The conditions for separated binary polynomials which graphs are closed curves

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Abstract

The paper mainly discusses, based on the properties and related knowledge of continuous functions, the conditions that the graphs of separated binary polynomials are closed curves by the method of separation of variables. It also studies the properties of closed curves in the ways of analytic geometry, and draws a sufficient and necessary condition to determine whether the graph of a separated binary polynomial is closed.

Keywords: separated polynomial, closed curve, separation of variable, stationary point

1. Introduction

A polynomial is called separated if it does not have cross terms, that is, its monomials are all in one variable. In this paper, we say a polynomial is closed if its graph is a closed curve. Moreover, if the graph of a polynomial consists of more than one closed curve or just a point, then we also say it is closed.

We have completely known the graphs of quadratic polynomials so far, and we know in what conditions the graphs are closed (they are ellipses). But, it is difficult to study higher degree polynomials since their graphs are influenced by many factors. This paper mainly studies higher degree polynomials and draws a conclusion to determine whether the graphs of the separated binary polynomials are closed.

Usually, we use algebraic method to investigate the graphs of polynomials. For instance, we can use algebraic method to transform a separated quadratic polynomial into $a(x-m)^2 + b(y-n)^2 = d$ and then discuss its graphs directly. However, this method is limited in higher degree cases. Based on the properties and related knowledge of continuous functions, we separate the variables of the higher degree polynomials and transform them into the form of two equal functions. Discussing in this way will avoid complicated and overloaded calculations. This point of view has brought us many advantages when we consider other problems. It is better to analyze a problem and think about the possible outcomes first rather than calculate it. This is also well in accordance with the thesis of Galois "Jump above calculations, group the operations, classify them according to their complexities rather than their appearance; this, I believe, is the mission of future mathematicians; this is the road I'm embarking in this work."

The theorem 3.3 in the paper can be extended to the analytic representation of closed curves, thereby defining closed curves in the sense of analytic geometry. This paper also gives

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a sufficient and necessary condition to determine the closedness of the graphs of polynomials, which also can be used to other aspects of investigations of the graphs of polynomials.

2. Main results

Theorem 2.1. *Suppose that $f(x)$ and $g(y)$ are continuous functions defined on intervals $[x_1, x_2]$ and $[y_1, y_2]$, respectively. Both the ranges of $f(x)$ and $g(y)$ are $[f(x)_{\min}, g(y)_{\max}]$, and $f(x)_{\min} = g(y_1) = g(y_2)$, $g(y)_{\max} = f(x_1) = f(x_2)$, where $f(x)_{\min}, g(y)_{\max}$ are the minimum and maximum of $f(x)$ and $g(y)$ in the intervals, respectively. Then the graph of the equation defined by*

$$F(x, y) = f(x) - g(y) = 0 \tag{2.1}$$

is a closed curve on $[x_1, x_2] \times [y_1, y_2]$. Moreover, if there is no rectangular subregion satisfying the above conditions, then the graph of equation (2.1) has only one closed curve on $[x_1, x_2] \times [y_1, y_2]$; otherwise, the graph of equation (2.1) on $[x_1, x_2] \times [y_1, y_2]$ will be the form that closed curves in the bigger closed curves.

Theorem 2.2. *The graph of the separated binary polynomial*

$$a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \cdots + a_1 x + b_m y^m + b_{m-1} y^{m-1} + \cdots + b_1 y = d \tag{2.2}$$

is only one or more than one closed curve if and only if function (2.2) has a solution (x, y), $a_n b_m > 0$ and n, m are even numbers.

3. The proofs of the main results

To use the method of separation of variables to proof our main results, we begin with quadratic polynomials and cubic polynomials.

Lemma 3.1. *The graph of the separated quadratic and binary polynomial*

$$a_2 x^2 + a_1 x + b_2 y^2 + b_1 y = d \tag{3.1}$$

is closed if and only if $a_2 b_2 > 0$ and $\frac{a_1^2}{4a_2} + \frac{b_1^2}{4b_2} + d \geq 0$.

