ON A LOWER BOUND FOR THE ENERGY FUNCTIONAL FOR A CERTAIN FAMILY OF LAGRANGIAN TORI IN $\mathbb{C}P^2$

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Student: Aknazar Kazhymurat
Teacher: Dauren Aitmukhamet
NIS Almaty for Physics and Mathematics, Kazakhstan
Mentor: Andrey E. Mironov
Novosibirsk State University, Russia

Abstract

Abstract. In this project we study the energy functional on the set of Lagrangian tori in $\mathbb{C}P^2$. The energy functional has been introduced in [2] as integral of the potential of 2D periodic Schrödinger operator associated to Lagrangian torus. It has been conjectured in [2] that the Clifford torus is the unique global minimum of energy functional (the statement is later referred to as the energy conjecture). Due to geometric interpretation of energy functional as linear combination of the volume and Willmore functionals, this conjecture can be seen as the $\mathbb{C}P^2$ analogue of the well-known Willmore conjecture for tori in $\mathbb{R}^3$, recently proved in [18].

The energy conjecture has been verified for two families of Hamiltonian-minimal Lagrangian tori in [2]. Results of [5] and [23] imply the conjecture for minimal Lagrangian tori of sufficiently high spectral genus and non-embedded minimal Lagrangian tori, respectively.

In the present work we prove the energy conjecture for a family of Hamiltonian-minimal Lagrangian tori in $\mathbb{C}P^2$ constructed in [4]. In sharp distinction with cases considered in [2], the value of the energy functional for these tori *can not* be calculated exactly. The proof relies on analytic bounds for certain elliptic integrals arising from the induced metric of tori.

Possible directions of further work are:

1. Consider local behaviour of the energy functional. Are the critical points of the energy functional governed by an integrable PDE, akin to Tzizeica equation describing minimal Lagrangian tori? The same questions for critical points under Hamiltonian variations.

2. Is there an analogue of the energy conjecture for other Kähler-Einstein surfaces? The case of K3 surface is of special interest as minimal Lagrangian tori in K3 can be related to elliptic fibrations (for instance [20]) making the conjecture amenable to algebro-geometric analysis.

3. Examples of monotone Lagrangian tori with trivial Floer cohomology were constructed in [21]. Do there exist critical points of the energy functional with trivial Floer cohomology?

Introduction

Lagrangian submanifolds $\Sigma \subset \mathbb{C}P^2$ are well-studied objects in symplectic geometry. The topology of Lagrangian embeddings is quite restrictive: there are no Lagrangian embedded spheres [7], Klein bottles [8] and closed orientable surfaces with negative Euler characteristic [9]. Lagrangian immersions admit richer topology; there are examples of Lagrangian spheres [19], Klein bottles [3] and closed orientable surfaces of arbitrary odd genus [10].

In this article, we study Lagrangian tori in $\mathbb{C}P^2$. Firstly, the class of Lagrangian tori is quite wide as any oriented nullhomologous Lagrangian surface is necessarily a torus (by adjunction). Secondly, Lagrangian tori in $\mathbb{C}P^2$ provide local models for singularities of special Lagrangians in Calabi-Yau 3-folds relevant to SYZ conjecture [16]. Thirdly, Lagrangian tori in $\mathbb{C}P^2$ are related to integrable PDE and thus form a showcase for an interesting interplay of symplectic geometry and mathematical physics (for example [12]). Let us spell it out in some detail.

It has been noted in [1] that one can naturally associate a 2D periodic Schrödinger operator with any Lagrangian torus in $\mathbb{C}P^2$. More precisely, a Lagrangian torus $\Sigma \subset \mathbb{C}P^2$ with induced metric

$$ds^2 = 2e^{v(x,y)}(dx^2 + dy^2) \tag{1}$$

can be realized as the image of a composition of maps

$$r : \mathbb{R}^2 \to S^5 \xrightarrow{\mathscr{H}} \mathbb{C}P^2,$$

where r is a horizontal lift of the Lagrangian immersion and $\mathscr{H}$ is the Hopf projection. The vector function r solves 2D periodic Schrödinger equation

$$Lr = 0, \qquad L = (\partial_x - \frac{i\beta_x}{2})^2 + (\partial_y - \frac{i\beta_y}{2})^2 + V(x,y), \qquad V = 4e^v + \frac{1}{4}(\beta_x^2 + \beta_y^2) + \frac{i}{2}\Delta\beta,$$

where β is the Lagrangian angle (defined below).

