On Quasiinvariant Polynomials

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Abstract

Symmetric polynomials are polynomials that are invariant under the action of the symmetric group, and they play an integral role in mathematics. The space of quasiinvariant polynomials, polynomials that are invariant under the action of the symmetric group to a certain order, were introduced by Feigin and Veselov. These spaces are modules over the ring of symmetric polynomials, and their Hilbert series in fields of characteristic 0 were also computed by Feigin and Veselov.

In this paper, we study the Hilbert series of these spaces in fields of positive characteristic. Braverman, Etingof, and Finkelberg recently introduced spaces of twisted quasiinvariant polynomials, a generalization of quasiinvariant polynomials in which the space is twisted by a monomial. We extend some of their results to spaces twisted by a product of smooth functions and compute the Hilbert series of the space in certain cases.

1 Introduction

A polynomial in variables $x_1, \ldots, x_n$ is *symmetric* if permuting the variables does not change it. Another way to view symmetric polynomials is as invariant polynomials under the action of of the symmetric group. A natural generalization of symmetric polynomials then arises: if $s_{i,j}$ is the operator on polynomials that swaps the variables x_i and x_j, then we may consider polynomials G such that $G - s_{i,j}(G)$ vanishes to some order at $x_i = x_j$. Notably, if G is symmetric in x_i and x_j, then $G - s_{i,j}(G)$ vanishes to infinite order. These polynomials may be viewed as quasiinvariant polynomials of the symmetric group.

Definition. Let k be a field, n be a positive integer, and m be a nonnegative integer. We say that a polynomial $G \in k[x_1, \ldots, x_n]$ is m-quasiinvariant if

$$(x_i - x_j)^{2m+1} \mid G(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n) - G(x_1, \ldots, x_j, \ldots, x_i, \ldots, x_n)$$

for all $1 \leq i, j \leq n$. Denote by Q_m the set of all m-quasiinvariant polynomials over k in n variables.

Note that Q_m is a module over the ring of symmetric polynomials over k in n variables. Also, as Q_m is a space of polynomials, it has an increasing filtration by degree. Thus, we may define a Hilbert series and a Hilbert polynomial to encapsulate the structure of Q_m.

The motivation for studying quasiivariant polynomials arises from their relation with integral systems. In 1971, Calogero first solved the problem in mathematical physics of determining the energy spectrum of a one-dimensional system of quantum-mechanical particles with inversely quadratic potentials [1]. Moser later on connected the classical variant of his problem with integrable Hamiltonian systems and showed that the classical analogue is indeed integrable [2]. These so-called Calogero-Moser systems have been of great interest to mathematicians as they connect many different fields including algebraic geometry, representation theory, deformation theory, homological algebra, and Poisson geometry [3].

They have been widely studied since their introduction and even today mathematicians are proving results about them and their variants [4], [5].

Quasiinvariant polynomials are deeply related with solutions of quantum Calogero-Moser systems as well as representations of Cherednik algebras [6]. As such, the structure of Q_m, in particular freeness as a module, and its corresponding Hilbert series and polynomials have been extensively investigated by mathematicians. Introduced by Feigin and Veselov in 2001, their Hilbert series and lowest degree non-symmetric elements have subsequently been computed by Felder and Veselov [7]. In 2010, Berest and Chalykh generalized the idea to quasiinvariant polynomials over an arbitrary complex reflection group [8]. Recently in 2016, Braverman, Etingof, and Finkelberg proved freeness results and computed the Hilbert series of a generalization of Q_m twisted by monomial factors [9]. Our goal is to extend the investigation of Q_m and its various generalizations.

In Section 2, we investigate quasiinvariant polynomials in fields with characteristic p. In particular, we determine the Hilbert series when $n = 2$ and provide sufficient conditions for which the Hilbert series in characteristic p is greater than in characteristic 0 for $n > 2$. We conjecture that our sufficient conditions are also necessary. We also make conjectures about the properties of the Hilbert series.

In Section 3, we investigate a generalization of the twisted quasiinvariants. In [9], Braverman, Etingof and Finkelberg introduced the space of quasiinvariants twisted by monomial factors, again a module over the ring of symmetric polynomials. They proved freeness results and computed the corresponding Hilbert series. We generalize their work to the space of quasiinvariants which are twisted by arbitrary smooth functions and determine the Hilbert series in certain cases when there are two variables.

In Section 4, we discuss future directions for our research, in particular considering spaces of polynomial differential operators and q-deformations.

2 Quasiinvariant polynomials in fields of nonzero characteristic

Much of the previous research on quasiinvariant polynomials has been done in fields of characteristic zero. The general approach in [10] is to use representations of spherical rational Cherednik algebras [9]. In the case of fields with positive characteristic, we take a different approach.

Let k be $\mathbb{F}_p$, and Q_m the set of all m-quasiinvariant polynomials over k in n variables. To begin, we define the Hilbert series of Q_m.

Definition. Let the Hilbert series of Q_m be

$$H_m(t) = \sum_{d \geq 0} t^d \cdot \dim Q_{m,d}$$

where $Q_{m,d}$ is the k vector subspace of Q_m consisting of polynomials with degree d.

By the Hilbert basis theorem, Q_m is a finitely generated module over the ring of symmetric polynomials. Thus, we may write

$$H_m(t) = \frac{G_m(t)}{\prod_{i=1}^{n}(1-t^i)}$$

where $G_m(t)$ is the Hilbert polynomial associated with $H_m(t)$ and the terms in the denominator correspond to the elementary symmetric polynomials that generate the ring of symmetric polynomials in n variables.

We are mainly concerned with the difference between the Hilbert series of Q_m in characteristic p and characteristic 0. The following proposition states that the Hilbert series of Q_m is at least as large in the former case as in the latter case.

