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The L_p capacitary Minkowski problem for polytopes ☆

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Sufficient conditions are given for the existence of solutions to the discrete L_p Minkowski problem for $\mathfrak{p}$-capacity when $0 < p < 1$ and $1 < \mathfrak{p} < 2$.

1. Introduction

The setting for this article is Euclidean n-space, $\mathbb{R}^n$. A *convex body* in $\mathbb{R}^n$ is a compact convex set with nonempty interior. The Brunn-Minkowski theory of convex bodies,

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also called the mixed volume theory, which was developed by Minkowski, Aleksandrov, Fenchel, and many others, centers around the study of geometric functionals of convex bodies and the differentials of these functionals. Usually, the differentials of these functionals produce *new* geometric measures. This theory depends heavily on analytic tools such as the cosine transform on the unit sphere $\mathbb{S}^{n-1}$ and Monge-Ampère type equations.

A Minkowski problem is a *characterization* problem for a geometric measure generated by convex bodies: It asks for necessary and sufficient conditions in order that a given measure arises as the measure generated by a convex body. The solution to a Minkowski problem, in general, amounts to solving a fully nonlinear partial differential equation. The study of Minkowski problems has a long history and strong influence on both the Brunn-Minkowski theory and fully nonlinear partial differential equations. For details, see, e.g., [45, Chapter 8].

The L_p Minkowski problem for volume was originated in the 90s of last century, and it significantly generalized the classical Minkowski problem and was intensively investigated.

1.1. L_p surface area measures and the L_p Minkowski problem for volume

The L_p Brunn-Minkowski theory (see, e.g., [45, Sections 9.1 and 9.2]) is an extension of the classical Brunn-Minkowski theory, in which the L_p surface area measure introduced by Lutwak [37] is one of the most fundamental notions.

Let K be a convex body in $\mathbb{R}^n$ with the origin in its interior and $p \in \mathbb{R}$. Its L_p *surface area measure* $S_p(K,\cdot)$ is a finite Borel measure on $\mathbb{S}^{n-1}$, defined for Borel $\omega \subseteq \mathbb{S}^{n-1}$ by

$$S_p(K,\omega) = \int\limits_{x\in \mathrm{g}_K^{-1}(\omega)} (x\cdot \mathrm{g}_K(x))^{1-p}\, d\mathcal{H}^{n-1}(x),$$

where $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure; $\mathrm{g}_K : \partial' K \to \mathbb{S}^{n-1}$ is the Gauss map defined on the set $\partial' K$ of those points of ∂K that have a unique outer normal. Alternatively, $S_p(K,\cdot)$ can be defined by

$$S_p(K,\omega) = \int\limits_{\omega} h_K^{1-p}(u)\, dS(K,u), \tag{1.1}$$

where $h_K : \mathbb{R}^n \to \mathbb{R}$, $h_K(x) = \max\{x\cdot y : y \in K\}$ is the *support function* of K; $dS(K,\cdot)$ is the classical *surface area measure* of K.

Tracing the source, the L_p surface area measure resulted from the differential of volume functional of L_p combinations of convex bodies.

In 1962, Firey [28] introduced the notion of L_p *sum* of convex bodies. Let K, L be convex bodies with the origin in their interiors and $1 \leqslant p < \infty$. Their L_p sum $K+_pL$ is the compact convex set with support function $h_{K+_pL} = (h_K^p + h_L^p)^{1/p}$. For $t > 0$, the L_p *scalar* *multiplication* $t \cdot_p K$ is the set $t^{1/p}K$. Note that $K +_1 L = K + L = \{x + y : x \in K, y \in L\}$ is the *Minkowski sum* of K and L.

Using the L_p combination, Lutwak [37] established the following L_p variational formula

$$\left.\frac{dV(K +_p t \cdot_p L)}{dt}\right|_{t=0^+} = \frac{1}{p} \int\limits_{\mathbb{S}^{n-1}} h_L^p(u)\, dS_p(K, u), \tag{1.2}$$

where V is the n-dimensional volume, i.e., Lebesgue measure in $\mathbb{R}^n$. When $p = 1$, it reduces to the celebrated *Aleksandrov variational formula*

$$\left.\frac{dV(K + tL)}{dt}\right|_{t=0^+} = \int\limits_{\mathbb{S}^{n-1}} h_L(u)\, dS(K, u). \tag{1.3}$$

The integral in (1.3), divided by the factor n, is called the *first mixed volume* $V_1(K, L)$ of K and L. That is,

$$V_1(K, L) = \frac{1}{n} \int\limits_{\mathbb{S}^{n-1}} h_L(u)\, dS(K, u),$$

which is a generalization of the well-known volume formula

$$V(K) = \frac{1}{n} \int\limits_{\mathbb{S}^{n-1}} h_K(u)\, dS(K, u). \tag{1.4}$$

The L_p Minkowski problem for volume. *Suppose μ is a finite Borel measure on $\mathbb{S}^{n-1}$ and $p \in \mathbb{R}$. What are the necessary and sufficient conditions on μ such that μ is the L_p surface area measure $S_p(K, \cdot)$ of a convex body K in $\mathbb{R}^n$?*

The L_1 Minkowski problem is precisely the classical Minkowski problem. More than a century ago, Minkowski himself [42] solved this problem when the given measure is either discrete or has a continuous density. Aleksandrov [2,3] and Fenchel-Jessen [27] independently solved the problem for arbitrary measures: If μ is not concentrated on any great subsphere of $\mathbb{S}^{n-1}$, then μ is the surface area measure of a convex body if and only if its centroid is at the origin o; i.e., $\int_{\mathbb{S}^{n-1}} u\, d\mu(u) = o$.