The existence of operator L allows us to introduce the energy functional on the set of Lagrangian tori in $\mathbb{C}P^2$ [2]:

$$E(\Sigma) = \frac{1}{2}\int_\Sigma V\,dx \wedge dy.$$

The energy functional admits following geometric interpretation [2]:

$$E(\Sigma) = A(\Sigma) + \frac{1}{8}W(\Sigma), \qquad A(\Sigma) = \int_\Sigma d\sigma, \qquad W(\Sigma) = \int_\Sigma |H|^2\,d\sigma,$$

where $d\sigma = 2e^v dx \wedge dy$ is the induced area element and H is the mean curvature vector.

For the Clifford torus Σ_{Cl}, whose vector function is

$$r(x,y) = \Big(\frac{1}{\sqrt{3}}e^{2\pi i x}, \frac{1}{\sqrt{3}}e^{2\pi i(-\frac{1}{2}x+\frac{\sqrt{3}y}{2})}, \frac{1}{\sqrt{3}}e^{2\pi i(-\frac{1}{2}x-\frac{\sqrt{3}y}{2})}\Big),$$

energy equals

$$E(\Sigma_{Cl}) = \frac{4\pi^2}{3\sqrt{3}}.$$

In [2] following conjecture has been proposed.

Conjecture 1. *The minimum of energy functional in the class of Lagrangian tori in $\mathbb{C}P^2$ is attained on the Clifford torus.*

In [23] a similar functional on the class of Lagrangian tori in $\mathbb{C}P^2$ was introduced. Results of [23] imply conjecture 1 for non-embedded minimal Lagrangian tori.

In [2] conjecture 1 has been verified for two families of Hamiltonian-minimal (i.e. critical points of the volume functional under Hamiltonian deformations) Lagrangian tori: homogeneous tori and tori constructed in [3].

A homogeneous torus $\Sigma_{r_1,r_2,r_3} \subset \mathbb{C}P^2$, $r_1^2+r_2^2+r_3^2 = 1$, $r_i > 0$ is determined by the following vector function

$$r(x,y) = \big(r_1 e^{2\pi i x}, r_2 e^{2\pi i(a_1 x + b_1 y)}, r_3 e^{2\pi i(a_2 x+b_2 y)}\big),$$

with some restrictions on a_i, b_i. Following inequality holds

$$E(\Sigma_{r_1,r_2,r_3}) = \frac{\pi^2(1-r_1^2)(1-r_2^2)(1-r_3^2)}{2r_1r_2r_3} \geq \frac{4\pi^2}{3\sqrt{3}},$$

and equality is attained only for the Clifford torus (which is, by coincidence, the only *minimal* homogeneous torus).

The second family of tori $\Sigma_{m,n,k} \subset \mathbb{C}P^2, m,n,k \in \mathbb{Z}, m \geq n > 0, k < 0$ has form $\mathcal{H}(\tilde{\Sigma}_{m,n,k})$ where

$$\tilde{\Sigma}_{m,n,k} = \Big\{(u_1e^{2\pi i m y}, u_2 e^{2\pi i n y}, u_3 e^{2\pi i k y})\Big\} \subset S^5,$$

and numbers u_1, u_2, u_3 satisfy constraints:

$$u_1^2+u_2^2+u_3^2 = 1, \qquad mu_1^2+nu_2^2+ku_3^2 = 0.$$

Parameters m,n,k should be chosen so that the involution

$$(u_1,u_2,u_3) \longrightarrow (u_1\cos(m\pi), u_2\cos(n\pi), u_3\cos(k\pi))$$

on the surface $mu_1^2 + nu_2^2 + ku_3^2 = 0$ preserves its orientation (otherwise $\mathcal{H}(\tilde{\Sigma}_{m,n,k})$ is homeomorphic to Klein bottle [3]).

