THE ASYMMETRIC COLONEL BLOTTO GAME

Author: Yifan Zhu
Mentor: Simon Rubinstein-Salzedo
Coach: Hongjun Zhao
Affiliation: Shanghai Foreign Language School, Shanghai, China

THE ASYMMETRIC COLONEL BLOTTO GAME

Abstract. This paper explores the Nash equilibria of a variant of the Colonel Blotto game, which we call the Asymmetric Colonel Blotto game. In the Colonel Blotto game, two players simultaneously distribute forces across n battlefields. Within each battlefield, the player that allocates the higher level of force wins. The payoff of the game is the proportion of wins on the individual battlefields. In the asymmetric version, the levels of force distributed to the battlefields must be nondecreasing. In this paper, we find a family of Nash equilibria for the case with three battlefields and equal levels of force and prove the uniqueness of the marginal distributions. We also find the unique equilibrium payoff for all possible levels of force in the case with two battlefields, and obtain partial results for the unique equilibrium payoff for asymmetric levels of force in the case with three battlefields.

1. Introduction

In this section we discuss the background and origins of the Asymmetric Colonel Blotto game.

The Colonel Blotto game, which originates with Borel in [Bor53], is a constant-sum game involving two players, A and B, and n independent battlefields. A distributes a total of X_A units of force among the battlefields, and B distributes a total of X_B units of force among the battlefields, in such a way that each player allocates a nonnegative amount of force to each battlefield. The player who sends the higher level of force to a particular battlefield wins that battlefield. The payoff for the whole game is the proportion of the wins on the individual battlefields.

Roberson in [Rob06] characterizes the unique equilibrium payoffs for all (symmetric and asymmetric) configurations of the players' aggregate levels of force, and characterizes the complete set of equilibrium univariate marginal distributions for most of these configurations for the Colonel Blotto game.

A possible variant of the Colonel Blotto game, which has not been studied before, is the Asymmetric Colonel Blotto game, where the forces distributed among the battlefields must be in non-decreasing order.

The Asymmetric Colonel Blotto game is a constant-sum game involving two players, A and B, and n independent battlefields. A distributes X_A units of force among the battlefields in a nondecreasing manner and B distributes X_B units of force among the battlefields in a non-decreasing manner. Each player distributes forces without knowing the opponent's distribution. The player who provides the higher amount of force to a battlefield wins that battlefield. If both players deploy the same amount of force to a battlefield, we declare that battlefield to be a draw, and the payoff of that battlefield is equally distributed among the two players.[1] The payoff for each player is the proportion of battlefields won.

In this paper, we study the Nash equilibria and equilibrium payoffs of Asymmetric Colonel Blotto games.

In Section 3, we find a family of equilibria for the game with three battlefields and equal levels of force, and we prove the uniqueness of the marginal distribution functions. We also prove that in any equilibrium strategies for a game with equal levels of force and at least three battlefields, there are no atoms in the marginal distributions.

In Section 4, we find the unique equilibrium payoffs of all cases of the Asymmetric Colonel Blotto game involving only two battlefields, and in Section 5 we find the unique equilibrium payoffs in the case of three battlefields in certain cases. We conclude with Section 6, where we discuss the difficulties in extending our work to the case of $n \geq 4$ battlefields.

2. The model

In this section we introduce the model and related concepts. The definitions in this section are adaptations from those in [Rob06] to the asymmetric version.

[1] As we show in Theorem 3.5, Nash equilibria of games with equal levels of force do not contain atoms, so the probability that the two players place equal force on some battlefield is 0. Thus we may, if we choose, use a different tie-breaking rule without altering the result in this case.

2.1. **Players.** Two players, A and B, simultaneously allocate their forces X_A and X_B across n battlefields in a nondecreasing manner. Each player distributes forces without knowing the opponent's distribution. The player who provides the higher level of force to a battlefield wins that battlefield, gaining a payoffof $\frac{1}{n}$. If both players deploy the same level of force to a battlefield, that battlefield is a draw and both players gain a payoffof $\frac{1}{2n}$. The payo ff for each player is the proportion of battlefields won, or equivalently, the sum of the payoffs across all the battlefields.[2]

Player i sends x_i^k units to the kth battlefield. For player i, the set of feasible allocations of force across the n battlefields in the Asymmetric Colonel Blotto game is denoted by $\mathfrak{B}_i$:

$$\mathfrak{B}_i = \left\{ \mathbf{x} \in \mathbb{R}^n \;\middle|\; \sum_{j=1}^{n} x_i^j = X_i, 0 \leq x^1 \leq x^2 \leq \cdots \leq x^n \right\}.$$

Definition 2.1. Given an n-variate cumulative distribution function H, for every $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ such that $x_k \leq y_k$ for all $k \in \{1, \ldots, n\}$, the H-*volume* of the n-box $[x_1, y_1] \times \cdots \times [x_n, y_n]$ is,

$$V_H([\mathbf{x}, \mathbf{y}]) = \underset{n\ x_n}{\Delta^{y_n}} \underset{n-1\ x_{n-1}}{\Delta^{y_{n-1}}} \ldots \underset{2\ x_2}{\Delta^{y_2}} \underset{1\ x_1}{\Delta^{y_1}} H(\mathbf{t}),$$

where

$$\underset{k\ x_k}{\Delta^{y_k}} H(\mathbf{t}) = H(t_1, \ldots, t_{k-1}, y_k, t_{k+1}, \ldots, t_n) - H(t_1, \ldots, t_{k-1}, x_k, t_{k+1}, \ldots, t_n).$$

Intuitively, the H-volume of a n-box just measures the probability that a point within that n-box will be chosen given the cumulative distribution function H.

Definition 2.2. The *support* of an n-variate cumulative distribution function H is the complement of the union of all open sets of $\mathbb{R}^n$ with H-volume zero. Intuitively, the support of a mixed strategy is just the closure of the set of pure strategies that might be chosen.

2.2. **Strategies.** A mixed strategy, or a *distribution of force*, for player i is an n-variate cumulative distribution function (cdf) $P_i : \mathbb{R}_+^n \to [0, 1]$ with support in the set of feasible allocations of force $\mathfrak{B}_\mathrm{i}$. This means that if player i chooses strategy $(X^j)_{j=1}^n$, then the probability that $X^j \leq x^j$ $(j = 1, \ldots, n)$ is $P_i(x^1, \ldots, x^n)$. P_i has *marginal* cumulative distribution functions $\left\{F_i^j\right\}_{j=1}^n$, one univariate marginal cumulative distribution function for each battle field j. $F_i^j(x^j)$ is the probability that $X^j \leq x^j$. Equivalently, $F_i^j(x) = P_i(X_i, X_i, \ldots, x, X_i, \ldots, X_i)$, where the jth argument is x, and the rest of the arguments are X_i, the player's entire allocation of force. We write $P_i = \left(F_i^j\right)_{j=1}^n$.

In the case where the mixed strategy is the combination of finite pure strategies, the mixed strategy P_i where $(i_j^1, i_j^2, \ldots, i_j^n)$ units of force are distributed the battlefields $1, 2, \ldots, n$ respectively with probability p_j is denoted by

$$P_i = \left\{ \left(\left(i_j^1, i_j^2, \ldots, i_j^n\right), p_j \right) \right\}.$$

Here $i_j^1 + i_j^2 + \cdots + i_j^n = X_i$ and $\sum_j p_j = 1$.

[2]That the payoff for each player is the sum of the payoffs across all the battlefields means that two different joint distributions are equivalent if they have the same marginal distributions. Hence, this definition makes it possible to separate a joint distribution into the marginal distributions and a n-copula later in this paper.

2.3. **The Asymmetric Colonel Blotto Game.** The *Asymmetric Colonel Blotto game* with n battlefields, denoted by

$$ACB(X_A, X_B, n),$$

is a one-shot game in which players simultaneously and independently announce distributions of force $(x_i^1, \ldots, x_i^n)$ subject to their budget constraints $\sum_{j=1}^n x_1^j = X_A$ and $\sum_{j=1}^n x_2^j = X_B$, $x_i^j \geq 0$ for each i, j, and such that $x_i^1 \leq x_i^2 \leq \cdots x_i^n$ for $i = 1, 2$. Each battlefield, providing a payoff of $\frac{1}{n}$, is won by the player that provides the higher allocation of force on that battlefield (and declared a draw if both players allocate the same level of force to a battlefield, each gaining a payoff of $\frac{1}{2n}$), and players' payoffs equal the sum of the payoffs over all the battlefields.

2.4. **Nash equilibrium.** Mixed strategies P_A and P_B form a Nash equilibrium if and only if neither player can increase payoff by changing to a different strategy.

Since this particular game is two-player and constant-sum, it has the interesting property that the equilibrium payoff is always unique:

Theorem 2.1. *The Nash equilibrium payoff for both players of any two-player and constant-sum game is unique.*

Proof. Suppose P_A and P_B is a pair of Nash equilibrium strategies. Let w_i be the payoff for player i. For any pair of Nash equilibrium strategies P'_A and P'_B, let w'_i be the payoff for player i.

Let us consider the payoff for both players when player A plays strategy P_A and player B plays strategy P'_B. Call the payoff for player A v_A and the payoff for player B v_B. Since P_A is a strategy in a Nash equilibrium, $w_A \geq v_A$. Similarly, $w'_B \geq v_B$. So $w_A + w'_B \geq v_A + v_B$. Since we are considering a constant sum game, $w_A + w'_B \geq v_A + v_B = w_A + w_B$. Hence, $w'_B \geq w_B$. Similarly, we must have $w_B \geq w'_B$. So $w_B = w'_B$. Similarly $w_A = w'_A$. ■

3. Optimal univariate marginal distributions for three battlefields

In this section we use copulas to separate the joint distributions of players into the marginal distributions and suitable copula. We also find and prove the unique univariate marginal distribution for $ACB(1, 1, 3)$.

