The Hardness of Finding Hamiltonian Cycle in Grids Graphs of Semiregular Tessellations

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1 Abstract

The Hamiltonian cycle problem is an important problem in graph theory and computer science. It is a special case of the famous traveling salesman problem. A significant amount of research has been done on the special cases of finding Hamiltonian cycles in subgraphs of triangular, square and hexagonal grids. However, there is little work on more complicated grids. In this paper, we investigate the hardness of Hamiltonian cycle problem in grid graphs of semiregular tessellations, which are tessellations formed by two or more kinds of regular polygons. There are only eight semiregular tessellations, and we prove that the Hamiltonian cycle problem in all of them are NP-complete by reducing from NP-complete problems, such as the Hamiltonian cycle problem in max degree 3 bipartite planar graphs. Knowing the NP-completeness of Hamiltonian cycle problem in semiregular grids indicates that there will not be any polynomial time algorithm that solves the Hamiltonian path problem in these tessellations if NP does not equal to P, which helps show the limits of efficient motion planning algorithms and provides new information about what makes problems computationally difficult to solve.

2 Introduction

This paper discusses the hardness of Hamiltonian cycle problem in graphs based on semiregular tessellation, which are tessellations formed by more than two kinds of regular polygons. There are a total of eight semiregular tessellations [1], which are shown in Figure 1, and we prove that the Hamiltonian cycle problems (HCP) in all of them are NP-complete. We prove their hardness by reducing from three NP-complete problems: HCP in planar max degree 3 bipartite graphs, HCP in hexagonal grids, and Tree-Residue Vertex Breaking problem.

There is prior research on HCPs in regular grid graphs that proves hardness using the same reductions. A paper in 1982 proved that the HCP in square grid is NP-complete by reducing from the HCP in planar max degree 3 bipartite graphs[2]. Then, a paper published in 2008 proved that the HCPs in triangular and hexagonal grid are NP-complete by the same reduction [3]. In June of 2017, a new paper proved that HCP in hexagonal thin grid graph is NP-complete by reducing from 6-Regular Tree-Residue Vertex Breaking problem [4]. These papers also show results on grid graphs with restrictions such as thin, polygonal, and solid. With all the interest in the computational complexity of the HCP in grid graphs, it is reasonable to ask whether we can generalise or adapt these results to different types of grids. This paper addresses this question and proves the hardness for the HCPs in all of the semiregular grids.

Although the hardness of the HCP in semiregular grids seems like an abstract question, it has many possible applications. Grids are natural structures that things may be formatted into. For example, the layout of buildings or modular structures used in space may form a network that follow the patterns of semiregular grids. If certain locations in such net work need to be visited for maintenance, and one does not want to visit any location twice to avoid wasting energy, one asks exactly for an Hamiltonian path algorithm. The NP-completeness of HCP in semiregular grids provides insight into creating such algorithm. Additionally, the results in this paper are useful in that we may be able to prove hardness of other problems by reducing from HCPs in semiregular grids.

This papered is mentored by Mr. Jayson Lynch. He made this paper possible by guiding and encouraging me. I owe him much gratitude for his contribution.

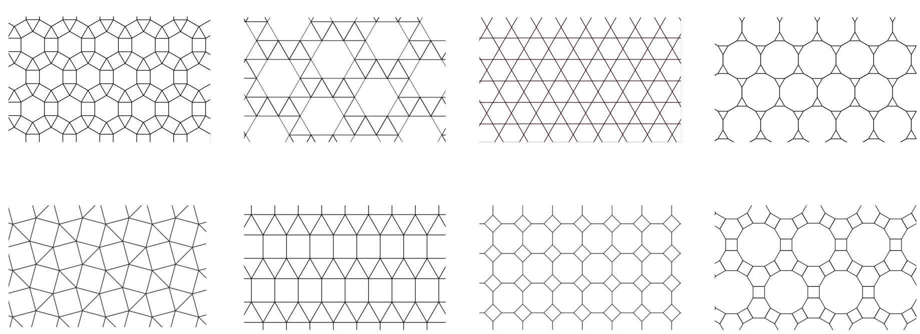


Figure 1: The Eight Semiregular Tessellations

3 Definitions

A *tessellation* is a tiling of a plane with polygons without overlapping. A *semiregular tessellation* is a tessellation which is formed by two or more kinds regular polygons of side length 1 and in which the corners of polygons are identically arranged.

An infinite lattice of semiregular tessellation is a lattice formed by taking the vertices of the regular polygons in the tessellation as the points of the lattice. A graph G is *induced* by the point set S if the vertices of G are the points in S and its edges connect vertices that are distance 1 apart. A *grid graph* of a semiregular tessellation, or a *semiregular grid*, is a graph induced by a subset of the infinite lattice formed by that tessellation.

A *Hamiltonian cycle* is a cycle that passes through each vertex of a graph exactly once. The *Hamiltonian cycle problem*, sometimes abbreviated as HCP, asks that given a graph, whether or not that graph admits a Hamiltonian cycle. The HCP in a semiregular tessellation asks, given a grid graph of that tessilation, whether it admits a Hamiltonian cycle.

4 Main Results

This section shows the hardness proofs of HCPs in eight semiregular tessellations. All of the proofs require reducing from some NP-complete problem to the HCPs in the semiregular tessellations. There are three NP-complete problems that we reduce from: the HCP in hexagonal grid, the HCP in planar max degree 3 bipartite graphs, and the Tree-Residue Vertex-Breaking problem. The main result section is therefore divided into three subsections, each of which introduces one of the three NP-complete problems and includes the hardness proofs that reduce from that NP-complete problem.