Proof. Since $a_2 b_2 > 0$, a_2, b_2 are both positive or both negative. Do not lose generality, suppose $a_2 > 0,\ b_2 > 0$ (if not, multiply by -1 on both sides of equation (3.1)).

Firstly, we separate the variables of equation (3.1), let

$$f(x) = a_2 x^2 + a_1 x,\ (a_2 > 0)$$

$$g(y) = d - b_2 y^2 - b_1 y,\ (b_2 > 0)$$

then equation (3.1) is equivalent to

$$f(x) = g(y).$$

It is easy to see that $f(x) = a_2 x^2 + a_1 x$ has a minimum value, suppose $f(x_m) = f(x)_{\min}$, and $g(y) = d - b_2 y^2 - b_1 y$ has a maximum value, suppose $g(y_m) = g(y)_{\max}$.

(1) To ensure the equation (3.1) has a solution, it must hold that

$$f(x)_{\min} \leq g(y)_{\max},$$

that is

$$\frac{a_1^2}{4a_2} + \frac{b_1^2}{4b_2} + d \geq 0.$$

(2) If $\frac{a_1^2}{4a_2} + \frac{b_1^2}{4b_2} + d = 0$, then graph of equation (3.1) is a point. If $\frac{a_1^2}{4a_2} + \frac{b_1^2}{4b_2} + d > 0$, then there exist x_1, x_2 ($x_1 < x_2$), y_1, y_2 ($y_1 < y_2$) such that

$$f(x_m) = f(x)_{\min} = g(y_1) = g(y_2),$$

and

$$g(y_m) = g(y)_{\max} = f(x_1) = f(x_2).$$

Then for each pair of x, y satisfying $f(x) = g(y)$, it is not difficult to check that $x_1 \leq x \leq x_2$, $y_1 \leq y \leq y_2$. Thus, we can draw the draft graph of $f(x) = g(y)$ which is tangent to the boundary of $[x_1, x_2] \times [y_1, y_2]$ at (x_1, y_m), (x_2, y_m), (x_m, y_1), (x_m, y_2), see figure 1.

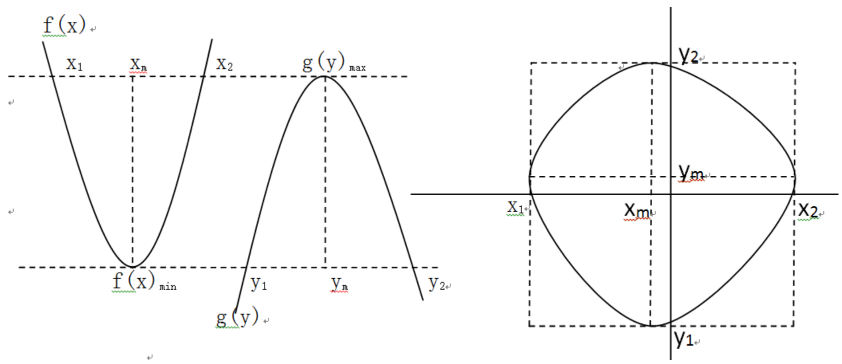

Figure 1: quadratic polynomial

To prove the graph of $f(x) = g(y)$ is closed, it suffices to show that the curves between the four tangent points are continuous. We show the curve between (x_m, y_2) and (x_2, y_m) is continuous for example. Consider the following two functions

$$f : [x_m, x_2] \longrightarrow [f(x_m), g(y_m)],$$

$$g^{-1} : [f(x_m), g(y_m)] \longrightarrow [y_m, y_2],$$

there are continuous and bijective, so, the composite

$$h = g^{-1} \circ f$$

is continuous and bijective. The curve between (x_m, y_2) and (x_2, y_m) of polynomial (3.1) is exactly the graph of function h, hence continuous. The other three parts are the same, we omit it here.

Conversely, if $a_2 b_2 \leq 0$ or $\frac{a_1^2}{4a_2} + \frac{b_1^2}{4b_2} + d < 0$, it is easy to check that the graph of polynomial (3.1) is not closed.