Let us first introduce the concept of copulas:

Definition 3.1. Let I denote the unit interval $[0, 1]$. An *n-copula* is a function C from I^n to I such that

(1) For all $\mathbf{x} \in I^n$, $C(\mathbf{x}) = 0$ if at least one coordinate of $\mathbf{x}$ is 0; and if all coordinates of $\mathbf{x}$ are 1 except x_k, then $C(\mathbf{x}) = x_k$.
(2) For every $\mathbf{x}, \mathbf{y} \in I^n$ such that $x_k \leq y_k$ for all $k \in \{1, \ldots, n\}$, the C-volume of the n-box $[x_1, y_1] \times \cdots \times [x_n, y_n]$ satisfies

$$V_C([\mathbf{x}, \mathbf{y}]) \geq 0.$$

The crucial property of n-copulas that we need is the following theorem of Sklar:

Theorem 3.1 (Sklar, [Skl59]). *Let H be an n-variate distribution function with univariate marginal distribution functions $F_1, F_2, \ldots, F_n$. Then there exists an n-copula C such that for all $\mathbf{x} \in \mathbb{R}^n$,*

$$H(x_1, \ldots, x_n) = C(F_1(x_1), \ldots, F_n(x_n)). \tag{1}$$

Conversely, if C is an n-copula and $F_1, F_2, \ldots, F_n$ are univariate distribution functions, then the function H defined by equation (1) is an n-variate distribution function with univariate marginal distribution functions $F_1, F_2, \ldots, F_n$.

The proof of this theorem can be found in [SS83].

This theorem establishes the equivalence between a joint distribution on the one hand, and a combination of a complete set of marginal distributions and a n-copula on the other hand. We will now show that the univariate marginal distribution functions and the n-copula are separate components of the players' best responses.

Proposition 3.2. *In the game $ACB(X_A, X_B, n)$, suppose that the opponent's strategy is fixed as the distribution P_{-i}, and that $X_A = X_B$. Then, in order for player i to maximize payoff under the constraint that the support of the chosen strategy must be in $\mathfrak{B}_i$, player i must solve an optimization problem. Given that there are no atoms in Nash equilibrium strategies (Theorem 3.5), we can write the Lagrangian for this optimization problem as*

$$\max_{\left\{F_i^j\right\}_{j=1}^n} \lambda_i \sum_{j=1}^n \left[\int_0^\infty \left[\frac{1}{n\lambda_i} F_{-i}^j(x) - x\right] dF_i^j\right] + \lambda_i X_i, \tag{2}$$

where the set of univariate marginal distribution functions $\left\{F_i^j\right\}_{j=1}^n$ satisfy the constraint that there exists an n-copula C such that the support of the n-variate distribution

$$C\left(F_i^1\left(x^1\right), \ldots, F_i^n\left(x^n\right)\right)$$

is contained in $\mathfrak{B}_i$.[3]

Proof. The payoff for player i given the opponent's marginal distribution functions $\left\{F_{-i}^j\right\}_{j=1}^n$ is the sum of the payoffs across all the battlefields:

$$\sum_{j=1}^n \int_0^\infty \frac{1}{n} F_{-i}^j(x)\, dF_i^j.$$

Here, the integral is the Riemann-Stieltjes integral, so the integrand is 0 for $x > X_i$. We also use the Riemann-Stieltjes integral for other integrals later in the paper.

$$\max_{P_i} \sum_{j=1}^n \int_0^\infty \frac{1}{n} F_{-i}^j(x)\, dF_i^j.$$

That P_i is contained in $\mathfrak{B}_i$ implies that the sum of the levels of force across all battlefields is X_i:

$$\sum_{j=1}^n \int_0^\infty x\; dF_i^j = X_i.$$

[3] Here we only maximize over the set of $\left\{F_i^j\right\}_{j=1}^n$ that satisfy the constraint, not all of them.

Hence, the Lagrangian is

$$\max_{P_i}\left[\sum_{j=1}^{n}\int_0^\infty \frac{1}{n}F_{-i}^j(x)dF_i^j-\lambda_i\left[\sum_{j=1}^{n}\int_0^\infty x\ dF_i^j-X_i\right]\right]$$

$$=\max_{\left\{F_i^j\right\}_{j=1}^{n}}\lambda_i\sum_{j=1}^{n}\left[\int_0^\infty\left[\frac{1}{n\lambda_i}F_{-i}^j(x)-x\right]dF_i^j\right]+\lambda_iX_i.$$

Finally, from Theorem 3.1 the n-variate distribution function P_i is equivalent to the set of univariate marginal distribution functions $\left\{F_i^j\right\}_{j=1}^{n}$ combined with an appropriate n-copula, C, so the result follows directly. ■

Theorem 3.3. *The unique Nash equilibrium univariate marginal distribution functions of the game $ACB(1,1,3)$ are for each player to allocate forces according to the following univariate distribution functions:*

$$F^1(u)=\begin{cases}3u & 0\le u\le\frac{1}{3}\\ 1 & \frac{1}{3}<u\le 1\end{cases}$$

$$F^2(u)=\begin{cases}0 & 0\le u<\frac{1}{6}\\ -\frac{1}{2}+3u & \frac{1}{6}\le u\le\frac{1}{2}\\ 1 & \frac{1}{2}<u\le 1\end{cases}$$

$$F^3(u)=\begin{cases}0 & 0\le u<\frac{1}{3}\\ -1+3u & \frac{1}{3}\le u\le\frac{2}{3}\\ 1 & \frac{2}{3}<u\le 1\end{cases}$$

The expected payoff for both players is $\frac{1}{2}$.

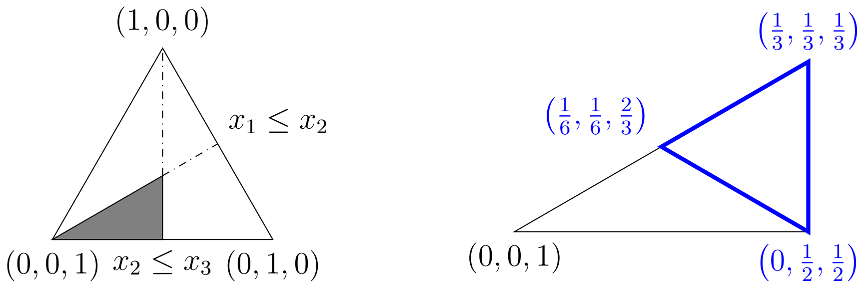
This means that any equilibrium strategies must have the marginal distributions described above, and that any joint distribution with support in $\mathfrak{B}_i$ with such marginal distributions is an equilibrium strategy.

Intuitively, it is easy to see why this particular set of marginal distributions might guarantee a Nash equilibrium. Since the distribution density is the same among the three battlefields, the payoff of a pure strategy $p=(a,b,c)$ remains constant at $\frac{1}{2}$ when it changes inside the region $0\le a\le\frac{1}{3}$, $\frac{1}{6}\le b\le\frac{1}{2}$, and $\frac{1}{3}\le c\le\frac{2}{3}$. A player can only hope to increase payo ff above that given by p by moving below the lower bound of the marginal distribution in some battlefield and staying inside the bounds of the marginal distributions in the other battlefields. However, this is impossible: a cannot be negative; any attempt to bring b below $\frac{1}{6}$ would result in c being above the upper bound $\frac{2}{3}$; c, as the biggest of the 3, cannot be below $\frac{1}{3}$. (The rigorous proof of this can be found in Lemma 3.6.)

Before we give the formal proof of this theorem, let us first examine some joint distributions that satisfy the conditions in Theorem 3.3.

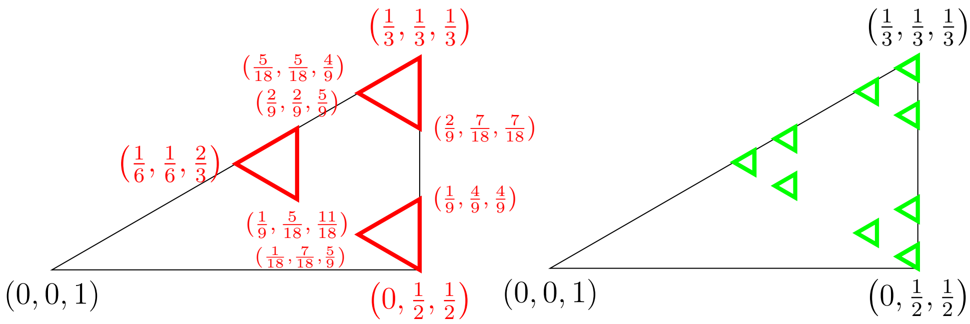
Consider the 3-variate distribution function P_1 that uniformly places mass $\frac{1}{3}$ on each of the three sides of the equilateral triangle with vertices $\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$, $\left(0,\frac{1}{2},\frac{1}{2}\right)$, and $\left(\frac{1}{6},\frac{1}{6},\frac{2}{3}\right)$ to $\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$ (Depicted in Figure 1b). Clearly its marginal distributions are those described in Theorem 3.3.

Similarly, as in Figure 1c, divide the original equilateral triangle into three smaller equilateral triangles with side lengths $\frac{1}{3}$ of the original, and let P_2 be the strategy that uniformly distribute on the sides of the smaller triangles. Clearly P_2 has the same marginal distributions as P_1, and is thus a joint distribution as described in Theorem 3.3. As shown in Figure 1d, we can continue this process on the smaller triangles (or only on some of the smaller triangles), and thus we obtain an countably-infinite family of joint distributions with marginal distributions as described in Theorem 3.3. Furthermore, given any two such suitable joint distributions, their weighted average is also a suitable joint distribution, and thus we obtain a continuum of suitable joint distributions.



(a) The support, $\mathfrak{B}_i$, is the shaded triangle, which is the triangle shown in Figures 1b to 1d.

(b) The strategy that distributes uniformly on the blue lines is an equilibrium strategy.



(c) The strategy that distributes uniformly on the red lines is an equilibrium strategy.