4.1 HCPs that Reduce from the HCP in Hexagonal Grid

This section proves that the HCPs in the 3.4.6.4 tessellation, 3.3.3.6 tessellation, 3.6.3.6 tessellation, and 3.12.12 tessellation are NP-complete by reducing from the HCP in hexagonal grid, which is proven to be NP-complete[3]. The reduction works in the following way: for any given Hexagonal grid graph G', which is sometimes referred to as the original graph, we can construct a simulated grid graph G of the target tessellation that has a Hamiltonian cycle if and only if G' has a Hamiltonian cycle. The grid graph G is constructed by using gadgets to represent vertices and edges of the original graph G'.

4.1.1 3.4.6.4 Tessellation

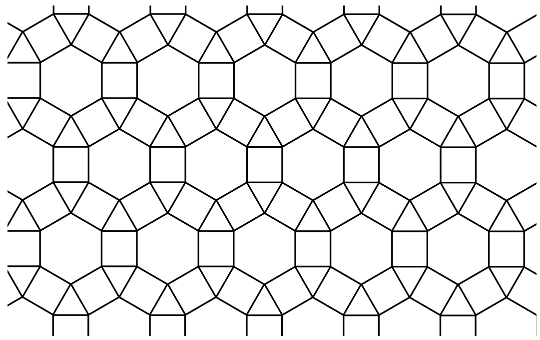


Figure 2: 3.4.6.4 Tessellation

Theorem 1. *The HCP in grid graphs of the 3.4.6.4 tessellation is NP-complete.*

Proof. We can reduce the HCP in this tessellation from the HCP in hexagonal grids. Let G' be a hexagonal grid graph. Then, we can construct a grid graph G of the 3.4.6.4. tessellation in this way: for every edge in G', we use the edge gadget shown in Figure 3; for every vertices, we use the vertex gadgets shown in Figure 4. Note that because hexagonal grid G' is bipartite, we design different vertex gadgets for the even and odd vertex. For example, the graph of two hexagons can be simulated by the graph below, as shown in Figure 5.

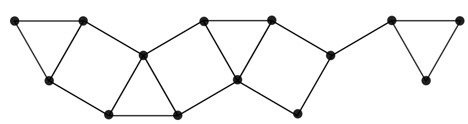


Figure 3: Edge Gadget

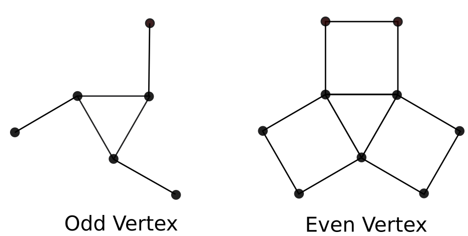


Figure 4: Vertex Gadgets

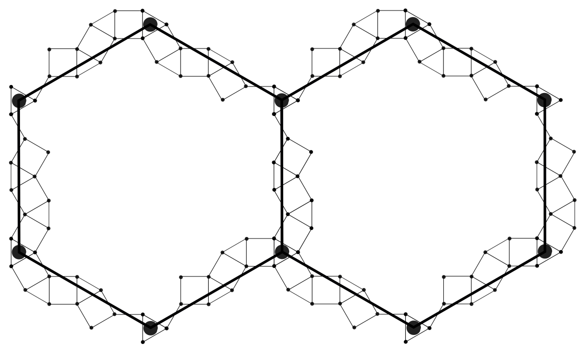


Figure 5: A Simulated Graph

Now, we will show that the original graph G' has a Hamiltonian cycle C' if and only if the simulated graph G has a Hamiltonian cycle C. If the G' has a Hamiltonian cycle C', for any taken edge in it, we go through the corresponded edge gadget in G with the cross path in Figure 6; for any untaken edge, we go through the corresponded edge gadget with the return path. Because the simulated vertices in G are triangles (K_3), there is always a path to take the simulated vertex by entering from one point and leaving at the other. Therefore, if there is a Hamiltonian cycle C' in the original graph G', then there is a Hamiltonian cycle C in the simulated graph G.

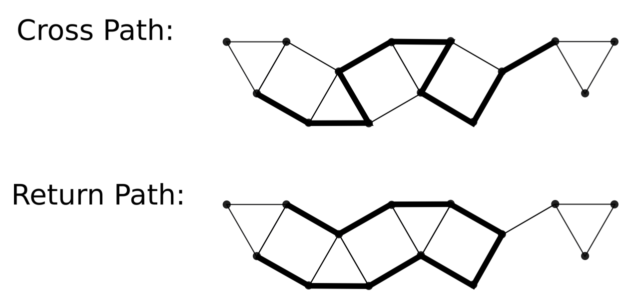


Figure 6: Two Kinds of Paths

The essential difference between the cross path and the return path is that a cross path starts and finishes at different ends of an edge while the return path starts and finishes in the same end. Note that the return and cross paths are the only two paths which go through the edge gadget and visit all of its vertices. The odd vertex gadget is connected to the edge gadget through a single edge connection which prevents the return path from entering the odd vertex gadget. If a Hamiltonian cycle C exists in the simulated graph G, it is not hard to see that each odd vertex gadget in G must be connected to two cross paths and the even vertex gadgets can either be connected to two cross paths or two cross paths and a return path. Then, we can find a cycle C' in the original graph G'

by making each edge gadget with a cross path in C a taken edge in C'. Thus, if there is a cycle C in the simulated graph G, there is a cycle C' in the original graph G'. Figure 7 is an example of a Hamiltonian cycle C in the simulated graph. This way, we showed the original graph G' has a Hamiltonian cycle C' if and only if the simulated graph G has a Hamiltonian cycle C. Thus, we can reduce the HCP in hexagonal graphs, which is proven to be NP-complete, to the HCP in the 3.4.6.5 tessellation and the proof is complete.