Recall that the L_p Minkowski problem for *volume* for $p < 1$ was publicized by a series of talks by Erwin Lutwak in the 1990's, and appeared in print in Chou and Wang [22] for the first time. The case $p = 0$ is the so called logarithmic Minkowski problem. In [11], the authors posed the subspace concentration condition and completely solved the even logarithmic Minkowski problem. Additional references regarding the logarithmic Minkowski problem can be found in, e.g., [7–10,20,33,46–48,50]. If $0 < p < 1$, the L_p Minkowski problem is essentially solved by Chen, Li and Zhu [19]. See also [12,51], for more details. It is worth mentioning that in the very recent work [6], the authors discussed the case $-n < p < 1$ for an absolutely continuous measure and provided an almost optimal sufficient condition for the case $0 < p < 1$.

Since for strictly convex bodies with smooth boundaries, the density of the surface area measure with respect to the Lebesgue measure is just the reciprocal of the Gauss curvature of closed convex hypersurface, analytically, the classical Minkowski problem is equivalent to solving a Monge-Ampère equation. Establishing the regularity of the solution is difficult and has led to a long series of highly influential works, see, e.g., Lewy [36], Nirenberg [43], Cheng-Yau [21], Pogorelov [44], Caffarelli [13,14].

By now, the L_p Minkowski problem for volume has been investigated and achieved great developments. See, e.g., [4,11,17,18,22,32,34,37–39,41,46,50,51]. Its solutions have been applied to establish sharp affine isoperimetric inequalities, such as the affine Moser-Trudinger and the affine Morrey-Sobolev inequalities, the affine L_p Sobolev-Zhang inequality, etc. See, e.g., [40,49], for more details.

1.2. L_p $\mathfrak{p}$-capacitary measures and the L_p Minkowski problem for $\mathfrak{p}$-capacity

Without a doubt, the Minkowski problem for electrostatic $\mathfrak{p}$-capacity is an extremely important variant among Minkowski problems. Recall that for $1 < \mathfrak{p} < n$, the electrostatic $\mathfrak{p}$-capacity of a compact set K in $\mathbb{R}^n$ is defined by

$$C_{\mathfrak{p}}(K) = \inf \left\{ \int_{\mathbb{R}^n} |\nabla u|^{\mathfrak{p}} \, dx : u \in C_c^\infty(\mathbb{R}^n) \text{ and } u \geqslant \chi_K \right\},$$

where $C_c^\infty(\mathbb{R}^n)$ denotes the set of smooth functions with compact supports, and χ_K is the characteristic function of K. The quantity $C_2(K)$ is the classical electrostatic (or Newtonian) capacity of K.

For convex bodies K and L, via the variation of capacity functional $C_2(K)$, there appears the classical Hadamard variational formula

$$\left.\frac{dC_2(K+tL)}{dt}\right|_{t=0^+} = \int_{\mathbb{S}^{n-1}} h_L(u) \, d\mu_2(K,u) \tag{1.5}$$

and its special case, the Poincaré capacity formula

$$C_2(K) = \frac{1}{n-2} \int_{\mathbb{S}^{n-1}} h_K(u) \, d\mu_2(K,u). \tag{1.6}$$

Here, $\mu_2(K,\cdot)$ is called the *electrostatic capacitary measure* of K.

In his celebrated article [35], Jerison pointed out the resemblance between the Poincaré formula (1.6) and the volume formula (1.4) and also a resemblance between their variational formulas (1.5) and (1.3). Thus, he initiated to study the Minkowski problem for electrostatic capacity: *Given a finite Borel measure μ on $\mathbb{S}^{n-1}$, what are the necessary and sufficient conditions on μ such that μ is the electrostatic capacitary measure $\mu_2(K,\cdot)$ of a convex body K in $\mathbb{R}^n$?*

Jerison [35] solved, in full generality, the Minkowski problem for capacity. He proved the necessary and sufficient conditions for existence of a solution, which are unexpectedly identical to the corresponding conditions in the classical Minkowski problem. Uniqueness was settled by Caffarelli, Jerison and Lieb [16]. The regularity part of the proof depends on the ideas of Caffarelli [15] for the regularity of solutions to the Monge-Ampère equation.