(d) The strategy that distributes uniformly on the green lines is an equilibrium strategy.

FIGURE 1. Equilibrium strategies.

Given these joint distributions that have marginal distributions as characterized by Theorem 3.3, we have the following theorem:

Theorem 3.4. *For the unique set of equilibrium univariate marginal distribution functions* $\left\{F_i^j\right\}_{j=1}^3$ *characterized in Theorem 3.3, there exists a* 3*-copula* C *such that the support of the* 3*-variate distribution function*

$$C\left(F_i^1\left(x^1\right), F_i^2\left(x^2\right), F_i^3\left(x^3\right)\right)$$

is contained in $\mathfrak{B}_i$.

Proof. Consider the 3-variate distribution function P_1 that uniformly places mass $\frac{1}{3}$ on each of the three sides of the equilateral triangle with vertices $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$, $\left(0, \frac{1}{2}, \frac{1}{2}\right)$, and $\left(\frac{1}{6}, \frac{1}{6}, \frac{2}{3}\right)$ to $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ (depicted in Figure 1b). Clearly its marginal distributions are those described in Theorem 3.3, and its support is in $\mathfrak{B}_i$. Hence, according to Sklar's theorem (Theorem 3.1), for the unique set of equilibrium univariate marginal distribution functions $\left\{F_i^j\right\}_{j=1}^3$ characterized in Theorem 3.3, there exists a 3-copula C such that the support of the 3-variate distribution function $C\left(F_i^1\left(x^1\right), F_i^2\left(x^2\right), F_i^3\left(x^3\right)\right)$ is contained in $\mathfrak{B}_i$.

■

Before we provide the formal proof of Theorem 3.3, we first seek to provide some intuition for the outline of the proof, which takes inspiration from the proofs in [Rob06] and [BKdV96].

From equation (2) in Proposition 3.2, we know that in an Asymmetric Colonel Blotto game $ACB(1,1,3)$, each player's Lagrangian can be written as

$$\max_{\left\{F_i^j\right\}_{j=1}^3} \lambda_i \sum_{j=1}^{3} \left[\int_0^\infty \left[\frac{1}{3\lambda_i} F_{-i}^j(x) - x\right] dF_i^j\right] + \lambda_i X_i,$$

subject to the constraint that there exists an n-copula, C, such that the support of the n-variate distribution $C\left(F_i^1\left(x^1\right), \ldots, F_i^n\left(x^n\right)\right)$ is contained in $\mathfrak{B}_i$. If there exists a suitable 3-copula, then, for different j, F_i^j is independent. So equation (3) is the maximization of three independent sums, hence the sum of three independent maximizations:

$$\max_{\left\{F_i^j\right\}_{j=1}^3} \lambda_i \sum_{j=1}^{3} \left[\int_0^\infty \left[\frac{1}{3\lambda_i} F_{-i}^j(x) - x\right] dF_i^j\right] + \lambda_i X_i$$
$$= \sum_{j=1}^{3} \max_{F_i^j} \lambda_i \int_0^\infty \left[\frac{1}{3\lambda_i} F_{-i}^j(x) - x\right] dF_i^j + \lambda_i X_i.$$

Hence we have reduced the maximization problem over a joint distribution to separate maximization problems over univariate distributions, which can be easily solved.

Note that each separate maximization problem has the same form as that of an all-pay auction. An all-pay auction is an auction where several players simultaneously call out a bid for a prize, and all bidders pay regardless of who wins the prize; the prize is awarded to the highest bidder. In an all-pay auction with two bidders, let F_i represent bidder i's distribution of the bid, and v_i represent the value of the auction for bidder i. Each bidder i's problem is

$$\max_{F_i} \int_0^\infty \left[v_i F_{-i}(x) - x\right] dF_i.$$

In the separate maximization problems for the Asymmetric Colonel Blotto game, the quantity $\frac{1}{3\lambda_i}$ acts as the value v_i for the all-pay auctions. Lemma 3.13 establishes the uniqueness of the Lagrange multipliers, hence the uniqueness of the value v_i.

A potential issue that arises is whether the constraint that the strategy P_i must be in $\mathfrak{B}_i$ leads to equilibria outside those characterized by Theorem 3.3. From Sklar's Theorem (Theorem 3.1), we know that the joint distribution P_i is equivalent to a set of marginal distributions $\left\{F_i^j\right\}_{j=1}^3$, together with a suitable 3-copula C. So if a suitable 3-copula exists, the constraint that P_i be in $\mathfrak{B}_i$ places no restraint on the set of potential univariate marginal distribution functions, $\left\{F_i^j\right\}_{j=1}^3$; instead, this constraint and the set of univariate marginal distributions places a restraint on the set of feasible 3-copulas. Since Theorem 3.4 establishes the existence of suitable 3-copula, this is not an issue.

On the other hand, the restriction on the 3-copula implies that the set of equilibrium 3-variate distributions for the game forms a strict subset of the set of all 3-variate distribution functions with univariate marginal distribution functions characterized by Theorem 3.3.

The proof of Theorem 3.3 under the assumption that suitable 3-copula exists is contained in the results that fill up the rest of this section. The proof takes inspiration from the proofs found in [Rob06] and [BKdV96].

First, for the form of the Lagrangian in Proposition 3.2 to be accurate, we need show that there are no atoms in any Nash equilibrium strategies. The following theorem proves this in the more general case of equal levels of force for both players and any n number of battlefields where $n \geq 3$:

Theorem 3.5. *If $n \geq 3$, then Nash equilibrium strategies for $ACB(1,1,n)$ cannot contain atoms.*

Proof. Suppose that we have an equilibrium strategy P_A with an atom in battlefield j on a_j.

Let $p = (b_1, b_2, \dots, b_{j-1}, b_j = a_j, b_{j+1}, \dots, b_n)$ be any pure strategy in the support of P_A that contains playing a_j on battlefield j. The general idea of this proof will be to find a pure strategy p' that does strictly better against P_A than p, thus reaching a contradiction that P_A cannot be an equilibrium strategy as we supposed.

Let $f_k(a)$ denote the possibility of choosing a on battlefield k in P_A. If b_k is any point that is not an atom and b_k is greater than b_{k-1} (or greater than 0 in the case of b_1), then consider the pure strategy p' that plays ε lower on battlefield k and plays $\varepsilon' = \frac{\varepsilon}{n-1}$ higher on all other battlefields. We can always find sufficiently small positive ε and ε' such that there is no atom between b_k and $b_k - \varepsilon$ in P_A. So the payoff of p' against P_A minus the payoffof p against P_A is at least

$$\frac{1}{n} f_k(a_j) - \delta$$

for any $\delta > 0$. Hence, p' does strictly better than p against P_A.

Therefore, for P_A to be an equilibrium strategy, all such b_k that are not atoms must be equal to b_{k-1} (or 0 if $k = 1$). So every pure strategy p in the support of P_A containing the atom a_j on battlefield j must be of the following form: a series of zeros in the first few battlefields (possibly none), an atom, the same level of force in the next few battlefields (also possibly none), another atom, the same level of force (as in the previous atom) in the next few battlefields, and so forth.

Now, one of the following statements must be true:

(1) All p in the support of P_A containing the atom a_j on battlefield j is played with probability 0.
(2) There exists some $p = (c_1, \dots, c_n)$ in the support of P_A containing the atom a_j on battlefield j that is played with a positive probability, hence every c_k is an atom on battlefield k.

Suppose that statement 1 is true. For a_j to be played with some positive probability, there must be a continuum of such p. Hence there must also be a continuum of atoms, which is clearly impossible. So statement 2 must be true. Let $q = (c_1, \dots, c_n)$ be such a pure strategy in the support of P_A where every c_k is an atom on battlefield k. Some casework is needed here:

(1) All the c_k are the same. Then they must all be $\frac{1}{n}$. In any pure strategy where player A plays $\frac{1}{n}$ on the first battlefield, he must also play $\frac{1}{n}$ on all the other battlefields. So $f_1\left(\frac{1}{n}\right) \leq f_k\left(\frac{1}{n}\right)$ for all $k \geq 2$. Hence,
$$f_1\left(\frac{1}{n}\right) < \sum_{k=2}^{n} f_k\left(\frac{1}{n}\right).$$
Consider the pure strategy q' that plays $\left(\frac{1}{n} - \varepsilon\right)$ on battlefield 1 and plays $\left(\frac{1}{n} + \frac{\varepsilon}{n-1}\right)$ on all the other battlefields. We can find a sufficiently small positive ε such that there are no atoms between $\frac{1}{n}$ and $\left(\frac{1}{n} - \varepsilon\right)$ on battlefield 1. The payoffof q' against P_A minus the payoffof q against P_A is at least
$$\frac{1}{n} \cdot \left(\sum_{k=2}^{n} f_k\left(\frac{1}{n}\right) - f_1\left(\frac{1}{n}\right)\right) - \delta$$
for any $\delta > 0$. So q' does strictly better against P_A than q.

(2) All the c_k fall into exactly two values, d_0 and d_1. ($d_0 < d_1$) Suppose q contains m battlefields with level of force d_0 and then $(n-m)$ battlefields with level of force d_1.i

(a) If $d_0 = 0$, then $d_1 = \frac{1}{n-m}$. Given any pure strategy in the support of P_A that plays d_1 on battlefield $(m+1)$, it must also play d_1 on all the battlefields after that, and play 0 on the battlefields 1 to m. So $f_{m+1}(d_1) \leq f_k(c_k)$ where $k \neq m+1$. Hence,
$$\sum_{k \neq m+1} f_k(c_k) > f_{m+1}(d_1 = c_{m+1}).$$
Consider the pure strategy q' that plays $d_1 - \varepsilon$ on battlefield $(m+1)$ and plays $c_k + \frac{\varepsilon}{n-1}$ on battlefield k for all $k \neq m+1$. We can find a sufficiently small positive ε such that there are no atoms between d_1 and $d_1 - \varepsilon$ on battlefield $(m+1)$. The payoffof q' against P_A minus the payoffof q against P_A is at least:
$$\frac{1}{n} \cdot \left(\sum_{k \neq m+1} f_k(c_k) - f_{m+1}(d_1)\right) - \delta$$
for any $\delta > 0$. So q' does strictly better against P_A than q.