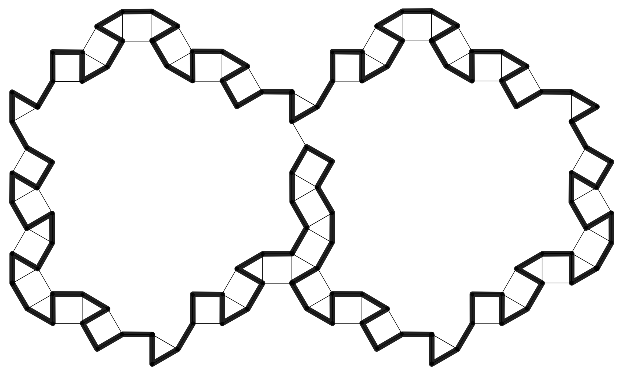


Figure 7: Cycle Example

□

4.1.2 3.3.3.3.6 Tessellation

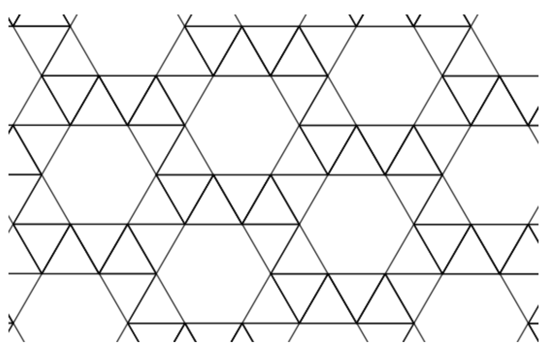


Figure 8: 3.3.3.3.6 Tessellation

Theorem 2. *The HCP in the grid graphs of the 3.3.3.3.6 tessellation is NP-complete.*

Proof. Similar to the 3.4.6.4 tessellation in Section 4.1.1, the NP-completeness of the HCP tessellation can also be proven by reducing from the HCP in hexagonal grid. We use the gadgets shown in Figure 9 to simulate the vertices and edges of the hexagonal grid. Now, we can construct a simulated graph G for any hexagonal grid G'. For example, the graph formed by two hexagons can be simulated by the grid in Figure 10.

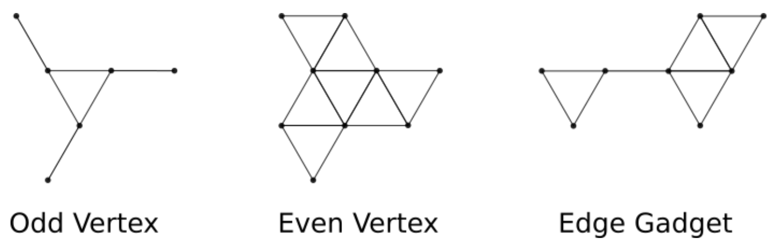


Figure 9: Gadgets

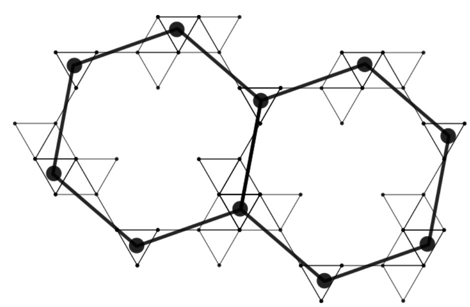
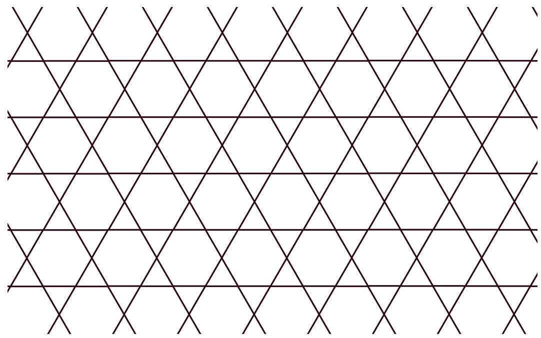


Figure 10: An Example of a Simulated Graph

Similar to the gadgets used in Section 4.1.1, there are two kinds of traversals for the edge gadget: a cross path that goes from one end to the other end and the return path that begins and finishes on the same end. The following reasoning on why G has a Hamiltonian cycle if and only G' has a Hamiltonian cycle is identical to that of the previous section. If a hexagonal grid G' has a Hamiltonian cycle, we can create a Hamiltonian cycle in G by going through the edge gadgets of the taken edges with cross paths and and the edge gadgets of the untaken edges with return paths. If there is a Hamiltonian cycle in G, each vertex gadget of G must be connected to exactly two cross paths, indicating that there exists a Hamiltonian cycle in G. The reduction is then complete. □

4.1.3 3.6.3.6 Tessellation

Theorem 3. *The HCP in the grid graphs of the 3.6.3.6 tessellation is NP-complete.*



Proof. We prove that the HCP in this tessellation is NP-complete by reducing from HCP in hexagonal grid. Using the following vertex gadgets and edge gadget, shown in Figures 11 and Figure 12, for any hexagonal grid G' we can construct a simulated graph G in the tessellation.

