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THE L_p MINKOWSKI PROBLEM FOR THE ELECTROSTATIC $\mathfrak{p}$-CAPACITY

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Abstract

Existence and uniqueness of the solution to the L_p Minkowski problem for the electrostatic $\mathfrak{p}$-capacity are proved when $p > 1$ and $1 < \mathfrak{p} < n$. These results are nonlinear extensions of the very recent solution to the L_p Minkowski problem for $\mathfrak{p}$-capacity when $p = 1$ and $1 < \mathfrak{p} < n$ by Colesanti et al. and Akman et al., and the classical solution to the Minkowski problem for electrostatic capacity when $p = 1$ and $\mathfrak{p} = 2$ by Jerison.

1. Introduction

The setting for this article is the Euclidean n-space, $\mathbb{R}^n$. A *convex body* in $\mathbb{R}^n$ is a compact convex set with nonempty interior. The Brunn-Minkowski theory of convex bodies, which was developed by Minkowski, Aleksandrov, Fenchel, and many others, centers around the study of geometric functionals of convex bodies and the differentials of these functionals. Usually, the differentials of these functionals produce *new* geometric measures. The theory depends heavily on analytic tools such as the cosine transform on the unit sphere $\mathbb{S}^{n-1}$ and Monge-Ampère type equations.

A Minkowski problem is a characterization problem for a geometric measure generated by convex bodies: It asks for necessary and sufficient conditions in order that a given measure arises as the measure generated by a convex body. The solution of a Minkowski problem, in general, amounts to solving a degenerate fully nonlinear partial differential equation. The study of Minkowski problems has a long history and strong influence on both the Brunn-Minkowski theory and fully nonlinear partial differential equations, see [**67**].

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The classical Brunn-Minkowski theory begins with the variation of volume functional.

1.1. Volume, surface area measure and the classical Minkowski problem. Without doubt, the most fundamental geometric functional in the Brunn-Minkowski theory is the volume functional. Via the variation of volume functional, it produces the most important geometric measure: surface area measure.

Specifically, if K and L are convex bodies in $\mathbb{R}^n$, then there exists a finite Borel measure $S(K,\cdot)$ on the unit sphere $\mathbb{S}^{n-1}$ known as the *surface area measure* of K, so that

$$\left.\frac{dV(K+tL)}{dt}\right|_{t=0^+} = \int_{\mathbb{S}^{n-1}} h_L(\xi)dS(K,\xi), \tag{1.1}$$

where V is the n-dimensional volume (i.e., Lebesgue measure in $\mathbb{R}^n$); the convex body $K+tL=\{x+ty : x\in K, y\in L\}$ is the *Minkowski sum* of K and tL; $h_L : \mathbb{S}^{n-1}\to\mathbb{R}$ is the support function of L, defined by $h_L(\xi)=\max\{\xi\cdot x : x\in L\}$, with $\xi\cdot x$ denoting the standard inner product of ξ and x in $\mathbb{R}^n$.

Formula (1.1), called the Aleksandrov variational formula, suggests that the surface area measure can be viewed as the differential of volume functional. The surface area measure $S(K,\cdot)$ can be defined directly, for each Borel set $\omega\subset\mathbb{S}^{n-1}$, by

$$S(K,\omega)=\mathcal{H}^{n-1}(\mathrm{g}_K^{-1}(\omega)), \tag{1.2}$$

where $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure. Here, $\mathrm{g}_K : \partial' K\to\mathbb{S}^{n-1}$ is the Gauss map defined on $\partial' K$ of those points of ∂K that have a unique outer normal and is hence defined $\mathcal{H}^{n-1}$-a.e. on ∂K. The integral in (1.1), divided by the ambient dimension n, is called the *first mixed volume* $V_1(K,L)$ of (K,L), i.e.,

$$V_1(K,L)=\frac{1}{n}\int_{\mathbb{S}^{n-1}} h_L(\xi)dS(K,\xi),$$

which is a generalization of the well-known volume formula

$$V(K)=\frac{1}{n}\int_{\mathbb{S}^{n-1}} h_K(\xi)dS(K,\xi). \tag{1.3}$$

The classical Minkowski problem, which characterizes the surface area measure, is one of the cornerstones of the Brunn-Minkowski theory of convex bodies. It reads: *Given a finite Borel measure μ on $\mathbb{S}^{n-1}$, what are the necessary and sufficient conditions on μ so that μ is the surface area measure $S(K,\cdot)$ of a convex body K in $\mathbb{R}^n$?* More than a century ago, Minkowski himself [**62**] solved this problem for the case when the given measure is either discrete or has a continuous density.

Aleksandrov [**2**], [**3**] and Fenchel-Jessen [**22**] independently solved the problem in 1938 for arbitrary measures: If μ is not concentrated on any closed hemisphere of $\mathbb{S}^{n-1}$, then μ is the surface area measure of a convex body if and only if its centroid is at the origin o, i.e., $\int_{\mathbb{S}^{n-1}} \xi d\mu(\xi) = o$.

Since for strictly convex bodies with smooth boundaries, the density of the surface area measure with respect to the spherical Lebesgue measure is just the reciprocal of the Gauss curvature, the classical Minkowski problem in differential geometry is to characterize the Gauss curvature of closed convex hypersurfaces. Analytically, the classical Minkowski problem is equivalent to solving a degenerate Monge-Ampère equation. Establishing the regularity of the solution to the Minkowski problem is difficult and has led to a long series of highly influential works, see, e.g., Lewy [**43**], Nirenberg [**64**], Cheng and Yau [**15**], Pogorelov [**65**], Caffarelli [**9**, **10**].

1.2. L_p surface area measures and the L_p Minkowski problem for volume. The L_p Brunn-Minkowski theory is an extension of the classical Brunn-Minkowski theory; see [**23**, **45**, **46**, **48**–**50**, **52**, **55**–**58**, **73**]. In the L_p theory, the L_p surface area measure introduced by Lutwak [**48**] is one of the most fundamental notions.

For an index $p \in \mathbb{R}$ and a convex body K in $\mathbb{R}^n$ with the origin in its interior, the L_p *surface area measure* $S_p(K, \cdot)$ of K is a Borel measure on $\mathbb{S}^{n-1}$ defined, for each Borel set $\omega \subset \mathbb{S}^{n-1}$, by

$$S_p(K, \omega) = \int\limits_{x \in \mathrm{g}_K^{-1}(\omega)} (x \cdot \mathrm{g}_K(x))^{1-p} d\mathcal{H}^{n-1}(x).$$

The measure $S_p(K, \cdot)$ can also be defined directly, for each Borel set $\omega \subset \mathbb{S}^{n-1}$, by

$$S_p(K, \omega) = \int\limits_{\omega} h_K(\xi)^{1-p} dS(K, \xi). \tag{1.4}$$

Note that $S_1(K, \cdot)$ is just the surface area measure $S(K, \cdot)$. The measure $n^{-1}S_0(K, \cdot)$ is the cone-volume measure of K, which is the only $\mathrm{SL}(n)$ invariant measure among all the L_p surface area measures. In recent years, the cone-volume measure has been greatly investigated, e.g., [**5**, **30**, **46**, **47**, **63**, **66**, **69**, **74**]. The measure $S_2(K, \cdot)$ is called the quadratic surface area measure of K, which was studied in [**45**] and [**53**, **54**, **60**]. Applications of the L_p surface area measure to affine isoperimetric inequalities were given in, e.g., [**13**, **51**, **52**, **57**].

In 1962, Firey [**23**] introduced the L_p *sum* of convex bodies. Let $1 \leq p < \infty$. If K and L are compact convex sets containing the origin, their L_p sum $K +_p L$ is the compact convex set with support function

$h_{K+_pL} = \left(h_K^p + h_L^p\right)^{1/p}$. See also, [**23**, **28**, **48**, **61**, **78**]. Clearly, $K +_1 L = K + L$. For $t > 0$, the L_p scalar multiplication $t \cdot_p K$ is the set $t^{1/p}K$.

Using the L_p combination, Lutwak [**48**] established the L_p variational formula for volume

$$\left.\frac{dV(K +_p t \cdot_p L)}{dt}\right|_{t=0^+} = \frac{1}{p} \int_{\mathbb{S}^{n-1}} h_L(\xi)^p dS_p(K, \xi), \tag{1.5}$$

where K is a convex body with the origin in its interior, and L is a compact convex set containing the origin. Clearly, (1.5) reduces to (1.1) when $p = 1$. Formula (1.5) suggests that the L_p surface area measure can be viewed as the differential of volume functional of L_p combinations of convex bodies.

Lutwak [**48**] initiated the following L_p Minkowski problem.

***L_p* Minkowski problem for volume.** *Suppose μ is a finite Borel measure on $\mathbb{S}^{n-1}$ and $p \in \mathbb{R}$. What are the necessary and sufficient conditions on μ so that μ is the L_p surface area measure $S_p(K, \cdot)$ of a convex body K in $\mathbb{R}^n$?*

The L_1 Minkowski problem is just the classical Minkowski problem. The L_0 Minkowski problem, also known as the *logarithmic Minkowski problem*, characterizes the cone-volume measure and is regarded as the most important case. Andrews [**4**] proved Firey's conjecture [**24**] that convex surfaces moving by their Gauss curvature become spherical as they contract to points. A major breakthrough was made by Böröczky, Lutwak, Yang and Zhang [**8**], who established the sufficient and necessary conditions for the existence of a solution to the even logarithmic Minkowski problem. The L_{-n} Minkowski problem is the centro-affine Minkowski problem. See Chou and Wang [**16**], and Zhu [**71**, **73**].

By now, the L_p Minkowski problems for volume have been intensively investigated and achieved great developments. See, e.g., [**14**, **16**, **34**, **39**, **41**, **44**, **48**, **50**, **56**, **68**, **73**]. Their solutions have been applied to establish sharp affine isoperimetric inequalities, such as the affine Moser-Trudinger and the affine Morrey-Sobolev inequalities, the affine L_p Sobolev-Zhang inequality, etc. See, e.g., [**7**, **17**, **35**, **36**, **55**, **59**, **70**], for more details.

1.3. $\mathfrak{p}$-capacitary measures and the Minkowski problem for $\mathfrak{p}$-capacity. It is worth mentioning that the Minkowski problem for electrostatic $\mathfrak{p}$-capacity is doubtless an extremely important variant among Minkowski problems. For $1 < \mathfrak{p} < n$, the *electrostatic $\mathfrak{p}$-capacity* of a compact set K in $\mathbb{R}^n$ is defined by

$$\mathrm{C}_{\mathfrak{p}}(K) = \inf \left\{ \int_{\mathbb{R}^n} |\nabla u|^{\mathfrak{p}} dx : u \in C_c^{\infty}(\mathbb{R}^n) \text{ and } u \geq \chi_K \right\},$$

where $C_c^{\infty}(\mathbb{R}^n)$ denotes the set of functions from $C^{\infty}(\mathbb{R}^n)$ with compact supports, and χ_K is the characteristic function of K. The quantity $\mathrm{C}_2(K)$ is the classical electrostatic (or Newtonian) capacity of K.

For convex bodies K and L, via the variation of capacity functional $\mathrm{C}_2(K)$, there appears the classical Hadamard variational formula

$$\left.\frac{d\mathrm{C}_2(K+tL)}{dt}\right|_{t=0^+} = \int_{\mathbb{S}^{n-1}} h_L(\xi) d\mu_2(K,\xi) \tag{1.6}$$

and its special case, the Poincaré capacity formula

$$\mathrm{C}_2(K) = \frac{1}{n-2} \int_{\mathbb{S}^{n-1}} h_K(\xi) d\mu_2(K,\xi). \tag{1.7}$$

Here, the new measure $\mu_2(K,\cdot)$ is a finite Borel measure on $\mathbb{S}^{n-1}$, called the *electrostatic capacitary measure* of K. Formula (1.6) suggests that the electrostatic capacitary measure can be viewed as the differential of capacity functional.

In his celebrated article [**40**], Jerison pointed out the *resemblance* between the Poincaré capacity formula (1.7) and the volume formula (1.3) and also a resemblance between their variational formulas (1.6) and (1.1). Therefore, he initiated to study the Minkowski problem for electrostatic capacity: *Given a finite Borel measure μ on $\mathbb{S}^{n-1}$, what are the necessary and sufficient conditions on μ so that μ is the electrostatic capacitary measure $\mu_2(K,\cdot)$ of a convex body K in $\mathbb{R}^n$?*

Jerison himself [**40**] solved, in full generality, the above Minkowski problem. He proved that the necessary and sufficient conditions for existence of a solution are unexpectedly identical to the corresponding conditions in the classical Minkowski problem. The uniqueness part was settled by Caffarelli, Jerison and Lieb [**12**]. The regularity part of the proof depends on the ideas of Caffarelli [**11**] for regularity of solutions to Monge-Ampère equation.

Jerison's work inspired much subsequent research on this topic. In the very recent article [**20**], Colesanti, Nyström, Salani, Xiao, Yang and Zhang extended Jerison's work to electrostatic $\mathfrak{p}$-capacity. Let K, L be convex bodies in $\mathbb{R}^n$ and $1 < \mathfrak{p} < n$. Colesanti et al. established the Hadamard variational formula for $\mathfrak{p}$-capacity

$$\left.\frac{d\mathrm{C}_{\mathfrak{p}}(K+tL)}{dt}\right|_{t=0^+} = (\mathfrak{p}-1) \int_{\mathbb{S}^{n-1}} h_L(\xi) d\mu_{\mathfrak{p}}(K,\xi) \tag{1.8}$$

and therefore the Poincaré $\mathfrak{p}$-capacity formula

$$\mathrm{C}_{\mathfrak{p}}(K) = \frac{\mathfrak{p}-1}{n-\mathfrak{p}} \int_{\mathbb{S}^{n-1}} h_K(\xi) d\mu_{\mathfrak{p}}(K,\xi). \tag{1.9}$$

Here, the new measure $\mu_{\mathfrak{p}}(K,\cdot)$ is a finite Borel measure on $\mathbb{S}^{n-1}$, called the *electrostatic* $\mathfrak{p}$*-capacitary measure* of K. Formula (1.8) suggests that $\mu_{\mathfrak{p}}(K,\cdot)$ can be viewed as the differential of $\mathfrak{p}$-capacity functional.

Consequently, the Minkowski problem for $\mathfrak{p}$-capacity was posed [**20**]: *Given a finite Borel measure μ on $\mathbb{S}^{n-1}$, what are the necessary and sufficient conditions on μ so that μ is the $\mathfrak{p}$-capacitary measure $\mu_{\mathfrak{p}}(K,\cdot)$ of a convex body K in $\mathbb{R}^n$?*

Colesanti et al. [**20**] proved the uniqueness of the solution when $1 < \mathfrak{p} < n$, and existence when $1 < \mathfrak{p} < 2$ with an extra technical condition. Very recently, the existence for the case $2 < \mathfrak{p} < n$ was solved by Akman, Gong, Hineman, Lewis and Vogel [**1**].

1.4. L_p $\mathfrak{p}$-capacitary measures and the L_p Minkowski problem for $\mathfrak{p}$-capacity. By reviewing the Minkowski problems for volume and capacity respectively, we find that they have been intensively investigated along two parallel tracks, and their similarities are greatly highlighted therein. However, compared with a series of remarkable results on L_p Minkowski problem for volume, the general L_p Minkowski problem for capacity is hardly ever proposed yet. The time is ripe to initiate the research on general L_p Minkowski problem for capacity.

In this article, we aim to generalize the Minkowski problem for $\mathfrak{p}$-capacity to general L_p Minkowski problems for $\mathfrak{p}$-capacity. In some sense, this is the first paper to push the Minkowski problem for $\mathfrak{p}$-capacity to L_p stage. Here, it is worth mentioning that to comply with the habits, we stick to using the terminology "L_p" Minkowski problem in our paper. But to avoid the confusion, we use "$\mathfrak{p}$-capacity", instead of "p-capacity", to distinguish the "p" in "L_p".

In light of the significant role of L_p surface area measure $S_p(K,\cdot)$ in the L_p theory of convex bodies, we introduce the important geometric measure: L_p $\mathfrak{p}$*-capacitary measure.*

Definition. Let $p \in \mathbb{R}$ and $1 < \mathfrak{p} < n$. For a convex body K in $\mathbb{R}^n$ with the origin in its interior, the L_p $\mathfrak{p}$*-capacitary measure* $\mu_{p,\mathfrak{p}}(K,\cdot)$ of K is a finite Borel measure on $\mathbb{S}^{n-1}$ defined, for each Borel set $\omega \subseteq \mathbb{S}^{n-1}$, by

$$\mu_{p,\mathfrak{p}}(K,\omega) = \int_{\omega} h_K(\xi)^{1-p} d\mu_{\mathfrak{p}}(K,\xi).$$

Later, we shall see that the L_p $\mathfrak{p}$-capacitary measure $\mu_{p,\mathfrak{p}}(K,\cdot)$ arises as the variation of $\mathfrak{p}$-capacity functional $\mathrm{C}_{\mathfrak{p}}$ of L_p sum of convex bodies.