□

Now we study the graph of cubic binary polynomial

$$a_3x^3 + a_2x^2 + a_1x + b_3y^3 + b_2y^2 + b_1y = d.$$

We mainly consider the case that the graph of the cubic polynomial is partially closed.

Theorem 3.2. *Let*

$$a_3x^3 + a_2x^2 + a_1x + b_3y^3 + b_2y^2 + b_1y = d \tag{3.2}$$

be a separated binary and cubic polynomial,

$$f(x) = a_3x^3 + a_2x^2 + a_1x$$

and

$$g(y) = d - b_3y^3 - b_2y^2 - b_1y.$$

Then the graph of (3.2) is partially closed if and only if $f(x)$ and $g(y)$ both have two stationary points and the function values of the stationary points of $f(x)$ and $g(y)$ are arranged alternately.

Proof. The derivative function of $f(x) = a_3x^3 + a_2x^2 + a_1x$ is $f'(x) = 3a_3x^2 + 2a_2x + a_1$. If $\Delta = 4a_2^2 - 12a_3a_1 > 0$, that is $a_2^2 - 3a_3a_1 > 0$, then $f'(x)$ has two roots, it follows that $f(x)$ has two stationary points. If $a_2^2 - 3a_3a_1 \le 0$, then $f(x)$ is monotone.

As the same reason, if $b_2^2 - 3b_3b_1 > 0$, then $g(x)$ has two stationary points; if $b_2^2 - 3b_3b_1 \le 0$, then $b(x)$ is monotone.

In order to prove the necessity, we discuss the graph of equation (3.2) in several cases.

(1) The stationary points of $f(x)$ and $g(y)$ are both less than or equal to 1. Then

$$a_2^2 - 3a_3a_1 \le 0, \text{ and } b_2^2 - 3b_3b_1 \le 0,$$

it follows that both $f'(x) = 0$ and $g'(y) = 0$ have at most one root. Since $f'(x)$ and $g'(y)$ are quadratic functions, we have $f'(x) \le 0$ or $f'(x) \ge 0$ and $g'(y) \le 0$ or $g'(y) \ge 0$ for all x, y. Hence, $f(x)$ and $g(y)$ are monotone functions.

If $f(x)$ and $g(y)$ are both increasing (or, decreasing) functions, then the map that sends each x to y whenever (x, y) satisfies $f(x) = g(y)$ is continuous and bijective, and it is easy to check that if $f(x_1) = g(y_1)$, $f(x_2) = g(y_2)$, then $x_1 > x_2$ implies $y_1 > y_2$. Hence, the graph of equation (3.2) is increasing.

If one of $f(x)$ and $g(y)$ is increasing and the other is decreasing, then it is easy to check that if $f(x_1) = g(y_1)$, $f(x_2) = g(y_2)$, then $x_1 > x_2$ implies $y_1 < y_2$. Hence, the graph of equation (3.2) is decreasing.

Consequently, the graph of equation (3.2) does not have a closed part.

(2) One of $f(x)$ and $g(y)$ has two stationary points and the other does not have stationary points . Do not lose generality, suppose that $f(x)$ has two stationary points x_1, x_2 $(x_1 < x_2)$. Then we get three monotone functions when we restrict $f(x)$ on $(-\infty, x_1]$, $[x_1, x_2]$ and $[x_2, +\infty)$ respectively. Since $g(x)$ is monotone, the three functions above together with $g(y)$ generate three subcases similar to that in (1) respectively, which correspond to three continuous parts of the graph of equation (3.2), see figure 2. The joint points of three continuous parts are $(x_1, g^{-1}f(x_1))$ and $(x_2, g^{-1}f(x_2))$. The graph of equation (3.2), formed by the three continuous parts, is continuous but not closed.

(3) One of $f(x)$ and $g(y)$ has two stationary points and the other has only one stationary point. Similar to (2), it is not difficult to check that the graph of equation (3.2) does not have a closed part.