In [2] it is proved that $E(\Sigma_{m,n,k}) > E(\Sigma_{Cl})$.

In the case of *minimal* Lagrangian tori function $v(x,y)$ satisfies Tzizeica equation (as mentioned in [4]). Smooth periodic solutions of this equation are finite-gap, i.e. can be expressed in terms of theta-function of the Prym variety of the spectral curve [22]. Conjecture 1 is a corollary of results of [5] for minimal tori corresponding to spectral curves of sufficiently high genus.

Y. Oh [13] has formulated a related conjecture: Clifford torus minimizes area in its Hamiltonian isotopy class. E. Goldstein [15] has proved a weaker version of Oh's conjecture using the computation of the Floer cohomology of Σ_{Cl} in [17]. In fact, Goldstein's estimates combined with Biran-Cornea narrow-wide dichotomy [21] imply following statement.

Proposition 1. *A monotone Lagrangian torus $\mathbb{C}P^2$ has trivial Floer cohomology if its induced volume satisfies $A(\Sigma) < \frac{4\pi}{\sqrt{3}}$.*

Unfortunately, the problem of giving uniform upper estimates for the volume of tori considered in this paper appears to be quite difficult so we can not say much about their symplectic topology.

The primary aim of the present work is to verify conjecture 1 for a family of Hamiltonian-minimal Lagrangian tori invariant under S^1-group of isometries of $\mathbb{C}P^2$, constructed in [4] (independently in [6]). In sharp distinction with cases considered in [2], the value of the energy functional for these tori *can not* be calculated exactly, due to discontinuous behaviour of one of the periods of tori.

Let $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{Z}, b = -\alpha_1 - \alpha_2 - \alpha_3, c = \alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3, c_1 = -\alpha_1\alpha_2\alpha_3$, $a_1 > a_2 > 0$ be some real numbers satisfying the inequalities (4), (5) (see below). Following theorem has been proved in [4].

Theorem 1. *The mapping $\psi : \mathbb{R}^2 \to \mathbb{C}P^2$ defined by the formula*

$$\psi(x,y) = \left(F_1(x)e^{i(G_1(x)+\alpha_1 y)} : F_2(x)e^{i(G_2(x)+\alpha_2 y)} : F_3(x)e^{i(G_3(x)+\alpha_3 y)}\right),$$

is a conformal Hamiltonian minimal Lagrangian immersion, where

$$F_i = \sqrt{\frac{2e^v + \alpha_{i+1}\alpha_{i+2}}{(\alpha_i - \alpha_{i+1})(\alpha_i - \alpha_{i+2})}}, \qquad G_i = \alpha_i \int_0^x \frac{c_2 - ae^v}{2\alpha_i e^v - c_1}\, dz,$$

$$2e^{v(x)} = a_1\left(1 - \frac{a_1 - a_2}{a_1}\mathrm{sn}^2\left(x\sqrt{a_1 + a_3}, \frac{a_1 - a_2}{a_1 + a_3}\right)\right) \tag{2}$$

(index i runs modulo 3), $\mathrm{sn}(x)$ *is the Jacobi's elliptic function,* c_2 *is a real root of* (3), $a_3 = \frac{c_1^2+c_2^2}{a_1 a_2}$.

Moreover, if the rationality constraints (8) are met, ψ is a doubly periodic mapping and the image of the plane is a Hamiltonian minimal Lagrangian torus $\Sigma_M \subset \mathbb{C}P^2$.

The principal result of the present work is following theorem.

Theorem 2. *The inequality*

$$E(\Sigma_M) > E(\Sigma_{Cl})$$

holds if $\alpha_1 - \alpha_3, \alpha_2 - \alpha_3$ *are relatively prime.*

The theorem 2 thus confirms the conjecture 1.