(b) If $d_0 > 0$, then at least one of the following two must be true:

(i)
$$f_{m+1}(c_{m+1}) < \sum_{k \neq m+1} f_k(c_k).$$

(ii)
$$\sum_{k=m+1}^{n} f_k(c_k) > \sum_{k=1}^{m} f_k(c_k).$$

Similar to the arguments above, if the first one is true, then we can construct a q' by playing ε lower on battlefield $(m+1)$ and ε' higher on all the other battlefields; if the second one is true, then we can construct a q' by playing ε

higher on battlefield $(m+1)$ and all the battlefields after that, and playing ε' lower on battlefields 1 to m. In either case, q' does strictly better than q against P_A.

(3) All the c_k fall into at least three different values. So from these values we can choose two different values that are not zero. Then we apply the proof in item 2b and obtain the needed pure strategy q'.

In all the cases, a contradiction is reached, showing that P_A cannot be an equilibrium strategy. ■

In the following discussions, let $P = \{F^j\}_{j=1}^3$ be any joint distribution characterized in Theorem 3.3, and let $P' = \{f^j\}_{j=1}^3$ be any equilibrium strategy. Our goal is to prove that P is an equilibrium strategy, and that P and P' have the same marginal distributions.

Lemma 3.6. *Suppose $p = (a, b, c)$ is any pure strategy in $\mathfrak{B}_A = \mathfrak{B}_B = \mathfrak{B}$. Then the payoff of p against P is $\frac{1}{2}$ if $0 \le a \le \frac{1}{3}$, $\frac{1}{6} \le b \le \frac{1}{2}$, and $\frac{1}{3} \le c \le \frac{2}{3}$; and the payoff is less than $\frac{1}{2}$ otherwise.*

Proof. Suppose A plays the mixed strategy P and B plays the pure strategy $p = (a, b, c)$, where $0 \le a \le b \le c$ and $a + b + c = 1$. Then, let $W(a, b, c)$ be the payoff for B. So

$$W(a,b,c) = \frac{1}{3}\left(F^1(a) + F^2(b) + F^3(c)\right)$$

Our goal is to find the maximum value of $W(a, b, c)$ in $\mathfrak{B}$ and to show that it is no greater than 0.

Clearly, $0 \le a \le \frac{a+b+c}{3} = \frac{1}{3}$, so $F^1(a) = 3a$. And $b \le \frac{b+c}{2} \le \frac{1}{2}$

- If $b < \frac{1}{6}$, then $c = 1 - a - b \ge 1 - 2b > \frac{2}{3}$, so $F^2(b) = 0$ and $F^3(c) = 1$. And $a \le b < \frac{1}{6}$

$$\begin{aligned} W(a,b,c) &= \frac{1}{3}(3a + 0 + 1) \\ &< \frac{1}{3}(3 \cdot \frac{1}{6} + 0 + 1) \\ &= \frac{1}{2}. \end{aligned}$$

  So $W(a, b, c) < \frac{1}{2}$.

- If $b \ge \frac{1}{6}$, then $F^2(b) = -\frac{1}{2} + 3b$. Since $c \ge \frac{1}{3}$, $F^3(c) \le 3c - 1$.

$$\begin{aligned} W(a,b,c) &\le \frac{1}{3}(3a - \frac{1}{2} + 3b + 3c - 1) \\ &= (a + b + c) - \frac{1}{2} \\ &= \frac{1}{2}. \end{aligned}$$

  Equality holds if and only if $F^3(c) = 3c - 1$, which is equivalent to $\frac{1}{3} \le c \le \frac{2}{3}$. In this case $0 \le a \le \frac{1}{3}$, $\frac{1}{6} \le b \le \frac{1}{2}$, and $\frac{1}{3} \le c \le \frac{2}{3}$.

  Otherwise, equality does not hold, and the payoff is less than $\frac{1}{2}$.

■

Lemma 3.7. *Any joint strategy P as characterized in Theorem 3.3 is a Nash equilibrium strategy.*

Proof. We know that the game $ACB(1,1,3)$ is symmetrical and has constant sum 1, and since Lemma 3.6 indicates that P gives a payoff of at least $\frac{1}{2}$ against any pure strategy, so P must be an equilibrium strategy. ■

Let $\bar{s}^1 = \frac{1}{3}$, $\underline{s}^1 = 0$, $\bar{s}^2 = \frac{1}{2}$, $\underline{s}^2 = \frac{1}{6}$, $\bar{s}^3 = \frac{1}{3}$, $\underline{s}^3 = \frac{2}{3}$. Clearly, $\bar{s}^j$ is just the upper bound of P on battlefield j, and $\underline{s}^j$ is the lower bound.

Lemma 3.8. $F^j(x^j) = \frac{x^j - \underline{s}^j}{\bar{s}^j - \underline{s}^j}$ *for* $\underline{s}^j \le x^j \le \bar{s}^j$ *and all* j.

Proof. This is self-evident from the representation of F^j in Theorem 3.3:

$$F^1(u) = \begin{cases} 3u & 0 \le u \le \frac{1}{3} \\ 1 & \frac{1}{3} < u \le 1 \end{cases}$$

$$F^2(u) = \begin{cases} 0 & 0 \le u < \frac{1}{6} \\ -\frac{1}{2} + 3u & \frac{1}{6} \le u \le \frac{1}{2} \\ 1 & \frac{1}{2} < u \le 1 \end{cases}$$

$$F^3(u) = \begin{cases} 0 & 0 \le u < \frac{1}{3} \\ -1 + 3u & \frac{1}{3} \le u \le \frac{2}{3} \\ 1 & \frac{2}{3} < u \le 1. \end{cases}$$

■

Lemma 3.9. *If $x < \underline{s}^j$, then $f^j(x) = 0$. If $x > \bar{s}^j$, then $f^j(x) = 1$. Or, in other words, P' does not place any strategy outside $[\underline{s}^j, \bar{s}^j]$.*

Proof. Since $ACB(1,1,3)$ is a two player symmetric constant sum 1 game, every pure strategy in the support of P', an equilibrium strategy, must give the unique equilibrium payoff ,$\frac{1}{2}$, when played against another equilibrium strategy, P. From Lemma 3.6 we know that a pure strategy p only gives payoff $\frac{1}{2}$ against P when p plays a level of force between $\underline{s}^j$ and $\bar{s}^j$ on battlefield j for all j. So P' cannot play any strategy outside that range. ■

Corollary 3.10. $f^j(\underline{s}^j) = 0$ *and* $f^j(\bar{s}^j) = 1$.

Proof. Theorem 3.5 implies that f^j is continuous. This, together with Lemma 3.9, gives the desired result. ■

Let us recall player i's optimization problem for $ACB(1,1,3)$ (equation (2) in Proposition 3.2):

$$\max_{\{F_i^j\}_{j=1}^3} \lambda_i \sum_{j=1}^{3} \left[\int_0^\infty \left[\frac{1}{3\lambda_i} F_{-i}^j(x) - x \right] dF_i^j \right] + \lambda_i X_i \tag{3}$$

where the set of univariate marginal distribution functions $\left\{F_i^j\right\}_{j=1}^3$ satisfy the constraint that there exists a 3-copula C such that the support of the 3-variate distribution

$$C\left(F_i^1\left(x^1\right), F_i^2\left(x^2\right), F_i^3\left(x^3\right)\right)$$

is contained in $\mathfrak{B}_i$.

From the lemmas above, we can add some further restrictions to it. From Lemma 3.9, we know that P_i must be played within $\left[\underline{s^j}, \overline{s^j}\right]$ for every battlefield j. From Lemma 3.7, we know that P is an equilibrium strategy, so P_i must be a best response against P and vice versa. Since Theorem 3.4 establishes the existence of suitable 3-copula, we can disregard that restriction for now and focus on the rest.

For different j, F_i^j is independent. So equation (3) is the maximization of three independent sums, hence the sum of three independent maximizations:

$$\max_{\{F_i^j\}_{j=1}^3} \lambda_i \sum_{j=1}^3 \left[\int_0^\infty \left[\frac{1}{3\lambda_i} F_{-i}^j(x) - x\right] dF_i^j\right] + \lambda_i X_i$$
$$= \sum_{j=1}^3 \max_{F_i^j} \lambda_i \int_0^\infty \left[\frac{1}{3\lambda_i} F_{-i}^j(x) - x\right] dF_i^j + \lambda_i X_i.$$

The term $\lambda_i X_i$ is just a constant, so we could throw that away. Thus the problem for player i becomes:

$$\max_{F_i^j} \lambda_i \int_0^\infty \left[\frac{1}{3\lambda_i} F_{-i}^j(x) - x\right] dF_i^j$$

for all battlefields j, under the constraint that P_A is a best response against P, P is a best response against P_A, and P_A is played within $[\underline{s}^j, \bar{s}^j]$. Let us set $P_A = P' = \{f^j\}_{j=1}^3$. Since we assume the existence of a suitable 3-copula, the different f^j can be considered independent and the different maximizations for different battlefields can also be considered independent. Hence, f^j and F^j form an equilibrium for all j.

Let $B_i^j(x_i^j, F_{-i}^j) = \lambda_i \left(\frac{1}{3\lambda} F_{-i}^j\left(x_i^j\right) - x_i^j\right)$. This is the payoff for player i by playing x_i^j when player $-i$ plays F_{-i}^j in the maximization problem for battlefield j.

Lemma 3.11. $B_i^j(x^j, f^j) = \lambda_i \left(\frac{1}{3\lambda_i} f^j\left(x^j\right) - x^j\right)$ *is constant for all* $\underline{s}^j \leq x^j \leq \bar{s}^j$.