Each edge gadget have two kinds of traversals: return paths and cross paths.

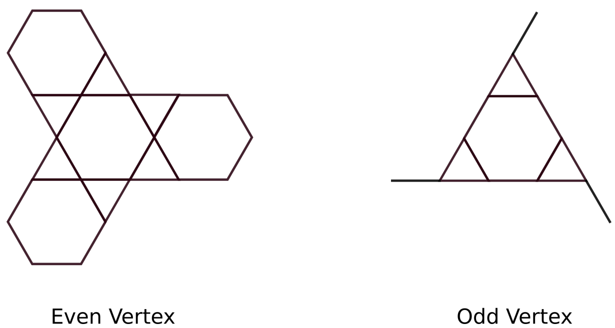


Figure 11: Vertex Gadgets

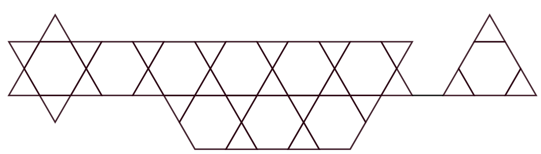


Figure 12: Edge Gadget

Return paths begin and finish on the same end of the edge while cross paths start and finish on different ends. With some inspection, it is clear that return paths and cross paths are the only two kinds of traversals allowed in the edge gadget. Figure 13 shows a possible return path. Different from those of previous tessellations, the edge gadget here has two kinds of cross paths as shown in Figure 14. Although the two kinds of cross paths start out the same from the odd vertex gadget on the right, they finish in the even vertex on the left differently. The way a cross path connects to an even vertex gadgets dictates which direction it can go next. The upper cross path must turns clockwise when going through the even vertex, allowing it to connect to an upper cross path while the lower one must turn counter-clockwise, allowing it to connect to a lower cross path. By choosing the correct kind of cross paths, any pair of the three edges of the even vertex gadget can be taken by compatible cross paths. By inspection, we can easily see that odd vertex gadget can connect to any pair of the three edges in two cross paths as well.

Now, we will show that the simulated graph G has a Hamiltonian cycle if and

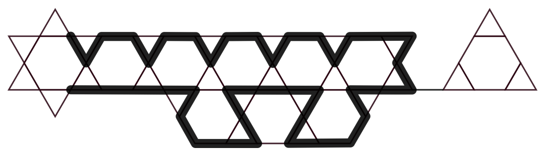


Figure 13: Return Path

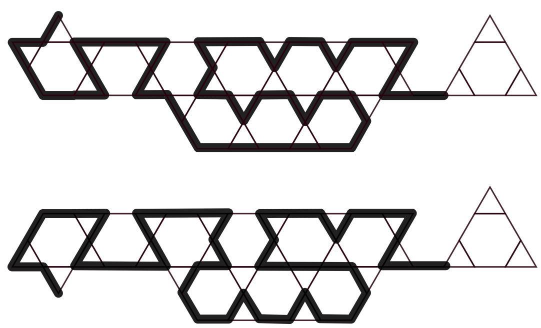


Figure 14: Two Kinds of Cross Paths

only if the original graph G' has a cycle. If the original hexagonal grid G' has a cycle C', then we go through the edges gadgets representing taken edges in C' with a cross path and those representing untaken edges with a return path. Note that we need to use the correct kind of cross paths so that the choice matches the turning at the vertex. Then, there is a also a Hamiltonian cycle in G. If the simulated graph G has a Hamiltonian cycle, each vertex gadget must be connected to exactly two cross paths, which indicate that there is a Hamiltonian cycle in G'. The reduction therefore works.

□

4.1.4 3.12.12 Tessellation

Theorem 4. *The HCP in the grid graphs of the 3.12.12 tessellation is NP-complete.*

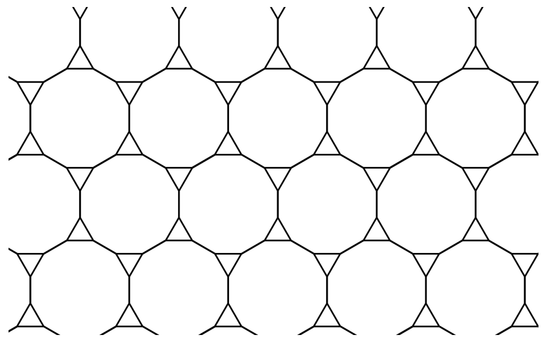


Figure 15: 12.3.12 Tessellation

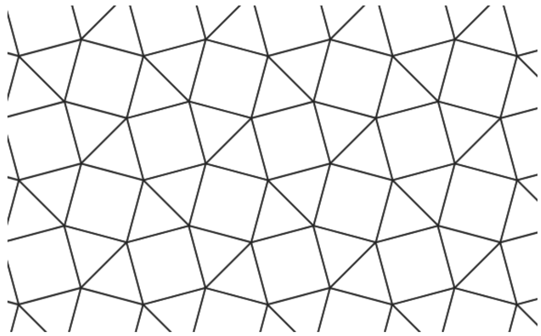
Proof. This tessellation is composed of dodecagons and triangles. For a hexagonal grid G', we construct a simulated graph G in the tessellation by using the triangles as vertex of G' and the edges in between triangles as the edges of G'. If a Hamiltonian cycle exists in G, each triangle must be connected to two paths that form a 120^o angle. Then, there must also be a Hamiltonian cycle in the hexagonal grid G'. If there is a Hamiltonian path in the hexagonal grid G', then there exist one in G. The reduction is thus complete.