0.1 The proof of the theorem 2

Lagrangianity of Σ, horizontality of the mapping $r : \mathbb{R}^2 \to S^5$ and the form of the induced metric (1) imply

$$R = \begin{pmatrix} r \\ \frac{r_x}{|r_x|} \\ \frac{r_y}{|r_y|} \end{pmatrix} \in U(3).$$

The Lagrangian angle $\beta(x, y)$ is defined by the equation $e^{i\beta} = \det R$. The mean curvature vector field can be expressed in terms of the Lagrangian angle $H = J\nabla\beta$ where J is the complex structure on $\mathbb{C}P^2$. For minimal tori $\beta = \mathrm{const}$. As demonstrated in [11] in the case of Hamiltonian minimal tori β is a linear function in the conformal coordinates x, y.

Let us consider the Hamiltonian minimal immersion ψ [4] defined in the theorem 2.

The equation

$$(a_1 - a_2)^2 x^4 + 2(a_1^3 a_2^2 + a_1^2 a_2^3 + (a_1^2 a_2 + a_1 a_2^2) b c_1 + (a_1^2 + a_2^2) c_1^2 + 2 a_1^2 a_2^2 c) x^2 +$$

$$+ ((a_1 + a_2) c_1^2 - a_1^2 a_2^2 + a_1 a_2 b c_1)^2 = 0. \quad (3)$$

has a real root $x = c_2$ iff following inequalities are satisfied

$$P = a_1^3 a_2^2 + a_1^2 a_2^3 + (a_1^2 a_2 + a_1 a_2^2) b c_1 + (a_1^2 + a_2^2) c_1^2 + 2 a_1^2 a_2^2 c \leqslant 0, \quad (4)$$

$$P^2 - (a_1 - a_2)^2 ((a_1 + a_2) c_1^2 - a_1^2 a_2^2 + a_1 a_2 b c_1)^2 \geqslant 0. \quad (5)$$

Recall that $\mathrm{sn}(u,k) = \sin\theta$ where

$$u(\theta) = \int_0^\theta \frac{d\phi}{\sqrt{1-k^2\sin^2\phi}}. \tag{6}$$

The function $\mathrm{sn}^2(u)$ is periodic with period $2u(\frac{\pi}{2})$ (see, for instance, [14]). Therefore $v(x)$ has period

$$T = \frac{2u\left(\frac{\pi}{2}\right)}{\sqrt{a_1+a_3}}. \tag{7}$$

Further we assume that $(\alpha_1-\alpha_3, \alpha_2-\alpha_3) = 1$.

The immersion $\psi : \mathbb{R}^2 \to \mathbb{C}P^2$ is doubly periodic if there exists $\tau \in \mathbb{R}$ such that

$$\lambda_1 = \frac{G_1(T)-G_3(T)+(\alpha_1-\alpha_3)\tau}{2\pi}, \qquad \lambda_2 = \frac{G_2(T)-G_3(T)+(\alpha_2-\alpha_3)\tau}{2\pi} \in \mathbb{Q}. \tag{8}$$

Then the vectors of period can be expressed as follows

$$e_1 = (0, 2\pi), \qquad e_2 = N(T,\tau),$$

where N is some natural number. If the condition (8) is met, $\Sigma_M \subset \mathbb{C}P^2$ is an immersed torus with Lagrangian angle $\beta = ax+by$ where

$$a = \frac{bc_1 + a_1a_3 + a_2a_3 - a_1a_2}{c_2}. \tag{9}$$

Following equality holds

$$|H|^2 = \frac{1}{2}e^{-v}(a^2+b^2).$$

Let us find lower bounds for $W(\Sigma_M)$ and $A(\Sigma_M)$.

Using (7) and $a_3 > 0$ we arrive at the inequalities

$$u(\frac{\pi}{2}) > \frac{\pi}{2}, \qquad T > \frac{\pi}{\sqrt{a_1+a_3}}.$$

Thus

$$W(\Sigma_M) = \int_{\Sigma_M} |H|^2\,d\sigma = \int_\Lambda \frac{1}{2}e^{-v}(a^2+b^2)2e^v\,dx\wedge dy = 2\pi NT(a^2+b^2).$$

Therefore, following lower bound for $W(\Sigma_M)$ holds

$$W(\Sigma_M) > 2\pi^2\frac{a^2+b^2}{\sqrt{a_1+a_3}}. \tag{10}$$

Following lemma provides a lower bound for $A(\Sigma_M)$.