Proof. Since F^j is an equilibrium strategy against f^j, every strategy in the support of F gives a constant payo ffagainst f^j. Since the support of F^j is $[\underline{s}^j, \bar{s}^j]$, the result directly follows. ■

Lemma 3.12. $B_i^j(x^j, f^j) = \lambda_i \left(\frac{1}{3\lambda_i} f^j\left(x^j\right) - x^j\right) = -\lambda_i \underline{s}^j = \frac{1}{3} - \lambda_i \bar{s}^j$ *for all* $\underline{s}^j \leq x^j \leq \bar{s}^j$.

Proof. From Corollary 3.10, $B_i^j(\underline{s}^j, f^j) = -\lambda_i \underline{s}^j$, and $B_i^j(\bar{s}^j, f^j) = \frac{1}{3} - \lambda_i \bar{s}^j$. The result directly follows from Lemma 3.11. ■

Lemma 3.13. $\lambda_i = 1$ *for all* i.

Proof. From Lemma 3.12, we have $\lambda_i = \frac{1}{3(\bar{s}^j - \underline{s}^j)}$. Note that $(\bar{s}^j - \underline{s}^j)$ is always $\frac{1}{3}$ for all j, so $\lambda_i = 1$. ■

Lemma 3.14. $f^j(x^j) = F^j(x^j)$ *for all* j *and all* x^j.

Proof. From Lemmas 3.12 and 3.13, we have $f^j(x^j) = \frac{x^j - \underline{s}^j}{\bar{s}^j - \underline{s}^j}$ for $\underline{s}^j \leq x^j \leq \bar{s}^j$ and all j. From Lemma 3.8, the value of f^j coincides with the value of F^j here. Corollary 3.10 ensures that f^j and F^j are the same elsewhere. ■

With these lemmas, we can prove the uniqueness of the marginal distributions in the Nash equilibria of the game $ACB(1,1,3)$. We restate the theorem here for convenience.

Theorem 3.3. *The unique Nash equilibrium univariate marginal distribution functions of the game $ACB(1,1,3)$ are for each player to allocate forces according to the following univariate distribution functions:*

$$F^1(u) = \begin{cases} 3u & 0 \leq u \leq \frac{1}{3} \\ 1 & \frac{1}{3} < u \leq 1 \end{cases}$$

$$F^2(u) = \begin{cases} 0 & 0 \leq u < \frac{1}{6} \\ -\frac{1}{2} + 3u & \frac{1}{6} \leq u \leq \frac{1}{2} \\ 1 & \frac{1}{2} < u \leq 1 \end{cases}$$

$$F^3(u) = \begin{cases} 0 & 0 \leq u < \frac{1}{3} \\ -1 + 3u & \frac{1}{3} \leq u \leq \frac{2}{3} \\ 1 & \frac{2}{3} < u \leq 1 \end{cases}$$

The expected payoff for both players is $\frac{1}{2}$.

This means that any equilibrium strategies must have the marginal distributions described above, and that any joint distribution with support in $\mathfrak{B}_i$ with such marginal distributions is an equilibrium strategy.

Proof of Theorem 3.3. From Lemma 3.7 we know that every joint distribution with marginal distribution functions as characterized above is a Nash equilibrium strategy, hence the second part of the theorem is proved.

Lemma 3.14 establishes the uniqueness of marginal distributions of Nash equilibrium strategies, and proves that these marginal distributions are exactly those characterized above. Hence, we have proven the first part of the theorem. ■

4. Unique equilibrium payoffs of the game $ACB(X_A, X_B, 2)$

In this section we find the unique equilibrium payoffs of all cases of the Asymmetric Colonel Blotto game involving only two battlefields.

Suppose without loss of generality that $X_A = 1$ and $X_B = t \leq 1$.

Let $W_n(t)$ denote the payoff for A in a Nash equilibrium in such a game with n battlefields. From Theorem 2.1, we know that $W_n(t)$ is well-defined.

Theorem 4.1.

$$W_2(t) = \tfrac{k+2}{2k+2}, \quad \tfrac{2k}{2k+1} \leq t < \tfrac{2k+2}{2k+3} \quad \textit{where } k = 0, 1, 2, \ldots$$

See Figure 2 for a graphical representation.

The proof for $W_2(t)$ and constructions of Nash equilibriums can be found later in this section. Before we go on to prove this, let us first prove a lemma regarding the Asymmetric Colonel Blotto game with two battlefields:

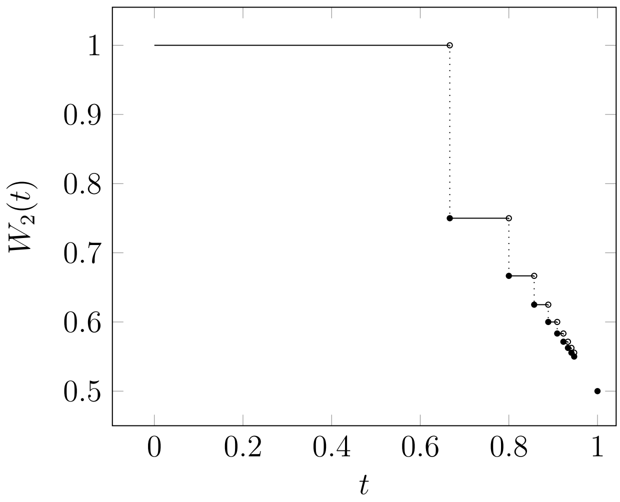


FIGURE 2. $W_2(t)$, the payoff for Asymmetric Colonel Blotto Game with 2 battlefields

Lemma 4.2. *Suppose that $X_A > X_B$. If player A deploys the pure strategy $(a, X_A - a)$ and B deploys the pure strategy $(b, X_B - b)$, then*

$$W_A = \begin{cases} \frac{1}{2} & a-b<0 \textit{ or } a-b > X_A - X_B \\ \frac{3}{4} & a-b=0 \textit{ or } a-b = X_A - X_B \\ 1 & 0 < a-b < X_A - X_B \end{cases}$$

where W_A is the payoff for player A.

Proof.

- If $a-b<0$, then $X_A - a > X_B - b$, so $W_A = \frac{1}{2}$.
- If $a-b=0$, then $X_A - a > X_B - b$, so $W_A = \frac{3}{4}$.
- If $0 < a-b < X_A - X_B$, then $X_A - a > X_B - b$, so $W_A = 1$.
- If $a-b = X_A - X_B > 0$, then $W_A = \frac{3}{4}$.
- If $a-b > X_A - X_B$, then $X_A - a < X_B - b$, so $W_A = \frac{1}{2}$.

Hence,

$$W_A = \begin{cases} \frac{1}{2} & a-b<0 \text{ or } a-b > X_A - X_B \\ \frac{3}{4} & a-b=0 \text{ or } a-b = X_A - X_B \\ 1 & 0 < a-b < X_A - X_B. \end{cases}$$

■

With the help of Lemma 4.2, we can prove Theorem 4.1:

Proof of Theorem 4.1.

(1) Suppose that $t < \frac{2}{3}$. In this case player A can simply overwhelm player B in all the battlefields. Take $P_A = \left(\left(\frac{1}{3}, \frac{2}{3}\right), 1\right)$ and P_B to be any strategy. P_A and P_B form a Nash equilibrium and $W_2(t) = 1$.

Given any pure strategy $(x, t-x)$ of B, we must have $x \le t-x$, so $x \le \frac{t}{2}$, $\frac{1}{3} > \frac{t}{2} \ge x$, and $\frac{2}{3} > t \ge t - x$. Thus in this case, the payoff to B is 0. This means that B cannot increase payoff regardless of the strategy (or mixed strategy) chosen. On the other hand, the payoffof A is 1, which is the maximum possible value, so clearly neither can A increase payoff by changing strategy. Hence, $P_A = \left(\left(\frac{1}{3}, \frac{2}{3}\right), 1\right)$ and any P_B form a Nash equilibrium and $W_2(t) = 1$.

(2) Suppose k is such that $\frac{2k}{2k+1} \leq t < \frac{2k+2}{2k+3}$, where $k \in \mathbb{Z}^+$. Take

$$P_A = \left\{ \left((\varepsilon + j(1-t), 1 - \varepsilon - j(1-t)), \frac{1}{k+1} \right) \middle| 0 \leq j \leq k \right\}$$

and

$$P_B = \left\{ \left((j(1-t), t - j(1-t)), \frac{1}{k+1} \right) \middle| 0 \leq j \leq k \right\},$$

where ε is such that

$$\frac{2k+1}{2} t - k < \varepsilon < \min\left(1 - t, tk - k + \frac{1}{2}\right). \tag{4}$$

The notation here just means that player A plays pure strategy

$$(\varepsilon + j(1-t), 1 - \varepsilon - j(1-t))$$

with probability $\frac{1}{k+1}$ for all j such that $0 \leq j \leq k$; and player B plays pure strategy

$$(j(1-t), t - j(1-t))$$

with probability $\frac{1}{k+1}$ for all j such that $0 \leq j \leq k$. Then we claim that P_A and P_B form a Nash equilibrium and $W_2(t) = \frac{k+2}{2k+2}$.

First we will show that these mixed strategies are legitimate. If $t < \frac{2k+2}{2k+3}$, then $\frac{2k+3}{2} t < k + 1$, so $\frac{2k+1}{2} t - k < 1 - t$. Now $\frac{t}{2} < \frac{1}{2}$, so $\frac{2k+1}{2} t - k < tk - k + \frac{1}{2}$, which in turn implies that

$$\frac{2k+1}{2} \cdot t - k \geq k - k = 0.$$

Hence, a positive ε satisfying equation (4) exists. Further, we need to check that the level of force distributed on the first battlefield, x_1, is less than or equal to the force distributed on the second battlefield, x_2; or, equivalently, for player i, we need to check that $x_1 \leq \frac{X_i}{2}$. First, let's check player A's strategy. Since $j \leq k$, we must have

$$\varepsilon + j(1-t) \leq \varepsilon + k(1-t).$$

Then we plug in the upper bound of ε in equation (4) and get

$$\varepsilon + k(1-t) < t \cdot k - k + \frac{1}{2} + k(1-t) = \frac{1}{2}.$$

So $\varepsilon + j(1-t) < \frac{1}{2}$. Now let's check player B's strategy. We already know that $t \geq \frac{2k}{2k+1}$, rearrange and we would get

$$k(1-t) \leq \frac{t}{2}.$$

Since $j(1-t) \leq k(1-t)$, we must have

$$j(1-t) \leq \frac{t}{2}.$$

So both P_A and P_B are legitimate mixed strategies.