Jerison's work inspired much subsequent research on this topic. In [24], the authors extended Jerison's work to electrostatic $\mathfrak{p}$-capacity, $1<\mathfrak{p}<n$, and established the Hadamard variational formula for $\mathfrak{p}$-capacity,

$$\left.\frac{dC_{\mathfrak{p}}(K+tL)}{dt}\right|_{t=0^+}=(\mathfrak{p}-1)\int\limits_{\mathbb{S}^{n-1}} h_L(u)\,d\mu_{\mathfrak{p}}(K,u) \tag{1.7}$$

and therefore the Poincaré $\mathfrak{p}$-capacity formula

$$C_{\mathfrak{p}}(K)=\frac{\mathfrak{p}-1}{n-\mathfrak{p}}\int\limits_{\mathbb{S}^{n-1}} h_K(u)\,d\mu_{\mathfrak{p}}(K,u). \tag{1.8}$$

Here, the new measure $\mu_{\mathfrak{p}}(K,\cdot)$ is called the *electrostatic $\mathfrak{p}$-capacitary measure* of K.

Naturally, the Minkowski problem for $\mathfrak{p}$-capacity was posed [24]: *Given a finite Borel measure μ on $\mathbb{S}^{n-1}$, what are the necessary and sufficient conditions on μ such that μ is the $\mathfrak{p}$-capacitary measure $\mu_{\mathfrak{p}}(K,\cdot)$ of a convex body K in $\mathbb{R}^n$?*

In [24], the authors proved the uniqueness of the solution when $1<\mathfrak{p}<n$, and existence and regularity when $1<\mathfrak{p}<2$. Very recently, the existence for $2<\mathfrak{p}<n$ was solved by M. Akman, J. Gong, J. Hineman, J. Lewis, and A. Vogel [1].

Inspired by the developed L_p Minkowski problem for volume, D. Zou and G. Xiong [52] initiated to research the following *L_p Minkowski problem for $\mathfrak{p}$-capacitary measure.*

Let $p\in\mathbb{R}$ and $1<\mathfrak{p}<n$. For a convex body K in $\mathbb{R}^n$ with the origin in its interior, its *L_p $\mathfrak{p}$-capacitary measure* $\mu_{p,\mathfrak{p}}(K,\cdot)$ is a finite Borel measure on $\mathbb{S}^{n-1}$, defined for Borel $\omega\subseteq\mathbb{S}^{n-1}$ by

$$\mu_{p,\mathfrak{p}}(K,\omega)=\int\limits_{\omega} h_K^{1-p}(u)\,d\mu_{\mathfrak{p}}(K,u).$$

Similar to the L_p surface area measure $S_p(K,\cdot)$, $\mu_{p,\mathfrak{p}}(K,\cdot)$ is also resulted from the variation of $\mathfrak{p}$-capacity functional of L_p sum of convex bodies. Specifically, if K,L are convex bodies in $\mathbb{R}^n$ with the origin in their interiors, then for $1\leqslant p<\infty$,

$$\left.\frac{dC_{\mathfrak{p}}(K+_p t\cdot_p L)}{dt}\right|_{t=0^+}=\frac{\mathfrak{p}-1}{p}\int\limits_{\mathbb{S}^{n-1}} h_L^p(u)\,d\mu_{p,\mathfrak{p}}(K,u).$$

The L_p Minkowski problem for $\mathfrak{p}$-capacity. *Suppose μ is a finite Borel measure on $\mathbb{S}^{n-1}$, $p \in \mathbb{R}$ and $1 < \mathfrak{p} < n$. What are the necessary and sufficient conditions on μ such that μ is the L_p $\mathfrak{p}$-capacitary measure $\mu_{p,\mathfrak{p}}(K, \cdot)$ of a convex body K in $\mathbb{R}^n$?*

In [52], Zou and Xiong completely solved the case when $p > 1$ and $1 < \mathfrak{p} < n$. In this article, we focus on the case $0 < p < 1$ and $1 < \mathfrak{p} < 2$, and prove the following.

Theorem 1.1. *Let μ be a finite positive Borel measure on $\mathbb{S}^{n-1}$ which is not concentrated on any closed hemisphere, $0 < p < 1$ and $1 < \mathfrak{p} < 2$. Suppose μ is discrete and satisfies $\mu(\{-u\}) = 0$ whenever $\mu(\{u\}) > 0$, for $u \in \mathbb{S}^{n-1}$.*

(1) If $p+\mathfrak{p} \neq n$, then there exists an n-dimensional polytope P such that $\mu_{p,\mathfrak{p}}(P, \cdot) = \mu$;

(2) If $p + \mathfrak{p} = n$, then there exists an n-dimensional polytope P and a constant $\lambda > 0$ such that $\lambda\mu_{p,\mathfrak{p}}(P, \cdot) = \mu$.

This article is organized as follows. In Section 2, we collect some basic facts on the theory of convex bodies. In Section 3, we study an extremal problem under translation transforms. After clarifying the relationship between this extremal problem and our concerned L_p Minkowski problem for capacity in Section 4, we present the proof of the main result in Section 5.

2. Preliminaries

2.1. Basics of convex bodies

For quick reference, we collect some basic facts on the theory of convex bodies. Good references are the books by Gardner [29], Gruber [31] and Schneider [45].

Write $x \cdot y$ for the standard inner product of $x, y \in \mathbb{R}^n$. Let B be the standard unit ball of $\mathbb{R}^n$. Denote by $\mathcal{K}^n$ the set of convex bodies in $\mathbb{R}^n$, and by $\mathcal{K}_o^n$ the set of convex bodies with the origin o in their interiors.