Proposition 1. *$\dim Q_{m,d}$ in $\mathbb{F}_p$ is at least as large as in $\mathbb{C}$ for each choice of m, n (the number of variables), and d.*

Proof. Suppose that $F = \sum_{i_1+\cdots+i_n=d} a_{i_1,\dots,i_n} x_1^{i_1}\cdots x_n^{i_n}$ is in $Q_{m,d}$. Then, either $d < 2m+1$, in which case we must have F symmetric or $(x_i - x_j)^{2m+1}$ would divide a nonzero polynomial with degree d for some choice of i and j, a contradiction. This means that the dimensions are equal in either characteristic. Otherwise, we have that

$$F - s_{i,j}F = (x_i - x_j)^{2m+1}\left(\sum_{j_1+\ldots+j_n=d-(2m+1)} b_{i,j,j_1,\dots,j_n} x_1^{j_1}\ldots x_n^{j_n}\right)$$

for each pair i, j. These yield a system of linear equations in the undetermined coefficients of F and $\frac{F-s_{i,j}F}{(x_i-x_j)^{2m+1}}$, which is invariant of the field we are considering. It then follows from considering the null-space that the dimension of the solution space over a field of characteristic p is at least the dimension over a field of characteristic 0. $\square$

However, there are only finitely many primes for which it is strictly greater for each m.

Theorem 2. *For each m, there are only finitely many primes p for which the Hilbert series of Q_m is greater in $\mathbb{F}_p$ than in $\mathbb{C}$.*

Proof. Let $P = \mathbb{Z}[x_1, \dots, x_n]$, $Q = \bigoplus_{1\le i<j\le n} P/(x_i - x_j)^{2m+1}P$, and h be the linear map from P to Q defined as

$$h(F) = \bigoplus_{1\le i<j\le n} (1 - s_{i,j})F.$$

Note that $\mathrm{Ker}(h)$ coincides with Q_m by definition. Set $M = \mathrm{Coker}(h)$ as the cokernal of h in Q and note that if Q_m over $\mathbb{F}_p$ has a higher dimension than Q_m over $\mathbb{C}$ for some degree of the polynomials, then M must have p-torsion. To prove that there are only finitely many such primes p, we use the following lemma:

Lemma 3 (Grothendiek generic freeness lemma)**.** *For a Noetherian integra domain A, a finitely generated A-algebra B, and a finitely generated B-module M, there exists a nonzero element f of A such that the localization M_r is a free A_r module.*

We apply this in the case where $A = \mathbb{Z}$, $B = \mathbb{Z}[x_1, \ldots, x_n]^{S_n}$ and $M = \mathrm{Coker}(h)$. It is easy to see that these satisfy the conditions for A, B, and M in the lemma. Thus there exists an integer $r \in \mathbb{Z} \setminus \{0\}$ such that M_r is free over $\mathbb{Z}_r$. As M has no p-torsion for any $p \nmid r$, M has no p-torsion for all but finitely many primes p so the Hilbert series in $\mathbb{F}_p$ is the same as in $\mathbb{C}$. $\square$

We now determine the primes for which it is greater. First, we examine the case when $n = 2$.

Proposition 4. *When $n = 2$, the Hilbert series for Q_m over characteristic p coincides with that of characteristic 0. It is $\frac{1+t^{2m+1}}{(1-t)(1-t^2)}$ over all fields.*

Proof. We claim that the dimension of $Q_{m,d}$ over $\mathbb{C}$ is equal to the dimension of $Q_{m,d}$ over $\mathbb{F}_p$. By Proposition 1, it suffices to show that for each m and d, the dimension of $Q_{m,d}$ over $\mathbb{C}$ is at least the dimension of $Q_{m,d}$ over $\mathbb{F}_p$. Consider a basis $f_1, \ldots, f_k \in \mathbb{F}_p[x, y]$ of $Q_{m,d}$ over $\mathbb{F}_p$. We will show the existence of $F_1, \ldots, F_k \in \mathbb{Z}[x, y]$ of $Q_{m,d}$ over $\mathbb{C}$ such that $F_i \equiv f_i \pmod{p}$ for all i. This means that $F_1, \ldots, F_k$ are linearly independent, as otherwise there exist relatively prime integers $n_1, \ldots, n_k$ with $n_1 F_1 + \cdots + n_k F_k = 0$. Taking the equation modulo p yields $n_1 f_1 + \cdots + n_k f_k \equiv 0 \pmod{p}$, a contradiction of $f_1, \ldots, f_k$ forming a basis of $Q_{m,d}$ as not all of $n_1, \ldots, n_k$ are divisible by p.

To do so, let $f = f_i$ and suppose that $f(x, y) - f(y, x) = (x - y)^{2m+1} g(x, y)$ for some $g(x, y) \in \mathbb{F}_p[x, y]$. Consider $G(x, y) \in \mathbb{Z}[x, y]$ such that $G \equiv g \pmod{p}$. Let $f(x, y) = \sum_{i=0}^{d} a_i x^i y^{d-i}$ and suppose that $G(x, y)(x-y)^{2m+1} = \sum_{i=1}^{d} B_i x^i y^{d-i}$ with $a_i \in \mathbb{F}_p$ and $B_i \in \mathbb{Z}$. We have that $a_i - a_{d-i} \equiv B_i \pmod{p}$. Note that $g(x, y)$, and thus $G(x, y)$ is symmetric, so $G(x, y)(x - y)^{2m+1}$ is anti-symmetric, which implies that $B_i + B_{d-i} = 0$ for all i. Now, define $F(x, y) = \sum_{i=1}^{d} A_i x^i y^{d-i}$, where $A_i \equiv a_i \pmod{p}$ for $i \leq \frac{d}{2}$ and $A_i = A_{d-i} + B_i$ for $i > \frac{d}{2}$. Note that for $i > \frac{d}{2}$, we have that $A_i \equiv A_{d-i} + B_i \equiv a_i \pmod{p}$, so this F satisfies $F \equiv f \pmod{p}$. It remains to check the quasiinvariance condition. However, note that $F(x,y) - F(y,x) = \sum_{i=1}^{d}(A_i - A_{d-i})x^i y^{d-i} = \sum_{i=1}^{d} B_i x^i y^{d-i} = G(x,y)(x-y)^{2m+1}$ by definition, so we are done.