Lemma 1. *The inequality*
$$A(\Sigma_M) > \pi^2 \frac{a_1 + a_2}{\sqrt{a_1 + a_3}}$$
is true.

Proof of the lemma 1. We have
$$A(\Sigma_M) = \int_{\Sigma_M} d\sigma = \int_{\Lambda} 2e^{v(x)}\, dx \wedge dy = 2\pi \int_0^{NT} 2e^{v(x)} dx \geqslant 2\pi \int_0^T 2e^{v(x)}\, dx =$$
$$= 2\pi \int_0^T a_1 \left(1 - \frac{a_1 - a_2}{a_1} \mathrm{sn}^2 \left(x\sqrt{a_1 + a_3}, \frac{a_1 - a_2}{a_1 + a_3}\right)\right) dx =$$
$$= \frac{2\pi a_1}{\sqrt{a_1 + a_3}} \int_0^{2u(\frac{\pi}{2})} \left(1 - \frac{a_1 - a_2}{a_1} \mathrm{sn}^2 \left(u, \frac{a_1 - a_2}{a_1 + a_3}\right)\right) du.$$
Using (6) we arrive at
$$\int_0^T 2e^{v(x)}\, dx = \frac{a_1}{\sqrt{a_1 + a_3}} \int_0^{\pi} \frac{1 - \frac{a_1 - a_2}{a_1} \sin^2\theta}{\sqrt{1 - \left(\frac{a_1 - a_2}{a_1 + a_3}\right)^2 \sin^2\theta}}\, d\theta.$$

As $0 < \frac{a_1 - a_2}{a_1 + a_3} < 1$, following estimate is true
$$\int_0^T 2e^{v(x)} dx > \frac{a_1}{\sqrt{a_1 + a_3}} \int_0^{\pi} \left(1 - \frac{a_1 - a_2}{a_1} \sin^2\theta\right) d\theta = \frac{\pi(a_1 + a_2)}{2\sqrt{a_1 + a_3}}.$$
Lemma 1 is proved.

The inequalities (4), (5) are invariant under simultaneous change of sign $\alpha_1, \alpha_2, \alpha_3$ and their permutations. If $\alpha_1, \alpha_2, \alpha_3$ are all of the same sign, the inequality (4) has no positive solutions. Therefore we assume without loss of generality that $\alpha_1 \geqslant \alpha_2 \geqslant 0 \geqslant \alpha_3$.

Lemma 2. *If $\alpha_1 \geqslant \alpha_2 \geqslant 0 \geqslant \alpha_3$ and $a_1 > a_2 > 0$, the inequalities (4) and (5) are satisfied simultaneously iff*
$$-\alpha_2\alpha_3 \leqslant a_2 < a_1 \leqslant -\alpha_1\alpha_3. \tag{11}$$

Proof of the lemma 2. Denote
$$Q(x) = -(x + \alpha_1\alpha_2)(x + \alpha_1\alpha_3)(x + \alpha_2\alpha_3).$$

Then (3) assumes the form

$$(a_1-a_2)^2\left(x^2-\left(\frac{a_1\sqrt{Q(a_2)}-a_2\sqrt{Q(a_1)}}{a_1-a_2}\right)^2\right)\left(x^2-\left(\frac{a_1\sqrt{Q(a_2)}+a_2\sqrt{Q(a_1)}}{a_1-a_2}\right)^2\right)=0.$$

This equation has a positive root iff $Q(a_1)\geqslant 0, Q(a_2)\geqslant 0$. This is equivalent to $-\alpha_2\alpha_3\leqslant a_2<a_1\leqslant -\alpha_1\alpha_3$. Lemma 2 is proved.