$\mathcal{K}^n$ is often equipped with the *Hausdorff metric* δ_H, which is defined for compact convex sets K, L by

$$\delta_H(K, L) = \max\{|h_K(u) - h_L(u)| : u \in \mathbb{S}^{n-1}\}.$$

For compact convex sets K and L, they are said to be *homothetic*, if $K = sL + x$, for some $s > 0$ and $x \in \mathbb{R}^n$. The *reflection* of K is $-K = \{-x : x \in K\}$.

Write $\operatorname{int} K$ and $\operatorname{bd} K$ for the interior and boundary of a set K, respectively. Write $\operatorname{relint} K$ and $\operatorname{relbd} K$ for the *relative interior* and *relative boundary* of K, that is, the interior and boundary of K relative to its affine hull, respectively.

Denote by $C(\mathbb{S}^{n-1})$ the set of continuous functions defined on $\mathbb{S}^{n-1}$, which is equipped with the metric induced by the maximal norm. Write $C_+(\mathbb{S}^{n-1})$ for the set of strictly positive functions in $C(\mathbb{S}^{n-1})$. For nonnegative $f, g \in C(\mathbb{S}^{n-1})$ and $t \geqslant 0$, define

$$f +_p t \cdot g = \left(f^p + t g^p\right)^{1/p},$$

where, without confusion and for brevity, we omit the subscript p under the dot thereafter. If in addition $f > 0$ and g is nonzero, the definition holds when $t > -\left(\frac{\min_{\mathbb{S}^{n-1}} f}{\max_{\mathbb{S}^{n-1}} g}\right)^{1/p}$.

For nonnegative $f \in C(\mathbb{S}^{n-1})$, define

$$[f] = \bigcap_{u \in \mathbb{S}^{n-1}} \{x \in \mathbb{R}^n : x \cdot u \leqslant f(u)\}.$$

The set is called the *Aleksandrov body* (also known as the *Wulff shape*) associated with f. Obviously, $[f]$ is a compact convex set containing the origin. For a compact convex set containing the origin, say K, we have $K = [h_K]$. If $f \in C_+(\mathbb{S}^{n-1})$, then $[f] \in \mathcal{K}_o^n$.

The *Aleksandrov convergence lemma* reads: If the sequence $\left\{f_j\right\}_j \subseteq C_+(\mathbb{S}^{n-1})$ converges uniformly to $f \in C_+(\mathbb{S}^{n-1})$, then $\lim_{j\to\infty}[f_j] = [f]$.

For $K \in \mathcal{K}^n$ and $u \in \mathbb{S}^{n-1}$, the *support hyperplane* $H(K,u)$ is defined by

$$H(K,u) = \{x \in \mathbb{R}^n : x \cdot u = h(K,u)\}.$$

The *half-space* $H^-(K,u)$ in the direction u is defined by

$$H^-(K,u) = \{x \in \mathbb{R}^n : x \cdot u \leqslant h(K,u)\}.$$

The *support set* $F(K,u)$ in the direction u is defined by

$$F(K,u) = K \cap H(K,u).$$

Suppose the unit vectors $u_1, \ldots, u_N$ are not concentrated on any closed hemisphere of $\mathbb{S}^{n-1}$, $N \geqslant n+1$. Let $P(u_1, \ldots, u_N)$ be the set with $P \in P(u_1, \ldots, u_N)$ such that for fixed $a_1, \ldots, a_N \geqslant 0$,

$$P = \bigcap_{k=1}^{N} \{x \in \mathbb{R}^n : x \cdot u_k \leqslant a_k\}.$$

Obviously, for $P \in P(u_1, \ldots, u_N)$, P has at most N facets, i.e., $(n-1)$ dimensional faces, and the outer normals of P are a subset of $\{u_1, \ldots, u_N\}$. Let $P_N(u_1, \ldots, u_N)$ be the subset of $P(u_1, \ldots, u_N)$ such that a polytope $P \in P_N(u_1, \ldots, u_N)$, if $P \in P(u_1, \ldots, u_N)$ and P has exactly N facets.

2.2. Basic facts on $\mathfrak{p}$-capacity

This part lists some necessary facts on $\mathfrak{p}$-capacity. For more details, see, e.g., [24,26, 35].

Let $1 < \mathfrak{p} < n$. The $\mathfrak{p}$-capacity $C_\mathfrak{p}$ has the following properties. First, it is increasing with respect to set inclusion; that is, if $E \subseteq F$, then $C_\mathfrak{p}(E) \leqslant C_\mathfrak{p}(F)$. Second, it is positively homogeneous of degree $(n - \mathfrak{p})$, i.e., $C_\mathfrak{p}(sE) = s^{n-\mathfrak{p}}C_\mathfrak{p}(E)$, for $s > 0$. Third, it is rigid motion invariant, i.e., $C_\mathfrak{p}(gE + x) = C_\mathfrak{p}(E)$, for $x \in \mathbb{R}^n$ and $g \in \mathrm{O}(n)$.