Hence, the dimension, and thus the series, is invariant of the field. It is known from [9] that the series is $\frac{1+t^{2m+1}}{(1-t)(1-t^2)}$, as desired. $\square$

When $n > 2$, the series differs greatly for many primes. In this case, we have found certain sufficient conditions for when the Hilbert series in characteristic p is greater.

Theorem 5. *Let $m \geq 0$ and $n \geq 3$ be integers. Let p be a prime such that exist integers $a > 0$ and $k \geq 0$ with*

$$\frac{mn(n-2) + \binom{n}{2}}{n(n-2)k + \binom{n}{2} - 1} \leq p^a \leq \frac{mn}{nk+1}.$$

Then the Hilbert series of Q_m with n variables in $\mathbb{F}_p$ is greater than the Hilbert series in $\mathbb{C}$.

Proof. The following formula, due to [6], gives the Hilbert polynomial for Q_m in $\mathbb{C}$:

$$n! t^{m\binom{n}{2}} \sum_{\text{Young diagrams}} \prod_{i=1}^{n} t^{m(\ell_i - a_i) + \ell_i} \frac{1-t^i}{h_i(1-t^{h_i})}.$$

Here, the sum is over Young diagrams with n boxes, a_i denotes the number of boxes to the right of the ith box, ℓ_i denotes the number of boxes below the ith box, and $h_i = a_i + \ell_i - 1$. It is not hard to see that the formula gives that the Hilbert polynomial is of the form $1 + (n-1)t^{mn+1} + \ldots$, where the exponents are sorted in ascending order. This implies that all polynomials in Q_m with degree at most mn are symmetric, and Q_m as a module over symmetric polynomials has a generator of degree $mn+1$. Denote this generator by P_m.

To show that the Hilbert series is greater in $\mathbb{F}_p$, we consider the following non-symmetric polynomial:

$$F = P_k^{p^a} \prod_{1 \leq i < j \leq n} (x_i - x_j)^{2b}.$$

Here, set $b = \frac{2m+1-p^a(2k+1)}{2}$, so $\deg F = p^a(nk+1) + 2b\binom{n}{2} = p^a(nk+1) + \binom{n}{2}(2m+1-p^a(2k+1)) = \binom{n}{2}(2m+1) + p^a(1 - \binom{n}{2} - n(n-2)k) \leq \binom{n}{2}(2m+1) - (mn(n-2) + \binom{n}{2}) = mn < \deg P_m$.

Hence, if we show that $F \in Q_m$, then as $\deg F < \deg P_m$ we obtain a greater Hilbert series in $\mathbb{F}_p$, in particular in the coefficient of $t^{\deg F}$. To do that, note that b is an integer when p is odd and a half-integer when $p = 2$. Either way, $\prod(x_i - x_j)^{2b}$ is a symmetric polynomial in $\mathbb{F}_p$, so we have that $(1-s_{i,j})(P_k^{p^a} \prod(x_i - x_j)^{2b}) = ((1-s_{i,j})P_k)^{p^a} \prod(x_i - x_j)^{2b}$ by the fact that $(u+v)^{p^a} = u^{p^a} + v^{p^a}$ in $\mathbb{F}_p$. Hence, as $(x_i - x_j)^{2k+1}$ divides P_k by assumption, we have that $(x_i - x_j)^{p^a(2k+1)+2b} = (x_i - x_j)^{2m+1}$ divides $(1-s_{i,j})F$. Hence, F is in Q_m, so this produces a generator of Q_m of lower degree in $\mathbb{F}_p$ and thus a larger Hilbert series in $\mathbb{F}_p$, as desired. $\square$