□

4.2 HCPs that Reduce from the HCP in Planar Max Degree 3 Bipartite Graph

This section prove that the HCPs in 3.3.4.3.4 tessellation and 3.3.3.4.4. tessellation are NP-complete by reducing from the HCP in planar max degree 3 bipartite graph, which is proven to be NP-complete [2]. Similar to that of the last section, the reduction works in this way: for any given Planar Max Degree 3 Bipartite Graph G', which is sometimes referred to as the original graph, we can construct a grid graph G of the tessellation that has a Hamiltonian cycle if and only if G' has a Hamiltonian cycle. When constructing G, we again use gadgets to simulate the edges and vertices of the original graph G.

4.2.1 3.3.4.3.4 Tessellation



Theorem 5. *The HCP in the grid graphs of the 3.3.4.3.4 tessellation is NP-complete.*

Proof. This tessellation can be viewed as a square grid with some extra diagonals. We directly use the gadgets of the square grid proof in the 1982 paper for constructing G [2]. The edge, even vertex and odd vertex gadgets are shown below. Note that these gadgets are identical to the square grid gadgets except they have some extra edges. In creating the simulated graph G based on a planar max degree 3 bipartite graph G', we go through the same process as that in the square grid reduction: first create a parity-preserving embedding of the max degree 3 bipartite graph; then replace the edges and vertices of the embedding with respective gadgets [2].

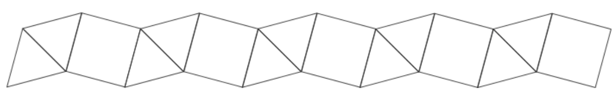


Figure 16: Edge Gadget

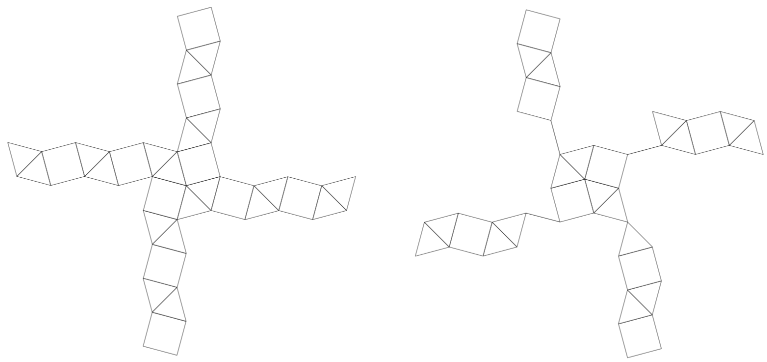


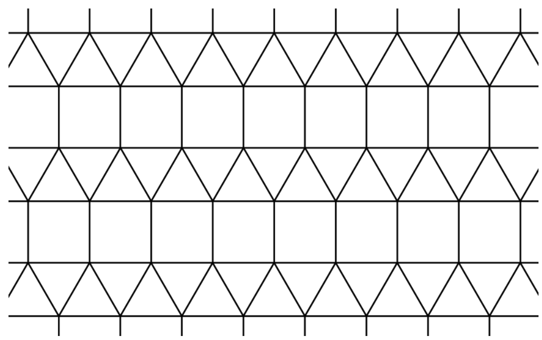
Figure 17: Vertex Gadgets

It is not hard to see that there are only two kinds of traversals for the edge gadget: cross paths and a return paths. Although there are more than one kinds cross path due to the extra edges, they have the essential characteristic of starting from one end of the gadgets and finishing at the other end (unlike the return path that begins and finishes at the same end). Another difference from the square grid reduction is that the odd vertex gadgets connect to the bottom edge gadget through a single point rather than a single edge as the other edge gadgets. As a single edge connection does, this single point connection also prevents a return path from entering the odd vertex gadget. The single edge and single point connections are called pin connections.

Now we will show that G has a Hamiltonian cycle if and only G' has a Hamiltonian cycle. The 1982 paper shows that the simulated square grid constructed with these gadgets has a Hamiltonian cycle if the original graph G' has a Hamiltonian cycle [2]. Because for any original graph G', G has all the edges that the simulated square grid has, if G' has a cycle, G must also have a cycle. If there exists a Hamiltonian cycle in G, the odd vertex gadgets must be connected to two cross paths because the inner points of the gadget cannot be reached by a return path. Consequently, the even vertex must also be connected to two cross paths. Then, the original graph G' must have a cycle as well. The reduction therefore works.

□

4.2.2 3.3.3.4.4 Tessellation



Theorem 6. *The HCP in the grid graphs of the 3.3.3.4.4 tessellation is NP-complete.*

Proof. Similar to the 3.3.4.3.4 Tessellation, this tessellation can also be considered as a square grid with extra diagonals. Because its resemblance to square grid, we use the square grid gadgets on it as well. However, if we use the same reduction from the square grid proof in 1982, an extra diagonal may disable a pin connection, being an extra edge that connects the odd vertex gadget with the edge gadget. Then, a return path can enter into the odd vertices through this disabled pin connection, which is not allowed in order for the reduction to work.The connection to the upper edge gadget in an odd vertex gadget shown in Figure 18 is an example of a disabled pin connection. Therefore, we cannot just apply the reduction in the 1982 paper on this tessellation.