Let $K \in \mathcal{K}^n$. The $\mathfrak{p}$-capacitary measure $\mu_\mathfrak{p}(K, \cdot)$ has the following properties. First, it is positively homogeneous of degree $(n - \mathfrak{p} - 1)$, i.e., $\mu_\mathfrak{p}(sK, \cdot) = s^{n-\mathfrak{p}-1}\mu_\mathfrak{p}(K, \cdot)$, for $s > 0$. Second, it is translation invariant, i.e., $\mu_\mathfrak{p}(K + x, \cdot) = \mu_\mathfrak{p}(K, \cdot)$, for $x \in \mathbb{R}^n$. Third, its centroid is at the origin, i.e., $\int_{\mathbb{S}^{n-1}} u\, d\mu_\mathfrak{p}(K, u) = o$. Moveover, it is absolutely continuous with respect to the surface area measure $S(K, \cdot)$.

For convex bodies K_j, $K \in \mathcal{K}^n$, $j \in \mathbb{N}$, if $K_j \to K \in \mathcal{K}^n$, then $\mu_\mathfrak{p}(K_j, \cdot) \to \mu_\mathfrak{p}(K, \cdot)$ weakly, as $j \to \infty$. This important fact was proved in [24, p. 1550].

Let $K \in \mathcal{K}^n_o$ and $f \in C(\mathbb{S}^{n-1})$. There is a $t_0 > 0$ such that $h_K + tf \in C_+(\mathbb{S}^{n-1})$, for $|t| < t_0$. The Aleksandrov body $[h_K + tf]$ is continuous in $t \in (-t_0, t_0)$. The Hadamard variational formula for $\mathfrak{p}$-capacity (see [24, Theorem 1.1]) states that

$$\left.\frac{dC_\mathfrak{p}([h_K + tf])}{dt}\right|_{t=0} = (\mathfrak{p} - 1) \int\limits_{\mathbb{S}^{n-1}} f(u)\, d\mu_\mathfrak{p}(K, u). \tag{2.1}$$

For $K, L \in \mathcal{K}^n$, the *mixed* $\mathfrak{p}$*-capacity* $C_\mathfrak{p}(K, L)$ (see [24, p. 1549]) is defined by

$$C_\mathfrak{p}(K, L) = \frac{1}{n - \mathfrak{p}} \left.\frac{dC_\mathfrak{p}(K + tL)}{dt}\right|_{t=0^+} = \frac{\mathfrak{p} - 1}{n - \mathfrak{p}} \int\limits_{\mathbb{S}^{n-1}} h_L(u)\, d\mu_\mathfrak{p}(K, u). \tag{2.2}$$

When $L = K$, it reduces to the Poincaré $\mathfrak{p}$-capacity formula (1.8). From the weak convergence of $\mathfrak{p}$-capacitary measures, it follows that $C_\mathfrak{p}(K, L)$ is continuous in (K, L).

The $\mathfrak{p}$-capacitary Brunn-Minkowski inequality, proved by Colesanti-Salani [25], reads: If $K, L \in \mathcal{K}^n$, then

$$C_\mathfrak{p}(K + L)^{\frac{1}{n-\mathfrak{p}}} \geqslant C_\mathfrak{p}(K)^{\frac{1}{n-\mathfrak{p}}} + C_\mathfrak{p}(L)^{\frac{1}{n-\mathfrak{p}}}, \tag{2.3}$$

with equality if and only if K and L are homothetic. When $\mathfrak{p} = 2$, the inequality was first established by Borell [5], and the equality condition was shown by Caffarelli, Jerison and Lieb [16]. For more details, see, e.g., Colesanti [23] and Gardner and Hartenstine [30].

The inequality (2.3) is equivalent to the following $\mathfrak{p}$-capacitary Minkowski inequality

$$C_\mathfrak{p}(K, L) \geqslant C_\mathfrak{p}(K)^{n-\mathfrak{p}-1}C_\mathfrak{p}(L), \tag{2.4}$$

with equality if and only if K and L are homothetic. See [24, p. 1549] for more details.

2.3. Basic facts on Aleksandrov bodies

For nonnegative $f \in C(\mathbb{S}^{n-1})$, define

$$C_{\mathfrak{p}}(f) = C_{\mathfrak{p}}([f]).$$

Obviously, $C_{\mathfrak{p}}(h_K) = C_{\mathfrak{p}}(K)$, for a compact set K that contains the origin.

By the Aleksandrov convergence lemma and the continuity of $C_{\mathfrak{p}}$ on $\mathcal{K}^n$, the functional $C_{\mathfrak{p}} : C_+(\mathbb{S}^{n-1}) \longrightarrow (0, \infty)$ is continuous.

Let $0 < p < \infty$ and $1 < \mathfrak{p} < n$. For $K \in \mathcal{K}_o^n$ and nonnegative $f \in C(\mathbb{S}^{n-1})$, define

$$C_{p,\mathfrak{p}}(K, f) = \frac{\mathfrak{p} - 1}{n - \mathfrak{p}} \int\limits_{\mathbb{S}^{n-1}} f(u)^p h_K(u)^{1-p}\, d\mu_{\mathfrak{p}}(K, u).$$

For brevity, write $C_{\mathfrak{p}}(K, f)$ for $C_{1,\mathfrak{p}}(K, f)$. Obviously, $C_{p,\mathfrak{p}}(K, h_K) = C_{\mathfrak{p}}(K)$.

Lemma 2.1. *Let $0 < p < \infty$ and $1 < \mathfrak{p} < n$. If $f \in C_+(\mathbb{S}^{n-1})$, then*

$$C_{p,\mathfrak{p}}([f], f) = C_{\mathfrak{p}}([f]) = C_{\mathfrak{p}}(f).$$

Proof. Note that $h_{[f]} \leqslant f$. A basic fact established by Aleksandrov is that $h_{[f]} = f$, a.e., with respect to $S([f], \cdot)$. That is, $S([f], \{h_{[f]} < f\}) = 0$. Since $\mu_{\mathfrak{p}}([f], \cdot)$ is absolutely continuous with respect to $S([f], \cdot)$, it follows that $\mu_{\mathfrak{p}}([f], \{h_{[f]} < f\}) = 0$. Combining this fact and the inequality $h_{[f]} \leqslant f$, it follows that

$$C_{p,\mathfrak{p}}([f], f) - C_{\mathfrak{p}}(f) = \frac{\mathfrak{p} - 1}{n - \mathfrak{p}} \int\limits_{h_{[f]} < f} (f^p - h_{[f]}^p) h_{[f]}^{1-p}\, d\mu_{\mathfrak{p}}([f], \cdot) = 0,$$

as desired. $\square$

Lemma 2.2. *Let $0 < p < \infty$ and μ be a finite positive Borel measure on $\mathbb{S}^{n-1}$, which is not concentrated on any closed hemisphere. If Q is a compact convex set in $\mathbb{R}^n$ containing the origin and $\dim Q \geqslant 1$, then $0 < \int_{\mathbb{S}^{n-1}} h_Q^p\, d\mu < \infty$.*

Proof. That the integral is finite is clear, since μ is finite and h_Q is nonnegative and bounded. To prove the positivity of the integral, we can take a line segment $Q_0 \subseteq Q$, which is of the form $Q_0 = l\{tu_0 : 0 \leqslant t \leqslant 1\}$, with $0 < l < \infty$ and $u_0 \in \mathbb{S}^{n-1}$. Since μ is not concentrated on any closed hemisphere, it implies that $\mu(\{u \in \mathbb{S}^{n-1} : u \cdot u_0 > 0\}) > 0$. Therefore,

$$\int\limits_{\mathbb{S}^{n-1}} h_Q^p\, d\mu \geqslant \int\limits_{\mathbb{S}^{n-1}} h_{Q_0}^p\, d\mu = l \int\limits_{\{u \in \mathbb{S}^{n-1} : u \cdot u_0 > 0\}} (u \cdot u_0)^p\, d\mu(u) > 0,$$

as desired. $\square$

Lemma 2.3. *Let $I \subseteq \mathbb{R}$ be an interval containing 0 in its interior, and let $h_t(u) = h(t,u) : I \times \mathbb{S}^{n-1} \longrightarrow (0,\infty)$ be continuous, such that the convergence in*

$$h'(0,u) = \lim_{t\to 0} \frac{h(t,u) - h(0,u)}{t}$$

is uniform on $\mathbb{S}^{n-1}$. Then

$$\left.\frac{dC_{\mathfrak{p}}(h_t)}{dt}\right|_{t=0} = (\mathfrak{p}-1) \int\limits_{\mathbb{S}^{n-1}} h'(0,u)\, d\mu_{\mathfrak{p}}([h_0],u). \tag{2.5}$$

Lemma 2.4. *The convergence $\lim_{i\to\infty} K_i = K$ in $\mathcal{K}^n$ is equivalent to the following conditions taken together:*

(1) each point in K is the limit of a sequence $\{x_i\}_i$ with $x_i \in K_i$ for $i \in \mathbb{N}$;

(2) the limit of any convergent sequence $\{x_{i_j}\}_j$ with $x_{i_j} \in K_{i_j}$ for $j \in \mathbb{N}$ belongs to K.

For the proof of the above two lemmas, see [24, Theorem 5.2] and [45, Theorem 1.8.8], respectively.

3. An extremal problem for $F_p(Q,x)$ under translation transforms

Suppose $c_1, \ldots, c_N \in (0,\infty)$ and the unit vectors $u_1, \ldots, u_N$ are not concentrated on any closed hemisphere of $\mathbb{S}^{n-1}$. Let

$$\mu = \sum_{k=1}^{N} c_k \delta_{u_k}(\cdot)$$

be the discrete measure on $\mathbb{S}^{n-1}$. For $Q \in P(u_1, \ldots, u_N)$ and $0 < p < 1$, define the functional $F_p(Q,\cdot) : Q \longrightarrow \mathbb{R}$ by

$$F_p(Q,x) = \sum_{k=1}^{N} c_k (h_Q(u_k) - x \cdot u_k)^p. \tag{3.1}$$

We show there exists a unique point $x_Q \in \operatorname{int} Q$ such that $F_p(Q,x)$ attains the maximum.

Lemma 3.1. *Let $Q \in P(u_1, \ldots, u_N)$ and $0 < p < 1$. Then there exists a unique point $x_Q \in \operatorname{relint} Q$ such that*

$$F_p(Q,x_Q) = \max_{x\in Q} F_p(Q,x).$$

In addition, if Q is an n-dimensional polytope, then

$$\sum_{k=1}^{N} c_k (h_Q(u_k) - x_Q \cdot u_k)^{p-1} u_k = o. \tag{3.2}$$

Proof. First, we prove the uniqueness of the maximal point. Assume $x_1, x_2 \in \operatorname{relint} Q$ and

$$F_p(Q, x_1) = F_p(Q, x_2) = \max_{x \in Q} F_p(Q, x).$$

Using the definition (3.1), the Jensen inequality and the above assumption, we have

$$\begin{aligned}
F_p(Q, \frac{1}{2}(x_1 + x_2)) &= \sum_{k=1}^{N} c_k (h_Q(u_k) - \frac{1}{2}(x_1 + x_2) \cdot u_k)^p \\
&= \sum_{k=1}^{N} c_k (\frac{1}{2}(h_Q(u_k) - x_1 \cdot u_k) + \frac{1}{2}(h_Q(u_k) - x_2 \cdot u_k))^p \\
&\geqslant \frac{1}{2} \sum_{k=1}^{N} c_k (h_Q(u_k) - x_1 \cdot u_k)^p + \frac{1}{2} \sum_{k=1}^{N} c_k (h_Q(u_k) - x_2 \cdot u_k)^p \\
&= \frac{1}{2} F_p(Q, x_1) + \frac{1}{2} F_p(Q, x_2) \\
&= \max_{x \in Q} F_p(Q, x).
\end{aligned}$$

Since Q is convex, $\frac{1}{2}(x_1 + x_2) \in Q$. So the equality in the third line holds. By the equality condition of the Jensen inequality, we have

$$h_Q(u_k) - x_1 \cdot u_k = h_Q(u_k) - x_2 \cdot u_k, \quad \text{for } k = 1, \ldots, N.$$

That is,

$$x_1 \cdot u_k = x_2 \cdot u_k, \quad \text{for } k = 1, \ldots, N.$$

Since the unit vectors $u_1, \ldots, u_N$ are not concentrated on any closed hemisphere, it follows that $x_1 = x_2$, which proves the uniqueness.

Second, we prove the existence of the maximal point. Since $F_p(Q, x)$ is continuous in $x \in Q$ and Q is compact, so $F_p(Q, x)$ attains its maximum at a point of Q, say x_Q. In the following, we prove $x_Q \in \operatorname{relint} Q$.

Assume $x_Q \in \operatorname{relbd} Q$, the boundary of Q relative to its affine hull. Fix $y_0 \in \operatorname{relint} Q$. Let $u_0 = \frac{y_0 - x_Q}{|y_0 - x_Q|}$. Then for sufficiently small $\delta > 0$, it follows that $x_Q + \delta u_0 \in \operatorname{relint} Q$. Next, we aim to show that

$$\begin{aligned}
&F_p(Q, x_Q + \delta u_0) - F_p(Q, x_Q) \\
&\quad = \sum_{k=1}^{N} c_k (h_Q(u_k) - x_Q \cdot u_k - \delta u_0 \cdot u_k)^p - \sum_{k=1}^{N} c_k (h_Q(u_k) - x_Q \cdot u_k)^p
\end{aligned}$$

is positive, which will contradict the maximality of F_p at x_Q. Consequently, $x_Q \in \operatorname{relint} Q$.

For this aim, we divide $\{u_1, \ldots, u_N\}$ into three parts and let

$$\begin{aligned}
U_1 &= \{u_k : x_Q \cdot u_k = h_Q(u_k), u_k \cdot u_0 \neq 0, k \in \{1, \ldots, N\}\},\\
U_2 &= \{u_k : x_Q \cdot u_k = h_Q(u_k), u_k \cdot u_0 = 0, k \in \{1, \ldots, N\}\},\\
U_3 &= \{u_k : x_Q \cdot u_k < h_Q(u_k), k \in \{1, \ldots, N\}\}.
\end{aligned}$$

We first claim U_1 is nonempty.

Assume U_1 is empty. Since $x_Q \in \operatorname{relbd} Q$, there exists a $u_{i_0} \in \{u_1, \ldots, u_N\}$ such that $x_Q \cdot u_{i_0} = h_Q(u_{i_0})$. So, $u_{i_0} \in U_2$, and therefore U_2 is nonempty. In light of the fact that $u_1, \ldots, u_N$ are not concentrated on any closed hemisphere, U_3 is nonempty.

Choose a point $x_Q - \delta u_0$ for $\delta > 0$. Then, $x_Q - \delta u_0 \notin Q$.