It follows from the proof of the lemma 2 that if $\alpha_3=0$ or $\alpha_1=\alpha_2$ inequalities (4), (5) are not satisfied for $a_1>a_2$. Therefore we assume without loss of generality

$$\alpha_1>\alpha_2\geqslant 0>\alpha_3. \tag{12}$$

The inequality (10) and lemma 1 imply

$$E(\Sigma_M)>\pi^2\frac{a_1+a_2+\frac{a^2+b^2}{4}}{\sqrt{a_1+a_3}}.$$

Let us prove $E(\Sigma_M)>E(\Sigma_{Cl})$. We will consider two cases: $\alpha_2>0$ and $\alpha_2=0$.

Assume $\alpha_2>0$.

If $(a_1+a_2)a_3\geqslant\frac{7}{4}(a_1a_2-bc_1)$ then

$$a^2=\frac{((a_1+a_2)a_3-(a_1a_2-bc_1))^2}{c_2^2}\geqslant\frac{9}{49}(a_1+a_2)^2\frac{a_3^2}{c_2^2}=\frac{9}{49}(a_1+a_2)^2\frac{a_3}{a_1a_2}\frac{c_1^2+c_2^2}{c_2^2}\geqslant\frac{9}{49}(a_1+a_2)^2\frac{a_3}{a_1a_2}.$$

As $a_1>a_2\geqslant 1$ and $(a_1+a_2)^2>4a_1a_2$ we have

$$E(\Sigma_M)>\pi^2\frac{a_1+a_2+\frac{9(a_1+a_2)^2a_3}{196a_1a_2}}{\sqrt{a_1+a_3}}>\pi^2\frac{a_1+\frac{9a_3}{49}}{\sqrt{a_1+a_3}}=\pi^2\sqrt{a_1}\frac{1+\frac{9a_3}{49a_1}}{\sqrt{1+\frac{a_3}{a_1}}}>\pi^2\frac{1+\frac{9a_3}{49a_1}}{\sqrt{1+\frac{a_3}{a_1}}}.$$

Note that for positive x we have $\frac{1+\frac{9x}{49}}{\sqrt{1+x}}>\frac{4}{3\sqrt{3}}$ holds. Consequently, $E(\Sigma_M)>E(\Sigma_{Cl})$.

Now consider the case

$$(a_1+a_2)a_3<\frac{7}{4}(a_1a_2-bc_1).$$

We analyse two cases: $\alpha_1>-\frac{3}{2}\alpha_2\alpha_3$ and $\alpha_1\leqslant-\frac{3}{2}\alpha_2\alpha_3$.

If $\alpha_1>-\frac{3}{2}\alpha_2\alpha_3$ then

$$\alpha_1<-3b=3(\alpha_1+\alpha_2+\alpha_3),$$

as $\alpha_1 > -\frac{3}{2}(\alpha_2+\alpha_3)$. From (11)

$$-\frac{bc_1}{a_1+a_2} = \frac{b\alpha_1\alpha_2\alpha_3}{a_1+a_2} < \frac{b(3b)\alpha_2\alpha_3}{2\alpha_2\alpha_3} = \frac{3}{2}b^2.$$

Hence

$$E(\Sigma_M) > \pi^2\frac{a_1+a_2+\frac{b^2}{4}}{\sqrt{a_1+a_3}} > \pi^2\frac{a_1+a_2+\frac{b^2}{4}}{\sqrt{a_1+\frac{7}{4}\frac{a_1a_2}{a_1+a_2}-\frac{7}{4}\frac{bc_1}{a_1+a_2}}} > \pi^2\frac{a_1+a_2+\frac{b^2}{4}}{\sqrt{a_1+\frac{7}{4}a_2+\frac{21}{8}b^2}} >$$

$$> \pi^2\frac{a_1+a_2+\frac{b^2}{4}}{\sqrt{\frac{7}{4}a_1+\frac{7}{4}a_2+\frac{21}{8}b^2}} = \pi^2\sqrt{\frac{4(a_1+a_2)}{7}}\frac{1+\frac{b^2}{4(a_1+a_2)}}{\sqrt{1+\frac{3}{2}\frac{b^2}{a_1+a_2}}} > \pi^2\sqrt{\frac{8}{7}}\frac{1+\frac{b^2}{4(a_1+a_2)}}{\sqrt{1+\frac{3}{2}\frac{b^2}{a_1+a_2}}} > E(\Sigma_{Cl}).$$

The last inequality can be seen by considering the function $f(x) = \sqrt{\frac{8}{7}}\frac{1+\frac{x}{4}}{\sqrt{1+\frac{3}{2}x}}$ for $x > 0$.