Suppose A chooses some pure strategy $p'_A = (x, 1-x)$. Set $a = \left\lfloor \frac{x}{1-t} \right\rfloor$. Hence,

$$(1-t)a \le x < (1-t)(a+1)$$

where $0 \le a \le k+1$. Now let us expand P_B into pure strategies in the calculation of $W_A\left(p'_A, P_B\right)$:

$$\begin{aligned}
(k+1)W_A(p'_A, P_B) &= \sum_{j=0}^{k} W_A\left((x, 1-x), (j(1-t), t-j(1-t))\right) \\
&\le \sum_{j=0}^{a-2} W_A\left((x, 1-x), (j(1-t), t-j(1-t))\right) \\
&\quad + \sum_{j=a+1}^{k} W_A\left((x, 1-x), (j(1-t), t-j(1-t))\right) \\
&\quad + W_A\left((x, 1-x), ((a-1)(1-t), t-(a-1)(1-t))\right) \\
&\quad + W_A\left((x, 1-x), (a(1-t), t-a(1-t))\right).
\end{aligned}$$

There is a $\le$ sign on the second line since if $a = k+1$, there is one additional non-negative term on the right, $W_A\left((x, 1-x), ((k+1)(1-t), t-(k+1)(1-t))\right)$, compared with the original formula.

First let us consider the sum

$$\sum_{j=0}^{a-2} W_A\left((x, 1-x), (j(1-t), t-j(1-t))\right).$$

Here,

$$\begin{aligned}
x - j(1-t) &\ge x - (a-2)(1-t) \\
&\ge a(1-t) - (a-2)(1-t) \\
&> (1-t).
\end{aligned}$$

Hence, according to Lemma 4.2,

$$W_A\left((x, 1-x), (j(1-t), t-j(1-t))\right) = \frac{1}{2}$$

if $0 \le j \le a-2$. Thus,

$$\sum_{j=0}^{a-2} W_A\left((x, 1-x), (j(1-t), t-j(1-t))\right) = \frac{a-1}{2}.$$

Then let us consider the sum

$$\sum_{j=a+1}^{k} W_A\left((x, 1-x), (j(1-t), t-j(1-t))\right).$$

Here,

$$\begin{aligned} x - j(1-t) &\leq x - (a+1)(1-t) \\ &< (a+1)(1-t) - (a+1)(1-t) \\ &< 0. \end{aligned}$$

Hence, according to Lemma 4.2,

$$W_A\left((x, 1-x), (j(1-t), t - j(1-t))\right) = \frac{1}{2}$$

if $a+1 \leq j \leq k$. Thus,

$$\sum_{j=a+1}^{k} W_A\left((x, 1-x), (j(1-t), t - j(1-t))\right) = \frac{k-a}{2}.$$

If $x = a(1-t)$, then according to Lemma 4.2,

$$\begin{aligned} &W_A\left((x, 1-x), ((a-1)(1-t), t - (a-1)(1-t))\right) \\ &\qquad + W_A\left((x, 1-x), (a(1-t), t - a(1-t))\right) = \frac{3}{4} + \frac{3}{4} = \frac{3}{2}. \end{aligned}$$

If $x > a(1-t)$, then according to Lemma 4.2,

$$\begin{aligned} &W_A\left((x, 1-x), ((a-1)(1-t), t - (a-1)(1-t))\right) \\ &\qquad + W_A\left((x, 1-x), (a(1-t), t - a(1-t))\right) = \frac{1}{2} + 1 = \frac{3}{2}. \end{aligned}$$

Hence, $W_A(p'_A, P_B) \leq \frac{1}{k+1} \cdot \frac{k+2}{2} = \frac{k+2}{2k+2}$. So A cannot increase payoff above $\frac{k+2}{2k+2}$ by changing strategy. Now let's consider player B's strategy. Suppose B chooses some pure strategy $p'_B = (y, t-y)$. Set $b = \lceil \frac{y-\varepsilon}{1-t} \rceil$. Hence,

$$(1-t)(b-1) + \varepsilon < y \leq (1-t)b + \varepsilon$$

where $0 \leq b \leq k$.

(a) In the case where $0 \leq b \leq k-1$, let's expand P_A into pure strategies in the calculations of $W_A\left(P_A, p'_B\right)$:

$$\begin{aligned} (k+1)W_A(P_A, p'_B) &= \sum_{j=0}^{k} W_A\left((j(1-t)+\varepsilon, 1 - j(1-t) - \varepsilon), (y, t-y)\right) \\ &= \sum_{j=0}^{b-1} W_A\left((j(1-t)+\varepsilon, 1 - j(1-t) - \varepsilon), (y, t-y)\right) \\ &\qquad + \sum_{j=b+2}^{k} W_A\left((j(1-t)+\varepsilon, 1 - j(1-t) - \varepsilon), (y, t-y)\right) \\ &\qquad + W_A\left((b(1-t)+\varepsilon, 1 - b(1-t) - \varepsilon), (y, t-y)\right) \\ &\qquad + W_A\left(((b+1)(1-t)+\varepsilon, 1 - (b+1)(1-t) - \varepsilon), (y, t-y)\right). \end{aligned}$$

First let us consider the sum

$$\sum_{j=0}^{b-1} W_A\left((j(1-t)+\varepsilon, 1-j(1-t)-\varepsilon), (y, t-y)\right).$$

Here,

$$j(1-t)+\varepsilon \leq (b-1)(1-t)+\varepsilon < y.$$

Hence, according to Lemma 4.2,

$$W_A\left((j(1-t)+\varepsilon, 1-j(1-t)-\varepsilon), (y, t-y)\right) = \frac{1}{2}$$

if $0 \leq j \leq b-1$. Thus,

$$\sum_{j=0}^{b-1} W_A\left((j(1-t)+\varepsilon, 1-j(1-t)-\varepsilon), (y, t-y)\right) = \frac{b}{2}.$$

Then let us consider the sum

$$\sum_{j=b+2}^{k} W_A\left((j(1-t)+\varepsilon, 1-j(1-t)-\varepsilon), (y, t-y)\right).$$

Here,

$$\begin{aligned} j(1-t)+\varepsilon-y &\geq (b+2)(1-t)+\varepsilon-y \\ &\geq (b+2)(1-t)+\varepsilon-b(1-t)-\varepsilon \\ &> 1-t. \end{aligned}$$

Hence, according to Lemma 4.2,

$$W_A\left((j(1-t)+\varepsilon, 1-j(1-t)-\varepsilon), (y, t-y)\right) = \frac{1}{2}$$

if $b+2 \leq j \leq k$. Thus,

$$\sum_{j=b+2}^{k} W_A\left((j(1-t)+\varepsilon, 1-j(1-t)-\varepsilon), (y, t-y)\right) = \frac{k-b-1}{2}.$$

If $y = b(1-t)+\varepsilon$, then according to Lemma 4.2,

$$\begin{aligned} &W_A\left((b(1-t)+\varepsilon, 1-b(1-t)-\varepsilon), (y, t-y)\right) \\ &\qquad + W_A\left(((b+1)(1-t)+\varepsilon, 1-(b+1)(1-t)-\varepsilon), (y, t-y)\right) = \frac{3}{4}+\frac{3}{4} = \frac{3}{2}. \end{aligned}$$

If $y < b(1-t)+\varepsilon$, then according to Lemma 4.2,

$$\begin{aligned} &W_A\left((b(1-t)+\varepsilon, 1-b(1-t)-\varepsilon), (y, t-y)\right) \\ &\qquad + W_A\left(((b+1)(1-t)+\varepsilon, 1-(b+1)(1-t)-\varepsilon), (y, t-y)\right) = 1+\frac{1}{2} = \frac{3}{2}. \end{aligned}$$

Therefore, in either case, $W_A(P_A, p'_B) = \frac{k+2}{2k+2}$.

(b) If $b = k$, then

$$k(1-t)+\varepsilon > k(1-t)+\frac{2k+1}{2}t-k=\frac{1}{2}t \geq y.$$

So

$$\begin{aligned}(k+1)W_A(P_A,p'_B) &= \sum_{j=0}^{k} W_A\left((j(1-t)+\varepsilon, 1-j(1-t)-\varepsilon),(y,t-y)\right)\\ &= \sum_{j=0}^{k-1} W_A\left((j(1-t)+\varepsilon, 1-j(1-t)-\varepsilon),(y,t-y)\right)\\ &\quad + W_A\left((k(1-t)+\varepsilon, 1-k(1-t)-\varepsilon),(y,t-y)\right).\end{aligned}$$

Similarly,

$$\sum_{j=0}^{k-1} W_A\left((j(1-t)+\varepsilon, 1-j(1-t)-\varepsilon),(y,t-y)\right)=\frac{k}{2}.$$

And since $(1-t)(k-1)+\varepsilon < y < (1-t)k+\varepsilon$,

$$W_A\left((k(1-t)+\varepsilon, 1-k(1-t)-\varepsilon),(y,t-y)\right)=1.$$

Hence, $W_A(P_A,p'_B) \geq \frac{k+2}{2k+2}$, which means that B cannot increase payoff above $\frac{k}{2k+2}$ by changing strategy. Hence, P_A and P_B form a Nash equilibrium, and the equilibrium payoff for A is $W_2(t)=\frac{k+2}{2k+2}$.