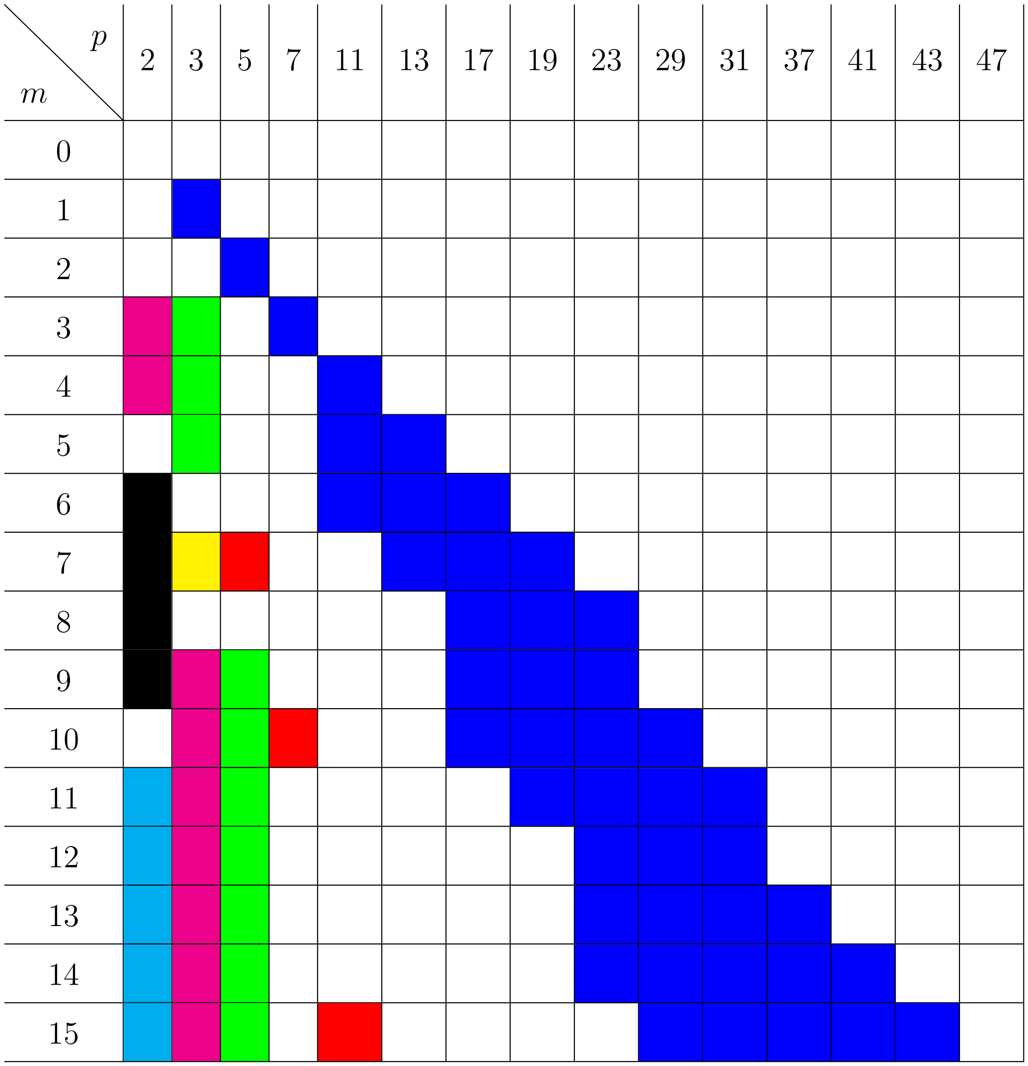
Remark. Let $k = 0$ in the inequality in Theorem 5. Then, we see that all primes p with a power between roughly $2m$ and mn satisfy the inequality. These primes p satisfy the property that Q_m in $\mathbb{F}_p$ has a larger Hilbert series than in $\mathbb{C}$.

Conjecture 1. *The sufficient condition we have given in Theorem 5 is also necessary. If the Hilbert series of Q_m in $\mathbb{F}_p$ is greater than the Hilbert series in $\mathbb{C}$, then there exist integers $a > 0$ and $k \geq 0$ with*

$$\frac{mn(n-2) + \binom{n}{2}}{n(n-2)k + \binom{n}{2} - 1} \leq p^a \leq \frac{mn}{nk+1}.$$

In particular, if $p > mn$, then the Hilbert series in $\mathbb{F}_p$ is the same as in $\mathbb{C}$.

This is supported by computer calculations, especially in the case of $n = 3$. They suggest that the Hilbert series takes a form depending on the smallest non-symmetric element of Q_m which is described by the proof of Theorem 5 and hence follows the inequality. The following table summarizes the results of our computer program verification for $m \leq 15, p \leq 50$. The colored boxes represent the instances in which the series was greater in $\mathbb{F}_p$ than in $\mathbb{C}$, and these match up with what the conjecture predicts exactly. The different colors represent the values of a and k that make the condition hold true. Blue corresponds to $a = 1, k = 0$, red corresponds to $a = 1, k = 1$, yellow corresponds to $a = 1, k = 2$, green corresponds to $a = 2, k = 0$, magenta corresponds to $a = 3, k = 0$, black corresponds to $a = 4, k = 0$, and cyan corresponds to $a = 5, k = 0$.



Through our programs, we have found that when $n = 3$, the Hilbert series takes the form

$$\frac{1 + 2t^d + 2t^{6m+3-d} + t^{6m+3}}{(1-t)(1-t^2)(1-t^3)}.$$

for small p, where d is the degree of the smallest non-symmetric generator of Q_m in $\mathbb{F}_p$. In particular, this smallest non-symmetric polynomial in Q_m is of the form $P_k^{p^a} \prod (x_i - x_j)^{2b}$ where the P_k are as described in Theorem 5. Furthermore, we conjecture that Q_m is a free module over the ring of symmetric polynomials for $n = 3$ in any field.

In [6], the authors prove some properties of Hilbert series and polynomials in $\mathbb{C}$, specifically their maximal term and symmetry. We believe that similar results still hold in $\mathbb{F}_p$, and this is supported by our computer calculations for when $n = 3$.

Conjecture 2. *The largest degree term in the Hilbert polynomial is always $t^{\binom{n}{2}(2m+1)}$. Furthermore, when p is an odd prime Q_m is a free module over the ring of the symmetric polynomials of rank $n!$, and the the Hilbert polynomial is symmetric.*

Remark. The condition that p is odd appears to be necessary. Indeed, a computer calculation shows that when $n = 4$, $m = 1$ and $p = 2$, the Hilbert series is

$$1 + t + 2t^2 + 3t^3 + 8t^4 + 9t^5 + 15t^6 + 23t^7 + 38t^8 + 50t^9 + 71t^{10} + \cdots =$$
$$\frac{1 + 3t^4 + 3t^7 + 5t^8 + 3t^9 - t^{10} + \cdots}{(1-t)(1-t^2)(1-t^3)(1-t^4)},$$

and the negative coefficient implies that the module cannot be free. In particular, computing the polynomial up to t^{18} also demonstrates that it is not symmetric.