(4) Both of $f(x)$ and $g(y)$ have two stationary points but the function values of the stationary points are not arranged alternately, see figure 3. Suppose that x_1, x_2 $(x_1 < x_2)$ are stationary points of $f(x)$, and y_1, y_2 $(y_1 < y_2)$ are stationary points of $g(y)$. If $f(x_1) >$

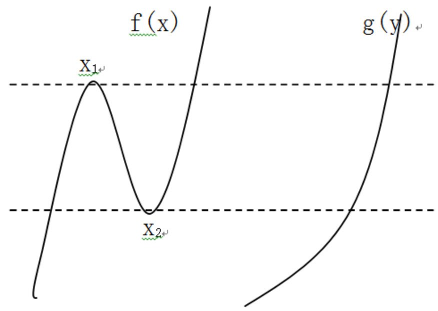

Figure 2:

$f(x_2) > g(y_1) > g(y_2)$, then it can be seen as two subcases of that in (2). So, the graph of equation (3.2) does not have a closed part. If $f(x_1) > g(y_1) > g(y_2) > f(x_2)$, it is also not difficult to check that the graph of equation (3.2) does not have a closed part.

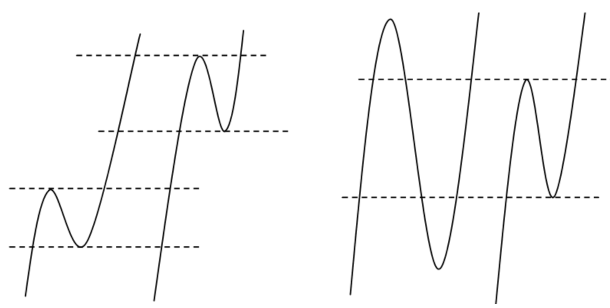

Figure 3:

Now we prove the sufficiency. Suppose that x_1, x_2 $(x_1 < x_2)$ are stationary points of $f(x)$, y_1, y_2 $(y_1 < y_2)$ are stationary points of $g(y)$ and $f(x_1) > g(y_1) > f(x_2) > g(y_2)$, see figure 4.

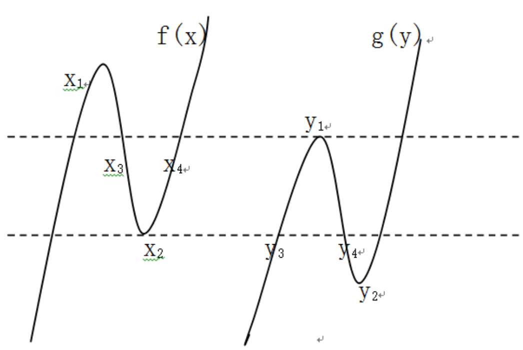

Figure 4:

Suppose that $f(x_3) = f(x_4) = g(y_1)$, $f(x_2) = g(y_3) = g(y_4)$, then $f(x), g(y)$ are continuous and monotone on (x_3, x_2), (x_2, x_4), (y_3, y_1) and (y_1, y_4), respectively. Now we consider the graph of the polynomial (3.2) on $[x_3, x_4] \times [y_3, y_4]$. Similar to the quadratic case in Lemma 3.1, the four points (x_3, y_1), (x_4, y_1), (x_2, y_3) and (x_2, y_4) are on the boundary of $[x_3, x_4] \times [y_3, y_4]$, and the curves between the four points are actually composites of monotone functions, it follows that graph of the polynomial (3.2) on $[x_3, x_4] \times [y_3, y_4]$ is a closed curve. Moreover, there are no x, y satisfying $f(x) = g(y)$ outside of the intervals $[x_3, x_4]$ and $[y_3, y_4]$, so, the closed curve will be separated from other parts of the graph of the polynomial (3.2).

This completes the proof. $\square$

Now we come to the first main result of this section.