(3) Finally, suppose that $t=1$. Take any mixed strategy P_A and any mixed strategy P_B. Then they form a Nash equilibrium with $W_2(t)=\frac{1}{2}$. To see this, suppose A plays the pure strategy $P'_A=(a,1-a)$ and B plays the pure strategy $P'_B=(b,1-b)$.
- If $a=b$, clearly $W_A(P'_A,P'_B)=\frac{1}{2}$.
- If $a<b$, then $1-a>1-b$, so $W_A(P'_A,P'_B)=\frac{1}{2}$. Similarly, if $a>b$, $W_A(P'_A,P'_B)=\frac{1}{2}$.

Hence, the payoff is $\frac{1}{2}$ regardless of the pure strategies that both players play. As a result, the payoff is also $\frac{1}{2}$ regardless of what mixed strategies that the two players play.

■

5. Unique equilibrium payoffs of the game $ACB(X_A, X_B, 3)$

In this section we find the unique equilibrium payoffs of some cases of the Asymmetric Colonel Blotto game involving three battlefields. The results that follow are ordered by ascending values of t.

Suppose without loss of generality that $X_A=1$ and $X_B=t\leq 1$. The case where $t=1$ is already solved in Section 3, and we have $W_3(1)=\frac{1}{2}$. In the following discussions, let the function $s(x)$ be defined as follows:

$$s(x)=\begin{cases}0 & x<0\\ \frac{1}{2} & x=0\\ 1 & x>0.\end{cases}$$

Theorem 5.1. *In the case where $t < \frac{6}{11}$, $w_3(t) = 1$.*

Proof. In this case, player A can simply overwhelm player B in all the battlefields.

Take $P_A = \{((\frac{2}{11}, \frac{3}{11}, \frac{6}{11}), 1)\}$ and P_B to be any strategy. P_A and P_B form a Nash equilibrium and $W_3(t) = 1$. ■

Theorem 5.2. *In the case where $\frac{6}{11} \leq t < \frac{18}{31}$, take*

$$P_A = \left\{ \left(\left(\frac{t}{3} + \varepsilon, \frac{t}{2} + \varepsilon, 1 - \frac{5}{6}t - 2\varepsilon \right), \frac{1}{3} \right), \left(\left(\frac{t}{3} + \varepsilon, 1 - \frac{4}{3}t - 2\varepsilon, t + \varepsilon \right), \frac{1}{3} \right), \left(\left(1 - \frac{3}{2}t - 2\varepsilon, \frac{t}{2} + \varepsilon, t + \varepsilon \right), \frac{1}{3} \right) \right\}$$

and

$$P_B = \left\{ \left((0, 0, t), \frac{1}{3} \right), \left(\left(0, \frac{t}{2}, \frac{t}{2} \right), \frac{1}{3} \right), \left(\left(\frac{t}{3}, \frac{t}{3}, \frac{t}{3} \right), \frac{1}{3} \right) \right\}$$

where $0 < \varepsilon < \frac{1}{2}\left(1 - \frac{31}{18}t\right)$.

P_A and P_B form a Nash equilibrium and $W_3(t) = \frac{8}{9}$.

Proof. Since $t < \frac{18}{31}$, a real ε satisfying the necessary condition must exist. From the range of t and ε, we can check that the strategies of A and B are legitimate, or in other words, the levels of force of the battlefields are nondecreasing.

Suppose A plays the pure strategy $p'_A = (a, b, c)$ where $a + b + c = 1$. Then,

$$\begin{aligned}
9W_A(p'_A, P_B) &= 3W_A((a, b, c), (0, 0, t)) \\
&\quad + 3W_A\left((a, b, c), \left(0, \frac{t}{2}, \frac{t}{2}\right)\right) \\
&\quad + 3W_A\left((a, b, c), \left(\frac{t}{3}, \frac{t}{3}, \frac{t}{3}\right)\right) \\
&= 2s(a - 0) + s\left(a - \frac{t}{3}\right) \\
&\quad + s(b - 0) + s\left(b - \frac{t}{2}\right) + s\left(b - \frac{t}{3}\right) \\
&\quad + s\left(c - \frac{t}{3}\right) + s\left(c - \frac{t}{2}\right) + s\left(c - \frac{t}{3}\right).
\end{aligned}$$

For $W_A(p'_A, P_B)$ to be more than $\frac{8}{9}$, none of the terms on the right can be 0. Hence, we must have

$$a \geq \frac{t}{3}, \qquad b \geq \frac{t}{2}, \qquad c \geq t.$$

So

$$1 = a + b + c \geq \frac{11}{6}t.$$

This is only possible when $t = \frac{11}{6}$. In this case, p'_A must be $(\frac{t}{3}, \frac{t}{2}, t)$, so $W_A(p'_A, P_B) = \frac{5}{6} < \frac{8}{9}$. Hence, $W_A(p'_A, P_B) \leq \frac{8}{9}$ for all pure strategies p'_A.

Suppose B plays the pure strategy $p'_B = (d, e, f)$ where $d + e + f = t$. Then,

$$\begin{aligned}
9W_A\left(P_A, p'_B\right) &= 3W_A\left(\left(\frac{t}{3}+\varepsilon, \frac{t}{2}+\varepsilon, 1-\frac{5}{6}t-2\varepsilon\right), (d,e,f)\right)\\
&+ 3W_A\left(\left(\frac{t}{3}+\varepsilon, 1-\frac{4}{3}t-2\varepsilon, t+\varepsilon\right), (d,e,f)\right)\\
&+ 3W_A\left(\left(1-\frac{3}{2}t-2\varepsilon, \frac{t}{2}+\varepsilon, t+\varepsilon\right), (d,e,f)\right).
\end{aligned}$$

Remember that $d \le \frac{t}{3}$, $e \le \frac{t}{2}$, and $f \le t$. So,

$$9W_A\left(P_A, p'_B\right) = 6 + s\left(1-\frac{3}{2}t-2\varepsilon-d\right) + s\left(1-\frac{4}{3}t-2\varepsilon-e\right) + s\left(1-\frac{5}{6}t-2\varepsilon-f\right).$$

From $\frac{6}{11} \le t < \frac{18}{31}$ and $0 < \varepsilon < \frac{1}{2}\left(1-\frac{31}{18}t\right)$, we can show that

$$\begin{aligned}
d \ge 1-\frac{3}{2}t-2\varepsilon &\Rightarrow e \le \frac{t-d}{2} \le \frac{5}{4}t+\varepsilon-\frac{1}{2} < 1-\frac{4}{3}t-2\varepsilon,\\
e \ge 1-\frac{4}{3}t-2\varepsilon &\Rightarrow f \le t-e \le \frac{7}{3}t-1+2\varepsilon < 1-\frac{5}{6}t-2\varepsilon,\\
\text{and } f \ge 1-\frac{5}{6}t-2\varepsilon &\Rightarrow d \le t-2f \le \frac{8}{3}t-2+4\varepsilon < 1-\frac{3}{2}t-2\varepsilon.
\end{aligned}$$

Hence, at least 2 terms on the right must be 1. So $W_A(P_A, p'_B) \ge \frac{8}{9}$ for all pure strategies p'_B.

To conclude, player A cannot increase payoff above $\frac{8}{9}$ by changing strategy, and player B can also not increase payoff above $\frac{1}{9}$ by changing strategy. So P_A, P_B form a Nash equilibrium and the equilibrium payoff $W_3(t) = \frac{8}{9}$. ■

Theorem 5.3. *When $\frac{3}{5} < t < \frac{30}{47}$, take*

$$P_A = \left\{\left(\left(\frac{30-22t}{75}, \frac{15-11t}{25}, \frac{11t}{15}\right), \frac{1}{2}\right), \left(\left(\frac{2t}{15}, \frac{13t}{30}, 1-\frac{17t}{30}\right), \frac{1}{2}\right)\right\}$$

and

$$P_B = \left\{\left(\left(\frac{t}{3}, \frac{t}{3}, \frac{t}{3}\right), \frac{1}{2}\right), \left((0,0,t), \frac{1}{2}\right)\right\}.$$

Then P_A and P_B form a Nash equilibrium and $W_3(t) = \frac{5}{6}$.

Proof. Let's begin with checking that the strategies distributed among the three battlefields are non-decreasing. First, let's check player A's first strategy. x_A^1 is obviously smaller than x_A^2 (x_i^j means the level of force player i distributes on battlefield j):

$$\frac{30-22t}{75} \le \frac{30-22t}{75} \cdot \frac{3}{2} = \frac{15-11t}{25}.$$

Since $t > \frac{3}{5}$ and $\frac{3}{5} > \frac{45}{88}$, we must have $5 < 88t$. Rearrange and we would get

$$\frac{15-11t}{25} < \frac{11t}{15}.$$

Now let's check player A's second strategy. Obviously, ${x_A^1}'$ is smaller than ${x_A^2}'$:

$$\frac{2t}{15} < \frac{13t}{30}.$$

Furthermore, since $t < 1$, we can rearrange and obtain

$$\frac{13t}{30} < 1 - \frac{17t}{30}.$$

Hence, P_A is legitimate. And clearly P_B is legitimate.

If player A plays the pure strategy $p'_A = (a, b, 1-a-b)$. Then,

$$\begin{aligned}6W_A(P'_A, P_B) = 3W_A&\left((a,b,1-a-b), \left(\frac{t}{3},\frac{t}{3},\frac{t}{3}\right)\right) + 3W_A\left((a,b,1-a-b),(0,0,t)\right)\\ &= s\left(a-\frac{t}{3}\right) + s\left(b-\frac{t}{3}\right) + s\left(1-a-b-\frac{t}{3}\right)\\ &\qquad + s\left(a-0\right) + s\left(b-0\right) + s\left(1-a-b-t\right).\end{aligned}$$

For $W_A(p'_A, P_B)$ to be greater than $\frac{5}{6}$, none of the six terms on the right can be 0. Hence, we obtain the following inequalities:

$$\begin{array}{rr} a \geq \frac{t}{3} & a \geq 0 \\ b \geq \frac{t}{3} & b \geq 0 \\ 1-a-b \geq \frac{t}{3} & 1-a-b \geq t. \end{array}$$

Add together $a \geq \frac{t}{3}$, $b \geq \frac{t}{3}$, $1-a-b \geq t$, and we get

$$1 \geq \frac{5}{3}\cdot t.$$

Hence, $t \geq \frac{3}{5}$, contradicting our hypothesis on t. Therefore, we must have $W_A(p'_A, P_B) \leq \frac{5}{6}$ given all pure strategy p'_A.