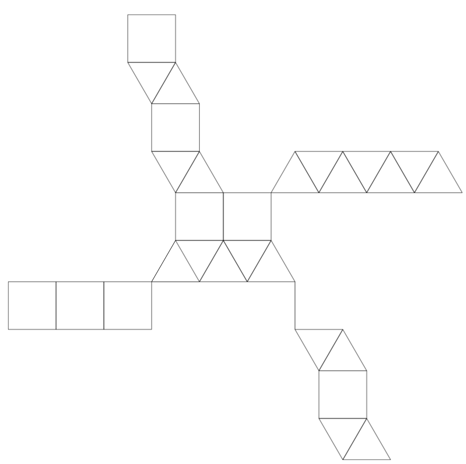


Figure 18: An Odd Vertex Gadget

Although one pin connection may be disabled in a odd vertex gadget, there remains three other functioning pin connections. Because the reduction only requires max degree three vertices, there are still ways to make the reduction work. We construct the simulated grid G in the following way. Given a parity preserving square grid embedding of the original max degree three bipartite graph G' as mentioned in the 1982 paper [2], we enlarge the embedding by a factor of 3 so that any single segment is at least three segments long and the parities of the vertices are preserved. We then adjust the embedding by replacing every disabled pin connection with a functioning pin connection. Figure 19 shows that if the upper connection is disabled, we use the left or right connection to replace it (the upper row represents embedding before adjustment while the lower row represents embedding after adjustment). Based on the adjusted embedding, we can then construct a simulated graph G using the square grid gadgets. Because the pin connections are all functioning in G, the reduction still works.

□

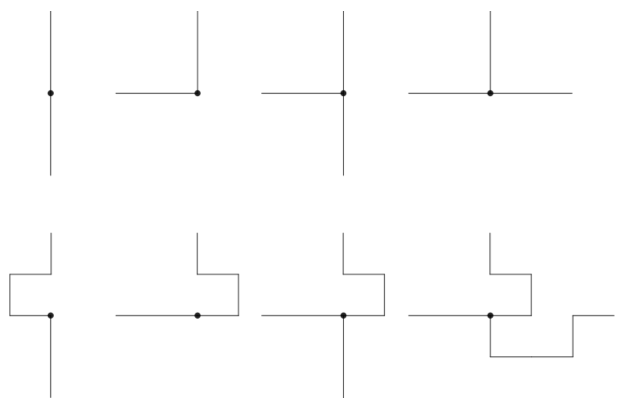


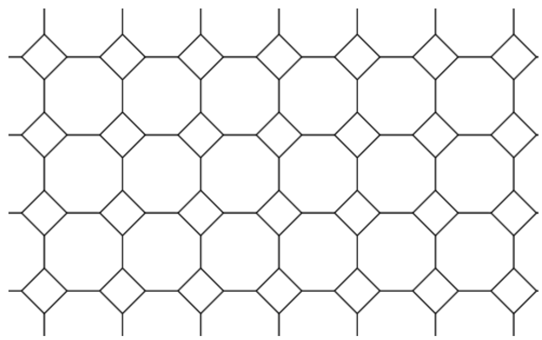
Figure 19: Embedding Adjustment

4.3 HCPs that Reduce from the Tree-Residue Vertex Breaking Problem

This section proves that the HCPs in 4.8.8 tessellation and 4.6.12 tessellation are NP-complete by reducing from the Tree-Residue Vertex Breaking Problem studied in [5]. Here, *breaking* a degree n vertex means turning the vertex into n degree one vertices that are at the ends of the n edges. Tree-Residue Vertex Breaking problem asks that given a planar multigraph M and with some of its vertices marked breakable, is it possible to break some of the breakable vertices so that the resulting graph is a tree. N-Regular Breakable Planar Tree-Residue Vertex-Breaking problem asks that given a planar multigraph with all the vertices degree n and breakable, is it possible to produce a tree from breaking some vertices. The HCPs in these section reduce from 4-Regular Breakable

Planar Tree-Residue Vertex-Breaking problem and 6-Regular Breakable Planar Tree-Residue Vertex-Breaking problem, both of which are NP-complete[5]. The reduction works in this fashion: for any graph M, we will construct a grid graph G of the tessellation so that G has a Hamiltonian cycle if and only if M is breakable.

4.3.1 4.8.8 Tessellation



Theorem 7. *The HCP in the grid graphs of the 4.8.8 tessellation is NP-complete.*

Proof. 4-Regular Breakable Planar Tree-Residue Vertex-Breaking problem reduce to the HCP in the 4.8.8 tessellation. When constructing a grid graph G of the 4.8.8 tessellation based on M, we first make a square grid embedding of M, using a method such as the one described in [6]. Then, for each vertex of M, we use the vertex gadget in Figure 21. For the edges in the embedding, we use the edge gadget formed by the boundary vertices of a three-octagon wide strip, as shown in Figure 20. Notice that the edge gadget can shift and turn easily. Due to this flexibility, we can form a graph G based on the embedding using the gadgets.