3 Twisted quasiinvariants

3.1 A generalization of quasiinvariants

In [9], Braverman, Etingof and Finkelberg introduced quasiinvariants twisted by a monomial $x_1^{a_1} \dots x_n^{a_n}$ where $a_1 \dots, a_n \in \mathbb{C}$. We further generalize this by allowing the twist to be a product of arbitrary smooth functions $f_1(x_1)f_2(x_2)\dots f_n(x_n)$.

Definition. Let m be a nonnegative integer. Fix one-variable complex smooth functions $f_1, f_2, \dots, f_n$ and define $Q_m(f_1, \dots, f_n)$ to be the space of polynomials $F \in \mathbb{C}[x_1, \dots, x_n]$ for which

$$\frac{(1 - s_{i,j})\,(f_1(x_1)\dots f_n(x_n)F)}{(x_i - x_j)^{2m+1}}$$

is smooth for all $1 \leq i < j \leq n$.

Remark. In [9], $f_i(x) = x^{a_i}$ for $a_i \in \mathbb{C}$ for all i, and $Q_m(f_1, \ldots, f_n)$ was denoted as $Q_m(a_1, \ldots, a_n)$. In cases of unambiguous use, we will shorten this to Q_m.

Similar to in [9], we believe that $Q_m(f_1, \ldots, f_n)$ is in general a free module.

Conjecture 3. *For generic $f_1, \ldots, f_n$, in particular when $\frac{f_i}{f_j}$ is not a monomial in $x_1, \ldots, x_n$, $Q_m(f_1, \ldots, f_n)$ is a free module over the ring of symmetric polynomials in $x_1, \ldots, x_n$.*

The definitions of the Hilbert series and polynomial remain the same.

3.2 Rationality of the logarithmic derivative

For the rest of this section, we will let g denote an arbitrary smooth function. Note that for any $f_1, \ldots, f_n$ and i, j, any $F \in (x_i - x_j)^{2m}\mathbb{C}[x_1, ..., x_n]$ has the property that $\frac{(1-s_{i,j})(f_1(x_1)...f_n(x_n)F)}{(x_i-x_j)^{2m+1}}$ is smooth. Firstly, we want to find out which choices of the f_i, f_j yield polynomials in Q_m that are not in $(x_i - x_j)^{2m}\mathbb{C}[x_1, \ldots,_n]$.

Proposition 6. *If F is divisible by $x_i - x_j$ and $\frac{(1-s_{i,j})(f_1(x_1)...f_n(x_n)F)}{(x_i-x_j)^{2(k+1)+1}}$ is smooth then $(x_i - x_j)^2 \mid F$ and $\frac{(1-s_{i,j})\left(f_1(x_1)...f_n(x_n)\frac{F}{(x_i-x_j)^2}\right)}{(x_i-x_j)^{2k+1}}$ is smooth.*

Proof. Let $F(x_i, x_j) = (x_i - x_j)G(x_i, x_j)$ for some polynomial G. Substituting, the condition becomes

$$f_i(x_i)f_j(x_j)G(x_i, x_j) + f_j(x_i)f_i(x_j) = (x_i - x_j)^{2k+2}g(x_i, x_j)$$

and setting $x_i = x_j = x$ gives

$$f_i(x)f_j(x)G(x, x) = 0$$

so $G(x, x) = 0$. This implies that $(x_i - x_j) \mid G$, so $(x_i - x_j)^2 \mid F$, as desired. The second part of the proposition follows by definition as $\frac{F}{(x_i-x_j)^2}$ is smooth. $\square$

Theorem 7. *Let $h_{i,j} = \frac{f_i}{f_j}$. If dlog$(h_{i,j})$ is not a rational function, then $Q_m \subset (x_i - x_j)^{2m+1}\mathbb{C}[x_1, \ldots, x_n]$.*

Proof. The proposition is trivial for $m = 0$. For $m > 0$, note that $F \in Q_m$ if and only if

$$f_i(x_i)f_j(x_j)F(x_i, x_j) - f_j(x_i)f_i(x_j)F(x_j, x_i) = (x_i - x_j)^{2m+1}g(x_i, x_j)$$

$$h_{i,j}(x_i)F(x_i, x_j) - h_{i,j}(x_j)F(x_j, x_i) = (x_i - x_j)^{2m+1}g(x_i, x_j).$$

Here, we treat the rest of the functions and variables as constants. Differentiating with respect to x_i, we have that

$$h_{i,j}(x_i)(\mathrm{dlog}(h_{i,j})(x_i)F(x_i, x_j) + F_1(x_i, x_j)) - h_{i,j}(x_j)F_2(x_j, x_i) = (x_i - x_j)^{2m}g(x_i, x_j)$$

where for a function $F(x, y)$ we define $F_1 = \frac{\partial F}{\partial x}$ and $F_2 = \frac{\partial F}{\partial y}$. Setting $x_i = x_j = x$, we have that

$$h_{i,j}(x)(\mathrm{dlog}(h_{i,j})(x)F(x, x) + F_1(x, x)) - h_{i,j}(x)F_2(x, x) = 0$$

$$\mathrm{dlog}(h_{i,j})(x)F(x, x) = F_2(x, x) - F_1(x, x)$$

which means that $F(x, x) = 0$. Otherwise, we would have

$$\mathrm{dlog}(h_{i,j})(x) = \frac{F_2(x, x) - F_1(x, x)}{F(x, x)}$$

which is a contradiction as the right hand side is a rational function. Hence, $(x_i - x_j) \mid F$. Now, by Proposition 6 we have $(x_i - x_j)^2 \mid F$ and $\frac{F}{(x_i - x_j)^2} \in Q_{m-1}$, which implies the desired result by a straightforward induction. □