Theorem 3.3. *Suppose that $f(x)$ and $g(y)$ are continuous functions defined on intervals $[x_1, x_2]$ and $[y_1, y_2]$, respectively. Both the ranges of $f(x)$ and $g(y)$ are $[f(x)_{\min}, g(y)_{\max}]$, and $f(x)_{\min} = g(y_1) = g(y_2)$, $g(y)_{\max} = f(x_1) = f(x_2)$, where $f(x)_{\min}, g(y)_{\max}$ are the minimum and maximum of $f(x)$ and $g(y)$ in the intervals, respectively. Then the equation defined by*

$$F(x, y) = f(x) - g(y) = 0 \tag{3.3}$$

is a closed curve on $[x_1, x_2] \times [y_1, y_2]$. Moreover, if there is no rectangular subregion satisfying the above conditions, then the graph of equation (3.3) has only one closed curve on $[x_1, x_2] \times [y_1, y_2]$; otherwise, the graph of equation (3.3) on $[x_1, x_2] \times [y_1, y_2]$ will be the form that closed curves in the bigger closed curves.

Proof. It is easy to see that a point (x, y) satisfying equation (3.3) must hold that

$$f(x) \leq g(y)_{\max}, \text{ and } g(y) \geq f(x)_{\min}.$$

Suppose $f(x_m) = f(x)_{\min}$ and $g(y_m) = g(y)_{\max}$. The graph of the equation (3.3) is included in $[x_1, x_2] \times [y_1, y_2]$ which is tangent to the boundary of $[x_1, x_2] \times [y_1, y_2]$ at (x_1, y_m), (x_2, y_m), (x_m, y_1) and (x_m, y_2). To prove the graph is closed, it suffices to show that the curves between the four points are continuous.

If $f(x)$ and $g(y)$ are monotone on $[x_1, x_m]$, $[x_m, x_2]$, $[y_1, y_m]$ and $[y_m, y_2]$, then it is similar to the case in Lemma 3.1, so, the graph is closed.

If $f(x)$ and $g(y)$ have stationary points, then it will be very complex, thus, we have to think about it in other ways. Our method is to study every point on the graph rather than discuss the maps of the graphs of $f(x)$ and $g(y)$ on the intervals. We prove the curve between the points (x_2, y_m) and (x_m, y_1) is continuous for example. Since $f(x), g(y)$ are continuous, then a point on them (discarding (x_2, y_m) and (x_m, y_1)) is either a point in a monotone interval ($f(x^-) < f(x) < f(x^+)$ or $f(x^+) < f(x) < f(x^-)$, we call it monotone point) or a stationary point (($f(x^-) < f(x) > f(x^+)$ or $f(x^-) > f(x) < f(x^+)$). Suppose (x, y) is a point such that $f(x) = g(y)$, we discuss x, y in several cases. In the sequel, when we say a endpoint it means a point like A or B in figure 5.

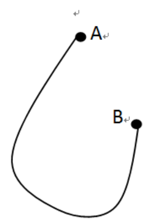

Figure 5:

(1) Monotone point to monotone point, see figure 6.

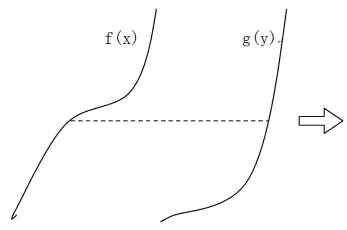

Figure 6:

In this case, x and y are increasing or decreasing at the same time, then the point (x, y) is on the monotone part of the graph of equation 3.3 (not endpoint).

(2) Monotone point to stationary point, see figure 7.

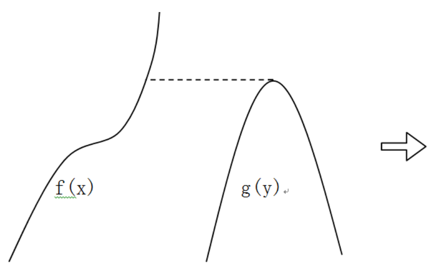

Figure 7:

In this case, one of x, y can both increase and decrease, but, correspondingly, the other variable can only increase or decrease . Then (x, y) is a stationary point in the graph of equation (3.3) (not endpoint).