Specifically, if K, L are convex bodies in $\mathbb{R}^n$ with the origin in their interiors, then

$$\left.\frac{d\mathrm{C}_{\mathfrak{p}}(K+_p t\cdot_p L)}{dt}\right|_{t=0^+} = \frac{(\mathfrak{p}-1)}{p}\int_{\mathbb{S}^{n-1}} h_L(\xi)^p d\mu_{p,\mathfrak{p}}(K,\xi),$$

where $1 \leq p < \infty$. See Corollary 3.1 for details.

Naturally, we pose the following L_p Minkowski problem for $\mathfrak{p}$-capacity.

L_p Minkowski problem for $\mathfrak{p}$-capacity. *Suppose μ is a finite Borel measure on $\mathbb{S}^{n-1}$, $1 < \mathfrak{p} < n$ and $p \in \mathbb{R}$. What are the necessary and sufficient conditions on μ so that μ is the L_p $\mathfrak{p}$-capacitary measure $\mu_{p,\mathfrak{p}}(K,\cdot)$ of a convex body K in $\mathbb{R}^n$?*

Recall that Jerison [40] solved the classical case when $p = 1$ and $\mathfrak{p} = 2$. Colesanti et al. [20] and Akman et al. [1] solved the case when $p = 1$ and $1 < \mathfrak{p} < n$. For $p \neq 1$, this problem is completely *new*.

1.5. Main results. Our main goal in this article is to solve the L_p Minkowski problem for $\mathfrak{p}$-capacity when $p > 1$. It is interesting that the necessary and sufficient conditions for existence and uniqueness of the solution is surprisingly *identical* to the conditions in the L_p Minkowski problem for volume when $p > 1$.

Theorem 1.1. *Suppose $1 < p < \infty$, $1 < \mathfrak{p} < n$, and that μ is a finite positive Borel measure on $\mathbb{S}^{n-1}$. If μ is not concentrated on any closed hemisphere, there exists a unique convex body K containing the origin, such that*

$$d\mu_{\mathfrak{p}}(K,\cdot) = c h_K^{p-1} d\mu,$$

where $c = 1$ if $p + \mathfrak{p} \neq n$, or $\mathrm{C}_{\mathfrak{p}}(K)$ if $p + \mathfrak{p} = n$. Furthermore, if in addition $p \geq n$, then K contains the origin in its interior, and $\mu_{p,\mathfrak{p}}(K,\cdot) = c\mu$.

Concerning the discrete L_p Minkowski problem for $\mathfrak{p}$-capacity, we prove the following.

Theorem 1.2. *Suppose $1 < p < \infty$, $1 < \mathfrak{p} < n$, and that μ is a finite positive Borel measure on $\mathbb{S}^{n-1}$. If μ is discrete and is not concentrated on any closed hemisphere, there exists a unique convex polytope K containing the origin in its interior, such that*

$$\mu_{p,\mathfrak{p}}(K,\cdot) = c\mu,$$

where $c = 1$ if $p + \mathfrak{p} \neq n$, or $\mathrm{C}_{\mathfrak{p}}(K)$ if $p + \mathfrak{p} = n$.

Concerning the even L_p Minkowski problem for $\mathfrak{p}$-capacity, we prove the following.

Theorem 1.3. *Suppose $1 < p < \infty$, $1 < \mathfrak{p} < n$, and that μ is a finite positive Borel measure on $\mathbb{S}^{n-1}$. If μ is even and is not concentrated on* *any great subsphere, there exists a unique origin-symmetric convex body* K*, such that*

$$\mu_{p,\mathfrak{p}}(K,\cdot) = c\mu,$$

where $c = 1$ *if* $p + \mathfrak{p} \neq n$*, or* $\mathrm{C}_{\mathfrak{p}}(K)$ *if* $p + \mathfrak{p} = n$*.*

We emphasize that, for $p > 1$, the L_p Minkowski problem for the $\mathfrak{p}$-capacity is considerably more complicated than the $p = 1$ case, requiring both new ideas and techniques. Our approach to this problem is rooted in the ideas and techniques from convex geometry. So its proof exhibits rich geometric flavour. Specifically, techniques developed Hug, Lutwak, Yang and Zhang [**39**], Klain [**42**] and Lutwak [**48**, **56**] are comprehensively employed. In particular, a crucial technique based on the interplay between two dual extremum problems of convex bodies is adopted, which was first used by Lutwak et al. [**58**] to establish the L_p John ellipsoids and then developed by the authors themselves in [**74**–**77**] to establish the Orlicz-John ellipsoids and Orlicz-Legendre ellipsoids.

It is worth mentioning that to remove the technical assumption that the given measure μ does not have a pair of antipodal point masses, we take an ingenious approximation tactics, which makes fully use of solutions to the L_p Minkowski problem for *volume*. For more details, see Section 5.

This article is organized as follows. In Section 2, we introduce the necessary notations and collect some basic facts on convex bodies, the $\mathfrak{p}$-capacity and the Aleksandrov body. In Section 3, we prove some basic results on the L_p $\mathfrak{p}$-capacitary measure. To characterize the uniqueness of $\mu_{p,\mathfrak{p}}(K,\cdot)$, the L_p Minkowski inequality for $\mathfrak{p}$-capacity is established. To solve the existence of solution to the L_p Minkowski problem for $\mathfrak{p}$-capacity, more tools and techniques are introduced and developed in Sections 4–6. Section 4 first focuses on two dual extremum problems for $\mathfrak{p}$-capacity (Problem 1 and Problem 2), then demonstrates clearly the relations of Problem 1, Problem 2, the normalized L_p Minkowski problem for $\mathfrak{p}$-capacity (Problem 3), and our concerned original problem: the L_p Minkowski problem for $\mathfrak{p}$-capacity (Problem 4). More technical preparations for approximations are provided in Section 6. The formal proofs of the main results are presented in Section 7.

2. Preliminaries

2.1. Basics of convex bodies. For quick reference, we collect some basic facts on convex bodies. Excellent references are the books by Gardner [**26**], Gruber [**31**] and Schneider [**67**].

Write $x \cdot y$ for the standard inner product of $x, y \in \mathbb{R}^n$. A compact convex set K in $\mathbb{R}^n$ is uniquely determined by its support function $h_K : \mathbb{R}^n \to \mathbb{R}$, given for $x \in \mathbb{R}^n$ by

$$h_K(x) = \max\{x \cdot y : y \in K\}.$$

The support function is positively homogeneous of degree 1, and is usually restricted to the sphere $\mathbb{S}^{n-1}$. The set of compact convex sets in $\mathbb{R}^n$ is equipped with the Hausdorff metric δ_H, which is defined for compact convex sets K, L by

$$\delta_H(K,L) = \max\left\{|\ h_K(\xi) - h_L(\xi)\ |: \xi \in \mathbb{S}^{n-1}\right\}.$$

Denote by $\mathcal{K}^n$ the set of convex bodies in $\mathbb{R}^n$. Write $\mathcal{K}_o^n$ for the set of convex bodies with the origin o in their interiors.

For compact convex sets K and L, they are said to be homothetic, provided $K = sL + x$, for some $s > 0$ and $x \in \mathbb{R}^n$. Their Minkowski sum is the set $K + L = \{x + y : x \in K, y \in L\}$. For $s > 0$, the set $sK = \{sx : x \in K\}$ is called a dilate of K. The reflection of K is $-K = \{-x : x \in K\}$.

Let $C(\mathbb{S}^{n-1})$ be the set of continuous functions defined on $\mathbb{S}^{n-1}$, which is equipped with the metric induced by the maximal norm. Write $C_+(\mathbb{S}^{n-1})$ for the set of strictly positive functions in $C(\mathbb{S}^{n-1})$. For nonnegative $f, g \in C(\mathbb{S}^{n-1})$ and $t \geq 0$, define

$$f +_p t \cdot g = (f^p + tg^p)^{1/p},$$

where, without confusion and for brevity, we omit the subscript p under the dot thereafter. If in addition $f > 0$ and g is nonzero, the definition holds when $t > -\left(\frac{\min_{\mathbb{S}^{n-1}} f}{\max_{\mathbb{S}^{n-1}} g}\right)^{1/p}$.

Recall that for compact convex sets K, L which contain the origin, the L_p combination $K +_p t \cdot L$, with $1 < p < \infty$ and $t \geq 0$, is the compact convex set defined by

$$h_{K +_p t \cdot L} = h_K +_p t \cdot h_L.$$

For nonnegative $f \in C(\mathbb{S}^{n-1})$, define

$$[f] = \bigcap_{\xi \in \mathbb{S}^{n-1}} \{x \in \mathbb{R}^n : x \cdot \xi \leq f(\xi)\}.$$

The set is called the Aleksandrov body (also known as Wulff shape) associated with f. Obviously, $[f]$ is a compact convex set containing the origin. For a compact convex set containing the origin, say K, we have $K = [h_K]$. If $f \in C^+(\mathbb{S}^{n-1})$, then $[f] \in \mathcal{K}_o^n$.

The Aleksandrov convergence lemma reads: If the sequence $\{f_j\}_j \subset C_+(\mathbb{S}^{n-1})$ converges uniformly to $f \in C_+(\mathbb{S}^{n-1})$, then $\lim_{j\to\infty} [f_j] = [f]$.

2.2. Basic facts on $\mathfrak{p}$-capacity. This part lists some necessary facts on $\mathfrak{p}$-capacity. For more details, see, e.g., [**20**, **21**, **29**, **40**].

Let $1 < \mathfrak{p} < n$. The $\mathfrak{p}$-capacity $\mathrm{C}_\mathfrak{p}$ has the following properties. First, it is increasing with respect to set inclusion; that is, if $E \subseteq F$, then $\mathrm{C}_\mathfrak{p}(E) \leq \mathrm{C}_\mathfrak{p}(F)$. Second, it is positively homogeneous of degree $(n - \mathfrak{p})$, i.e., $\mathrm{C}_\mathfrak{p}(sE) = s^{n-\mathfrak{p}}\mathrm{C}_\mathfrak{p}(E)$, for $s > 0$. Third, it is rigid invariant, i.e., $\mathrm{C}_\mathfrak{p}(gE + x) = \mathrm{C}_\mathfrak{p}(E)$, for $x \in \mathbb{R}^n$ and $g \in \mathrm{O}(n)$.

Let $K \in \mathcal{K}^n$. The $\mathfrak{p}$-capacitary measure $\mu_{\mathfrak{p}}(K,\cdot)$ has the following properties. First, it is positively homogeneous of degree $(n-\mathfrak{p}-1)$, i.e., $\mu_{\mathfrak{p}}(sK,\cdot) = s^{n-\mathfrak{p}-1}\mu_{\mathfrak{p}}(K,\cdot)$, for $s>0$. Second, it is translation invariant, i.e., $\mu_{\mathfrak{p}}(K+x,\cdot) = \mu_{\mathfrak{p}}(K,\cdot)$, for $x \in \mathbb{R}^n$. Third, its centroid is at the origin, i.e., $\int_{\mathbb{S}^{n-1}} \xi d\mu_{\mathfrak{p}}(K,\xi) = o$. Moreover, it is absolutely continuous with respect to the surface area measure $S(K,\cdot)$.

For convex bodies $K_j, K \in \mathcal{K}^n$, $j \in \mathbb{N}$, if $K_j \to K \in \mathcal{K}^n$, then $\mu_{\mathfrak{p}}(K_j,\cdot) \to \mu_{\mathfrak{p}}(K,\cdot)$ weakly, as $j \to \infty$. This important fact was proved by Colesanti et al. [**20**, p. 1550].

Let $K \in \mathcal{K}^n_o$ and $f \in C(\mathbb{S}^{n-1})$. There is $t_0 > 0$ so that $h_K + tf \in C_+(\mathbb{S}^{n-1})$, for $|t| < t_0$. The Aleksandrov body $[h_K + tf]$ is continuous in $t \in (-t_0, t_0)$. The Hadamard variational formula for $\mathfrak{p}$-capacity (see [**20**, Theorem 1.1]) states that

$$\left.\frac{d\mathrm{C}_{\mathfrak{p}}([h_K+tf])}{dt}\right|_{t=0} = (\mathfrak{p}-1)\int_{\mathbb{S}^{n-1}} f(\xi)d\mu_{\mathfrak{p}}(K,\xi). \tag{2.1}$$

For $K, L \in \mathcal{K}^n$, the *mixed* $\mathfrak{p}$*-capacity* $C_{\mathfrak{p}}(K,L)$ (see [**20**, p. 1549]) is defined by

$$\mathrm{C}_{\mathfrak{p}}(K,L) = \frac{1}{n-\mathfrak{p}}\left.\frac{d\mathrm{C}_{\mathfrak{p}}(K+tL)}{dt}\right|_{t=0^+} = \frac{\mathfrak{p}-1}{n-\mathfrak{p}}\int_{\mathbb{S}^{n-1}} h_L(\xi)d\mu_{\mathfrak{p}}(K,\xi). \tag{2.2}$$

When $L = K$, it reduces to the Poincaré $\mathfrak{p}$-capacity formula (1.9). From the weak convergence of $\mathfrak{p}$-capacitary measures, it follows that $\mathrm{C}_{\mathfrak{p}}(K,L)$ is continuous in (K,L).

The $\mathfrak{p}$-capacitary Brunn-Minkowski inequality, proved by Colesanti-Salani [**19**], reads: If $K, L \in \mathcal{K}^n$, then

$$\mathrm{C}_{\mathfrak{p}}(K+L)^{\frac{1}{n-\mathfrak{p}}} \geq \mathrm{C}_{\mathfrak{p}}(K)^{\frac{1}{n-\mathfrak{p}}} + C_{\mathfrak{p}}(L)^{\frac{1}{n-\mathfrak{p}}}, \tag{2.3}$$

with equality if and only if K and L are homothetic. When $\mathfrak{p} = 2$, the inequality was first established by Borell [**6**], and the equality condition was shown by Caffarelli, Jerison and Lieb [**12**]. For more details, see, e.g., Colesanti [**18**], Gardner [**25**], and Gardner and Hartenstine [**27**].

The $\mathfrak{p}$-capacitary Brunn-Minkowski inequality yields the $\mathfrak{p}$-capacitary Minkowski inequality,

$$\mathrm{C}_{\mathfrak{p}}(K,L)^{n-\mathfrak{p}} \geq \mathrm{C}_{\mathfrak{p}}(K)^{n-\mathfrak{p}-1}\mathrm{C}_{\mathfrak{p}}(L), \tag{2.4}$$

with equality if and only if K and L are homothetic. See [**20**, p. 1549] for its proof.

2.3. Basic facts on Aleksandrov bodies. For nonnegative $f \in C(\mathbb{S}^{n-1})$, define

$$\mathrm{C}_{\mathfrak{p}}(f) = \mathrm{C}_{\mathfrak{p}}([f]). \tag{2.5}$$

Obviously, $\mathrm{C}_\mathfrak{p}(h_K) = \mathrm{C}_\mathfrak{p}(K)$, for a compact convex set K that contains the origin.

By the Aleksandrov convergence lemma and the continuity of $\mathrm{C}_\mathfrak{p}$ on $\mathcal{K}^n$, we see that $\mathrm{C}_\mathfrak{p} : C_+(\mathbb{S}^{n-1}) \to (0, \infty)$ is continuous.

Let $1 \le p < \infty$ and $1 < \mathfrak{p} < n$. For $K \in \mathcal{K}^n_o$ and nonnegative $f \in C(\mathbb{S}^{n-1})$, define

$$\mathrm{C}_{p,\mathfrak{p}}(K, f) = \frac{\mathfrak{p}-1}{n-\mathfrak{p}} \int_{\mathbb{S}^{n-1}} f(\xi)^p h_K(\xi)^{1-p} d\mu_\mathfrak{p}(K, \xi). \tag{2.6}$$

For brevity, write $\mathrm{C}_\mathfrak{p}(K, f)$ for $\mathrm{C}_{1,\mathfrak{p}}(K, f)$. Obviously, $\mathrm{C}_{p,\mathfrak{p}}(K, h_K) = \mathrm{C}_\mathfrak{p}(K)$.

Lemma 2.1. *Let $1 \le p < \infty$ and $1 < \mathfrak{p} < n$. If $f \in C_+(\mathbb{S}^{n-1})$, then*

$$\mathrm{C}_{p,\mathfrak{p}}([f], f) = \mathrm{C}_\mathfrak{p}([f]) = \mathrm{C}_\mathfrak{p}(f).$$

Note that $\mathrm{C}_\mathfrak{p}(K, h_{[f]}) \le \mathrm{C}_\mathfrak{p}(K, f)$, for $K \in \mathcal{K}^n_o$ and $f \in C_+(\mathbb{S}^{n-1})$.