3.3 Hilbert series in two variables

Let $n = 2$, $x = x_1$, and $y = x_2$. Note that scaling f_1 and f_2 by some smooth function does not affect Q_m. Hence, we may multiply them both by $\frac{1}{f_2}$ and let $f = \frac{f_1}{f_2}$. For convenience, we use $Q_m(f)$ instead of $Q_m(f,1)$. Throughout this section, we will let $\operatorname{dlog}(f(x)) = \frac{p(x)}{q(x)}$ for relatively prime $p, q \in \mathbb{C}[x]$, as we have from Section 3.2 that either $Q_m = (x-y)^{2m}\mathbb{C}[x,y]$ or $\operatorname{dlog}(f(x))$ is a rational function. For convenience, we will also set $F_x = \frac{\partial F}{\partial x}$, $F_y = \frac{\partial F}{\partial y}$, and $F_{xy} = \frac{\partial^2 F}{\partial x \partial y}$.

Lemma 8. *If $F(x,y) \in Q_m(f)$, then $p(x)F_y(x,y)+q(x)F_{xy}(x,y) \in Q_{m-1}\left(\frac{f}{q}\right)$ and $-p(y)F_x(x,y)+q(y)F_{xy}(x,y) \in Q_{m-1}(fq)$.*

Proof. We begin with our quasiinvariant condition, which in our case of $n=2$ is

$$f(x)F(x,y) - f(y)F(y,x) = (x-y)^{2m+1}g(x,y).$$

Differentiating by x and then y, we obtain

$$f'(x)F_y(x,y) + f(x)F_{xy}(x,y) - f'(y)F_y(y,x) + f(y)F_{xy}(y,x) = (x-y)^{2m+1}g(x,y).$$

By the definition of p and q, this is equivalent to

$$\frac{f(x)}{q(x)}\left(p(x)F_y(x,y) + q(x)F_{xy}(x,y)\right) - \frac{f(y)}{q(y)}\left(p(y)F_y(y,x) + q(y)F_{xy}(y,x)\right) = (x-y)^{2m-1}g(x,y)$$

which is exactly the quasiinvariant condition that is desired.

Dividing our quasiinvariant condition by $f(x)f(y)$ gives

$$\frac{1}{f(x)}F(y,x) - \frac{1}{f(y)}F(x,y) = (x-y)^{2m+1}g(x,y).$$

Thus, $-p(x)F_x(y,x) + q(x)F_{xy}(y,x) \in Q_{m-1}\left(\frac{1}{fq}\right)$ by the above. Expanding the quasiinvariant condition and multiplying by $f(x)f(y)q(x)q(y)$, we obtain the equivalent statement $-p(y)F_x(x,y) + q(y)F_{xy}(x,y) \in Q_{m-1}(fq)$, as desired. $\square$

Now, we specialize to the case in which $f(x) = \prod_{i=1}^{k}(x-a_i)^{b_i}$ for arbitary complex numbers a_i, b_i. Note that in this case $\operatorname{dlog}(f) = \sum_{i=1}^{k} \frac{b_i}{x-a_i}$.

Definition. For a nonnegative integer m and a complex number z, denote

$$d_m(z) = \begin{cases} \min(m, |z|) & \text{if } z \in \mathbb{Z} \\ m & \text{otherwise} \end{cases}$$

and

$$d_m(f) = \sum_{i=1}^{k} d_m(b_i)$$

where $f(x) = \prod_{i=1}^{k}(x-a_i)^{b_i}$ for $a_1, \ldots, a_k, b_1, \ldots, b_k \in \mathbb{C}$ with $a_1, \ldots, a_k$ pairwise distinct.

Lemma 9. *We have that*

$$\prod_{i=1}^{k}(x-a_i)^{d_m(b_i)} \mid F(x,x)$$

for any $F \in Q_m(f)$.

Proof. We proceed using induction, with the base case of $m = 0$ clearly true. It suffices to prove this divisibility for each $(x-a_i)^{d_m(b_i)}$. By the inductive hypothesis and Lemma 8, we have that $(x-a_i)^{d_{m-1}(b_i-1)} \mid p(x)F_y(x,x)+q(x)F_{xy}(x,x)$ and $(x-a_i)^{d_{m-1}(b_i+1)} \mid -p(x)F_x(x,x) + q(x)F_{xy}(x,x)$. It is easy to see that $d_m(b_i) - 1 \leq d_{m-1}(b_i), d_{m-1}(b_i - 1), d_{m-1}(b_i + 1)$.

Thus, $(x-a_i)^{d_m(b_i)-1} \mid (x-a_i)^{d_{m-1}(b_i)} \mid F(x,x)$ as F is also in $Q_{m-1}(f)$. From the other two divisibilities we also obtain $(x-a_i)^{d_m(b_i)-1} \mid p(x)F_y(x,x)+q(x)F_{xy}(x,x), -p(x)F_x(x,x)+q(x)F_{xy}(x,x)$, so $(x-a_i)^{d_m(b_i)-1} \mid p(x)(F_x(x,x)+F_y(x,x)) = p(x)\frac{\mathrm{d}F(x,x)}{\mathrm{d}x}$. As p and q are relatively prime, we must have $(x-a_i)^{d_m(b_i)-1} \mid \frac{\mathrm{d}F(x,x)}{\mathrm{d}x}$ which together with $(x-a_i)^{d_m(b_i)-1} \mid F(x,x)$ implies $(x-a_i)^{d_m(b_i)} \mid F(x,x)$, as desired. □