(3) Stationary point to stationary point (bottom to top), see figure 8.

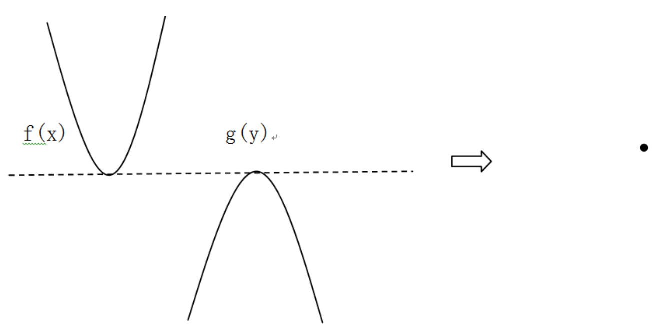

Figure 8:

This case is similar to that in Theorem 3.2, and (x, y) in the graph of equation (3.3) is an isolated point (not endpoint).

(4) Stationary point to stationary point (bottom to bottom or top to top), see figure 9.

In this case, when x is increasing or decreasing, then y is increasing and decreasing correspondingly. There exist four points (x, y) like this in the graph of equation (3.3) (not endpoint).

Consequently, there does not exist a endpoint in the curve from (x_2, y_m) to (x_m, y_1), which still holds for the other three curves. So, the graph of equation (3.3) is closed. $\square$

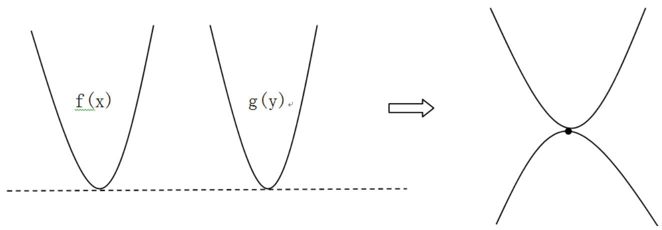

Figure 9:

Now we discuss the graph of the separated four-degree and binary polynomial

$$a_4x^4 + a_3x^3 + a_2x^2 + a_1x + b_4y^4 + b_3y^3 + b_2y^2 + b_1y = d, \ (a_4, b_4 > 0). \tag{3.4}$$

Denote

$$f(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x,$$
$$g(y) = d - b_4y^4 - b_3y^3 - b_2y^2 - b_1y,$$

then polynomial (3.4) will be

$$f(x) = g(y).$$

Since the derivative functions of $f(x), g(y)$ are cubic polynomials, they have at most three stationary points. We prove in the following that if $f(x)$ has three stationary points then there will be two minimal values and one maximal value. Firstly, we have

$$f'(x) = 4a_4x^3 + 3a_3x^2 + 2a_2x + a_1,$$

$$f''(x) = 12a_4x^2 + 6a_3x + 2a_2.$$

By assumption, $f'(x) = 0$ has three roots, so, $f''(x) = 0$ has two roots, suppose they are x_1, x_2, see figure 10. Then there exists $x_0 \in (x_1, x_2)$ such that $f'(x_0) = 0$, and it holds that $f''(x_0) < 0$, thus, $f(x_0)$ is a maximal value of $f(x)$. It is easy to see that the values of $f(x)$ of the other two roots of $f'(x_0) = 0$ are the minimal values.

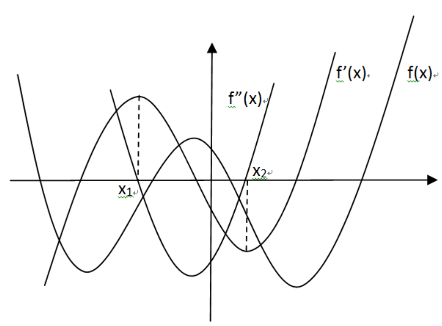

Figure 10:

As the same reason, if $g(y)$ has three stationary points, then there will be two maximal values and one minimal value. We denote the maximal values and minimal values of $f(x), g(y)$ by $f(x)_{\max}$, $g(y)_{\max_1}$, $g(y)_{\max_2}$ and $f(x)_{\min_1}$, $f(x)_{\min_2}$, $g(y)_{\min}$, respectively.

We discuss some special cases that might arise in the following.

(1) If $f(x)$ has three stationary points, $g(y)$ has only one maximal value and

$$f(x)_{\max} > g(y)_{\max} > \max\{f(x)_{\min_1},\ f(x)_{\min_2}\},$$

there are x_1, x_2, x_3, x_4 $(x_1 < x_2 < x_3 < x_4)$ satisfying $f(x) = g(y)_{\max}$, and y_1, y_2 satisfying $g(y) = f(x)_{\min_1}$, and y_3, y_4 satisfying $g(y) = f(x)_{\min_2}$. Suppose that the minimum value of $f(x)$ on $[x_1, x_2]$ and $[x_3, x_4]$ are $f(x)_{\min_1}$ and $f(x)_{\min_2}$, then the graph of $f(x) = g(y)$ on $[x_1, x_2] \times [y_1, y_2]$ and $[x_3, x_4] \times [y_3, y_4]$ are closed curves, see figure 11.

(2) If $f(x)$ and $g(x)$ both have three stationary points, and

$$f(x)_{\max} > g(y)_{\max_{1,2}} > f(x)_{\min_{1,2}} > g(y)_{\min},$$

the graph of $f(x) = g(y)$ has at most four closed curves, see figure 12.

(3) If $f(x)$ and $g(x)$ both have three stationary points, and

$$g(y)_{\max_1} > f(x)_{\max} > g(y)_{\max_2} > f(x)_{\min_1} > g(y)_{\min} > f(x)_{\min_2},$$

then there will be closed curve in bigger closed curve, see figure 13.

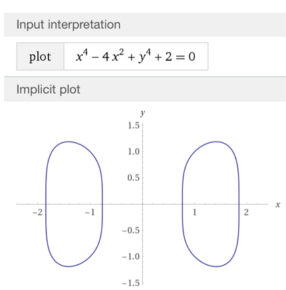

Figure 11:

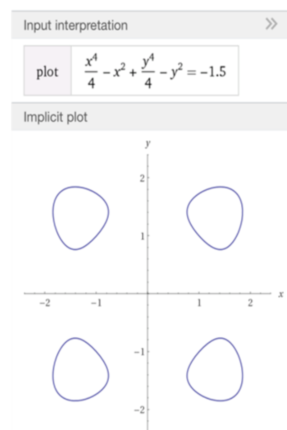

Figure 12:

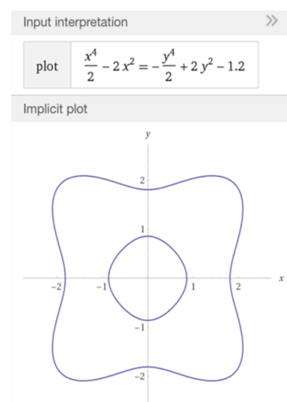

Figure 13:

Now we are ready to prove the second main result of this paper.

Theorem 3.4. *The graph of the separated binary polynomial*

$$a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + b_m y^m + b_{m-1}y^{m-1} + \cdots + b_1 y = d,\ (a_n, b_m \neq 0) \quad (3.5)$$

is only one or more than one closed curve if and only if equation (3.5) has a solution (x, y), $a_n b_m > 0$ and n, m are even numbers.

Proof. We separate the variables first, let

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x,$$

$$g(y) = d - b_m y^m - b_{m-1}y^{m-1} - \cdots - b_1 y,$$

then the equation (3.5) is

$$f(x) = g(y).$$

If one of m, n is odd, suppose m is odd, then $f(x)$ approaches the different limits when x approaches positive infinite and negative infinite. So, there exists the case that $f(x)$ and $g(y)$ approach positive infinite or negative infinite at the same time, it follows that the graph of $f(x) = g(y)$ will approach infinite. Hence, the graph of $f(x) = g(y)$ is not closed. If $a_n b_m < 0$, then the graph of $f(x) = g(y)$ will also approach infinite, hence not closed.

Now suppose that equation (3.5) has a solution, $a_n b_m > 0$ and n, m are even numbers. Do not lose generality, suppose $a_n > 0$ and $b_m > 0$. Then $f(x)$ has a minimum value, denoted by $f(x)_{\min}$, and $g(y)$ has a maximum value, denoted by $g(y)_{\max}$. It is easy to see that if x, y satisfies $f(x) = g(y)$ then it holds that

$$f(x) \leq g(y)_{\max}, \text{ and } g(y) \geq f(x)_{\min}.$$

Consider the equation

$$f(x) = g(y)_{\max},$$

if $f(x_0)$ is tangent to the line $y = g(y)_{\max}$, then we say $f(x) = g(y)_{\max}$ has two roots at x_0. Hence, the equation $f(x) = g(y)_{\max}$ has an even number of roots, suppose they are

$$x_1 \leq x_2 \leq \cdots \leq x_p.$$

The equation $g(y) = f(x)_{\min}$ also has an even number of roots, suppose they are

$$y_1 \leq y_2 \leq \cdots \leq y_q.$$

It is not difficult to see that $f(x) \leq g(y)_{\max}$ whenever $x \in [x_{2j-1}, x_{2j}]$, but not whenever $x \in [x_{2j}, x_{2j+1}]$, where $1 \leq j \leq \frac{p}{2}$. Since $f(x)$ is continuous, $f(x)$ has a minimum value (stationary point) in $[x_{2j-1}, x_{2j}]$, denoted by $f(x)_{\min_j}$. Then $f(x)_{\min_j} \geq f(x)_{\min}$.

It also holds that $g(y) \geq f(x)_{\min}$ whenever $y \in [y_{2i-1}, y_{2i}]$. Then $g(y)$ has a maximum value in $[y_{2j-1}, y_{2j}]$, say $g(y)_{\max_i}$, and it holds that $g(y)_{\max_i} \leq g(y)_{\max}$.

(1) If $f(x)_{\min_j} > g(y)_{\max_i}$, then there is no graph in $[x_{2j-1}, x_{2j}] \times [y_{2i-1}, y_{2i}]$.

(2) If $f(x)_{\min_j} = g(y)_{\max_i}$, then the graph in $[x_{2j-1}, x_{2j}] \times [y_{2i-1}, y_{2i}]$ of equation (3.5) is a point.

(3) If $f(x)_{\min_j} < g(y)_{\max_i}$, then it is actually the case in Lemma 3.1, it follows that the graph in $[x_{2j-1}, x_{2j}] \times [y_{2i-1}, y_{2i}]$ of equation (3.5) is a closed curve, see figure 14.

Consequently, the graph of equation (3.5) is a closed curve or more than one closed curve. This completes the proof.

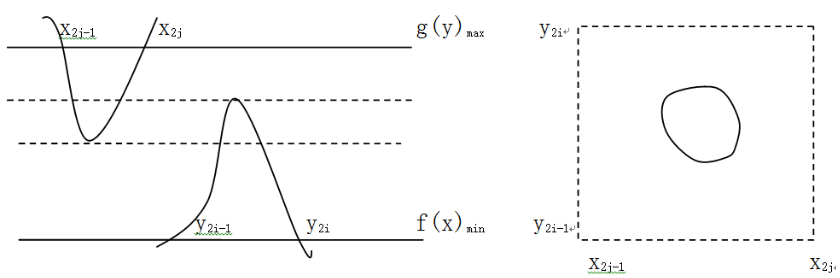

Figure 14:

□

Remark 3.5. We use the method of separation of variables to discuss the separated polynomials, but whether it is can be extended to the general polynomials is a question, which will be discussed in the future. In addition, we think the Theorem 3.3 may be useful in other problems and fields. For instance, it could be useful to the existence problems of continuous functions.

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