Proof. Note that $h_{[f]} \le f$. A basic fact established by Aleksandrov is that $h_{[f]} = f$, a.e. with respect to $S([f], \cdot)$. That is, $S([f], \{h_{[f]} < f\}) = 0$. Since $\mu_\mathfrak{p}([f], \cdot)$ is absolutely continuous with respect to $S([f], \cdot)$, it follows that $\mu_\mathfrak{p}([f], \{h_{[f]} < f\}) = 0$. Combining this fact and the inequality $h_{[f]} \le f$, it follows that

$$\mathrm{C}_{p,\mathfrak{p}}([f], f) - \mathrm{C}_\mathfrak{p}(f) = \frac{\mathfrak{p}-1}{n-\mathfrak{p}} \int_{\{f > h_{[f]}\}} \left(f^p - h_{[f]}^p \right) h_{[f]}^{1-p} d\mu_\mathfrak{p}([f], \cdot) = 0,$$

as desired. q.e.d.

The following important result is the Theorem 5.2 [20] established by Colesanti et al.

Lemma 2.2. *Let $I \subset \mathbb{R}$ be an interval containing 0 in its interior, and let $h_t(\xi) = h(t, \xi) : I \times \mathbb{S}^{n-1} \to (0, \infty)$ be continuous, such that the convergence in*

$$h'(0, \xi) = \lim_{t \to 0} \frac{h(t, \xi) - h(0, \xi)}{t}$$

is uniform on $\mathbb{S}^{n-1}$. Then

$$\left. \frac{d\mathrm{C}_\mathfrak{p}(h_t)}{dt} \right|_{t=0} = (\mathfrak{p}-1) \int_{\mathbb{S}^{n-1}} h'(0, \xi) d\mu_\mathfrak{p}([h_0], \xi).$$

The following lemma will be used in Sections 4 and 6.

Lemma 2.3. *Suppose μ is a finite positive Borel measure on $\mathbb{S}^{n-1}$ which is not concentrated on any closed hemisphere, and $0 < p < \infty$. If Q is a compact convex set in $\mathbb{R}^n$ containing the origin and $\dim(Q) \ge 1$, then $0 < \int_{\mathbb{S}^{n-1}} h_Q^p d\mu < \infty$.*

Proof. Since μ is finite and h_Q is nonnegative and bounded, the integral is obviously finite. To prove the positivity of the integral, we can take a line segment $Q_0 \subset Q$, which is of the form $Q_0 = l\{t\xi_0 : 0 \le t \le 1\}$, with $0 < l < \infty$ and $\xi_0 \in \mathbb{S}^{n-1}$. Since μ is not concentrated on any closed hemisphere, this implies that $\mu\left(\{\xi \in \mathbb{S}^{n-1} : \xi \cdot \xi_0 > 0\}\right) > 0$. Thus,

$$\begin{aligned}\int_{\mathbb{S}^{n-1}} h_Q^p d\mu &\ge \int_{\mathbb{S}^{n-1}} h_{Q_0}^p d\mu \\ &= l^p \int_{\{\xi\in\mathbb{S}^{n-1}:\, \xi\cdot\xi_0>0\}} (\xi\cdot\xi_0)^p d\mu(\xi) \\ &> 0,\end{aligned}$$

as desired. q.e.d.

3. The L_p $\mathfrak{p}$-capacitary measure $\mu_{p,\mathfrak{p}}(K,\cdot)$

3.1. The first L_p variational of $\mathfrak{p}$-capacity. Let $K \in \mathcal{K}_o^n$ and $1 \le p < \infty$. Consider a nonnegative and nonzero function $f \in C(\mathbb{S}^{n-1})$, and take the interval

$$I = \left(-(\min_{\mathbb{S}^{n-1}} h_K / \max_{\mathbb{S}^{n-1}} f)^{1/p}, \infty\right).$$

Since $h_t(\xi) = h(t,\xi) = (h_K +_p t\cdot f)(\xi) : I \times \mathbb{S}^{n-1} \to (0,\infty)$ is continuous and

$$\lim_{t\to 0} \frac{(h_K +_p t\cdot f) - h_K}{t} = \frac{f^p h_K^{1-p}}{p}$$

holds uniformly on $\mathbb{S}^{n-1}$, according to Lemma 2.2 and (2.6), we have

$$\left.\frac{d\mathrm{C}_{\mathfrak{p}}(h_K +_p t\cdot f)}{dt}\right|_{t=0^+} = \frac{n-\mathfrak{p}}{p}\mathrm{C}_{p,\mathfrak{p}}(K,f).$$

Note that it is precisely the Hadamard variational formula (2.1) when $p = 1$. For a compact convex set L containing the origin, letting $f = h_L$, we obtain the following.

Corollary 3.1. *Suppose $1 \le p < \infty$ and $1 < \mathfrak{p} < n$. If $K \in \mathcal{K}_o^n$ and L is a compact convex set containing the origin, then*

$$\left.\frac{d\mathrm{C}_{\mathfrak{p}}(K +_p t\cdot L)}{dt}\right|_{t=0^+} = \frac{\mathfrak{p}-1}{p}\int_{\mathbb{S}^{n-1}} h_L(\xi)^p h_K(\xi)^{1-p} d\mu_{\mathfrak{p}}(K,\xi).$$

Definition 3.2. Let $1 < \mathfrak{p} < n$ and $p \in \mathbb{R}$. For $K \in \mathcal{K}_o^n$, the finite Borel measure $\mu_{p,\mathfrak{p}}$ on $\mathbb{S}^{n-1}$, defined by

$$d\mu_{p,\mathfrak{p}}(K,\cdot) = h_K^{1-p} d\mu_{\mathfrak{p}}(K,\cdot),$$

is called the L_p $\mathfrak{p}$-*capacitary measure* $\mu_{p,\mathfrak{p}}(K,\cdot)$ of K.

Definition 3.3. Let $1 < \mathfrak{p} < n$ and $p \in \mathbb{R}$. For $K, L \in \mathcal{K}_o^n$, if $p \neq 0$, then define

$$\mathrm{C}_{p,\mathfrak{p}}(K,L) = \frac{\mathfrak{p}-1}{n-\mathfrak{p}} \int_{\mathbb{S}^{n-1}} h_L(\xi)^p h_K(\xi)^{1-p} d\mu_{\mathfrak{p}}(K,\xi),$$

and call it the L_p *mixed* $\mathfrak{p}$*-capacity* of (K, L).

If in addition $p > 0$ and L is a compact convex set containing the origin, the L_p mixed $\mathfrak{p}$-capacity $\mathrm{C}_{p,\mathfrak{p}}(K,L)$ is also well defined.

For $p = 0$, we call $\mu_{0,\mathfrak{p}}(K,\cdot)$ the *logarithmic* $\mathfrak{p}$*-capacitary measure* of K. Let $\bar{\mu}_{0,\mathfrak{p}}(K,\cdot)$ be the probability normalization of $\mu_{0,\mathfrak{p}}(K,\cdot)$. That is,

$$d\bar{\mu}_{0,\mathfrak{p}}(K,\cdot) = \frac{(\mathfrak{p}-1)h_K}{(n-\mathfrak{p})\mathrm{C}_{\mathfrak{p}}(K)} d\mu_{\mathfrak{p}}(K,\cdot). \tag{3.1}$$

Then, $\mathrm{C}_{p,\mathfrak{p}}(K,L)$ can be normalized as

$$\overline{\mathrm{C}}_{p,\mathfrak{p}}(K,L) = \left[\frac{\mathrm{C}_{p,\mathfrak{p}}(K,L)}{\mathrm{C}_{\mathfrak{p}}(K)}\right]^{1/p} = \left[\int_{\mathbb{S}^{n-1}} \left(\frac{h_L}{h_K}\right)^p d\bar{\mu}_{0,\mathfrak{p}}(K,\cdot)\right]^{1/p}.$$

For $p = 0$, let

$$\overline{\mathrm{C}}_{0,\mathfrak{p}}(K,L) = \lim_{p\to 0} \overline{\mathrm{C}}_{p,\mathfrak{p}}(K,L) = \exp \int_{\mathbb{S}^{n-1}} \log \frac{h_L(\xi)}{h_K(\xi)} d\bar{\mu}_{0,\mathfrak{p}}(K,\xi),$$

and call it the *normalized logarithmic mixed* $\mathfrak{p}$*-capacity* of (K, L).

From the Definitions 3.2 and 3.3, several facts follow directly. For $\mu_{p,\mathfrak{p}}(K,\cdot)$, we have $\mu_{1,\mathfrak{p}}(K,\cdot) = \mu_{\mathfrak{p}}(K,\cdot)$, and $\frac{\mathfrak{p}-1}{n-\mathfrak{p}}\mu_{0,\mathfrak{p}}(K,\mathbb{S}^{n-1}) = \mathrm{C}_{\mathfrak{p}}(K)$. For $\mathrm{C}_{p,\mathfrak{p}}(K,L)$, we have $\mathrm{C}_{1,\mathfrak{p}}(K,L) = \mathrm{C}_{\mathfrak{p}}(K,L)$, $\mathrm{C}_{p,\mathfrak{p}}(K,K) = \mathrm{C}_{\mathfrak{p}}(K)$ and $\mathrm{C}_{p,\mathfrak{p}}(K,h_L) = \mathrm{C}_{p,\mathfrak{p}}(K,L)$. Besides, $\mathrm{C}_{p,\mathfrak{p}}(gK,gL) = \mathrm{C}_{p,\mathfrak{p}}(K,L)$, for $g \in \mathrm{O}(n)$. Similar to the proof of Theorem 3.1 in [**74**], $\mathrm{C}_{p,\mathfrak{p}}(K,L)$ is continuous in (K, L, p).

As the L_p mixed volume $V_p(K,L)$ and the L_p surface area measure $S_p(K,\cdot)$ significantly extend the first mixed volume $V_1(K,L)$ and the classical surface area measure $S(K,\cdot)$ in convex geometry, respectively, $\mathrm{C}_{p,\mathfrak{p}}(K,L)$ and $\mu_{p,\mathfrak{p}}(K,\cdot)$ are precisely the L_p extensions of the mixed $\mathfrak{p}$-capacity $\mathrm{C}_{\mathfrak{p}}(K,L)$ and the $\mathfrak{p}$-capacitary measure $\mu_{\mathfrak{p}}(K,\cdot)$, respectively.

From Definition 3.2, together with the weak convergence and the positive homogeneity of $\mu_{\mathfrak{p}}$, we obtain the following results.

Lemma 3.4. *Suppose $K_j, K \in \mathcal{K}_o^n$, $j \in \mathbb{N}$. Let $1 \le p < \infty$ and $1 < \mathfrak{p} < n$. If $K_j \to K$, then $\mu_{p,\mathfrak{p}}(K_j,\cdot) \to \mu_{p,\mathfrak{p}}(K,\cdot)$ weakly, as $j \to \infty$.*

Lemma 3.5. *Suppose $K \in \mathcal{K}_o^n$. Let $-\infty < p < \infty$ and $1 < \mathfrak{p} < n$. Then, $\mu_{p,\mathfrak{p}}(sK,\cdot) = s^{n-\mathfrak{p}-p}\mu_{p,\mathfrak{p}}(K,\cdot)$, for $s > 0$*

3.2. The L_p Minkowski inequality for $\mathfrak{p}$-capacity. In this part, we show that associated with $\mathrm{C}_{p,\mathfrak{p}}(K,L)$, there is a natural L_p extension of the $\mathfrak{p}$-capacitary Minkowski inequality. Then we use it to extend the $\mathfrak{p}$-capacitary Brunn-Minkowski inequality to the L_p setting. It is noted that the L_p Brunn-Minkowski type inequality for $\mathfrak{p}$-capacity was previously established in [**78**] by the authors' L_p transference principle.

Theorem 3.6. *Let $1 < p < \infty$ and $1 < \mathfrak{p} < n$. If $K \in \mathcal{K}_o^n$ and $f \in C_+(\mathbb{S}^{n-1})$, then*

$$\mathrm{C}_{p,\mathfrak{p}}(K,f)^{n-\mathfrak{p}} \geq \mathrm{C}_{\mathfrak{p}}(K)^{n-\mathfrak{p}-p}\mathrm{C}_{\mathfrak{p}}(f)^p, \tag{3.2}$$

with equality if and only if K and $[f]$ are dilates.

Proof. From (2.6) combined with (3.1), the fact $f \geq h_{[f]}$, Jensen's inequality, (3.1) again combined with (2.2), and finally the $\mathfrak{p}$-capacitary Minkowski inequality (2.4) combined with (2.5), it follows that

$$\begin{aligned}
\left[\frac{\mathrm{C}_{p,\mathfrak{p}}(K,f)}{\mathrm{C}_{\mathfrak{p}}(K)}\right]^{1/p} &= \left[\int_{\mathbb{S}^{n-1}} \left(\frac{f}{h_K}\right)^p d\bar{\mu}_{0,\mathfrak{p}}(K,\cdot)\right]^{1/p} \\
&\geq \left[\int_{\mathbb{S}^{n-1}} \left(\frac{h_{[f]}}{h_K}\right)^p d\bar{\mu}_{0,\mathfrak{p}}(K,\cdot)\right]^{1/p} \\
&\geq \int_{\mathbb{S}^{n-1}} \frac{h_{[f]}}{h_K} d\bar{\mu}_{0,\mathfrak{p}}(K,\cdot) \\
&= \frac{\mathrm{C}_{p,\mathfrak{p}}(K,[f])}{\mathrm{C}_{\mathfrak{p}}(K)} \\
&\geq \frac{\mathrm{C}_{\mathfrak{p}}(f)^{1/(n-\mathfrak{p})}}{\mathrm{C}_{\mathfrak{p}}(K)^{1/(n-\mathfrak{p})}},
\end{aligned}$$

as desired. In the next, we prove the equality condition.

Assume the equality holds in (3.2). By equality conditions of $\mathfrak{p}$-capacitary Minkowski inequality, there exist $x \in \mathbb{R}^n$ and $s > 0$, such that $[f] = sK + x$. By the equality conditions of Jensen inequality, $\mathrm{C}_{\mathfrak{p}}(K,[f])h_K(\xi) = \mathrm{C}_{\mathfrak{p}}(K)h_{[f]}(\xi)$, for $\mu_{\mathfrak{p}}(K,\cdot)$-a.e. $\xi \in \mathbb{S}^{n-1}$. Hence, for $\mu_{\mathfrak{p}}(K,\cdot)$-a.e. $\xi \in \mathbb{S}^{n-1}$,

$$\left(s\mathrm{C}_{\mathfrak{p}}(K) + \frac{\mathfrak{p}-1}{n-\mathfrak{p}}x \cdot \int_{\mathbb{S}^{n-1}} \xi d\mu_p(K,\xi)\right) h_K(\xi) = \mathrm{C}_{\mathfrak{p}}(K)(sh_K(\xi) + x\cdot\xi).$$

Since the centroid of $\mu_{\mathfrak{p}}(K,\cdot)$ is at the origin, this implies that $x\cdot\xi = 0$, for $\mu_{\mathfrak{p}}(K,\cdot)$-a.e. $\xi \in \mathbb{S}^{n-1}$. Note that the $\mathfrak{p}$-capacitary measure $\mu_{\mathfrak{p}}(K,\cdot)$ is not concentrated on any great subsphere of $\mathbb{S}^{n-1}$. Hence, $x = o$, which in turn implies that K and $[f]$ are dilates.

Conversely, assume that K and $[f]$ are dilates, say, $K = s[f]$, for some $s > 0$. From (2.6), the fact that $\mu_\mathfrak{p}(s[f], \cdot) = s^{n-\mathfrak{p}-1}\mu_\mathfrak{p}([f], \cdot)$, Lemma 2.1, and the fact that $\mathrm{C}_\mathfrak{p}(K) = \mathrm{C}_\mathfrak{p}(s[f]) = s^{n-\mathfrak{p}}\mathrm{C}_\mathfrak{p}([f]) = \mathrm{C}_\mathfrak{p}(f)$, it follows that

$$\begin{aligned}\mathrm{C}_{p,\mathfrak{p}}(K, f) &= \mathrm{C}_{p,\mathfrak{p}}(s[f], f)\\ &= s^{n-\mathfrak{p}-p}\mathrm{C}_{p,\mathfrak{p}}([f], f)\\ &= s^{n-\mathfrak{p}-p}\mathrm{C}_\mathfrak{p}([f])\\ &= \mathrm{C}_\mathfrak{p}(K)^{\frac{n-\mathfrak{p}-p}{n-\mathfrak{p}}}\mathrm{C}_\mathfrak{p}(f)^{\frac{p}{n-\mathfrak{p}}}.\end{aligned}$$

This completes the proof. q.e.d.

Theorem 3.6 directly yields that for $K, L \in \mathcal{K}_o^n$,

$$\mathrm{C}_{p,\mathfrak{p}}(K, L)^{n-\mathfrak{p}} \geq \mathrm{C}_\mathfrak{p}(K)^{n-\mathfrak{p}-p}\mathrm{C}_p(L)^p, \tag{3.3}$$

with equality if and only if K and L are dilates.