If $\alpha_1 \leqslant -\frac{3}{2}\alpha_2\alpha_3$, the inequalities (11) and (12) imply

$$-bc_1 \leqslant -2\alpha_1^2\alpha_2\alpha_3 < \frac{9}{2}a_1a_2^2.$$

Therefore

$$E(\Sigma_M) > \pi^2\frac{a_1+a_2}{\sqrt{a_1+\frac{7}{4}\frac{a_1a_2-bc_1}{a_1+a_2}}} = \pi^2\frac{(a_1+a_2)\sqrt{a_1+a_2}}{\sqrt{a_1(a_1+a_2)+\frac{7}{4}a_1a_2-\frac{7}{4}bc_1}} >$$

$$> \pi^2\frac{(a_1+a_2)\sqrt{a_1+a_2}}{\sqrt{a_1^2+\frac{11}{4}a_1a_2+\frac{63}{8}a_1a_2^2}} > \pi^2\frac{(a_1+a_2)\sqrt{a_1+a_2}}{\sqrt{a_1^3+\frac{11}{4}a_1^2a_2+\frac{63}{8}a_1a_2^2}} = \pi^2\frac{(1+\frac{a_2}{a_1})\sqrt{1+\frac{a_2}{a_1}}}{\sqrt{1+\frac{11}{4}\frac{a_2}{a_1}+\frac{63}{8}\frac{a_2^2}{a_1^2}}} > E(\Sigma_{Cl}).$$

Let us consider the case $\alpha_2 = 0$. Introduce $p = -\alpha_1\alpha_3$, $x = \frac{a_1}{p}$, $y = \frac{a_2}{p}$. Note that $0 < y < x \leqslant 1$ due to (12). Then inequalities (4), (5) assume following form

$$p^5x^2y^2(x+y-2) \leqslant 0, \qquad 4p^{10}x^4y^4(1-x)(1-y) \geqslant 0.$$

The equation (3) implies

$$c_2^2 = p^3x^2y^2\frac{2-x-y\pm\sqrt{(2-x-y)^2-(x-y)^2}}{(x-y)^2}. \tag{13}$$

As $2-x-y > 0$ we have $\sqrt{(2-x-y)^2-(x-y)^2} = (2-x-y)\sqrt{1-\frac{(x-y)^2}{(2-x-y)^2}}$. Note that by

Bernoulli inequality

$$1-\frac{(x-y)^2}{(2-x-y)^2}\leqslant\sqrt{1-\frac{(x-y)^2}{(2-x-y)^2}}\leqslant 1-\frac{(x-y)^2}{2(2-x-y)^2}.$$

Consequently,

$$2-x-y-\frac{(x-y)^2}{2-x-y}\leqslant\sqrt{(2-x-y)^2-(x-y)^2}\leqslant 2-x-y-\frac{(x-y)^2}{2(2-x-y)}. \tag{14}$$

Consider two cases: sign '+' and '-' in (13). For the '-' sign (13) and (14) imply the inequalities

$$p^3\frac{x^2y^2}{2(2-x-y)}\leqslant c_2^2\leqslant p^3\frac{x^2y^2}{2-x-y}.$$

As $c_1=0$ we have following bound for a_3

$$a_3=\frac{c_2^2}{a_1a_2},\qquad p\frac{xy}{2(2-x-y)}\leqslant a_3\leqslant p\frac{xy}{2-x-y}.$$