In fact, this lemma is sharp in the sense that there exists F such that the divisibility becomes equality. To prove that, we utilize the following lemma:

Lemma 10. *If $F \in Q_m(f)$ and $G \in Q_m(g)$, then $FG \in Q_m(fg)$.*

Proof. We have that

$$\frac{f(x)g(x)F(x,y)G(x,y)-f(x)g(y)F(y,x)G(y,x)}{(x-y)^{2m+1}}$$
$$=g(x)G(x,y)\cdot\frac{f(x)F(x,y)-f(y)F(y,x)}{(x-y)^{2m+1}}$$
$$+f(y)F(y,x)\cdot\frac{g(x)G(x,y)-g(y)G(y,x)}{(x-y)^{2m+1}}$$

is smooth. □

Lemma 11. *There exists $P_m \in Q_m(f)$ with*

$$P_m(x,x)=\prod_{i=1}^{k}(x-a_i)^{d_m(b_i)}.$$

Proof. Note that by Lemma 10 it suffices to show this for when $k=1$ as for $k>1$ we can take the product of all such P_m in $Q_m\left((x-a_1)^{d_m(b_1)}\right),\ldots,Q_m\left((x-a_k)^{d_m(b_k)}\right)$. Shifting, we may also assume that $a_1=0$. Now, note that if $z=b_1$ is an integer less than m, we can simply take $P_m=y^z$. Otherwise, we claim that we can take

$$P_m(x,y)=\frac{\sum_{i=0}^{m}\binom{m-z}{i}\binom{m+z}{m-i}x^iy^{m-i}}{\binom{2m}{m}}.$$

Indeed, note that we have that $P_m(x,x)=x^m$ by Vandermonde's identity, so it suffices to show that P_m, or equivalently the numerator of P_m, is in Q_m. We proceed using induction, with the base case of $m=0$ obvious. For the inductive step, note that we wish to show that

$$\sum_{i=0}^{m}\binom{m-z}{i}\binom{m+z}{m-i}(x^{i+z}y^{m-i}-x^{m-i}y^{i+z})$$

vanishes at $x=y$ to order $2m+1$. Differentiating with respect to x and setting $x=y$, we would like to show that

$$\sum_{i=0}^{m}\binom{m-z}{i}\binom{m+z}{m-i}(2i-m+z)=0.$$

As we have

$$\sum_{i=0}^{m}\binom{m-z}{i}\binom{m+z}{m-i}i=\sum_{i=1}^{m}(m-z)\binom{m-z-1}{i-1}\binom{m+z}{m-i}=(m-z)\binom{2m-1}{m-1}$$

and

$$\sum_{i=0}^{m}\binom{m-z}{i}\binom{m+z}{m-i}(i-m)=-\sum_{i=0}^{m-1}(m+z)\binom{m-z}{i}\binom{m+z-1}{m-i-1}=-(m+z)\binom{2m-1}{m-1}$$

by Vandermonde's identity, the expression reduces to

$$-2z\binom{2m-1}{m-1}+\sum i=0^m\binom{m-z}{i}\binom{m+z}{m-i}=-2z\binom{2m-1}{m-1}+z\binom{2m}{m}=0$$

as desired.

Differentiating by both x and y, it suffices to show that

$$\sum_{i=0}^{m}\binom{m-z}{i}\binom{m+z}{m-i}(i+z)(m-i)(x^{i+z-1}y^{m-i-1}-x^{m-i-1}y^{i+z-1})$$

vanishes at $x=y$ to order $2m-1$. But note that this expression is

$$\sum_{i=0}^{m-1}\binom{m-z}{i}\binom{m+z-1}{m-1-i}(i+z)(m+z)(x^{i+z-1}y^{m-i-1}-x^{m-i-1}y^{i+z-1})$$

$$=(m+z)\sum_{i=0}^{m-1}\binom{m-z}{i}\binom{m+z-2}{m-1-i}(m+z-1)(x^{i+z-1}y^{m-i-1}-x^{m-i-1}y^{i+z-1})$$

$$=(m+z)(m+z-1)\sum_{i=0}^{m-1}\binom{m-z}{i}\binom{m+z-2}{m-1-i}(x^{i+z-1}y^{m-i-1}-x^{m-i-1}y^{i+z-1})$$

which vanishes at $x=y$ to order $2m-1$ by the inductive hypothesis on $Q_{m-1}(x^{z-1})$, as desired. □

Lemma 12. *Let R denote the ring of symmetric polynomials in x and y. Then, for all $m>0$ we have that*

$$Q_m=RP_m+(x-y)^2Q_{m-1}.$$

Proof. Let $F(x, y)$ be an element of Q_m. By Lemma 9, $P_m(x, x) \mid F(x, x)$, so there exists a polynomial $g \in \mathbb{C}[x]$ with $P_m(x, x)g(x) = F(x, x)$. Now, consider the polynomial

$$F'(x, y) = F(x, y) - P_m(x, y)g\left(\frac{x+y}{2}\right)$$

which is in Q_m as $F, P_m \in Q_m$ and $g\left(\frac{x+y}{2}\right) \in R$. But now note that $F'(x, x) = F(x, x) - P_m(x, x)g(x) = 0$, so by Proposition 6, $F' \in (x-y)^2 Q_{m-1}$, which immediately implies the desired. □

Corollary. *We have that*

$$Q_m = RP_m + R(x-y)^2 P_{m-1} + \cdots + R(x-y)^{2m-2} P_1 + (x-y)^{2m} Q_0$$

for all m.

Now, we are finally ready to prove our main result of this section.