Corollary 3.7. *Suppose $K \in \mathcal{K}_o^n$, $1 < p < \infty$ and $1 < \mathfrak{p} < n$. Then*

$$\mu_{p,\mathfrak{p}}(K, \mathbb{S}^{n-1})^{n-\mathfrak{p}} \geq n^p\omega_n{}^p\left(\frac{n-\mathfrak{p}}{\mathfrak{p}-1}\right)^{(\mathfrak{p}-1)p}\mathrm{C}_\mathfrak{p}(K)^{n-\mathfrak{p}-p},$$

with equality if and only if K is an origin-symmetric ball.

Proof. Let L be the unit ball B in $\mathbb{R}^n$. Since $\mathrm{C}_\mathfrak{p}(B) = n\omega_n\left(\frac{n-\mathfrak{p}}{\mathfrak{p}-1}\right)^{\mathfrak{p}-1}$, from Theorem 3.6, the desired inequality with its equality condition is obtained. q.e.d.

Suppose $f_1, f_2, g \in C_+(\mathbb{S}^{n-1})$. From the definition of $f_1 +_p f_2$ and (2.6), it follows that

$$\mathrm{C}_{p,\mathfrak{p}}([g], f_1 +_p f_2) = \mathrm{C}_{p,\mathfrak{p}}([g], f_1) + \mathrm{C}_{p,\mathfrak{p}}([g], f_2).$$

This, together with Theorem 3.6, gives

$$\mathrm{C}_{p,\mathfrak{p}}([g], f_1 +_p f_2) \geq \mathrm{C}_\mathfrak{p}([g])^{\frac{n-\mathfrak{p}-p}{n-\mathfrak{p}}}\left(\mathrm{C}_\mathfrak{p}(f_1)^{\frac{p}{n-\mathfrak{p}}} + \mathrm{C}_\mathfrak{p}(f_2)^{\frac{p}{n-\mathfrak{p}}}\right),$$

with equality if and only if $[f_1]$ and $[f_2]$ are dilates of $[g]$. Letting $g = f_1 +_p f_2$, it yields an L_p extension of the Colesanti-Salani Brunn-Minkowski inequality.

Theorem 3.8. *Let $1 < p < \infty$ and $1 < \mathfrak{p} < n$. If $f_1, f_2 \in C_+(\mathbb{S}^{n-1})$, then*

$$\mathrm{C}_\mathfrak{p}(f_1 +_p f_2)^{p/(n-\mathfrak{p})} \geq \mathrm{C}_\mathfrak{p}(f_1)^{p/(n-\mathfrak{p})} + \mathrm{C}_\mathfrak{p}(f_2)^{p/(n-\mathfrak{p})},$$

with equality if and only if $[f_1]$ and $[f_2]$ are dilates.

Consequently, for $K, L \in \mathcal{K}_o^n$, we have

$$\mathrm{C}_\mathfrak{p}(K +_p L)^{p/(n-\mathfrak{p})} \geq \mathrm{C}_\mathfrak{p}(K)^{p/(n-\mathfrak{p})} + \mathrm{C}_\mathfrak{p}(L)^{p/(n-\mathfrak{p})}, \tag{3.4}$$

with equality if and only if K and L are dilates.

The L_p $\mathfrak{p}$-capacitary Brunn-Minkowski inequality (3.4) also yields the L_p $\mathfrak{p}$-capacitary Minkowski inequality (3.3). Indeed, consider the nonnegative concave function

$$h(t) = \mathrm{C}_{\mathfrak{p}}(K +_p t \cdot L)^{\frac{p}{n-\mathfrak{p}}} - \mathrm{C}_{\mathfrak{p}}(K)^{\frac{p}{n-\mathfrak{p}}} - t\mathrm{C}_{\mathfrak{p}}(L)^{\frac{p}{n-\mathfrak{p}}}.$$

Using Corollary 3.1 and (3.4), we have

$$\lim_{t \to 0^+} \frac{h(t) - h(0)}{t} = \mathrm{C}_{\mathfrak{p}}(K)^{\frac{p}{n-\mathfrak{p}}-1}\mathrm{C}_{p,\mathfrak{p}}(K, L) - \mathrm{C}_{\mathfrak{p}}(L)^{\frac{p}{n-\mathfrak{p}}} \geq 0.$$

If the equality holds on the right, then h must be linear, and therefore K, L must be dilates.

3.3. Uniqueness of the L_p $\mathfrak{p}$-capacitary measures. In this part, by using the L_p $\mathfrak{p}$-capacitary Minkowski inequality (3.3), we show the uniqueness of solution to the L_p Minkowski problem for $\mathfrak{p}$-capacity, which is closely related with the following question: *If $K, L \in \mathcal{K}_o^n$ are such that $\mu_{p,\mathfrak{p}}(K, \cdot) = \mu_{p,\mathfrak{p}}(L, \cdot)$, then is this the case that $K = L$?*

Theorems 3.9 (2) and 3.11 (2) affirm this question.

Theorem 3.9. *Suppose $K, L \in \mathcal{K}_o^n$ and $\mathcal{C}$ is a subset of $\mathcal{K}_o^n$ such that $K, L \in \mathcal{C}$. Let $1 < p < \infty$, $1 < \mathfrak{p} < n$ and $p + \mathfrak{p} \neq n$. Then the following assertions hold.*
(1) If $\mathrm{C}_{p,\mathfrak{p}}(K, Q) = \mathrm{C}_{p,\mathfrak{p}}(L, Q)$ for all $Q \in \mathcal{C}$, then $K = L$.
(2) If $\mu_{p,\mathfrak{p}}(K, \cdot) = \mu_{p,\mathfrak{p}}(L, \cdot)$, then $K = L$.
(3) If $\mathrm{C}_{p,\mathfrak{p}}(Q, K) = \mathrm{C}_{p,\mathfrak{p}}(Q, L)$ for all $Q \in \mathcal{C}$, then $K = L$.

Proof. Since $\mathrm{C}_{p,\mathfrak{p}}(K, K) = \mathrm{C}_{\mathfrak{p}}(K)$, by the assumption it follows that $\mathrm{C}_{p,\mathfrak{p}}(L, K) = \mathrm{C}_{\mathfrak{p}}(K)$. By (3.3),

$$\mathrm{C}_{\mathfrak{p}}(K)^{(n-\mathfrak{p}-p)/(n-\mathfrak{p})} \geq \mathrm{C}_{\mathfrak{p}}(L)^{(n-\mathfrak{p}-p)/(n-\mathfrak{p})},$$

with equality if and only if K and L are dilates. This inequality is reversed if interchanging K and L. So $\mathrm{C}_{\mathfrak{p}}(K) = \mathrm{C}_{\mathfrak{p}}(L)$. Since $\mathrm{C}_{\mathfrak{p}}$ is positively homogeneous of degree $(n - \mathfrak{p})$, it follows that $K = L$.

If $\mu_{p,\mathfrak{p}}(K, \cdot) = \mu_{p,\mathfrak{p}}(L, \cdot)$, then $\mathrm{C}_{p,\mathfrak{p}}(K, Q) = \mathrm{C}_{p,\mathfrak{p}}(L, Q)$ for all $Q \in \mathcal{C}$. So, (2) follows from (1) immediately. The third assertion can be proved similarly to (1). q.e.d.

Theorem 3.10. *Suppose $K, L \in \mathcal{K}_o^n$ are such that $\mu_{p,\mathfrak{p}}(K, \cdot) \leq \mu_{p,\mathfrak{p}}(L, \cdot)$. Let $1 < p < \infty$, $1 < \mathfrak{p} < n$, and $p + \mathfrak{p} \neq n$. Then the following assertions hold.*
(1) If $\mathrm{C}_{\mathfrak{p}}(K) \geq \mathrm{C}_{\mathfrak{p}}(L)$ and $p + \mathfrak{p} < n$, then $K = L$.
(2) If $\mathrm{C}_{\mathfrak{p}}(K) \leq \mathrm{C}_{\mathfrak{p}}(L)$ and $p + \mathfrak{p} > n$, then $K = L$.

Proof. From $\mathrm{C}_{p,\mathfrak{p}}(L, L) = \mathrm{C}_{\mathfrak{p}}(L)$, the assumption $\mu_{p,\mathfrak{p}}(K, \cdot) \leq \mu_{p,\mathfrak{p}}(L, \cdot)$, the inequality (3.3) and the assumptions in (1) or (2), we have

$$\begin{aligned}\mathrm{C}_{\mathfrak{p}}(L) &\geq \mathrm{C}_{p,\mathfrak{p}}(K,L)\\ &\geq \mathrm{C}_{\mathfrak{p}}(K)^{\frac{n-\mathfrak{p}-p}{n-\mathfrak{p}}}\mathrm{C}_{\mathfrak{p}}(L)^{\frac{p}{n-\mathfrak{p}}}\\ &\geq \mathrm{C}_{\mathfrak{p}}(L)^{\frac{n-\mathfrak{p}-p}{n-\mathfrak{p}}}\mathrm{C}_{\mathfrak{p}}(L)^{\frac{p}{n-\mathfrak{p}}}\\ &= \mathrm{C}_{\mathfrak{p}}(L).\end{aligned}$$

Thus $\mathrm{C}_{\mathfrak{p}}(K) = \mathrm{C}_{\mathfrak{p}}(L)$, and K and L are dilates. Hence, $K = L$. q.e.d.

When $p + \mathfrak{p} = n$, we obtain the following result.

Theorem 3.11. *Suppose $K, L \in \mathcal{K}_o^n$ and $\mathcal{C}$ is a subset of $\mathcal{K}_o^n$ such that $K, L \in \mathcal{C}$. Let $1 < p < \infty$ and $1 < \mathfrak{p} < n-1$. Then the following assertions hold.*
(1) If $\mathrm{C}_{n-\mathfrak{p},\mathfrak{p}}(K,Q) \geq \mathrm{C}_{n-\mathfrak{p},\mathfrak{p}}(L,Q)$ for all $Q \in \mathcal{C}$, then K and L are dilates.
(2) If $\mu_{n-\mathfrak{p},\mathfrak{p}}(K,\cdot) \geq \mu_{n-\mathfrak{p},\mathfrak{p}}(L,\cdot)$, then K and L are dilates.

Proof. Take $Q = K$. From the fact $\mathrm{C}_{n-\mathfrak{p},\mathfrak{p}}(K,K) = \mathrm{C}_{\mathfrak{p}}(K)$, the assumption in (1) and the inequality (3.3), we have

$$\mathrm{C}_{\mathfrak{p}}(K) \geq \mathrm{C}_{n-\mathfrak{p},\mathfrak{p}}(L,K) \geq \mathrm{C}_{\mathfrak{p}}(K).$$

Thus, all the equalities above hold, and K, L are dilates. With the first assertion in hand, the second is obtained directly. q.e.d.

4. Two dual extremum problems for $\mathfrak{p}$-capacity

Throughout this section, let $1 < p < \infty$ and $1 < \mathfrak{p} < n$. Suppose μ is a finite Borel measure on $\mathbb{S}^{n-1}$ which is not concentrated on any closed hemisphere. For a compact convex set Q containing the origin, define

$$F_p(Q) = \frac{\mathfrak{p}-1}{n-\mathfrak{p}} \int_{\mathbb{S}^{n-1}} h_Q^p \, d\mu.$$

To prove main results of this article, we start from the following extremum problems, which are closely connected with our concerned L_p Minkowski problem for $\mathfrak{p}$-capacity.

Problem 1. Among all convex bodies Q in $\mathbb{R}^n$ containing the origin, find one to solve the following constrained minimization problem

$$\inf_Q F_p(Q) \qquad \text{subject to} \qquad \mathrm{C}_{\mathfrak{p}}(Q) \geq 1.$$

Problem 2. Among all convex bodies Q in $\mathbb{R}^n$ containing the origin, find one to solve the following constrained maximization problem

$$\sup_Q \mathrm{C}_{\mathfrak{p}}(Q) \qquad \text{subject to} \qquad F_p(Q) \leq 1.$$

Some basic facts are observed. Let Q be a convex body in $\mathbb{R}^n$ containing the origin. By Lemma 2.3, it follows that

$$0 < F_p(Q) < \infty. \tag{4.1}$$

Since $F_p(\alpha Q) = \alpha^p F_p(Q)$ and $\mathrm{C}_{\mathfrak{p}}(\alpha Q) = \alpha^{n-\mathfrak{p}}\mathrm{C}_{\mathfrak{p}}(K)$, for $\alpha > 0$, two necessary conditions of solutions to Problems 1 and 2 are seen. If Q solves Problem 1, then $\mathrm{C}_{\mathfrak{p}}(Q) = 1$. Similarly, if Q solves Problem 2, then $F_p(Q) = 1$.

The following lemma shows the duality between Problems 1 and 2.

Lemma 4.1. *For Problems 1 and 2, the following assertions hold.*

(1) If convex body $\bar{K}$ solves Problem 1, then

$$K := F_p(\bar{K})^{-1/p}\bar{K}$$

solves Problem 2.

(2) If convex body K solves Problem 2, then

$$\bar{K} := \mathrm{C}_{\mathfrak{p}}(K)^{-1/(n-\mathfrak{p})}K$$

solves Problem 1.

Proof. (1) Assume $\bar{K}$ solves Problem 1. Let Q be a convex body containing the origin such that $F_p(Q) \le 1$. From the positive homogeneity of $\mathrm{C}_{\mathfrak{p}}$ together with that $\mathrm{C}_{\mathfrak{p}}(\bar{K}) = 1$, the assumption together with that $\mathrm{C}_{\mathfrak{p}}(\mathrm{C}_{\mathfrak{p}}(Q)^{-1/(n-\mathfrak{p})}Q) = 1$, the positive homogeneity of F_p, and finally that $F_p(Q) \le 1$, we have

$$\begin{aligned}
\mathrm{C}_{\mathfrak{p}}(K)^{p/(n-\mathfrak{p})} &= F_p(\bar{K})^{-1} \\
&\ge F_p\left(\mathrm{C}_{\mathfrak{p}}(Q)^{-1/(n-\mathfrak{p})}Q\right)^{-1} \\
&= F_p(Q)^{-1}\mathrm{C}_{\mathfrak{p}}(Q)^{p/(n-\mathfrak{p})} \\
&\ge \mathrm{C}_{\mathfrak{p}}(Q)^{p/(n-\mathfrak{p})}.
\end{aligned}$$

Thus, K solves Problem 2.

(2) Assume K solves Problem 2. Let Q be a convex body containing the origin such that $\mathrm{C}_{\mathfrak{p}}(Q) \ge 1$. From the positive homogeneity of F_p together with that $F_p(K) = 1$, the assumption together with that $F_p(F_p(Q)^{-1/p}Q) = 1$, the positive homogeneity of $\mathrm{C}_{\mathfrak{p}}$, and finally that $\mathrm{C}_{\mathfrak{p}}(Q) \ge 1$, we have

$$\begin{aligned}
F_p(\bar{K}) &= \mathrm{C}_{\mathfrak{p}}(K)^{-p/(n-\mathfrak{p})} \\
&\le \mathrm{C}_{\mathfrak{p}}\left(F_p(Q)^{-1/p}Q\right)^{-p/(n-\mathfrak{p})} \\
&= F_p(Q)\mathrm{C}_{\mathfrak{p}}(Q)^{-p/(n-\mathfrak{p})} \\
&\le F_p(Q).
\end{aligned}$$

Thus, $\bar{K}$ solves Problem 1. q.e.d.

The next lemma confirms the uniqueness of solution to Problem 2. Equivalently, the uniqueness of solution to Problem 1 is shown.

Lemma 4.2. *Suppose μ is a finite Borel measure on $\mathbb{S}^{n-1}$ and is not concentrated on any closed hemisphere. Let $1 < p < \infty$ and $1 < \mathfrak{p} < n$. If K_1 and K_2 are convex bodies in $\mathbb{R}^n$ containing the origin and solving Problem 2 for $(\mu, \mathfrak{p}, p)$, then $K_1 = K_2$.*

Proof. Consider the convex body $2^{-1}(K_1 + K_2)$. Clearly, it also contains the origin. Since

$$F_p\left(\frac{K_1 + K_2}{2}\right) \le \frac{F_p\left(K_1\right) + F_p\left(K_2\right)}{2} \le 1,$$

the convex body $2^{-1}(K_1+K_2)$ still satisfies the constraint in Problem 2. So,

$$\mathrm{C}_\mathfrak{p}\left(\frac{K_1 + K_2}{2}\right) \le \mathrm{C}_\mathfrak{p}(K_1) = \mathrm{C}_\mathfrak{p}(K_2).$$

Consequently,

$$\mathrm{C}_\mathfrak{p}\left(\frac{K_1 + K_2}{2}\right)^{1/(n-\mathfrak{p})} \le \frac{1}{2}\mathrm{C}_\mathfrak{p}(K_1)^{1/(n-\mathfrak{p})} + \frac{1}{2}\mathrm{C}_\mathfrak{p}(K_2)^{1/(n-\mathfrak{p})}. \tag{4.2}$$

On the other hand, the reverse of (4.2) always holds, since it is just the Colesanti-Salani Brunn-Minkowski inequality. So, equality occurs in (4.2), and $K_1 = \alpha K_2 + x$, for some $\alpha > 0$ and $x \in \mathbb{R}^n$. From that $\mathrm{C}_\mathfrak{p}(K_1) = \mathrm{C}_\mathfrak{p}(K_2)$ together with the positive homogeneity and translation invariance of $\mathrm{C}_\mathfrak{p}$, it follows that $\alpha = 1$. Thus, $K_1 = K_2 + x$.

