sum_{i\le x}H(i)<\frac{1}{2}\prod_{i=1}^{m}\frac{p_i}{p_i+1}x^2+2^{m-1}x.$$

$$|Y(x)|=\frac{1}{m!}\frac{\log^m x}{\prod_{i=1}^{m}\log p_i}+O(\log^{m-1}x).$$

Finally, we point out that the multiplicative functions $H(n), H'(n)$ have further research value.

Key words: sequence of the naturals, multiplicative function, partial sum, valued "1" point.

Section 1

The set of naturals is buried with numerous secrets. Among those, I find an interesting rule. Let { A_n } be the sequence of the naturals i.e. $\{A_n\} = \{1, 2, 3, ..., n, ...\}$.We obtain a new sequence { L_n } by eliminating all the factor 2 for every term of $\{A_n\}$ according to the following rule:

(1)If A_n has factor 2,then we substitute A_n with $A_n / 2$ until it does not have factor 2;

(2) If A_n does not have factor 2,then keep it unchanged.

Thus, we obtain a new sequence $\{L_n\}$ as follows:

1,1,3,1,5,3,7,1,9,5,11,3,13,7,15,1,17,9,19,5,21,11,23,3,25,13,27,7,29,15,31,1,33,17,35,9,37,19,39,5,41,… （1）

We can dig out some secrets if we divide it into groups with 1 being the first term of each group as follows:

(1),(1,3),(1,5,3,7),(1,9,5,11,3,13,7,15),(1,17,9,19,5,21,11,23,3,25,13,27,7,29,15,31),(1,33,17,35,9,37,19,39,5,41,……

Summing up each group, we derive that the sum of these groups are 1,4,16,64,… respectively. Thus we have the following conjecture：

Conjecture 1. The sum of the new sequence { L_n } from 2^{n-1} th term to $(2^n - 1)$ th term is 2^{2n-2}.

Since { L_n } contains lots of 3,5,…, we also can divide them by setting 3,5…be the first term of each group as follows:

(1,1),(3,1,5),(3,7,1,9,5,11),(3,13,7,15,1,17,9,19,5,21,11,23),(3,25,13,27,7,29,15,31,1,33,17,35,9,37,19,39,5,41,……

(1,1,3,1),(5,3,7,1,9),(5,11,3,13,7,,15,1,17,9,19),(5,21,11,23,3,25,13,27,7,29,15,31,1,33,17,35,9,37,19,39),(5,41,……

We find that if we divide $\{L_n\}$ by 3, the sums of second, third, fourth,…groups are 9,36,144,… respectively; if we divide it by 5, the sums of second, third, fourth,…groups are 25, 100, 400,… respectively;…

Hence, we put forward the second conjecture.

Conjecture 2. The sum of the new sequence $\{L_n\}$ from $k\times 2^{n-1}$th term to $(k\times 2^n-1)$ th term is $k^2\times 2^{2n-2}$,where $k\in N$.

The above transformation is eliminating all the factor 2 for every term of $\{A_n\}$. But what property does the new sequence have if we eliminate all the factor 3 for every term of $\{A_n\}$? Or even more generally, we consider the following question:

Question 1: Let p be a prime. We obtain a new sequence $\{L_n\}$ by transforming $\{A_n\}$ according to the rule ER$(\{p\})$ as follows:

(1)If A_n has factor p, then we substitute A_n with A_n/p until it does not have factor p;

(2) If A_n does not have factor p, then keep it unchanged.

What properties does $\{L_n\}$ have?

The rule ER$(\{p\})$ above can also be written as follows:

If $A_n=p^r\times X$, where $r\geq 0, X\geq 1$ are integers，$(X,p)=1$，then $L_n=X$.

For this question, we derive the following theorem.

Theorem 1. Let p be a prime. Let $\{L_n\}$ be the sequence generated by transforming $\{A_n\}$ according to the rule ER$(\{p\})$ of Question 1. Then for $k\in N, n\in N$,the following equality holds:

$$\sum_{i=kp^{n-1}}^{kp^n-1} L_i=\frac{(p-1)}{2}k^2p^{2n-1} \qquad (2)$$

Proof. The sum $\sum_{i=kp^{n-1}}^{kp^n-1} L_i$ is from kp^{n-1}th term to (kp^n-1)th term，which means the number of the terms is $kp^n - 1 - kp^{n-1} + 1 = k(p-1)p^{n-1}$.

Now, we claim that the $k(p-1)p^{n-1}$ terms are different from each other.

Indeed, assume that there are positive integers i and j such that $kp^{n-1} \le i < j \le kp^n - 1$, with $L_i = L_j$. Then by the definition of sequence{ L_n }, we have that $i = p^r \times L_i$, $j = p^s \times L_j$ where r and s are nonnegative integers. Under the assumption, we have $j = i \times p^{s-r}, s-r \ge 1$, which contradicts with $kp^{n-1} \le i < j \le kp^n - 1$.

Furthermore, those $k(p-1)p^{n-1}$ terms are just the positive integers no larger than kp^n except those multiples of p. In fact, the set of positive integers which are no larger than kp^n contains kp^{n-1} multiples of p. Thus, there are $k(p-1)p^{n-1}$ terms left exactly, after these kp^{n-1} multiples of p removed.