Theorem 13. *The Hilbert series for $Q_m(f)$ is*

$$\frac{t^{2m} + t^{2m+1} + \sum_{i=1}^{m} t^{2(m-i)+d_i(f)} - \sum_{i=1}^{m} t^{2(m-i)+d_i(f)+2}}{(1-t)(1-t^2)}.$$

Proof. Note that $Q_0 = \mathbb{C}[x, y]$, which is generated by $P_0 = 1$ and $x - y$. By the corollary of Lemma 12, Q_m is generated by

$$P_m,\ (x-y)^2 P_{m-1},\ \ldots,\ (x-y)^{2m-2} P_1,\ (x-y)^{2m},\ (x-y)^{2m+1}.$$

Let $g_{m,i} = (x-y)^{2(m-i)} P_i$ and $g_m = (x-y)^{2m+1}$. We claim that Q_m is generated by $g_m,\ g_{m,0},\ \ldots,\ g_{m,m}$ and m independent relations of the form

$$(x-y)^2 g_{m,m} = r_{m,m-1} g_{m,m-1} + \cdots + r_{m,0} g_{m,0} + r_m g_m$$

$$(x-y)^2 g_{m,m-1} = r_{m-1,m-2} g_{m,m-2} + \cdots + r_{m-1,0} g_{m,0} + r_{m-1} g_m$$

$$\dots$$

$$(x-y)^2 g_{m,1} = r_{1,0} g_{m,0} + r_1 g_m$$

for some $r_i, r_{i,j} \in R$. We proceed using induction, noting that Q_0 is generated by 1 and $x-y$ with no relations. For the inductive step, first note that as $g_{m,m} = P_m \in Q_m \subset Q_{m-1}$, there exist $p,\ p_0,\ \dots,\ p_{m-1} \in R$ with $g_{m,m} = p g_{m-1} + p_0 g_{m-1,0} + \dots + p_{m-1} g_{m-1,m-1}$. This yields a relation in the form of the first relation above by setting $r_m = p,\ r_{m,0} = p_0,\ \dots,\ r_{m,m-1} = p_{m-1}$. This equation is true as $g_m = (x-y)^2 g_{m-1},\ g_{m,0} = (x-y)^2 g_{m-1,0},\ \dots,\ g_{m,m-1} = (x-y)^2 g_{m-1,m-1}$ by definition. Now, suppose that $q,\ q_0,\ \dots,\ q_m \in R$ such that $q g_m + q_0 g_{m,0} + \dots + q_m g_{m,m} = 0$. Then, as $(x-y)^2 \mid g_m, g_{m,0}, \dots, g_{m,m-1}$ and $(x-y)^2 \nmid g_{m,m}$, we must have that $(x-y)^2 \mid q_m$. Let $q_m = (x-y)^2 q'_m$. Then, subtracting q'_m times the first relation from $q g_m + q_0 g_{m,0} + \dots + q_m g_{m,m} = 0$, we obtain a relation of the form $q' g_m + q'_0 g_{m,0} + \dots + q'_{m-1} g_{m,m-1} = 0$ with $q',\ q'_0,\ \dots,\ q'_{m-1} \in R$. Note that this relation is uniquely determined by the first generating relation we have so the first generating relation is independent of the rest of the relations. Furthermore, this relation is $(x-y)^2$ times a relation among the generators of Q_{m-1}. By the inductive hypothesis, such a relation is generated by $(x-y)^2$ times the $m-1$ independent generating relations of Q_1, which are by definition the last $m-1$ generating relations on our list. Hence, Q_m is generated by those $m+2$ elements and m independent relations among those elements, as desired.

For the Hilbert polynomial, note that the generators have degrees

$$d_m(f),\ 2 + d_{m-1}(f),\ \dots,\ 2m - 2 + d_1(f),\ 2m,\ 2m+1$$

and that the independent relations have degrees

$$2 + d_m(f),\ 4 + d_{m-1}(f),\ \dots,\ 2m + d_1(f),$$

which gives the Hilbert polynomial and series exactly as described in the theorem. $\square$

4 Future prospects

We hope to make progress in proving Conjectures 1, 2, and 3. As with our current results, we expect to make extensive use of computer programs to discover key properties of quasi-invariant polynomials and their Hilbert series. We expect that resolving Conjecture 1 in the case of $n = 3$ will require studying the modular representation theory of S_3. In general, the modular representation theory of S_n will also apply to Conjecture 1 and 2.

We will approach Conjecture 3 by adapting the approach of the authors of [9], namely by constructing a Cherednik-like algebra related to $f_1, \ldots, f_n$ and the quasiinvariant polynomials. In doing so, we also hope to compute the Hilbert series. Currently, we are trying to define the Cherednik-like algebra as certain polynomial differential operators on the space of n variable polynomials that fix related spaces.

Finally, we hope to study q-deformations of the spaces of twisted quasiinvariant polynomials. In [9], Braverman, Etingof, and Finkelberg study q-deformations of their special case and show that when Q_m is free, its q-deformation is a flat deformation. They conjecture that it is a flat deformation in general even when Q_m is not a free module. Here, $Q_{m,q}(f_1, \ldots, f_n)$ is defined as the set of polynomials F for which

$$\frac{(1 - s_{i,j})(f_1(x_1) \ldots f_n(x_n)F)}{\prod_{k=-m}^{m}(x_i - q^k x_j)}$$

is a smooth function for all $1 \leq i < j \leq n$. We hope to make progress in resolving this in the case that Braverman, Etingof, and Finkelberg consider, as well as the general case we have presented. We believe that q-analogues of some of our results hold. For example, the q-analogue of Theorem 5 would be that $Q_{m,q} \subset \prod_{k=-m}^{m}(x_i - q^k x_j)$ if $\frac{h_{i,j}(qx)}{h_{i,j}(x)}$ is not rational.

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