To complete the proof, it remains to prove that $x = o$.

Since

$$\mathrm{C}_\mathfrak{p}\left(\frac{K_1 + K_2}{2}\right) = \mathrm{C}_\mathfrak{p}(K_1) = \mathrm{C}_\mathfrak{p}(K_2),$$

the body $2^{-1}(K_1 + K_2)$ is also a solution to Problem 2. Necessarily, $F_p\left(2^{-1}(K_1 + K_2)\right) = 1$. In other words,

$$\int_{\mathbb{S}^{n-1}} \left(\frac{h_{K_1} + h_{K_2}}{2}\right)^p d\mu = \int_{\mathbb{S}^{n-1}} \frac{h_{K_1}^p + h_{K_2}^p}{2} d\mu.$$

Since these integrands are nonnegative continuous and

$$\left(\frac{h_{K_1} + h_{K_2}}{2}\right)^p \le \frac{h_{K_1}^p + h_{K_2}^p}{2},$$

it follows that for all $\xi \in \operatorname{supp} \mu$,

$$\frac{h_{K_2}(\xi) + (h_{K_2}(\xi) + (\xi \cdot x))}{2} = \left(\frac{h_{K_2}(\xi)^p + (h_{K_2}(\xi) + (\xi \cdot x))^p}{2}\right)^{1/p}. \tag{4.3}$$

Let $U = \{\xi \in \mathbb{S}^{n-1} : x \cdot \xi > 0\}$. Since μ is not concentrated on any closed hemisphere, it follows that $\mu(U \cap \text{supp}\ \mu) > 0$. If x is nonzero, then for any $\xi \in U \cap \text{supp}\ \mu$,

$$\frac{h_{K_2}(\xi) + (h_{K_2}(\xi) + (\xi \cdot x))}{2} < \left(\frac{h_{K_2}(\xi)^p + (h_{K_2}(\xi) + (\xi \cdot x))^p}{2}\right)^{1/p},$$

which will obviously contradict (4.3). Hence, $K_1 = K_2$. q.e.d.

Lemma 4.3. *Suppose μ is a discrete measure on $\mathbb{S}^{n-1}$ and is not concentrated on any closed hemisphere. Let $1 < p < \infty$ and $1 < \mathfrak{p} < n$. If K is a convex body in $\mathbb{R}^n$ containing the origin and solving Problem 2 for $(\mu, \mathfrak{p}, p)$, then K is a convex proper polytope containing the origin in its interior.*

We need to make some preparations. Represent the discrete measure μ by the form

$$\mu = \sum_{i=1}^{m} c_i \delta_{\xi_i}, \tag{4.4}$$

where $\xi_i \in \mathbb{S}^{n-1}$ and $c_i > 0$, for all i. Then, for a convex body Q containing the origin,

$$F_p(Q) = \frac{\mathfrak{p} - 1}{n - \mathfrak{p}} \sum_{i=1}^{m} c_i h_Q(\xi_i)^p. \tag{4.5}$$

For nonzero $y = (y_1, \dots, y_m) \in [0, \infty)^m$, define

$$P(y) = \bigcap_{i=1}^{m} \{x \in \mathbb{R}^n : x \cdot \xi_i \le y_i\}. \tag{4.6}$$

Proof. We first verify that K is a convex proper polytope. Let

$$h = (h_1, \dots, h_m) = (h_K(\xi_1), \dots, h_K(\xi_m)). \tag{4.7}$$

Since $K \subseteq P(h)$, we have $\mathrm{C}_{\mathfrak{p}}(K) \le \mathrm{C}_{\mathfrak{p}}(P(h))$. Since $F_p(P(h)) = F_p(K)$, the proper polytope $P(h)$ satisfies the constraint in Problem 2. Since the convex body K solves Problem 2, it follows that $\mathrm{C}_{\mathfrak{p}}(K) = \mathrm{C}_{\mathfrak{p}}(P(h))$. This in turn implies that $P(h)$ also solves Problem 2. By Lemma 4.2, it follows that $P(h) = K$.

In the following, we prove that $P(h)$ contains the origin in its interior. For this aim, we argue by contradiction and assume that $o \in \partial P(h)$. We will construct a new convex proper polytope $P(z)$, such that

$$o \in \text{int} P(z), \quad F_p(P(z)) \le 1, \quad \text{but} \quad \mathrm{C}_{\mathfrak{p}}(P(z)) > \mathrm{C}_{\mathfrak{p}}(P(h)). \tag{4.8}$$

To prove the existence of such $P(z)$, we adopt the elegant deformation technique, which was previously employed by Hug et al. [39].

Since $o \in \partial P(y)$, without loss of generality, we assume

$$h_1 = \dots = h_k = 0, \quad \text{and} \quad h_{k+1}, \dots, h_m > 0, \text{ for some } 1 \le k < m. \tag{4.9}$$

Let

$$c = \frac{\sum_{i=1}^{k} c_i}{\sum_{i=k+1}^{m} c_i} \tag{4.10}$$

and let $0 < t_0 < \min\{h_i^p/c : 1 \le i \le k\}^{1/p}$. For $0 \le t < t_0$, consider

$$y_t = (y_{1,t}, \dots, y_{m,t}) = \left(t, \dots, t, (h_{k+1}^p - ct^p)^{1/p}, \dots, (h_m^p - ct^p)^{1/p}\right). \tag{4.11}$$

Then, $P(y_0) = P(h)$. Moreover, since $y_t \in (0, \infty)^m$, for $0 < t < t_0$, the convex polytope $P(y_t)$ is proper and contains the origin in its interior by (4.6).

For further discussion, four facts are listed.

First, $P(y_t)$ is continuous in $t \in [0, t_0]$. In particular,

$$\lim_{t \to 0^+} P(y_t) = P(h). \tag{4.12}$$

Second, since the facet normals of the proper polytope $P(y_t)$ belong to $\{\xi_1, \dots, \xi_m\}$, and since $h_{P(y_t)}(\xi_i) \le y_{i,t}$ with equality if $S(P(y_t), \{\xi_i\}) > 0$, for $i = 1, \dots, m$, added the absolute continuity of $\mu_{\mathfrak{p}}(P(y_t), \cdot)$ with respect to $S(P(y_t), \cdot)$, we have

$$\mathrm{C}_{\mathfrak{p}}(P(y_t)) = \frac{\mathfrak{p} - 1}{n - \mathfrak{p}} \sum_{i=1}^{m} y_{i,t} \mu_{\mathfrak{p}}(P(y_t), \{\xi_i\}). \tag{4.13}$$

Third, for $t_1, t_2 \in [0, t_0]$, we have

$$\mathrm{C}_{\mathfrak{p}}(P(y_{t_1}), P(y_{t_2})) = \frac{\mathfrak{p} - 1}{n - \mathfrak{p}} \sum_{i=1}^{m} h_{P(y_{t_2})}(\xi_i) \mu_{\mathfrak{p}}(P(y_{t_1}), \{\xi_i\}). \tag{4.14}$$

Fourth, we have

$$\sum_{i=1}^{k} \mu_{\mathfrak{p}}(P(y_0), \{\xi_i\}) > 0. \tag{4.15}$$

Indeed, since there is at least one facet of $P(y_0)$ containing o,

$$\sum_{i=1}^{k} S(P(y_0), \{\xi_i\}) > 0.$$

Also, by Colesanti et al. [**20**, Lemma 2.18], there exists a positive constant c depending on n, $\mathfrak{p}$ and the radius of a ball containing $P(y_0)$, such that $\mu_{\mathfrak{p}}(P(y_0), \cdot) \ge c^{-\mathfrak{p}} S(P(y_0), \cdot)$. Thus, (4.15) holds.

From (4.14) combined with (4.13), (4.12) combined with the weak convergence of $\mathfrak{p}$-capacitary measures, and finally (4.15), it follows that

$$\begin{aligned}
&\frac{n-\mathfrak{p}}{\mathfrak{p}-1}\lim_{t\to 0^+}\frac{\mathrm{C}_\mathfrak{p}(P(y_t))-\mathrm{C}_\mathfrak{p}(P(y_t),P(y_0))}{t}\\
&\quad=\sum_{i=1}^{k}\lim_{t\to 0^+}\frac{t-0}{t}\mu_\mathfrak{p}(P(y_t),\{\xi_i\})\\
&\qquad+\sum_{i=k+1}^{m}\lim_{t\to 0^+}\frac{(h_i^p-ct^p)^{1/p}-h_i}{t}\mu_\mathfrak{p}(P(y_t),\{\xi_i\})\\
&\quad=\sum_{i=1}^{k}\mu_\mathfrak{p}(P(y_0),\{\xi_i\})\\
&\quad>0.
\end{aligned}$$

Hence, by the $\mathfrak{p}$-capacitary Minkowski inequality (2.4) and continuity of $\mathrm{C}_\mathfrak{p}(P(y_t))$ in t, it follows that

$$\begin{aligned}
&\mathrm{C}_\mathfrak{p}(P(y_0))^{\frac{n-\mathfrak{p}-1}{n-\mathfrak{p}}}\liminf_{t\to 0^+}\frac{\mathrm{C}_\mathfrak{p}(P(y_t))^{\frac{1}{n-\mathfrak{p}}}-\mathrm{C}_\mathfrak{p}(P(y_0))^{\frac{1}{n-\mathfrak{p}}}}{t}\\
&\quad=\liminf_{t\to 0^+}\frac{\mathrm{C}_\mathfrak{p}(P(y_t))-\mathrm{C}_\mathfrak{p}(P(y_t))^{\frac{n-\mathfrak{p}-1}{n-\mathfrak{p}}}\mathrm{C}_\mathfrak{p}(P(y_0))^{\frac{1}{n-\mathfrak{p}}}}{t}\\
&\quad\geq\liminf_{t\to 0^+}\frac{\mathrm{C}_\mathfrak{p}(P(y_t))-\mathrm{C}_\mathfrak{p}(P(y_t),P(y_0))}{t}\\
&\quad>0.
\end{aligned}$$

Consequently, for sufficiently small $t>0$, we have $\mathrm{C}_\mathfrak{p}(P(y_t)) > \mathrm{C}_\mathfrak{p}(P(y_0))$.

Now, take a sufficiently small $t>0$ and let $z=y_t$. To show that $P(z)$ is a desired convex polytope satisfying (4.8), it remains to verify $F_p(P(z))\leq 1$.

Indeed, from (4.5) combined with that $P(z)=P(y_t)$, the fact that $h_{P(y_t)}(\xi_i)\leq y_{i,t}$ for all i, (4.11), that $F_p(P(h))=\frac{\mathfrak{p}-1}{n-\mathfrak{p}}\sum_{i=k+1}^m c_i h_i^p$ (from (4.5), (4.6), (4.7) and (4.9)), (4.10), and finally that $P(h)$ is a solution to Problem 2, it follows that

$$\begin{aligned}
F_p(P(z))&=\frac{\mathfrak{p}-1}{n-\mathfrak{p}}\sum_{i=1}^{m}c_i h_{P(y_t)}(\xi_i)^p\\
&\leq\frac{\mathfrak{p}-1}{n-\mathfrak{p}}\sum_{i=1}^{m}c_i y_{i,t}^p\\
&=\frac{\mathfrak{p}-1}{n-\mathfrak{p}}\sum_{i=1}^{k}c_i t^p+\frac{\mathfrak{p}-1}{n-\mathfrak{p}}\sum_{i=k+1}^{m}c_i(h_i^p-ct^p)
\end{aligned}$$

$$= F_p(P(h)) + \frac{\mathfrak{p}-1}{n-\mathfrak{p}} \left(\sum_{i=1}^{k} c_i - c \sum_{i=k+1}^{m} c_i \right) t^p$$
$$= F_p(P(h))$$
$$= 1,$$

as desired. q.e.d.

The following is the normalized L_p Minkowski problem for $\mathfrak{p}$-capacity.

Problem 3. Among all convex bodies in $\mathbb{R}^n$ that contain the origin, find a convex body K such that

$$\frac{d\mu_{\mathfrak{p}}(K, \cdot)}{\mathrm{C}_{\mathfrak{p}}(K)} = h_K^{p-1} d\mu.$$

The following lemma presents relations between Problems 2 and 3.

Lemma 4.4. *Suppose μ is a finite Borel measure on $\mathbb{S}^{n-1}$ and is not concentrated on any closed hemisphere. Let $1 < p < \infty$, $1 < \mathfrak{p} < n$ and K be a convex body in $\mathbb{R}^n$ containing the origin. Then the following assertions hold.*

(1) If K contains the origin in its interior and solves Problem 2 for $(\mu, \mathfrak{p}, p)$, then it precisely solves Problem 3 for $(\mu, \mathfrak{p}, p)$.

(2) If K solves Problem 3 for $(\mu, \mathfrak{p}, p)$, then it solves Problem 2 for $(\mu, \mathfrak{p}, p)$.

Proof. First, assume $K \in \mathcal{K}_o^n$ is the solution to Problem 2. We prove that it also solves Problem 3. Take a nonnegative $f \in C(\mathbb{S}^{n-1})$. For $t \in (-t_0, t_0)$, let

$$K_t = [h_K + tf] \quad \text{and} \quad F_p(h_K + tf) = \frac{\mathfrak{p}-1}{n-\mathfrak{p}} \int_{\mathbb{S}^{n-1}} (h_K + tf)^p d\mu,$$

where $t_0 > 0$ is chosen so that $h_K + tf > 0$. For $t \in (-t_0, t_0)$, let

$$G(t) = \mathrm{C}_{\mathfrak{p}} \left(\frac{K_t}{F_p(h_K + tf)^{1/p}} \right).$$

Several observations are in order. First, $G(t)$ is continuous in t, since $\mathrm{C}_{\mathfrak{p}}$, K_t and $F_p(h_K + tf)$ are continuous in t. Second, $G(t) \le G(0) = \mathrm{C}_{\mathfrak{p}}(K)$, since $F_p(h_K + tf) \ge F_p(K_t) > 0$ and K solves Problem 2. Third, from

$$\left.\frac{dF_p(h_K + tf)}{dt}\right|_{t=0} = \frac{p(\mathfrak{p}-1)}{n-\mathfrak{p}} \int_{\mathbb{S}^{n-1}} f h_K^{p-1} d\mu$$

and

$$\left.\frac{d\mathrm{C}_{\mathfrak{p}}(K_t)}{dt}\right|_{t=0} = (\mathfrak{p}-1) \int_{\mathbb{S}^{n-1}} f d\mu_{\mathfrak{p}}(K, \cdot),$$

it follows that the derivative of G exists at $t = 0$. Thus, $G'(0) = 0$. Moreover,

$$\mathrm{C}_{\mathfrak{p}}(K)^{-1} \int_{\mathbb{S}^{n-1}} f d\mu_{\mathfrak{p}}(K, \cdot) = \int_{\mathbb{S}^{n-1}} f h_K^{p-1} d\mu.$$

Since the equation holds for any nonnegative $f \in C(\mathbb{S}^{n-1})$, it holds for any $f \in C(\mathbb{S}^{n-1})$. Thus, $\mathrm{C}_{\mathfrak{p}}(K)^{-1} d\mu_{\mathfrak{p}}(K, \cdot) = h_K^{p-1} d\mu$. Alternatively, K solves Problem 3.