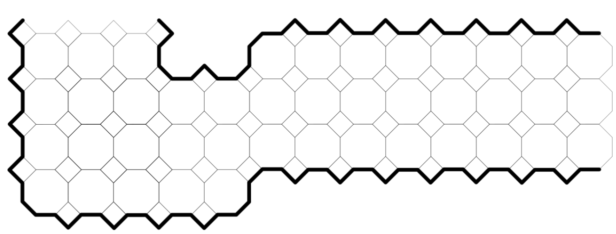


Figure 20: Edge Gadget with a turn

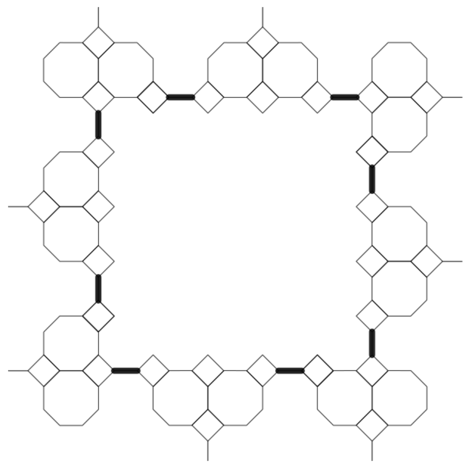


Figure 21: Vertex Gadget

Now, we will show why the constructed graph G has a Hamiltonian cycle if and only if M is breakable. Noticed that if G has a cycle C, all the sides on the edge gadgets must be in C, and the freedom is only in how to traverse the vertex gadgets. Figure 22 shows two solution to the vertex gadget. The four edge gadgets connect to the vertex on the four sides of it. Each edge gadgets has two separate strings of vertices that go into the vertex gadget. Note that there are eight single connection edges (bolded edges in Figure 21) in the vertex gadgets, each of which is in between a pair of adjacent strings. If a cycle exists and a path comes in from a string, the path must enter one of the two adjacent single edge connections and then connect with the path coming in from another string. Thus, for a vertex gadget, there are only two kinds of solution: one that has two strings of the same edge connected or one that has two strings of two adjacent edges connected. The first kind is illustrated by the solution on the left, which correspond to a broken vertex in M while the second kind is illustrated by the solution on the right side, which correspond to a unbroken vertex in M. To show that G has a Hamiltonian cycle if and only if M is breakable, we apply the reasoning used in the 2017 paper [4]. If M is breakable, then for every broken vertex in M, we traverse through the corresponding vertex gadget using the broken solution; for every unbroken vertex, we traverse through the corresponding vertex gadget using the unbroken solution. Note that after this procedure, the graph produced by breaking M is the same as the region inside the edge gadgets in G. If the graph produced by breaking M is indeed a tree, which is connected and acyclic, then the region inside the edges must also be connected and hole-free, which shows that there is a Hamiltonian cycle. If there is a Hamiltonian cycle in G, the region inside must by connected and hole-free, which then show that the graph M can be broken down to a tree. The reduction therefore works. $\square$

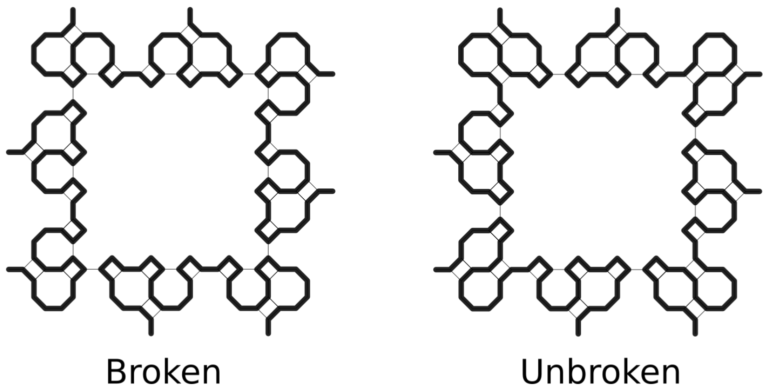
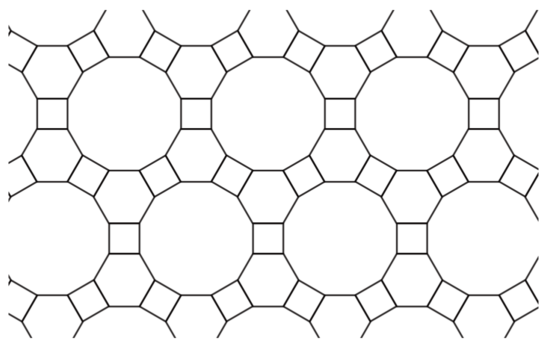


Figure 22: Solutions for the Vertex Gadget

4.3.2 4.6.12 Tessellation



Theorem 8. *The HCP in the grid graphs of the 4.6.12 tessellation is NP-complete.*

Proof. We prove that the HCP in 4.6.12 Tessellation is NP-complete by reducing from the 6-Regular Breakable Planar Tree-Residue Vertex-Breaking problem. When constructing a grid graph G in 4.8.8 tessellation based on the multigraph M, we first embed the multigraph in the triangular grid. Then, we use the vertex gadget shown in Figure 23 for every vertex in M and the edge gadget shown in Figure 24 for the edges in M. The edge gadget only includes the boundary vertices of the shape depicted in Figure 24. Because the turning demonstrated in 24 can have turning of 60 and 120 degrees, we can construct the induced subgraph G based on the triangular grid embedding.