These estimates and lemma 1 imply

$$A(\Sigma_M)\geqslant\pi^2\sqrt{p}\frac{x+y}{\sqrt{x+\frac{xy}{2-x-y}}}.$$

Following inequality holds

$$a=\frac{(a_1+a_2)a_3-a_1a_2}{c_2}\geqslant\frac{(xp+yp)p\frac{xy}{2(2-x-y)}-xyp^2}{c_2}\geqslant\sqrt{p}\left(\frac{x+y}{2(2-x-y)}-1\right)\sqrt{2-x-y}.$$

The estimate (10) implies

$$W(\Sigma_M)\geqslant 2\pi^2\frac{a^2}{\sqrt{a_1+a_3}}\geqslant 2\pi^2\sqrt{p}\left(\frac{x+y}{2(2-x-y)}-1\right)^2\frac{2-x-y}{\sqrt{x+\frac{xy}{2-x-y}}}.$$

Henceforth

$$E(\Sigma_M)\geqslant\pi^2\sqrt{p}\left(\frac{x+y}{\sqrt{x+\frac{xy}{2-x-y}}}+\frac{1}{4}\left(\frac{x+y}{2(2-x-y)}-1\right)^2\frac{2-x-y}{\sqrt{x+\frac{xy}{2-x-y}}}\right).$$

As $p \geqslant 1$ we have

$$E(\Sigma_M) \geqslant \pi^2 B_1(x,y), \qquad B_1(x,y) = \frac{16 - 7x^2 + 8x - 14yx + 8y - 7y^2}{16\sqrt{(2-x)(2-x-y)x}}.$$

Lemma 3. *If* $0 < y < x \leqslant 1$, *then* $B_1(x,y) > 1$.

Proof of the lemma 3. One can check by direct computation that there are no critical points $\partial_x B_1 = \partial_y B_1 = 0$ inside the triangle $0 < y < x \leqslant 1$ while on the boundary of the triangle $B_1(x,y) > 1$ holds. Lemma 3 is proved.

Therefore, $E(\Sigma_M) > E(\Sigma_{Cl})$ holds for the '-' sign in (13).

For the '+' sign in (13) (14) implies the inequalities

$$p^3 f(x,y) \leqslant c_2^2 \leqslant p^3 g(x,y),$$

where

$$f(x,y) = x^2y^2\frac{2(2-x-y) - \frac{(x-y)^2}{2-x-y}}{(x-y)^2}, \qquad g(x,y) = x^2y^2\frac{2(2-x-y) - \frac{(x-y)^2}{2(2-x-y)}}{(x-y)^2}.$$

Analogously one establishes the inequalities

$$p\frac{f(x,y)}{xy} \leqslant a_3 \leqslant p\frac{g(x,y)}{xy},$$

$$a \geqslant \sqrt{p}\frac{(x+y)\frac{f(x,y)}{xy} - xy}{\sqrt{g(x,y)}}.$$

The inequality (10) and lemma 1 imply

$$A(\Sigma_M) \geqslant \pi^2\sqrt{p}\frac{x+y}{\sqrt{x + \frac{g(x,y)}{xy}}},$$

$$W(\Sigma_M) \geqslant 2\pi^2\frac{a^2}{\sqrt{a_1 + a_3}} \geqslant 2\pi^2\sqrt{p}\frac{((x+y)\frac{f(x,y)}{xy} - xy)^2}{g(x,y)\sqrt{x + \frac{g(x,y)}{xy}}},$$

$$E(\Sigma_M) \geqslant \pi^2\sqrt{p}\frac{x+y+\frac{1}{4}\frac{((x+y)\frac{f(x,y)}{xy} - xy)^2}{g(x,y)}}{\sqrt{x + \frac{g(x,y)}{xy}}} \geqslant \pi^2 B_2(x,y),$$

where

$$B_2(x,y) = \frac{x + y + \frac{1}{4}\frac{((x+y)\frac{f(x,y)}{xy} - xy)^2}{g(x,y)}}{\sqrt{x + \frac{g(x,y)}{xy}}}.$$

The following lemma is established similarly to the lemma 3.

Lemma 4. *If* $0 < y < x \leqslant 1$, *then* $B_1(x,y) > 0.9$.

This finishes the proof of the theorem 2.

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