If player B plays the pure strategy $p'_B = (c, d, t-c-d)$, then,

$$\begin{aligned}6W_A\left(P_A, p'_B\right) = 3W_A&\left(\left(\frac{30-22t}{75}, \frac{15-11t}{25}, \frac{11t}{15}\right), (c,d,t-c-d)\right)\\ &+ 3W_A\left(\left(\frac{2t}{15}, \frac{13t}{30}, 1-\frac{17t}{30}\right), (c,d,t-c-d)\right)\\ = s&\left(\frac{30-22t}{75}-c\right) + s\left(\frac{15-11t}{25}-d\right) + s\left(\frac{11t}{15}-t+c+d\right)\\ &+ s\left(\frac{2t}{15}-c\right) + s\left(\frac{13t}{30}-d\right) + s\left(1-\frac{17t}{30}-t+c+d\right).\end{aligned}$$

Rearrange $t < \frac{30}{47}$ and we get

$$\frac{30-22t}{75} > \frac{t}{3} \geq c.$$

Rearrange $t < \frac{30}{47}$ and we also get

$$\frac{15-11t}{25} > \frac{t}{2} \geq d.$$

Similarly, rearrange $t < \frac{30}{47}$ and we get

$$1 - \frac{17t}{30} > t \geq t - c - d.$$

Hence,

$$6W_A\left(P_A, p_B'\right) = 3 + s\left(-\frac{4t}{15} + c + d\right) + s\left(\frac{2t}{15} - c\right) + s\left(\frac{13t}{30} - d\right)$$

- If $c < \frac{2t}{15}$,
  - and if $d \geq \frac{13t}{15}$, then $c + d > \frac{4t}{15}$. So

$$\begin{aligned} 6W_A\left(P_A, p_B'\right) &= 3 + s\left(-\frac{4t}{15} + c + d\right) + s\left(\frac{2t}{15} - c\right) + s\left(\frac{13t}{30} - d\right) \\ &\geq 3 + 1 + 1 + 0 \\ &= 5. \end{aligned}$$

  - and if $d < \frac{13t}{15}$, then

$$\begin{aligned} 6W_A\left(P_A, p_B'\right) &= 3 + s\left(-\frac{4t}{15} + c + d\right) + s\left(\frac{2t}{15} - c\right) + s\left(\frac{13t}{30} - d\right) \\ &\geq 3 + 0 + 1 + 1 \\ &= 5. \end{aligned}$$

- If $c = \frac{2t}{15}$, then, $d \leq \frac{t-c}{2} = \frac{13t}{30}$, and $c + d \geq 2c = \frac{4t}{15}$.
  - If $d = \frac{13t}{30}$, then $t - c - d = \frac{13t}{30}$. So

$$\begin{aligned} 6W_A\left(P_A, p_B'\right) &= 3 + s\left(-\frac{4t}{15} + c + d\right) + s\left(\frac{2t}{15} - c\right) + s\left(\frac{13t}{30} - d\right) \\ &= 3 + 1 + \frac{1}{2} + \frac{1}{2} \\ &= 5. \end{aligned}$$

  - If $d < \frac{13t}{30}$, then

$$\begin{aligned} 6W_A\left(P_A, p_B'\right) &= 3 + s\left(-\frac{4t}{15} + c + d\right) + s\left(\frac{2t}{15} - c\right) + s\left(\frac{13t}{30} - d\right) \\ &\geq 3 + \frac{1}{2} + \frac{1}{2} + 1 \\ &= 5. \end{aligned}$$

- If $c > \frac{2t}{15}$, then, $c + d \geq 2c > \frac{4t}{15}$, so

$$\begin{aligned} 6W_A\left(P_A, p_B'\right) &= 3 + s\left(-\frac{4t}{15} + c + d\right) + s\left(\frac{2t}{15} - c\right) + s\left(\frac{13t}{30} - d\right) \\ &\geq 3 + 1 + 1 + 0 \\ &= 5. \end{aligned}$$

Hence, $W_A(P_A, p_B') \geq \frac{5}{6}$ for any pure strategy p_B'.

To conclude, player A cannot increase payoff above $\frac{5}{6}$ by changing strategy, and player B can also not increase payoff above $\frac{1}{6}$ by changing strategy. So P_A, P_B form a Nash equilibrium and the equilibrium payoff, $W_3(t)$, is $\frac{5}{6}$. ■

Remark. To guarantee a Nash equilibrium, player A can play any strategy

$$P_A = \left\{ \left((a,b,c), \frac{1}{2} \right), \left((d,e,f), \frac{1}{2} \right) \right\}$$

satisfying

$$\begin{gathered} a+b+c=1, a \leq b \leq c, d \leq e \leq f, d+e+f=1, \\ a > \frac{t}{3}, b > \frac{t}{2}, f > t, \\ \text{and } 2d+c \geq t, d+2e \geq t. \end{gathered}$$

The strategy $\left\{ \left(\left(\frac{30-22t}{75}, \frac{15-11t}{25}, \frac{11t}{15} \right), \frac{1}{2} \right), \left(\left(\frac{2t}{15}, \frac{13t}{30}, 1 - \frac{17t}{30} \right), \frac{1}{2} \right) \right\}$ is only one of the possible ones.

Theorem 5.4. $W_3(\frac{2}{3}) \leq \frac{4}{5}$.

Proof. Let B play the strategy

$$P_B = \left\{ \left(\left(0, \frac{1}{16}, \frac{29}{48} \right), \frac{1}{5} \right), \left(\left(0, 0, \frac{2}{3} \right), \frac{1}{5} \right), \left(\left(\frac{1}{16}, \frac{1}{16}, \frac{13}{24} \right), \frac{1}{5} \right), \left(\left(\frac{1}{8}, \frac{13}{48}, \frac{13}{48} \right), \frac{1}{5} \right), \left(\left(\frac{5}{24}, \frac{11}{48}, \frac{11}{48} \right), \frac{1}{5} \right) \right\}$$

and let A play any pure strategy p, then we verified using a computer that the payoff for A, $W_A(p, P_B)$, is at most $\frac{4}{5}$. ■

Theorem 5.5. $W_3(\frac{5}{6}) \geq \frac{2}{3}$.

Proof. Let A play the strategy $P_A = \left(\frac{1}{6}, \frac{1}{3}, \frac{1}{2} \right)$, and let B play any pure strategy p, then we verified using a computer that the payoff for A, $W_A(P_A, p)$, is at least $\frac{2}{3}$. ■

Note that in this case where $t \neq 1$ and in the general case for $n = 2$ (Section 4), there are Nash equilibrium strategies with atoms. This is behavior very different from what we saw in Section 3, and also very different from what we proved about Nash equilibria of the game $ACB(1,1,n)$ where $n \geq 3$ in Theorem 3.5.

6. Open problems

Still much remains unknown about the Asymmetric Colonel Blotto game in the general case. For example, what would a Nash equilibrium for the game $ACB(1,1,4)$ look like? Or a Nash equilibrium for the game $ACB(1,1,n)$ where $n \geq 5$? The methods used to prove the uniqueness of the marginal distributions of the game $ACB(1,1,3)$ in Section 3 cannot be used here since these methods only prove the uniqueness of the marginal distributions given a Nash equilibrium strategy, so these methods do not work when we cannot find a Nash equilibrium in the first place. It is hard to find a Nash equilibrium for the game $ACB(1,1,4)$, although one can be approximated by means of computer simulation. What's more, we can show that the marginal distributions of Nash equilibrium strategies cannot be uniform. This makes it difficult to guess the correct Nash equilibrium strategy.

Another problem is to determine how the unique equilibrium payoff varies in the game $ACB(1,t,n)$ as t varies continuously in the general case. As we have shown in Section 4, $W_2(t)$ is locally constant and discontinuous as a function of t. This is quite a surprising result, as it indicates that there are phase changes in the game $ACB(1,t,2)$ as t changes. Our partial results in Section 5 also indicate that $W_3(t)$ is a discontinuous function (Theorem 5.1 and Theorem 5.2). Computer simulation of discrete cases also indicates that sometimes it is not differentiable where the function itself is continuous. Maybe the phase changes in this case correspond to discontinuous jumps in the equilibrium strategies. This can be illustrated by the drastic difference between the equilibrium strategies in Section 5 and those in Section 3. Is it possible to find all the critical values of t where these phase changes occur?

Yet another fundamental question left unanswered is the existence of Nash equilibria for the game $ACB(X_A, X_B, n)$ in the general case. We have discussed Nash equilibria in special cases, but we have not given a proof that guarantees the existence of Nash equilibria in the general case.

References

[BKdV96] Michael R. Baye, Dan Kovenock, and Casper G. de Vries. The all-pay auction with complete information. *Econom. Theory*, 8(2):291–305, 1996.

[Bor53] Emile Borel. The theory of play and integral equations with skew symmetric kernels. *Econometrica*, 21:97–100, 1953.

[Rob06] Brian Roberson. The Colonel Blotto game. *Econom. Theory*, 29(1):1–24, 2006.

[Skl59] M. Sklar. Fonctions de répartition à n dimensions et leurs marges. *Publ. Inst. Statist. Univ. Paris*, 8:229–231, 1959.

[SS83] B. Schweizer and A. Sklar. *Probabilistic metric spaces*. North-Holland Series in Probability and Applied Mathematics. North-Holland Publishing Co., New York, 1983.