Note that the sum of these kp^{n-1} multiples of p removed is $p \times \sum_{i=1}^{kp^{n-1}} i$, so we have

$$\sum_{i=kp^{n-1}}^{kp^n-1} L_i = \sum_{i=1}^{kp^n} i - p \times \sum_{i=1}^{kp^{n-1}} i = \frac{1}{2}\left(1+kp^n\right)kp^n - p \times \frac{1}{2}\left(1+kp^{n-1}\right)kp^{n-1}$$

$$= \frac{1}{2}\left(kp^n\right)^2 - \frac{1}{2}kp^n kp^{n-1} = \frac{(p-1)}{2}k^2 p^{2n-1}$$ □

Theorem 2 is derived from Theorem 1.

Theorem 2. Let p be a prime. Let { L_n } be the sequence generated by transforming $\{A_n\}$ according to the rule ER$(\{p\})$of Question 1. Then for $k, n \in N$,the following equality holds:

$$\sum_{i=1}^{kp^n} L_i = \sum_{i=1}^{k} L_i + \frac{k^2 p(p^{2n}-1)}{2(p+1)} \qquad (3)$$

Proof. $\sum_{i=1}^{kp^n} L_i = \sum_{i=1}^{k} L_i + \sum_{j=1}^{n} \sum_{i=kp^{j-1}+1}^{kp^j} L_i$

$$=\sum_{i=1}^{k}L_i+\sum_{j=1}^{n}\frac{(p-1)}{2}k^2p^{2j-1}$$

$$=\sum_{i=1}^{k}L_i+\sum_{j=1}^{n}\frac{1}{2}\left(1-\frac{1}{p}\right)k^2p^{2j}$$

$$=\sum_{i=1}^{k}L_i+\frac{1}{2}\left(1-\frac{1}{p}\right)k^2\frac{p^2(1-p^{2n})}{1-p^2}$$

$$=\sum_{i=1}^{k}L_i+\frac{k^2p(p^{2n}-1)}{2(p+1)}. \qquad \square$$

For the case k＝1，the following corollary holds：

Corollary 1. Let p be a prime. Let { L_n } be the sequence generated by transforming $\{A_n\}$ according to the rule ER $(\{p\})$ of Question 1. Then for $n\in N$,the following equality holds:

（1） $\sum_{i=p^{n-1}}^{p^n-1}L_i=\frac{(p-1)}{2}p^{2n-1}$

（2） $\sum_{i=1}^{p^n-1}L_i=\frac{p(p^{2n}-1)}{2(p+1)}$

For the case p＝2，both conjecture 1 and conjecture 2 are correct.

Section 2

In first section, we study the properties of the new sequence generated by eliminating all the factor p in $\{A_n\}$. Naturally, we wish to generalize it for the case of eliminating more primes. In this section, we consider the case of given two primes.

Question 2. Let p and q be primes. We obtain a new sequence $\{H(n)\}$ by transforming $\{A_n\}$ according to the rule ER $(\{p,q\})$ as follows:

(1)If A_n has factor p, then we substitute A_n with A_n/p until it does not have factor p; If A_n has factor q, then we substitute A_n with A_n / q until it does not have factor q;

(2) If A_n does not have factors p and q, then keep it unchanged.

What properties does $\{H(n)\}$ have?

The rule ER$(\{p,q\})$ above can also be written as follows:

If $A_n = p^r q^s \times X$, where $r \geq 0, s \geq 0, X \geq 1$ are integers, $(X, pq) = 1$, then $H(n) = X$.

For instance, let p=2, q=3. Then the generated sequence $\{H(n)\}$ is:

1,1,1,1,5,1,7,1,1,5,11,1,13,7,5,1,17,1,19,5,7,11,23,1,25,13,1,7,29,5,31,1,11,17,35,1,37,19,13,5,41,7,43,11,…… (4)

This sequence becomes much more complicated. What can be seen is that more number 1 occurs in this new sequence. In the following part, we study the partial sum of this new sequence and the frequency of number 1.

Firstly, the number-theoretic function $H(n): N \to R$ is a completely multiplicative function (cf. [1], p77-79) where R is the set of real numbers. And the function has following properties.

（1） $H(p) = H(q) = H(1) = 1$;

$H(p') = p'$, when p' is a prime, $p' \neq p, p' \neq q$.

（2） $H(mn) = H(m)H(n), H(pn) = H(n), H(qn) = H(n)$;

（3） $H(n) = n$, when n does not have factor p or q.

Theorem 3. Let p and q be primes. Let $\{H(n)\}$ be the sequence generated by transforming $\{A_n\}$ according to the rule ER$(\{p,q\})$ of Question 2. Then for $k \in N$, the following equality holds:

$$\sum_{i=1}^{kpq} H(i) = \frac{k^2}{2} pq(p-1)(q-1) + \sum_{i=1}^{kp} H(i) + \sum_{i=1}^{kq} H(i) - \sum_{i=1}^{k} H(i) \quad (5)$$

Note that this equality can be written as the following briefer forms: