Generalization of an Asymptotic Formula for the Smarandache kn-digital Sequence

Applicant: Haoyuan Hu

Supervisor: Chengpeng Zhang

Shanghai, China

No.2 Secondary School Attached to East China Normal University

December 4th, 2017

Abstract

The sequence {a(3,n)} is called the Smarandache 3n-digital sequence, if the digital of a(3,n) can be partitioned into two groups such that the second is 3 times of the first. Smarandache kn-digital sequence {a(k,n)} in the base p is defined similarly. This paper studies an asymptotic formula for Smarandache kn-digital sequence $\sum_{1\le n\le N}\ln a_n = 2N\ln N + O(N)$, $N\to+\infty$, $1\le k\le 9$ (defined in base ten) and generalizes the conclusion by proving that the asymptotic formula is true for any positive integer k and p (p>1). Furthermore, this paper proves some more precise asymptotic formulas for k=1,2,3,4,5,6,8,9,10,11 (defined in base ten) and for general positive integer k and p, and conjectures a more precise asymptotic formula for k=7.

Key words

Smarandache sequence;Asymptotic formula;Base

Contents

1 Introduction

For any arbitrary positive integer k, the sequences $\{a_n\}$ is called the Smarandache kn-digital sequence, if the digital of a_n can be partitioned into two groups such that the second is k times of the first. This sequence was defined by Smarandache, F. (1993, 2006, cited in Gou, S. 2010). There are a number of subsequent works.

Wu(2008:120-122) considered Zhang Wenpeng's conjecture that the Smarandache 3n-digital sequence does not contain any square number. Although this conjecture is not completely solved, Wu did prove the following results:

(1) a_n is not a square if n is square-free.

(2) a_n is not a square if n is a square.

(3) If a_n is a square, then $n = 2^{2\alpha_1} \cdot 3^{2\alpha_2} \cdot 5^{2\alpha_3} \cdot 11^{2\alpha_4} \cdot n_1$ holds, where $(n_1, 330) = 1$.

Lu, P.(2009:5-7, cited in Chen, J. 2012) considered whether there is a square number in the Smarandache 5n-digital sequence and got a negative answer when n equals some special values.

By using elementary method, Gou, S.(2010) proved that for any arbitrary positive integer N large enough, the asymptotic formula $\sum_{1 \le n \le N} \ln a_n = 2N \ln N + O(N)$ holds when $k = 3$. (When $k = 3$, $\{a_n\} = \{13, 26, 39, 412, 515, 618, 721, 824, \cdots\}$.)

Chen, J.(2012:9-14) pointed out that equation $\sum_{1 \le n \le N} \ln a_n = 2N \ln N + O(N)$ still holds under the condition of $1 \le k \le 9$ when $N \to +\infty$.

This paper generalizes the above asymptotic formula $\sum_{1 \le n \le N} \ln a_n = 2N \ln N + O(N)$ to arbitrary k and arbitrary base p, and improves the estimates of the error term.

2 Theoretical Discussions

2.1 Lemmas and Simple Corollaries

2.1.1 Taylor series with the Peano form of the remainder

Let $f(x)$ be n times differentiable at x_0, then there must be a neighborhood of x_0, for any x in this neighborhood, the following holds:

$$f(x)=f(x_0)+f'(x_0)(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2+\cdots+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n+r_n(x) \quad ,$$

In the equation above, the remainder $r_n(x)$ equals $o\left((x-x_0)^n\right)$. When $x_0=0$, the above equation is called the Maclaurin series.

2.1.2 Lemma: $\ln(x+1)=x+O(x^2)$, when $x\to 0$

Because $\ln(x+1)'=\frac{1}{x+1}$ and $\ln(x+1)''=-\frac{1}{(x+1)^2}$, using 2.1.1 (let $x_0=0, n=2$), we get the target equation as follows.

$$\begin{aligned}\ln(x+1)&=\ln(0+1)+\frac{1}{0+1}(x-0)+\frac{-\frac{1}{(0+1)^2}}{2!}(x-0)^2+o\left((x-0)^2\right)\\&=x-\frac{1}{2}x^2+o(x^2)=x+O(x^2)\end{aligned}$$

2.1.3 Stirling's approximation

$$n!\sim\sqrt{2\pi n}\left(\frac{n}{\mathrm{e}}\right)^n\left(1+\frac{1}{12n}+\frac{1}{288n^2}-\frac{139}{51840n^3}+\cdots\right)$$

2.1.4 Lemma: $\lim_{N\to+\infty}\frac{(\ln N)^2}{N}=\lim_{N\to+\infty}\frac{\ln N}{N}=0$

Using L'Hospital's rule:

$$\lim_{N\to+\infty}\frac{(\ln N)^2}{N}=\lim_{N\to+\infty}\frac{2\ln N\cdot\frac{1}{N}}{1}=\lim_{N\to+\infty}\frac{2\ln N}{N}=\lim_{N\to+\infty}\frac{\frac{2}{N}}{1}=0$$

$$\lim_{N\to+\infty}\frac{\ln N}{N}=\lim_{N\to+\infty}\frac{\frac{1}{N}}{1}=0$$

This means that $(\ln N)^2$ and $\ln N$ are the lower order infinities of N .

2.1.5 A computational result of dislocation subtraction

Conclusion: $\sum_{t=1}^{M}t\cdot p^t=\frac{1}{p-1}M\cdot p^{M+1}-\frac{p}{(p-1)^2}\left(p^M-1\right)$

We denote that $S=\sum_{t=1}^{M}t\cdot p^t$,有 $p\cdot S=\sum_{t=1}^{M}t\cdot p^{t+1}=\sum_{k=2}^{M+1}(t-1)\cdot p^t$,

and we have: $(p-1)S=p\cdot S-S=M\cdot p^{M+1}-\frac{p}{p-1}\left(p^M-1\right)$,

which means that $S=\frac{1}{p-1}M\cdot p^{M+1}-\frac{p}{(p-1)^2}\left(p^M-1\right)$.

Specifically, when $p=10$, we have $S=\frac{1}{9}M\cdot10^{M+1}-\frac{10}{81}\left(10^M-1\right)$. We will directly use the computational result hereafter.

2.2 Proof When $k=3$ in Base 10

2.2.1 Identical deformation of the target equation

Let a_n be in the sequence, and assume that $3n$ has t digits ($n\epsilon\mathbb{Z}^+, t\epsilon\mathbb{Z}^+$), then $\frac{10^{t-1}}{3}\le n<\frac{10^t}{3}$. Because of the definition of the sequence $\{a_n\}$, we know that $a_n=n\cdot\left(10^t+3\right)$. When N is large enough, there exists a unique $M\in\mathbb{Z}^+$ such that

$\frac{10^M}{3} \le N < \frac{10^{M+1}}{3}$. This is because the intervals $J_t = \left(\frac{10^t}{3}, \frac{10^{t+1}}{3}\right], t = 0, 1, 2, \cdots$ are pair-wise disjoint, and their union is $\left(\frac{1}{3}, +\infty\right)$, which includes all positive integers, so N must be included in one of these intervals, which means that there must be a unique M. We will use the uniqueness of M directly hereafter. Assume that $3N$ has $(M+1)$ digits , so $a_N = N \cdot \left(10^{M+1} + 3\right)$. Now we have the following identical equation:

$$\prod_{1 \le n \le N} a_n = \prod_{n=1}^{3} a_n \cdot \prod_{n=4}^{33} a_n \cdots \prod_{n=\frac{1}{3}\left(10^{M-1}-1\right)+1}^{\frac{1}{3}\left(10^M-1\right)} a_n \cdot \prod_{n=\frac{1}{3}\left(10^M-1\right)+1}^{N} a_n$$

$$= N! \cdot (10+3)^3 \cdot (100+3)^{30} \cdots \left(10^M + 3\right)^{3 \cdot 10^{M-1}} \cdot \left(10^{M+1} + 3\right)^{N - \frac{1}{3}\left(10^M - 1\right)}$$

Take the natural logarithm of the both sides, and the equation becomes:

$$\sum_{1 \le n \le N} \ln a_n = \ln N! + \sum_{t=1}^{M} \ln \left(10^t + 3\right)^{3 \cdot 10^{t-1}} + \ln \left(10^{M+1} + 3\right)^{N - \frac{1}{3}\left(10^M - 1\right)}$$

$$= \ln N! + 3 \cdot \sum_{t=1}^{M} 10^{t-1} \cdot \ln \left(10^t + 3\right) + \left[N - \frac{1}{3}\left(10^M - 1\right)\right] \ln \left(10^{M+1} + 3\right) \cdots (1)$$

2.2.2 Estimation of $N!$

Using Stirling's approximation (Lemma 2.1.3):

$$n! \sim \sqrt{2\pi n} \left(\frac{n}{\mathrm{e}}\right)^n \left(1 + \frac{1}{12n} + \frac{1}{288n^2} - \frac{139}{51840n^3} + \cdots\right)$$

According to 2.1.2, we take the natural logarithm of the both sides, and we get the equation as follows.

$$\ln N! = \ln \sqrt{2\pi N} + N \ln N - N + \ln \left(1 + \frac{1}{12N} + \frac{1}{288N^2} - \frac{139}{51840N^3} + \cdots\right)$$

$$= \left(N + \frac{1}{2}\right) \ln N - N + \ln \sqrt{2\pi} + O\left(\frac{1}{N}\right)$$

$$= \left(N + \frac{1}{2}\right) \ln N - N + O(1) \cdots\cdots (2)$$

We will use equation (2) directly hereafter.

2.2.3 Estimation of $3\cdot\sum_{t=1}^{M}10^{t-1}\cdot\ln\left(10^{t}+3\right)$

When $x\to 0$, $\ln\left(x+1\right)=x+O\left(x^{2}\right)$ (Lemma 2.1.2). According to 2.1.5, we get the following equation.

$$3\cdot\sum_{t=1}^{M}10^{t-1}\cdot\ln\left(10^{t}+3\right)=3\cdot\sum_{t=1}^{M}10^{t-1}\cdot\left[\ln\left(10^{t}\right)+\ln\left(1+\frac{3}{10^{t}}\right)\right]$$
$$=3\cdot\sum_{t=1}^{M}10^{t-1}\cdot\left[t\cdot\ln 10+\frac{3}{10^{t}}+O\left(\frac{1}{10^{2t}}\right)\right]$$
$$=3\ln 10\cdot\sum_{t=1}^{M}t\cdot 10^{t-1}+\frac{9}{10}M+O\left(1\right)$$
$$=\frac{\ln 10}{3}M\cdot 10^{M}-\frac{\ln 10}{27}\left(10^{M}-1\right)+\frac{9}{10}M+O\left(1\right)\cdots\cdots\left(3\right)$$

2.2.4 Estimation of $\left[N-\frac{1}{3}\left(10^{M}-1\right)\right]\ln\left(10^{M+1}+3\right)$

Note that when $x\to 0$, $\ln\left(x+1\right)=x+O\left(x^{2}\right)$ (Lemma 2.1.2). Therefore,

$$\left[N-\frac{1}{3}\left(10^{M}-1\right)\right]\ln\left(10^{M+1}+3\right)=\left[N-\frac{1}{3}\left(10^{M}-1\right)\right]\left[\ln\left(1+\frac{3}{10^{M+1}}\right)+\ln\left(10^{M+1}\right)\right]$$
$$=\left[N-\frac{1}{3}\left(10^{M}-1\right)\right]\left[\frac{3}{10^{M+1}}+O\left(\frac{1}{10^{2M}}\right)+\left(M+1\right)\cdot\ln 10\right],$$

where $\left[N-\frac{1}{3}\left(10^{M}-1\right)\right]=O\left(N\right)$, $\frac{3}{10^{M+1}}+O\left(\frac{1}{10^{2M}}\right)=O\left(\frac{1}{N}\right)$, and $O\left(N\right)\cdot O\left(\frac{1}{N}\right)=O\left(1\right)$.

This means that:

$$\left[N-\frac{1}{3}\left(10^M-1\right)\right]\ln(10^{M+1}+3)$$
$$=\left[N-\frac{1}{3}\left(10^M-1\right)\right]\cdot\left[\frac{3}{10^{M+1}}+O\left(\frac{1}{10^{2M}}\right)+(M+1)\cdot\ln 10\right]$$
$$=\left[N-\frac{1}{3}\left(10^M-1\right)\right]\cdot(M+1)\cdot\ln 10+O(1)$$
$$=\ln 10\cdot\left(MN-\frac{1}{3}M\cdot 10^M+\frac{1}{3}M+N-\frac{1}{3}\cdot 10^M+\frac{1}{3}\right)+O(1)$$
$$=\ln 10\cdot\left(MN-\frac{1}{3}M\cdot 10^M+\frac{1}{3}M+N-\frac{1}{3}\cdot 10^M\right)+O(1)\cdots\cdots\cdots\cdots\cdots\cdots(4)$$

2.2.5 Summate and analyze the error terms

Finaly we substitute $(2)(3)(4)$ into (1):

$$\sum_{1\le n\le N}\ln a_n=\left[\left(N+\frac{1}{2}\right)\ln N-N+O(1)\right]$$
$$+\left[\frac{\ln 10}{3}M\cdot 10^M-\frac{\ln 10}{27}\left(10^M-1\right)+\frac{9}{10}M+O(1)\right]$$
$$+\left[\ln 10\cdot\left(MN-\frac{1}{3}M\cdot 10^M+\frac{1}{3}M+N-\frac{1}{3}\cdot 10^M\right)+O(1)\right]$$
$$=N\ln N+\frac{1}{2}\ln N-N+\frac{\ln 10}{3}\cdot M\cdot 10^M-\frac{\ln 10}{27}\cdot 10^M+\frac{9}{10}M$$
$$+\ln 10\cdot MN-\frac{\ln 10}{3}\cdot M\cdot 10^M+\frac{\ln 10}{3}\cdot M+\ln 10\cdot N-\frac{\ln 10}{3}\cdot 10^M+O(1)$$
$$=(N\ln N+MN\cdot\ln 10)+\frac{1}{2}\ln N+(\ln 10-1)N-\frac{10\ln 10}{27}\cdot 10^M$$
$$+\left(\frac{\ln 10}{3}+\frac{9}{10}\right)M+O(1)$$

then:

$$\sum_{1\le n\le N}\ln a_n=(N\ln N+MN\cdot\ln 10)+\left[(\ln 10-1)N-\frac{10\ln 10}{27}\cdot 10^M\right]$$
$$+\left[\left(\frac{\ln 10}{3}+\frac{9}{10}\right)M+\frac{1}{2}\ln N\right]+O(1)\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(5)$$

This is the asymptotic formula with an $O(1)$ error term when $\mathrm{k}=3$.

Because the quotients of M over $\ln N$ and 10^M over N are bounded, the following equations hold:

$$\lim_{N\to+\infty}\frac{N\ln N+MN\ln 10}{N\ln N}=2,$$

$$\lim_{N\to+\infty}\frac{\frac{1}{2}\ln N+(\ln 10-1)N-\frac{10\ln 10}{27}10^M+\left(\frac{\ln 10}{3}+\frac{9}{10}\right)M}{N}=O(1).$$

This means that: $\sum_{1\le n\le N}\ln a_n=2N\ln N+O(N)$.

2.3 For Some Specific k ($k=1,2,4,5,6,8,9,10,11$, in Base 10)

The following proof is similar to $k=3$, but only different in the classification of n according to how many digits $k\cdot n$ has in the base 10. Chen, J.(2012:9-14) proved the asymptotic formula to be true when $1\le k\le 9$, but in fact, the asymptotic formula is true even when $k=10,11$.

2.3.1 $k=1$

Assume n has t digits ($n\in\mathbb{Z}^+, t\in\mathbb{Z}^+$), then $10^{t-1}\le n<10^t$. Because of the definition of the sequence $\{a_n\}$, we have $a_n=n\cdot(10^t+1)$. For any N that is large enough, there exists a unique $M\in\mathbb{Z}^+$ such that $10^M\le N<10^{M+1}$.

By the same argument:

$$\prod_{1\le n\le N}a_n=\prod_{n=1}^{9}a_n\cdot\prod_{n=10}^{99}a_n\cdots\prod_{n=10^{M-1}}^{10^M-1}a_n\cdot\prod_{n=10^M}^{N}a_n$$

$$=N!\cdot(10+1)^9\cdot(100+1)^{90}\cdots(10^M+1)^{9\cdot 10^{M-1}}\cdot(10^{M+1}+1)^{N-(10^M-1)}$$

$$\sum_{1\le n\le N}\ln a_n=\ln N!+\sum_{t=1}^{M}\ln(10^t+1)^{9\cdot 10^{t-1}}+\ln(10^{M+1}+1)^{N-(10^M-1)}$$

$$=\ln N!+9\cdot\sum_{t=1}^{M}10^{t-1}\cdot\ln(10^t+1)+\left[N-(10^M-1)\right]\ln(10^{M+1}+1)$$

We have: $\ln N!=\left(N+\frac{1}{2}\right)\ln N-N+O(1)$

When $x \to 0$, $\ln(x+1) = x + O(x^2)$, which means that:

$$
\begin{aligned}
9\cdot\sum_{t=1}^{M}10^{t-1}\cdot\ln\left(10^t+1\right) &= 9\cdot\sum_{t=1}^{M}10^{t-1}\cdot\left[\ln\left(10^t\right)+\ln\left(1+\frac{1}{10^t}\right)\right] \\
&= 9\cdot\sum_{t=1}^{M}10^{t-1}\cdot\left[t\cdot\ln 10+\frac{1}{10^t}+O\left(\frac{1}{10^{2t}}\right)\right] \\
&= 9\cdot\ln 10\cdot\sum_{t=1}^{M}t\cdot 10^{t-1}+\frac{9}{10}M+O(1) \\
&= \ln 10\cdot M\cdot 10^M-\frac{\ln 10}{9}\left(10^M-1\right)+\frac{9}{10}M+O(1)
\end{aligned},
$$

and

$$
\begin{aligned}
&\left[N-\left(10^M-1\right)\right]\ln\left(10^{M+1}+1\right) = \left[N-\left(10^M-1\right)\right]\left[\ln\left(1+\frac{1}{10^{M+1}}\right)+\ln\left(10^{M+1}\right)\right] \\
&= \left[N-\left(10^M-1\right)\right]\left[\frac{1}{10^{M+1}}+O\left(\frac{1}{10^{2M}}\right)+(M+1)\cdot\ln 10\right] \\
&= \ln 10\cdot(M+1)\left[N-\left(10^M-1\right)\right]+O(1) \\
&= \ln 10\cdot\left(MN-M\cdot 10^M+M+N-10^M+1\right)+O(1) \\
&= \ln 10\cdot\left(MN-M\cdot 10^M+M+N-10^M\right)+O(1)
\end{aligned}.
$$

At last we have:

$$
\begin{aligned}
\sum_{1\le n\le N}\ln a_n &= \left[\left(N+\frac{1}{2}\right)\ln N-N+O(1)\right] \\
&\quad+\left[\ln 10\cdot M\cdot 10^M-\frac{\ln 10}{9}\cdot\left(10^M-1\right)+\frac{9}{10}M+O(1)\right] \\
&\quad+\left[\ln 10\cdot\left(MN-M\cdot 10^M+M+N-10^M\right)+O(1)\right] \\
&= N\ln N+\frac{1}{2}\ln N-N+\ln 10\cdot M\cdot 10^M-\frac{\ln 10}{9}\cdot 10^M+\frac{9}{10}M \\
&\quad+\ln 10\cdot MN-\ln 10\cdot M\cdot 10^M+\ln 10\cdot M+\ln 10\cdot N-\ln 10\cdot 10^M+O(1) \\
&= \left(N\ln N+MN\cdot\ln 10\right)+\frac{1}{2}\ln N+\left(\ln 10-1\right)N-\frac{10\ln 10}{9}\cdot 10^M \\
&\quad+\left(\ln 10+\frac{9}{10}\right)M+O(1)
\end{aligned},
$$

which means that:

$$\sum_{1\le n\le N}\ln a_n=\left(N\ln N+MN\cdot\ln 10\right)+\left[\left(\ln 10-1\right)N-\frac{10\ln 10}{9}\cdot 10^M\right]$$
$$+\left[\left(\ln 10+\frac{9}{10}\right)M+\frac{1}{2}\ln N\right]+O(1)\cdots\cdots\cdots\cdots(6)$$

This is the asymptotic formula with an $O(1)$ error term when $\mathrm{k}=1$.

Because the quotients of M over $\ln N$ and 10^M over N are bounded, the following equations hold:

$$\lim_{N\to+\infty}\frac{N\ln N+MN\ln 10}{N\ln N}=2,$$

$$\lim_{N\to+\infty}\frac{\left(\ln 10-1\right)N-\frac{10\ln 10}{9}10^M+\frac{1}{2}\ln N+\left(\ln 10+\frac{9}{10}\right)M}{N}=O(1),$$

which means that: $\sum_{1\le n\le N}\ln a_n=2N\ln N+O(N)$.

2.3.2 $k=2,4,5,6,8,9,10,11$

Because the proof for $k=2,4,5,6,8,9,10,11$ is tedious and highly similar to the proof of $k=1$ and $k=3$, the detailed proof is presented in **'5 Appendix'** and here only the results are presented below. **(Equations $(7)\sim(17)$ are also in '5 Appendix'.)**

For $k=2,4,5,6,8,9,10,11$, the asymptotic formula $\sum_{1\le n\le N}\ln a_n=2N\ln N+O(N)$ is true.

Furthermore, we get the following asymptotic formulas with an $O(1)$ error term, in which $M=\lfloor\log_{10}kN\rfloor$. ($\lfloor x\rfloor$ is the floor function of x)

For $k=2$,

$$\sum_{1\le n\le N}\ln a_n=\left(N\ln N+MN\cdot\ln 10\right)+\left[\left(\ln 10-1\right)N-\frac{5\ln 10}{9}\cdot 10^M\right]$$
$$+\left[\left(\ln 10+\frac{9}{10}\right)M+\frac{1}{2}\ln N\right]+O(1)$$

For $k=4$,

$$\sum_{1\le n\le N}\ln a_n=\left(N\ln N+MN\cdot\ln 10\right)+\left[\left(\ln 10-1\right)N-\frac{5\ln 10}{18}\cdot 10^M\right]$$
$$+\left[\left(\ln 10+\frac{9}{10}\right)M+\frac{1}{2}\ln N\right]+O(1)$$

For $k=5$,

$$\sum_{1\le n\le N}\ln a_n=\left(N\ln N+MN\cdot\ln 10\right)+\left[\left(\ln 10-1\right)N-\frac{2\ln 10}{9}\cdot 10^M\right]$$
$$+\left[\left(\ln 10+\frac{9}{10}\right)M+\frac{1}{2}\ln N\right]+O(1)$$

For $k=6$,

$$\sum_{1\le n\le N}\ln a_n=\left(N\ln N+MN\cdot\ln 10\right)+\left[\left(\ln 10-1\right)N-\frac{5\ln 10}{27}\cdot 10^M\right]$$
$$+\left[\left(\frac{3\ln 10}{2}+\frac{9}{10}\right)M+\frac{1}{2}\ln N\right]+O(1)$$

For $k=8$,

$$\sum_{1\le n\le N}\ln a_n=\left(N\ln N+MN\cdot\ln 10\right)+\left[\left(\ln 10-1\right)N-\frac{5\ln 10}{36}\cdot 10^M\right]$$
$$+\left[\left(\ln 10+\frac{9}{10}\right)M+\frac{1}{2}\ln N\right]+O(1)$$

For $k=9$,

$$\sum_{1\le n\le N}\ln a_n=\left(N\ln N+MN\cdot\ln 10\right)+\left[\left(\ln 10-1\right)N-\frac{10\ln 10}{81}\cdot 10^M\right]$$
$$+\left[\left(\frac{\ln 10}{9}+\frac{9}{10}\right)M+\frac{1}{2}\ln N\right]+O(1)$$

For $k=10$,

$$\sum_{1\le n\le N}\ln a_n=\left(N\ln N+MN\cdot\ln 10\right)+\left[\left(\ln 10-1\right)N-\frac{\ln 10}{9}\cdot 10^M\right]$$
$$+\left[\left(\ln 10+\frac{9}{10}\right)M+\frac{1}{2}\ln N\right]+O(1)$$

For $k=11$,

$$\sum_{1\le n\le N}\ln a_n=\left(N\ln N+\ln 10\cdot MN\right)+\left[\left(\ln 10-1\right)N-\frac{10\ln 10}{99}\cdot 10^M\right]$$
$$+\left[\frac{1}{2}\ln N+\left(\frac{9}{10}+\frac{\ln 10}{2}\right)M\right]+O(1)$$