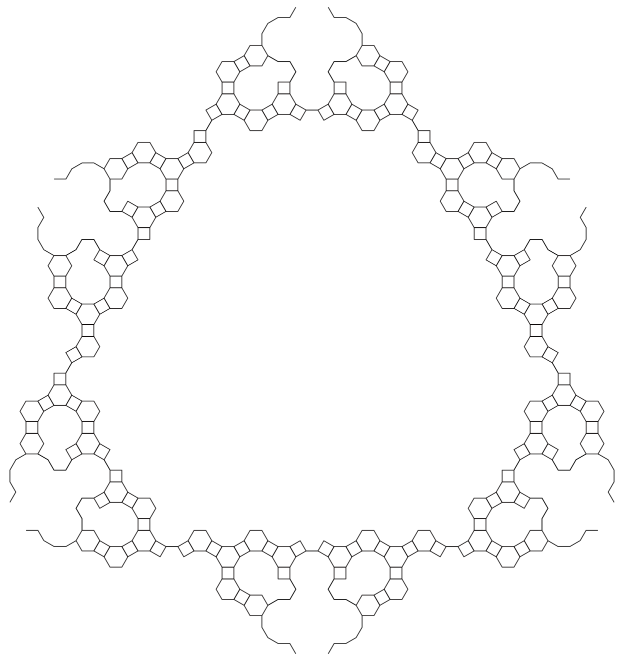


Figure 23: Vertex Gadget

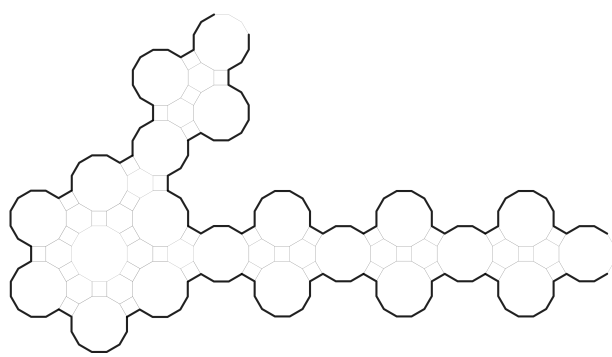


Figure 24: Edge Gadget with a turn

Now, we will show why the constructed graph G has a Hamiltonian cycle if and only if M is breakable. The traversals of the edge gadgets of 4.6.12 tessellation are already set and freedom is only in how to go through the vertex gadgets. The six edge gadgets connect to the vertex gadget on the six sides and each edge gadget consists of two strings of vertices. As mentioned in the 4.8.8 tessellation, because of the single edge connections between each pair of adjacent strings, there are only two kinds of traversals for a vertex gadget: the one that has two strings of the same edge connected or the one that has two strings of two adjacent edges connected. The first kind is illustrated by the solution in Figure 25, which corresponds to a broken vertex in M. The second kind is illustrated by the solution in Figure 26, which corresponds to an unbroken vertex in M. Just as the argument in 4.8.8 tessellation proof states, the region inside the edge gadgets represents the produced graph after breaking M. The reduction therefore works and the proof is complete.

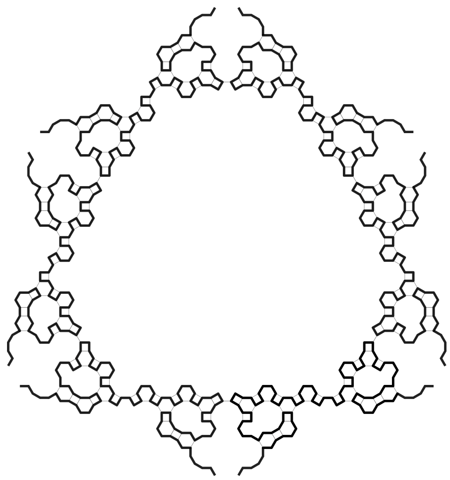


Figure 25: A Broken Vertex

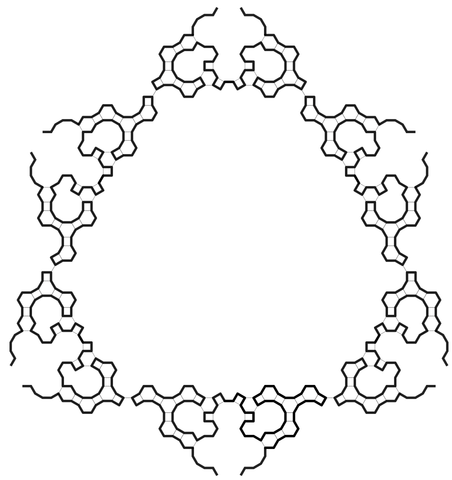


Figure 26: A Unbroken Vertex

□

5 Conclusion

We have shown that the HCPs in all of the eight semiregular tessellations are NP-complete. Knowing this, we then understand that it is impossible to come up with a polynomial-time algorithm for Hamiltonian cycle in semiregular grids assuming P is not equal to NP. Thus, if one wants to create an algorithm for this problem, it is better to look for an approximation algorithm rather than an exact algorithm. The results on semiregular tessellations presented in this paper suggest many other questions that are worth discussing. For example, we can think about why are the different semiregular grids suitable for different kinds of reductions. Can the HCPs that reduce from HCP planar max-degree 3 bipartite graph be easily reduced from Tree-Residue Vertex Breaking Problem, and vise versa? We can also study semiregular grids with specific restrictions such as thin, superthin, polygonal, and solid. After learning about semiregular tessellations, it is also reasonable to study and consider the hardness of HCPs in their dual graphs. If we are interested in motion planning in 3D space, the HCPs in 3D grids are also worth studying.

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