We now prove assertion (2). Assume K solves Problem 3. Let Q be a convex body containing the origin, such that $1 = \frac{\mathfrak{p}-1}{n-\mathfrak{p}} \int_{\mathbb{S}^{n-1}} h_Q^p d\mu$. It suffices to prove

$$\mathrm{C}_{\mathfrak{p}}(K) \geq \mathrm{C}_{\mathfrak{p}}(Q).$$

Since $\mathrm{C}_{\mathfrak{p}}(K) h_K^{p-1} d\mu = d\mu_{\mathfrak{p}}(K, \cdot)$, it follows that

$$\begin{aligned} 1 &= \frac{\mathfrak{p}-1}{n-\mathfrak{p}} \int_{\{h_K>0\}} h_Q^p d\mu + \frac{\mathfrak{p}-1}{n-\mathfrak{p}} \int_{\{h_K=0\}} h_Q^p d\mu \\ &\geq \frac{\mathfrak{p}-1}{n-\mathfrak{p}} \int_{\{h_K>0\}} h_Q^p d\mu \\ &= \frac{\mathfrak{p}-1}{n-\mathfrak{p}} \int_{\{h_K>0\}} \left(\frac{h_Q}{h_K}\right)^p \frac{h_K}{\mathrm{C}_{\mathfrak{p}}(K)} d\mu_{\mathfrak{p}}(K, \cdot). \end{aligned}$$

Meanwhile, by the Poincaré $\mathfrak{p}$-capacity formula, it gives that

$$\mathrm{C}_{\mathfrak{p}}(K) = \frac{\mathfrak{p}-1}{n-\mathfrak{p}} \int_{\{h_K>0\}} h_K d\mu_{\mathfrak{p}}(K, \cdot).$$

So, the measure $\frac{\mathfrak{p}-1}{(n-\mathfrak{p})\mathrm{C}_{\mathfrak{p}}(K)} h_K d\mu_{\mathfrak{p}}(K, \cdot)$ is a Borel probability measure on $\{h_K \neq 0\}$. Thus, from the Jensen inequality, the definition of mixed $\mathfrak{p}$-capacity (2.2), and the $\mathfrak{p}$-capacitary Minkowski inequality (2.4), it follows that

$$\begin{aligned} 1 &\geq \left(\frac{\mathfrak{p}-1}{(n-\mathfrak{p})\mathrm{C}_{\mathfrak{p}}(K)} \int_{\{h_K>0\}} \left(\frac{h_Q}{h_K}\right)^p h_K d\mu_{\mathfrak{p}}(K, \cdot) \right)^{1/p} \\ &\geq \frac{\mathfrak{p}-1}{(n-\mathfrak{p})\mathrm{C}_{\mathfrak{p}}(K)} \int_{\{h_K>0\}} \frac{h_Q}{h_K} h_K d\mu_{\mathfrak{p}}(K, \cdot) \\ &= \frac{\mathfrak{p}-1}{(n-\mathfrak{p})\mathrm{C}_{\mathfrak{p}}(K)} \int_{\{h_K>0\}} h_Q d\mu_{\mathfrak{p}}(K, \cdot) \\ &= \frac{\mathrm{C}_{\mathfrak{p}}(K, Q)}{\mathrm{C}_{\mathfrak{p}}(K)} - \frac{\mathfrak{p}-1}{(n-\mathfrak{p})\mathrm{C}_{\mathfrak{p}}(K)} \int_{\{h_K=0\}} h_Q d\mu_{\mathfrak{p}}(K, \cdot) \end{aligned}$$

$$\geq \left(\frac{\mathrm{C}_{\mathfrak{p}}(Q)}{\mathrm{C}_{\mathfrak{p}}(K)}\right)^{\frac{1}{n-\mathfrak{p}}} - \frac{\mathfrak{p}-1}{(n-\mathfrak{p})\mathrm{C}_{\mathfrak{p}}(K)} \int_{\{h_K=0\}} h_Q d\mu_{\mathfrak{p}}(K,\cdot).$$

Since $\mathrm{C}_{\mathfrak{p}}(K) h_K^{p-1} d\mu = d\mu_{\mathfrak{p}}(K,\cdot)$, it yields that

$$\int_{\{h_K=0\}} h_Q d\mu_{\mathfrak{p}}(K,\cdot) = \mathrm{C}_{\mathfrak{p}}(K) \int_{\{h_K=0\}} h_Q h_K^{p-1} d\mu = 0.$$

Hence,

$$1 \geq \left(\frac{\mathrm{C}_{\mathfrak{p}}(Q)}{\mathrm{C}_{\mathfrak{p}}(K)}\right)^{\frac{1}{n-\mathfrak{p}}},$$

as desired. q.e.d.

By Lemma 4.4 (2) and Lemma 4.2, we obtain the following.

Lemma 4.5. *Suppose $1 < \mathfrak{p} < n$, $1 < p < \infty$, and that μ is a finite Borel measure on $\mathbb{S}^{n-1}$ and is not concentrated on any closed hemisphere. If Problem 3 for $(\mu, \mathfrak{p}, p)$ has a solution, then such solution is unique.*

Recall that our original concerned L_p Minkowski problem for $\mathfrak{p}$-capacity is as follows.

Problem 4. Among all convex bodies in $\mathbb{R}^n$ that contain the origin, find a convex body K such that

$$d\mu_{\mathfrak{p}}(K,\cdot) = h_K^{p-1} d\mu.$$

The next lemma shows the equivalence between Problems 4 and 3.

Lemma 4.6. *Suppose K is a convex body in $\mathbb{R}^n$ containing the origin, and $p + \mathfrak{p} \neq n$. Then the following assertions hold.*

(1) If K solves Problem 3, then

$$K^* := \mathrm{C}_{\mathfrak{p}}(K)^{1/(p+\mathfrak{p}-n)} K$$

solves Problem 4.

(2) If K solves Problem 4, then

$$\tilde{K} := \mathrm{C}_{\mathfrak{p}}(K)^{-1/p} K$$

solves Problem 3.

Proof. We only prove the first assertion. Assertion (2) can be proved similarly.

From the positive homogeneity of $\mu_{\mathfrak{p}}$, that $d\mu_{\mathfrak{p}}(K,\cdot) = \mathrm{C}_{\mathfrak{p}}(K) h_K^{p-1} d\mu$, and the positive homogeneity of support functions, it follows that

$$\begin{aligned} d\mu_{\mathfrak{p}}\left(K^*,\cdot\right) &= d\mu_{\mathfrak{p}}\left(\mathrm{C}_{\mathfrak{p}}(K)^{1/(p+\mathfrak{p}-n)} K,\cdot\right) \\ &= \mathrm{C}_{\mathfrak{p}}(K)^{(n-\mathfrak{p}-1)/(p+\mathfrak{p}-n)} d\mu_{\mathfrak{p}}\left(K,\cdot\right) \end{aligned}$$

$$= \mathrm{C}_{\mathfrak{p}}(K)^{(n-\mathfrak{p}-1)/(p+\mathfrak{p}-n)} \mathrm{C}_{\mathfrak{p}}(K) h^{p-1}_{\mathrm{C}_{\mathfrak{p}}(K)^{1/(n-\mathfrak{p}-p)} K^*} d\mu$$
$$= h^{p-1}_{K^*} d\mu,$$

which proves the first assertion. q.e.d.

5. An approximation lemma: applications of L_p Minkowski problems for volume

In light of the important works [40] by Jerison and [20] by Colesanti et al., for a finite Borel measure μ on the sphere $\mathbb{S}^{n-1}$, we consider the following conditions.

(A_1) The measure μ is not concentrated on any closed hemisphere.
(A_2) The measure μ does not have a pair of antipodal point masses; that is, if $\mu(\{\xi\}) > 0$, then $\mu(\{-\xi\}) = 0$, for $\xi \in \mathbb{S}^{n-1}$.

In Section 7, to prove the existence of Theorem 1.1, we first deal with the discrete measure case under the conditions (A_1) and (A_2). In order to remove the condition (A_2) for general measures, we prove and make good use of the following Lemma 5.1, which is one of the crucial ingredients in our approximation strategy. Thanks to the already important developments on L_p Minkowski problems for volume, we can provide a geometric proof for Lemma 5.1.

Lemma 5.1. *Suppose μ is a finite positive Borel measure on $\mathbb{S}^{n-1}$ satisfying condition (A_1). Then there exists a sequence of finite and discrete measures $\{\mu_j\}_j$ on $\mathbb{S}^{n-1}$, which satisfy conditions (A_1) and (A_2), converging weakly to μ.*

To prove Lemma 5.1, we need to make some preparations. The next lemma is due to Hug et al. [39]. See also Chou and Wang [16].

Lemma 5.2. *Suppose $1 < p < \infty$ and that μ is a finite Borel measure on $\mathbb{S}^{n-1}$. Then the following assertions hold.*

(1) If μ is discrete and satisfies condition (A_1), there is a unique convex polytope $P \in \mathcal{K}^n_o$, such that $V(P)^{-1} S_p(P, \cdot) = \mu$.

(2) If μ satisfies condition (A_1) and $p \geq n$, there is a unique convex body $K \in \mathcal{K}^n_o$, such that $V(K)^{-1} S_p(K, \cdot) = \mu$.

Lemma 5.3. *Suppose K is a smooth and strictly convex body in $\mathbb{R}^n$ containing the origin in its interior. Then K can be approximated by a sequence of convex polytopes $\{P_j\}_j$ with the origin in their interiors and each $S_p(P_j, \cdot)$ satisfying condition (A_2), where $1 < p < \infty$.*

Proof. Let $\mu = V(K)^{-1} S_p(K, \cdot)$. Clearly, it satisfies conditions (A_1) and (A_2). Then, we can take a sequence of discrete measures $\{\mu_k\}_k$, which satisfy conditions (A_1) and (A_2), converging weakly to μ. By Lemma 5.2 (1), for each μ_k, there is a unique convex polytope Q_k with the origin in its interior, such that $\mu_k = V(Q_k)^{-1} S_p(Q_k, \cdot)$. In the proof of Lemma 5.2, Hug et al. [**39**] proved that $\{Q_k\}_k$ has a convergent subsequence $\{Q_{k_j}\}_j$, converging to K. Let $P_j = Q_{k_j}$, for each j. Then $\{P_j\}_j$ satisfies the requirements. q.e.d.

By (1.4) and the weak continuity of surface area measure, we obtain the following.

Lemma 5.4. *Suppose $K_j, K \in \mathcal{K}_o^n$, $j \in \mathbb{N}$. If $K_j \to K$, then $S_p(K_j,\cdot) \to S_p(K,\cdot)$ weakly, as $j \to \infty$.*

Proof of Lemma 5.1. Take a fixed $p \in [n,\infty)$. By Lemma 5.2 (2), there is a unique convex body K with the origin in its interior, such that $V(K)^{-1}S_p(K,\cdot) = \mu$. Consider the space of all compact convex sets in $\mathbb{R}^n$ with the Hausdorff metric. In this complete metric space, the set of smooth and strictly convex bodies is dense. See Schneider [**67**, Sections 2.7 and 3.4]. Thus, for each $j \in \mathbb{N}$, we can take a smooth and strictly convex body K_j with the origin in its interior, such that

$$\delta_H(K, K_j) \le 1/2j.$$

Meanwhile, by Lemma 5.3 we can take a convex polytope P_j with the origin in its interior, such that its L_p surface area measure $S_p(P_j,\cdot)$ satisfies condition (A_2), and

$$\delta_H(K_j, P_j) \le 1/2j.$$

So,

$$\delta_H(K, P_j) \le \delta_H(K, K_j) + \delta_H(K_j, P_j) \le 1/j.$$

Thus, $P_j \to K$, as $j \to \infty$. From Lemma 5.4 and the continuity of volume functional, it follows that

$$V(P_j)^{-1}S_p(P_j,\cdot) \to V(K)^{-1}S_p(K,\cdot),$$

weakly. For each j, let

$$\mu_j = V(P_j)^{-1}S_p(P_j,\cdot).$$

Then each μ_j is discrete and satisfies conditions (A_1) and (A_2). q.e.d.

6. More technical preparations for approximations

In light of the relations of Problems 2 and 3, we solve Problem 3 for general measures via the passage by solving Problem 2 in Section 7. For this aim, we need to make more preparatory works.

Throughout this section, let $1 < p < \infty$ and $1 < \mathfrak{p} < n$. Suppose μ and μ_j, $j \in \mathbb{N}$, are finite Borel measures on $\mathbb{S}^{n-1}$ and not concentrated on any closed hemisphere. Suppose $\mu_j \to \mu$ weakly, as $j \to \infty$. For each j, let K_j be the solution to Problem 3 for $(\mu_j, \mathfrak{p}, p)$, and

$$\bar{K}_j = \mathrm{C}_{\mathfrak{p}}(K_j)^{-1/(n-\mathfrak{p})}K_j. \tag{6.1}$$

Then, by Lemma 4.4 (2), K_j is the solution to Problem 2 for $(\mu_j, \mathfrak{p}, p)$; and $\bar{K}_j$ is the solution to Problem 1 for $(\mu_j, \mathfrak{p}, p)$, by Lemma 4.1 (2).

For a convex body Q in $\mathbb{R}^n$ containing the origin, let

$$F_{p,j}(Q)=\frac{\mathfrak{p}-1}{n-\mathfrak{p}}\int_{\mathbb{S}^{n-1}} h_Q^p d\mu_j \quad \text{and} \quad F_p(Q)=\frac{\mathfrak{p}-1}{n-\mathfrak{p}}\int_{\mathbb{S}^{n-1}} h_Q^p d\mu.$$

Lemma 6.1. *The sequences* $\{K_j\}_j$ *and* $\{\bar{K}_j\}_j$ *are bounded from above.*

Proof. For each j, there is a $\xi_j \in \mathbb{S}^{n-1}$ such that $h_{K_j}(\xi_j) = \max_{\mathbb{S}^{n-1}} h_{K_j}$. Since the segment joining the origin and $(\max_{\mathbb{S}^{n-1}} h_{K_j})\xi_j$ is contained in K_j, it follows that for all $\xi \in \mathbb{S}^{n-1}$

$$h_{K_j}(\xi) \geq (\max_{\mathbb{S}^{n-1}} h_{K_j})(\xi_j\cdot\xi)_+,$$

where $(\xi_j\cdot\xi)_+ = \max\{0,\xi_j\cdot\xi\}$. Thus,

$$\begin{aligned} 1 &= \frac{\mathfrak{p}-1}{n-\mathfrak{p}}\int_{\mathbb{S}^{n-1}} h_{K_j}^p d\mu_j \\ &\geq (\max_{\mathbb{S}^{n-1}} h_{K_j})^p\cdot\frac{\mathfrak{p}-1}{n-\mathfrak{p}}\int_{\mathbb{S}^{n-1}} (\xi_j\cdot\xi)_+^p d\mu_j(\xi) \\ &\geq (\max_{\mathbb{S}^{n-1}} h_{K_j})^p\cdot\frac{\mathfrak{p}-1}{n-\mathfrak{p}}\min_{\xi'\in\mathbb{S}^{n-1}}\int_{\mathbb{S}^{n-1}} (\xi'\cdot\xi)_+^p d\mu_j(\xi). \end{aligned}$$

Consider the function $\mathbb{R}^n\to\mathbb{R}$,

$$x\mapsto\left(\frac{\mathfrak{p}-1}{n-\mathfrak{p}}\int_{\mathbb{S}^{n-1}} (x\cdot\xi)_+^p d\mu_j(\xi)\right)^{1/p}.$$

By $((x+x')\cdot\xi)_+ \leq (x\cdot\xi)_+ + (x'\cdot\xi)_+$ and the Minkowski integral inequality, this function is convex; Since μ_j is not concentrated on any closed hemisphere, this function is strictly positive for any nonzero x. Thus, it is the support function of a unique convex body containing the origin in its interior, say $\Pi_{\mathfrak{p},p}\mu_j \in \mathcal{K}_o^n$. So, $\min_{\mathbb{S}^{n-1}} h_{\Pi_{\mathfrak{p},p}\mu_j} > 0$ and

$$\max_{\mathbb{S}^{n-1}} h_{K_j} \leq \frac{1}{\min_{\mathbb{S}^{n-1}} h_{\Pi_{\mathfrak{p},p}\mu_j}} < \infty.$$

Similarly, define the convex body $\Pi_{\mathfrak{p},p}\mu \in \mathcal{K}_o^n$ by

$$h_{\Pi_{\mathfrak{p},p}\mu}(x)=\left(\frac{\mathfrak{p}-1}{n-\mathfrak{p}}\int_{\mathbb{S}^{n-1}} (x\cdot\xi)_+^p d\mu(\xi)\right)^{1/p}.$$

Since $\mu_j\to\mu$ weakly, it follows that $h_{\Pi_{\mathfrak{p},p}\mu_j}\to h_{\Pi_{\mathfrak{p},p}\mu}$ uniformly on $\mathbb{S}^{n-1}$. Since the functions $h_{\Pi_{\mathfrak{p},p}\mu_j}$, $h_{\Pi_{\mathfrak{p},p}\mu}$ are strictly positive on $\mathbb{S}^{n-1}$, the sequence $\{h_{\Pi_{\mathfrak{p},p}\mu_j}\}_j$ on $\mathbb{S}^{n-1}$ is uniformly bounded from below by a constant $m > 0$. See, e.g., [**74**, Theorem 3.1] for its proof. Thus,

$$\sup_j \max_{\mathbb{S}^{n-1}} h_{K_j} \le \frac{1}{\inf_j \min_{\mathbb{S}^{n-1}} h_{\Pi_{p,\mathfrak{p}}}} \le \frac{1}{m} < \infty. \tag{6.2}$$

Consequently, the sequence $\{K_j\}_j$ is bounded from above.

To prove the boundedness of the sequence $\{\bar{K}_j\}_j$, two observations are listed. First, since $F_{p,j}\left((\frac{\mathfrak{p}-1}{n-\mathfrak{p}}|\mu_j|)^{-1/p}B\right) = 1$, where $|\mu_j| = \mu_j(\mathbb{S}^{n-1})$, it follows that

$$\mathrm{C}_\mathfrak{p}(K_j) \ge \mathrm{C}_\mathfrak{p}\left(\left(\frac{\mathfrak{p}-1}{n-\mathfrak{p}}\cdot|\mu_j|\right)^{-1/p} B\right). \tag{6.3}$$

Second, since $\mu_j \to \mu$ weakly, it follows that

$$\sup_j |\mu_j| < \infty. \tag{6.4}$$

Let

$$M = \left(\frac{\mathfrak{p}-1}{n-\mathfrak{p}}\right)^{1/p} \mathrm{C}_\mathfrak{p}(B)^{-1/(n-\mathfrak{p})} \sup_j |\mu_j|^{1/p} \sup_j \max_{\mathbb{S}^{n-1}} h_{K_j}. \tag{6.5}$$

From (6.1), (6.3), (6.5), and finally (6.2) together with (6.4), it follows that

$$\begin{aligned}\max_{\mathbb{S}^{n-1}} h_{\bar{K}_j} &= \frac{\max_{\mathbb{S}^{n-1}} h_{K_j}}{\mathrm{C}_\mathfrak{p}(K_j)^{1/(n-\mathfrak{p})}}\\ &\le \frac{\max_{\mathbb{S}^{n-1}} h_{K_j}}{\mathrm{C}_\mathfrak{p}\left(\left(\frac{\mathfrak{p}-1}{n-\mathfrak{p}}|\mu_j|\right)^{-1/p} B\right)^{1/(n-\mathfrak{p})}}\\ &\le M\\ &< \infty,\end{aligned}$$

which concludes that the sequence $\{\bar{K}_j\}_j$ is bounded from above. q.e.d.

Since $K_j = F_{p,j}(\bar{K}_j)^{-1/p}\bar{K}_j$, and $\mathrm{C}_\mathfrak{p}(K_j)h_{K_j}^{p-1}d\mu_j = d\mu_\mathfrak{p}(K_j,\cdot)$, we have

$$\mathrm{C}_\mathfrak{p}\left(\frac{\bar{K}_j}{F_{p,j}(\bar{K}_j)^{1/p}}\right) h^{p-1}_{\frac{\bar{K}_j}{F_{p,j}(\bar{K}_j)^{1/p}}} d\mu_j = d\mu_\mathfrak{p}\left(\frac{\bar{K}_j}{F_{p,j}(\bar{K}_j)^{1/p}},\cdot\right).$$

Since $\mathrm{C}_\mathfrak{p}(\bar{K}_j) = 1$, by the positive homogeneity of $\mathfrak{p}$-capacity, support function and $\mathfrak{p}$-capacitary measure, we obtain

$$h_{\bar{K}_j}^{p-1} d\mu_j = F_{p,j}(\bar{K}_j) d\mu_\mathfrak{p}(\bar{K}_j,\cdot). \tag{6.6}$$

Lemma 6.2. *If $\{\bar{K}_j\}_j$ converges to a compact convex set $\bar{K}$, then $\dim(\bar{K}) \ne n-1$.*

Proof. We argue by contradiction and assume that $\dim(\bar{K}) = n-1$.

Let $f \in C(\mathbb{S}^{n-1})$ be nonnegative. By Colesanti et al. [**20**, Lemma 2.18], there is a positive constant c depending on n, $\mathfrak{p}$ and M (given by (6.5)), such that $\mu_{\mathfrak{p}}(\bar{K}_j, \cdot) \geq c^{-\mathfrak{p}} S(\bar{K}_j, \cdot)$. Thus,

$$\int_{\mathbb{S}^{n-1}} f h_{\bar{K}_j}^{p-1} d\mu_j \geq c^{-\mathfrak{p}} F_{p,j}(\bar{K}_j) \int_{\mathbb{S}^{n-1}} f dS(\bar{K}_j, \cdot). \tag{6.7}$$

Now, several facts are in order. First, the convergence $\bar{K}_j \to \bar{K}$ is equivalent to the uniform convergence $h_{\bar{K}_j} \to h_{\bar{K}}$ on $\mathbb{S}^{n-1}$. Second, the uniform convergence $h_{\bar{K}_j} \to h_{\bar{K}}$ together with the weak convergence $\mu_j \to \mu$ yields the convergence $F_{p,j}(\bar{K}_j) \to F_p(\bar{K})$. Third, the convergence $\bar{K}_j \to \bar{K}$ again yields the weak convergence $S(\bar{K}_j, \cdot) \to S(\bar{K}, \cdot)$. Hence, letting $j \to \infty$, (6.7) yields the inequality

$$\int_{\mathbb{S}^{n-1}} f h_{\bar{K}}^{p-1} d\mu \geq c^{-\mathfrak{p}} F_p(\bar{K}) \int_{\mathbb{S}^{n-1}} f dS(\bar{K}, \cdot). \tag{6.8}$$

Recall that $\bar{K}$ is contained in an $(n-1)$-dimensional subspace with normal $\xi_0 \in \mathbb{S}^{n-1}$. So, $S(\bar{K}, \cdot) = V_{n-1}(\bar{K})(\delta_{\xi_0} + \delta_{-\xi_0})$, where $V_{n-1}(\bar{K})$ is the $(n-1)$-dimensional volume of $\bar{K}$. Let $\tilde{\mu}$ be the Borel measure on $\mathbb{S}^{n-1}$ given by $d\tilde{\mu} = h_{\bar{K}}^{p-1} d\mu$. And let $c' = c^{-\mathfrak{p}} F(\bar{K}) V_{n-1}(\bar{K})$. Then, c' is a positive constant (by Lemma 2.3) and is independent of f. The inequality (6.8) can be reformulated as

$$\int_{\mathbb{S}^{n-1}} f d\tilde{\mu} \geq c' \cdot (f(\xi_0) + f(-\xi_0)). \tag{6.9}$$

Consider the set

$$\mathfrak{F} = \left\{ f \in C(\mathbb{S}^{n-1}) : f \geq 0 \text{ and } f(\xi_0) = f(-\xi_0) = \frac{1}{2} \right\}.$$

Since the inequality (6.9) holds for all nonnegative continuous functions f, we obtain

$$\inf_{f \in \mathfrak{F}} \int_{\mathbb{S}^{n-1}} f d\tilde{\mu} \geq c' > 0.$$

On the other hand, since $d\tilde{\mu} = h_{\bar{K}}^{p-1} d\mu$ and $h_{\bar{K}}(\pm\xi_0) = 0$, it follows that

$$\int_{\mathbb{S}^{n-1}} f d\tilde{\mu} = \int_{\mathbb{S}^{n-1} \setminus \{\pm\xi_0\}} f d\tilde{\mu}.$$

Take a decreasing sequence $\{f_k\}_k \subset \mathfrak{F}$, which converges pointwise to $\chi_{\{\pm\xi_0\}}/2$. Then, $\lim_{k\to\infty} \int_{\mathbb{S}^{n-1}} f_k d\tilde{\mu} = 0$. Therefore,

$$\inf_{f \in \mathfrak{F}} \int_{\mathbb{S}^{n-1}} f d\tilde{\mu} = 0 < c'.$$

A contradiction occurs. Hence, $\dim(\bar{K}) \neq n-1$. q.e.d.

Lemma 6.3. *If $\{\bar{K}_j\}_j$ converges to a compact convex set $\bar{K}$, and in addition each μ_j is discrete and satisfies condition (A_2), then* $\dim(\bar{K}) = n$.

Proof. Let $\dim(\bar{K}) = k$. We argue by contradiction. In light of Lemma 6.2, we may assume $k < n-1$. Then, it necessarily has $n-\mathfrak{p} < k$. Otherwise, $\mathrm{C}_{\mathfrak{p}}(\bar{K}) = 0$, which is impossible because of the continuity of $\mathrm{C}_{\mathfrak{p}}$ and the fact that $\mathrm{C}_{\mathfrak{p}}(\bar{K}_j) = 1$ for each j.

By Lemma 6.1, we can take some $\rho > 0$ so that $\bar{K}_j \subseteq \rho B$. For each j, let $t_j = \delta_H\left(\bar{K}_j, \bar{K}\right)$. Then, $t_j > 0$ for each j; and $t_j \to 0$, as $j \to \infty$. We show that there appears

$$\lim_{j\to\infty} \mu_j(\mathbb{S}^{n-1}) = \infty, \tag{6.10}$$

which will contradict that $\mu_j \to \mu$ weakly. Then, the proof is completed.

The limit (6.10) follows from the recent work [**1**] of Akman et al.. We only need to repeat their argument on page 94, where E_j and E are replaced by $\bar{K}_j$ and $\bar{K}$, respectively. What we require modifying slightly is the proof of their inequality (13.48), i.e.,

$$t_j^{\psi-1}\nu_j(\partial E_j \cap aB) \leq C'\mu_j^*(\mathrm{g}_{\partial E_j}(\partial E_j \cap aB)). \tag{6.11}$$

This inequality still holds in the L_p setting. In fact, from Akman's et al. inequality (13.47), an application of Akman's et al. equation (13.39) to $f(y) = \|y\|_{\mathfrak{p}}$ for $y \in \mathbb{R}^n$, and our identity (6.6), it follows that

$$t_j^{\psi-1}\nu_j(\partial E_j \cap aB) \leq \frac{C'}{F_{p,j}(E_j)} \int\limits_{\mathrm{g}_{\partial E_j}(\partial E_j \cap aB)} h_{E_j}^{p-1} d\mu_j^*,$$

which reduces to (13.48) when $p = 1$. This inequality yields (6.11) when $p > 1$. Indeed, since $h_{E_j} \leq \rho$ (i.e., $E_j = \bar{K}_j \subseteq \rho B$), and $c := \min\{F_{p,j}(E_j), F_p(E) : j \in \mathbb{N}\} > 0$ (by the convergence $F_{p,j}(E_j) \to F_p(E)$ and Lemma 2.3), it follows that

$$\frac{1}{F_{p,j}(E_j)} \int\limits_{\mathrm{g}_{\partial E_j}(\partial E_j \cap aB)} h_{E_j}^{p-1} d\mu_j \leq \frac{\rho^{p-1}}{c}\mu_j^*(\mathrm{g}_{\partial E_j}(\partial E_j \cap aB)).$$

Repeating Akman's et al. argument after (13.48), we will arrive at (6.10). In fact, Akman et al. proved that there is a positive constant c' such that

$$\liminf_{j\to\infty} \nu_j(\partial E_j \cap aB) \geq c'.$$

This, together with (6.11) (i.e. Akman's et al. inequality (13.48)), directly gives

$$\infty = c' \lim_{j\to\infty} t_j^{\psi-1} \leq C' \lim_{j\to\infty} \mu_j(\mathbb{S}^{n-1}),$$

where C' is a positive constant independent of j, and $\psi - 1 = (1-n+k)/(\mathfrak{p}-1) < 0$. q.e.d.

Lemma 6.4. *If $\{\bar{K}_j\}_j$ converges to a compact convex set $\bar{K}$, and in addition $1 < \mathfrak{p} \le 2$, then $\dim(\bar{K}) = n$.*

Proof. The arguments here are similar to those of Colesanti et al. [**20**, p. 1571]. If $1 < \mathfrak{p} \le 2$ and $\dim(\bar{K}) \le n-2$, then $\dim(\bar{K}) \le n - \mathfrak{p}$ and thus $\mathcal{H}^{n-\mathfrak{p}}(\bar{K}) < \infty$. According to Evans and Gariepy [**21**, Theorem 3, p. 154]: If $\mathcal{H}^{n-\mathfrak{p}}(\bar{K}) < \infty$, then $\mathrm{C}_{\mathfrak{p}}(\bar{K}) = 0$. It follows that $\mathrm{C}_{\mathfrak{p}}(\bar{K}) = 0$. But this is impossible, because of the continuity of $\mathrm{C}_{\mathfrak{p}}$ and the fact that $\mathrm{C}_{\mathfrak{p}}(\bar{K}_j) = 1$ for each j. q.e.d.

Lemma 6.5. *Suppose $\{\bar{K}_j\}_j$ converges to a compact convex set $\bar{K}$. If in addition each μ_j is discrete and satisfies condition (A_2), or $1 < \mathfrak{p} \le 2$, then*

$$K := \left(\frac{\mathfrak{p}-1}{n-\mathfrak{p}} \int_{\mathbb{S}^{n-1}} h_{\bar{K}}^p d\mu \right)^{-1/p} \bar{K}$$

is the unique convex body solving Problem 3 for $(\mu, \mathfrak{p}, p)$.

Proof. From Lemma 6.3 or 6.4, together with (4.1) (or Lemma 2.3), it follows that K is a convex body containing the origin. Since

$$K_j = \left(\frac{\mathfrak{p}-1}{n-\mathfrak{p}} \int_{\mathbb{S}^{n-1}} h_{\bar{K}_j}^p d\mu_j \right)^{-1/p} \bar{K}_j,$$

and

$$\lim_{j\to\infty} \int_{\mathbb{S}^{n-1}} h_{\bar{K}_j}^p d\mu_j = \int_{\mathbb{S}^{n-1}} h_{\bar{K}}^p d\mu,$$

it follows that $K_j \to K$. Since $\mathrm{C}_{\mathfrak{p}}(K_j) h_{K_j}^{p-1} d\mu_j = d\mu_{\mathfrak{p}}(K_j, \cdot)$, and the uniform convergence $h_{K_j} \to h_K$ yields the convergence $\mathrm{C}_{\mathfrak{p}}(K_j) \to \mathrm{C}_{\mathfrak{p}}(K)$ and $\mu_{\mathfrak{p}}(K_j, \cdot) \to \mu_{\mathfrak{p}}(K, \cdot)$ weakly, it follows that $\mathrm{C}_{\mathfrak{p}}(K) h_K^{p-1} d\mu = d\mu_{\mathfrak{p}}(K, \cdot)$. Thus, K is a solution to Problem 3 for μ. The uniqueness of such body K is guaranteed by Lemma 4.5. q.e.d.

From Lemma 6.1, the Blaschke selection theorem and Lemma 6.5, we obtain the following main lemma in this section, which is a crucial tool to establish Theorem 1.1.

Lemma 6.6. *Suppose μ and μ_j, $j \in \mathbb{N}$, are finite Borel measures on $\mathbb{S}^{n-1}$ that are not concentrated on any closed hemisphere, and $\mu_j \to \mu$ weakly as $j \to \infty$. Let $1 < p < \infty$ and $1 < \mathfrak{p} < n$. For each j, assume Problem 3 for $(\mu_j, \mathfrak{p}, p)$ has a unique solution. If in addition each μ_j is discrete and satisfies condition (A_2), then Problem 3 for $(\mu, \mathfrak{p}, p)$ has a unique solution.*

Theorem 6.7. *Suppose μ and μ_j, $j \in \mathbb{N}$, are finite Borel measures on $\mathbb{S}^{n-1}$ that are not concentrated on any closed hemisphere. Let $1 < p < \infty$ and $1 < \mathfrak{p} \leq 2$. Assume K and K_j, $j \in \mathbb{N}$, are the unique solution to Problem 3 for $(\mu, \mathfrak{p}, p)$ and $(\mu_j, \mathfrak{p}, p)$, respectively. If $\mu_j \to \mu$ weakly, then $K_j \to K$, as $j \to \infty$.*

Proof. By Lemma 4.4 (2), K_j and K are the unique solution to Problem 2 for $(\mu_j, \mathfrak{p}, p)$ and $(\mu, \mathfrak{p}, p)$, respectively. By Lemma 6.1, $\{K_j\}_j$ is bounded from above. To prove $\lim_{j\to\infty} K_j = K$, it suffices to prove that each convergent subsequence $\{K_{j_l}\}_l$ of $\{K_j\}_j$ converges to K. For such subsequence, let $\bar{K}_{j_l} = \mathrm{C}_\mathfrak{p}(K_{j_l})^{-1/(n-\mathfrak{p})} K_{j_l}$, for $l \in \mathbb{N}$.

By Lemma 4.1 (2) and Lemma 4.2, $\bar{K}_{j_l}$ is the unique solution to Problem 1 for $(\mu_{j_l}, \mathfrak{p}, p)$. By Lemma 6.1, $\{\bar{K}_{j_l}\}_l$ is bounded from above. By the Blaschke selection theorem, $\{\bar{K}_{j_l}\}_l$ has a subsequence $\{\bar{K}_{j_{l_i}}\}_i$ converging to a compact convex set $\bar{K}_0$. By Lemma 6.5, the set $K_0 := \left(\frac{\mathfrak{p}-1}{n-\mathfrak{p}} \int_{\mathbb{S}^{n-1}} h_{\bar{K}_0}^p d\mu\right)^{-1/p} \bar{K}_0$ is the unique convex body solving Problem 3 for $(\mu, \mathfrak{p}, p)$. This in turn ensures $K_0 = K$. Therefore, $\lim_{i\to\infty} K_{j_{l_i}} = K$. Recall that $\{K_{j_l}\}_l$ is convergent. Hence, $\lim_{l\to\infty} K_{j_l} = K$. q.e.d.

Theorem 6.7 and Lemma 4.6 directly yield the following corollaries.

Corollary 6.8. *Suppose μ and μ_j, $j \in \mathbb{N}$, are finite Borel measures on $\mathbb{S}^{n-1}$ that are not concentrated on any closed hemisphere. Let $1 < p < \infty$ and $1 < \mathfrak{p} \leq 2$. Assume K and K_j, $j \in \mathbb{N}$, are convex bodies containing the origin such that $d\mu_\mathfrak{p}(K, \cdot) = c h_K^{p-1} d\mu$ and $d\mu_\mathfrak{p}(K_j, \cdot) = c_j h_{K_j}^{p-1} d\mu_j$, where $c = c_j = 1$ if $p + \mathfrak{p} \neq n$, or $c = \mathrm{C}_\mathfrak{p}(K)$ and $c_j = \mathrm{C}_\mathfrak{p}(K_j)$ if $p + \mathfrak{p} = n$. If $\mu_j \to \mu$ weakly, then $K_j \to K$, as $j \to \infty$.*

Corollary 6.9. *Suppose $K, K_j \in \mathcal{K}_o^n$, $j \in \mathbb{N}$, $1 < p < \infty$ and $1 < \mathfrak{p} \leq 2$. Then the following assertions hold.*

(1) If $p + \mathfrak{p} \neq n$ and $\mu_{p,\mathfrak{p}}(K_j, \cdot) \to \mu_{p,\mathfrak{p}}(K, \cdot)$ weakly, then $K_j \to K$, as $j \to \infty$.

(2) If $\mathrm{C}_\mathfrak{p}(K_j)^{-1} \mu_{n-\mathfrak{p},\mathfrak{p}}(K_j, \cdot) \to \mathrm{C}_\mathfrak{p}(K)^{-1} \mu_{n-\mathfrak{p},\mathfrak{p}}(K, \cdot)$ weakly, then $K_j \to K$, as $j \to \infty$.

7. Proofs of the L_p Minkowski problem for $\mathfrak{p}$-capacity

Throughout this section, let $1 < p < \infty$ and $1 < \mathfrak{p} < n$. We finish the proofs of Theorems 1.1, 1.2 and 1.3 by 4 steps.

Step I. We prove the existence of solution to Problem 3 for a discrete measure satisfying conditions (A_1) and (A_2). Precisely, we prove the following

Lemma 7.1. *Let μ be a finite Borel measure on $\mathbb{S}^{n-1}$. If μ is discrete and satisfies condition (A_1) and (A_2), then there exists a unique convex* *proper polytope P containing the origin in its interior, such that*

$$\frac{\mu_{p,\mathfrak{p}}(P,\cdot)}{\mathrm{C}_{\mathfrak{p}}(P)}=\mu.$$

Proof. Assume $\mu=\sum_{i=1}^{m}c_i\delta_{\xi_i}$, where $\xi_i\in\mathbb{S}^{n-1}$ and $c_i>0$, for all i. For a convex body Q containing the origin, let

$$F_p(Q)=\frac{\mathfrak{p}-1}{n-\mathfrak{p}}\sum_{i=1}^{m}c_ih_Q(\xi_i)^p.$$

In light of the relations of solutions to Problems 1, 2 and 3, for convenience, we aim to prove Problem 1 for $(\mu,\mathfrak{p},p)$ has a solution P_0, and P_0 is a convex proper polytope containing the origin.

First, we verify that there is a minimizing sequence $\{P_j\}_j$ for Problem 1, and each P_j is a convex proper polytope with facet normals belonging to $\{\xi_1,\ldots,\xi_m\}$. For this aim, take a minimizing sequence $\{Q_j\}_j$ of convex bodies for Problem 1. Corresponding to each body Q_j, we take the Wulff shape

$$P_j=\left\{x\in\mathbb{R}^n : x\cdot\xi_i\le h_{Q_j}(\xi_i), i=1,\ldots,m\right\}.$$

Since μ is not concentrated on any closed hemisphere and Q_j is a convex body containing the origin, P_j is a bounded convex polytope containing Q_j. Moreover, $h_{Q_j}(\xi_i)=h_{P_j}(\xi_i)$, for all i. So,

$$\mathrm{C}_{\mathfrak{p}}(P_j)\ge\mathrm{C}_{\mathfrak{p}}(Q_j)\ge 1\quad\text{and}\quad F_p(P_j)=F_p(Q_j).$$

Clearly, $\{P_j\}_j$ is a desired minimizing sequence for Problem 1.

Second, we show the boundedness of $\{P_j\}_j$. If so, then $\{P_j\}_j$ has a convergent subsequence $\{P_{j_k}\}_k$ by the Blaschke selection theorem, which converges to a convex polytope, say P_0.

Since $0<\mathrm{C}_{\mathfrak{p}}(B)<\infty$ and $\mathrm{C}_{\mathfrak{p}}\left(\mathrm{C}_{\mathfrak{p}}(B)^{-1/(n-\mathfrak{p})}B\right)=1$, it follows that

$$\begin{aligned}&\inf\left\{F_p(Q): o\in Q\in\mathcal{K}^n, \mathrm{C}_{\mathfrak{p}}(Q)\ge 1\right\}\\&\le M:=F_p\left(\mathrm{C}_{\mathfrak{p}}(B)^{-1/(n-\mathfrak{p})}B\right)<\infty.\end{aligned}$$

So, there is an index j_0 such that $F_p(P_j)\le M$ for all $j\ge j_0$. Without loss of generality, assume $F_p(P_j)\le M$ for all j. Since each c_i is positive and each P_j contains the origin, it follows that for all i and j,

$$\frac{\mathfrak{p}-1}{n-\mathfrak{p}}\min\left\{c_i : i=1,\ldots,m\right\}h_{P_j}(\xi_i)^p\le\frac{\mathfrak{p}-1}{n-\mathfrak{p}}\sum_{i=1}^{m}c_ih_{P_j}(\xi_i)^p\le M.$$

Thus,

$$h_{P_j}(\xi_i)\le\left(\frac{n-\mathfrak{p}}{\mathfrak{p}-1}\cdot\frac{M}{\min\left\{c_i : i=1,\ldots,m\right\}}\right)^{1/p}<\infty.$$

It implies that the minimizing sequence $\{P_j\}_j$ is bounded from above.

Third, we show $\dim(P_0)=n$. So, P_0 is a solution to Problem 1.

Since the facet normals of P_0 belong to the set $\{\xi_1,\dots,\xi_m\}$, by condition (A_2), no two vectors in $\{\xi_1,\dots,\xi_m\}$ are antipodal. This implies that $\dim(P_0)\neq n-1$. Moreover, $n-\mathfrak{p}<\dim(P_0)$. Otherwise, $\mathrm{C}_\mathfrak{p}(P_0)=0$, which is impossible because of the continuity of $\mathrm{C}_\mathfrak{p}$ and that $\mathrm{C}_\mathfrak{p}(P_{j_k})\geq 1$ for each k. So, we assume $n-\mathfrak{p}<\dim(P_0)<n-1$ and prove that P_0 is not a solution to Problem 1 for $(\mu,\mathfrak{p},p)$. Precisely, we prove that there is another distinct polytope Q with $\mathrm{C}_\mathfrak{p}(Q)=1$, but $F_p(Q)<F_p(P_0)$.

Since some of the values $h_{P_0}(\xi_1),\dots,h_{P_0}(\xi_m)$ are zero, w.l.o.g., we assume $h_{P_0}(\xi_i)=0$ for $1\leq i\leq i_0$, and $h_{P_0}(\xi_i)>0$ for $i_0+1\leq i\leq m$, where $3\leq i_0\leq m-2$ (see, e.g., [1]). Let $a=\frac{1}{4}\min\{h_{P_0}(\xi_i):i_0+1\leq i\leq m\}$, and take the Wulff shape

$$D=\{x\in\mathbb{R}^n : x\cdot\xi_i\leq a, i=1,\dots,m\}.$$

For small $t>0$, let

$$P_t=P_0+tD \quad \text{and} \quad \bar{P}_t=\mathrm{C}_\mathfrak{p}(P_t)^{-1/(n-\mathfrak{p})}P_t.$$

Two observations are in order. First, since P_0 is the limit of the minimizing sequence $\{P_{j_k}\}_k$, it necessarily implies that $\mathrm{C}_\mathfrak{p}(P_0)=1$. Second, P_t is continuous in t, and $P_t\to P_0$ as $t\to 0^+$. So, by the continuity of $\mathrm{C}_\mathfrak{p}$, $\bar{P}_t$ is continuous in t, and $\bar{P}_t\to P_0$ as $t\to 0^+$.

For small $t>0$, consider $F(t)=\frac{n-\mathfrak{p}}{\mathfrak{p}-1}F_p(\bar{P}_t)$, i.e.,

$$F(t)=\mathrm{C}_\mathfrak{p}(P_t)^{-p/(n-\mathfrak{p})}\sum_{i=1}^m c_i h_{P_t}(\xi_i)^p.$$

Then, $F(t)$ is positive and continuous in t; and $F(t)\to F(0)=F_p(P_0)$, as $t\to 0^+$. Moreover,

$$\begin{aligned}\frac{dF(t)}{dt}&=p\mathrm{C}_\mathfrak{p}(P_t)^{-p/(n-\mathfrak{p})}\sum_{i=1}^m c_i h_{P_t}(\xi_i)^{p-1}h_D(\xi_i)\\&\quad-\frac{p}{n-\mathfrak{p}}\mathrm{C}_\mathfrak{p}(P_t)^{-1-p/(n-\mathfrak{p})}\frac{d\mathrm{C}_\mathfrak{p}(P_t)}{dt}\sum_{i=1}^m c_i h_{P_t}(\xi_i)^p.\end{aligned}$$

Note that in Akman et al. [1, Proposition 13.2], it was proved that under the assumptions mentioned above, there appears

$$\lim_{t\to 0^+}\frac{d\mathrm{C}_\mathfrak{p}(P_t)}{dt}=\infty.$$

Thus, for some sufficiently small $t_0>0$ and for $0<t<t_0$,

$$\frac{dF(t)}{dt}<0.$$

Consequently, for sufficiently small $t>0$, we have $F(t)<F(0)$; i.e.,

$$F_p(\bar{P}_t)<F_p(P_0).$$

Recall that $\mathrm{C}_\mathfrak{p}(\bar{P}_t)=\mathrm{C}_\mathfrak{p}(P_0)=1$. Take $Q=\bar{P}_t$, as desired. This completes the third step.

Now, we turn to Problem 2 for $(\mu, \mathfrak{p}, p)$. Since P_0 is a convex proper polytope solving Problem 1, by (4.1) and Lemma 4.1 (1) we conclude that $P := F_p(P_0)^{-1/p} P_0$ is a convex proper polytope solving Problem 2. From Lemmas 4.2 and 4.3, it follows that such solution P is unique and contains the origin in its interior.

Finally, we return to Problem 3 for $(\mu, \mathfrak{p}, p)$. By Lemma 4.4 (1), the polytope P is the unique solution to Problem 3 for $(\mu, \mathfrak{p}, p)$. Since P contains the origin in its interior, the formula

$$\mathrm{C}_{\mathfrak{p}}(P)^{-1} d\mu_{\mathfrak{p}}(P, \cdot) = h_P^{p-1} d\mu$$

can be readily rewritten as that in Lemma 7.1. q.e.d.

Step II. We prove the existence of solution to Problem 3 for a general measure.

Lemma 7.2. *Problem 3 for $(\mu, \mathfrak{p}, p)$ has a unique solution. If in addition μ is discrete, the solution is a convex proper polytope containing the origin in its interior.*

Proof. By Lemma 5.1, we can choose a sequence of discrete measures $\{\mu_j\}_j$ satisfying conditions (A_1) and (A_2), converging weakly to μ. By Lemma 7.1, Problem 3 for each $(\mu_j, \mathfrak{p}, p)$ has a unique solution. By Lemma 6.6, it immediately implies that Problem 3 for each $(\mu, \mathfrak{p}, p)$ has a unique solution. If in addition μ is discrete, by Lemma 4.4 (2) and Lemma 4.3, such solution is a convex proper polytope containing the origin in its interior. q.e.d.

Step III. We prove the following

Lemma 7.3. *If in addition $p \geq n$, then the unique solution K to Problem 3 for $(\mu, \mathfrak{p}, p)$ contains the origin in its interior, and therefore, $\mu_{p,\mathfrak{p}}(K, \cdot) = \mathrm{C}_{\mathfrak{p}}(K)\mu$.*

Proof. Let $\{\mu_j\}_j$ be a sequence of discrete measures on $\mathbb{S}^{n-1}$, which satisfy conditions (A_1) and (A_2), and converge to μ weakly. For each j, Problem 3 for $(\mu_j, \mathfrak{p}, p)$ has a unique solution P_j, and P_j is a convex polytope with the origin in its interior. Similar to the proof of Theorem 6.7, we have $P_j \to K$, as $j \to \infty$.

Several useful facts are listed. Firstly, $\sup_j |\mu_j| < \infty$. Secondly, $d\mu_j = \mathrm{C}_{\mathfrak{p}}(P_j)^{-1} h_{P_j}^{1-p} d\mu_{\mathfrak{p}}(P_j, \cdot)$, for each j. Thirdly, from the convergence $P_j \to K$ and Colesanti et al. [**20**, Lemma 2.18], there is a positive constant c_1 depending on n, $\mathfrak{p}$ and $\max\{h_{P_j}(\xi) : \xi \in \mathbb{S}^{n-1}, j \in \mathbb{N}\}$, such that $\mu_{\mathfrak{p}}(P_j, \cdot) \geq c_1^{-\mathfrak{p}} S_{\bar{P}_j}$. Finally, from the convergence $P_j \to K$ again and the continuity of $\mathfrak{p}$-capacity, it follows that $0 < \sup_j \mathrm{C}_{\mathfrak{p}}(P_j) < \infty$.

Hence,

$$\begin{aligned}\infty &> \sup_j |\mu_j| \\ &\geq |\mu_j| \\ &= \frac{1}{\mathrm{C}_{\mathfrak{p}}(P_j)} \int_{\mathbb{S}^{n-1}} h_{P_j}^{1-p} d\mu_{\mathfrak{p}}(P_j, \cdot) \\ &\geq c_2 \int_{\mathbb{S}^{n-1}} h_{P_j}^{1-p} dS_{P_j},\end{aligned}$$

where $c_2 = \frac{c_1^{-\mathfrak{p}}}{\sup_j \mathrm{C}_{\mathfrak{p}}(P_j)}$.

Assume the origin is on the boundary of K. We will conclude that $p < n$ by adapting the arguments of Hug et al. [**39**, p.713]. In fact, let $\xi_K \in \mathbb{S}^{n-1}$ be such that ∂K can be locally represented as the graph of a convex function over (a neighborhood of) $B_r := \xi_K^{\perp} \cap rB$, $r > 0$, and $x \cdot \xi_K \geq 0$ for any $x \in K$. Then there exists a subsequence $\{j_l\}_l$ of $\mathbb{N}$ tending to ∞, and a constant $c_3 > 0$ independent of j, such that

$$\lim_{l\to\infty} \int_{\mathbb{S}^{n-1}} h_{P_{j_l}}^{1-p} dS_{P_{j_l}} \geq c_3 \int_0^r t^{n-p-1} dt.$$

Hence,

$$\infty > \sup_j \{|\mu_j|\} \geq c_2 c_3 \int_0^r t^{n-p-1} dt,$$

which implies that $p < n$. q.e.d.

Step IV. We conclude the proofs of Theorems 1.1, 1.2 and 1.3.

In light of Lemma 4.6, when $\mathfrak{p} + p \neq n$, the solutions to Problems 3 and 4 for $(\mu, \mathfrak{p}, p)$ differ from each other only by a positive scalar. So, by Lemma 7.2 and Lemma 7.3, the solution to Problem 4 contains the origin in its interior if $p \geq n$, or μ is discrete. Furthermore, when μ is discrete, the solution is a convex proper polytope. So, Theorems 1.1 and 1.2 are proved.

For a finite even Borel measure μ on $\mathbb{S}^{n-1}$, it is not concentrated on any closed hemisphere, if and only if it is not concentrated on any great subsphere. Thus, by Theorem 1.1, there exists a unique convex body K containing the origin, such that $d\mu_{\mathfrak{p}}(K, \cdot) = c h_K^{p-1} d\mu$ with constant c given in Theorem 1.1. Since μ is even, for any Borel subset $\omega \subseteq \mathbb{S}^{n-1}$, it follows that

$$\int_{\omega} d\mu_{\mathfrak{p}}(-K, \cdot) = \int_{-\omega} d\mu_{\mathfrak{p}}(K, \cdot) = c \int_{-\omega} h_K^{p-1} d\mu = c \int_{\omega} h_{-K}^{p-1} d\mu.$$

By the uniqueness part of Theorem 1.1, it follows that $K = -K$. Thus, Theorem 1.3 is proved.

8. Open problem

Since the logarithmic Minkowski problem is the most important case, we pose the following

Logarithmic Minkowski problem for capacity. Suppose μ is a finite Borel measure on $\mathbb{S}^{n-1}$ and $1 < \mathfrak{p} < n$. What are the necessary and sufficient conditions on μ so that μ is the L_0 $\mathfrak{p}$-capacitary measure $\mu_{0,\mathfrak{p}}(K, \cdot)$ of a convex body K in $\mathbb{R}^n$?

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