On one hand, for any $u_k \in U_2$, it follows that $(x_Q - \delta u_0) \cdot u_k = x_Q \cdot u_k - \delta u_0 \cdot u_k = h_Q(u_k)$. Meanwhile, for any $u_k \in U_3$, since $x_Q \cdot u_k < h_Q(u_k)$, for sufficiently small $\delta > 0$,

$$(x_Q - \delta u_0) \cdot u_k = x_Q \cdot u_k - \delta u_0 \cdot u_k < h_Q(u_k).$$

Hence,

$$\begin{aligned}
x_Q - \delta u_0 \in & \bigcap_{u_k \in U_2 \cup U_3} \Big\{x \in \mathbb{R}^n : x \cdot u_k \leqslant h_Q(u_k)\Big\}\\
&= \bigcap_{k=1}^{N} \Big\{x \in \mathbb{R}^n : x \cdot u_k \leqslant h_Q(u_k)\Big\}\\
&= Q.
\end{aligned}$$

That is, $x_Q - \delta u_0 \in Q$. This is a contradiction. Hence, U_1 is nonempty.

Thus,

$$\begin{aligned}
&F_p(Q, x_Q + \delta u_0) - F_p(Q, x_Q)\\
&= \sum_{u_k \in U_1 \cup U_2 \cup U_3} c_k \big[(h_Q(u_k) - x_Q \cdot u_k - \delta u_0 \cdot u_k)^p - (h_Q(u_k) - x_Q \cdot u_k)^p\big]\\
&= \sum_{u_k \in U_1} c_k (-\delta u_0 \cdot u_k)^p\\
&\quad + \sum_{u_k \in U_3} c_k \big[(h_Q(u_k) - x_Q \cdot u_k - \delta u_0 \cdot u_k)^p - (h_Q(u_k) - x_Q \cdot u_k)^p\big]\\
&\geqslant \sum_{u_k \in U_1} c_k (-\delta u_0 \cdot u_k)^p\\
&\quad - \sum_{u_k \in U_3} c_k \Big|(h_Q(u_k) - x_Q \cdot u_k - \delta u_0 \cdot u_k)^p - (h_Q(u_k) - x_Q \cdot u_k)^p\Big|.
\end{aligned}$$

Since U_1 is nonempty, and $x_Q + \delta u_0 \in \operatorname{relint} Q$ for sufficiently small $\delta > 0$, it follows that for any $u_k \in U_1$,

$$-\delta u_0 \cdot u_k = h_Q(u_k) - (x_Q + \delta u_0) \cdot u_k > 0.$$

So,

$$\sum_{u_k \in U_1} c_k(-\delta u_0 \cdot u_k)^p > 0. \tag{3.3}$$

Let

$$C = \min_{u_k \in U_3} \left(h_Q(u_k) - x_Q \cdot u_k\right).$$

Then for any $u_k \in U_3$, it follows that

$$h_Q(u_k) - x_Q \cdot u_k \geqslant \min_{u_k \in U_3} (h_Q(u_k) - x_Q \cdot u_k) = C > 0.$$

So, for sufficiently small $\delta > 0$, we have

$$h_Q(u_k) - x_Q \cdot u_k - \delta u_0 \cdot u_k \geqslant \frac{C}{2} > 0.$$

By the concavity of t^p for $0 < p < 1$, we have

$$\Big|(h_Q(u_k) - x_Q \cdot u_k - \delta u_0 \cdot u_k)^p - (h_Q(u_k) - x_Q \cdot u_k)^p\Big| \leqslant p\left(\frac{C}{2}\right)^{p-1}\Big| - \delta u_0 \cdot u_k\Big|.$$

So,

$$\begin{aligned}\sum_{u_k \in U_3} c_k\Big|(h_Q(u_k) - x_Q \cdot u_k - \delta u_0 \cdot u_k)^p - (h_Q(u_k) - x_Q \cdot u_k)^p\Big| \\ \leq \delta p\left(\frac{C}{2}\right)^{p-1}\sum_{u_k \in U_3} c_k\Big|u_0 \cdot u_k\Big|.\end{aligned} \tag{3.4}$$

Thus, according to (3.3) and (3.4), it follows that

$$\begin{aligned}&F_p(Q, x_Q + \delta u_0) - F_p(Q, x_Q) \\ &\geqslant \sum_{u_k \in U_1} c_k(-\delta u_0 \cdot u_k)^p - \delta p\left(\frac{C}{2}\right)^{p-1}\sum_{u_k \in U_3} c_k\Big|u_0 \cdot u_k\Big| \\ &= \delta^p\left\{\sum_{u_k \in U_1} c_k(-u_0 \cdot u_k)^p - \delta^{1-p} p\left(\frac{C}{2}\right)^{p-1}\sum_{u_k \in U_3} c_k\Big|u_0 \cdot u_k\Big|\right\} > 0,\end{aligned}$$

for sufficiently small $\delta > 0$. So $x_Q \in \operatorname{relint} Q$. The existence is proved.

Finally, we prove (3.2). Since $F_p(Q,x)$ attains its maximum at the interior point x_Q, we have

$$\begin{aligned} 0 &= \left.\frac{\partial F_p(Q,x)}{\partial x_i}\right|_{x=x_Q} \\ &= \sum_{k=1}^{N} c_k p (h_Q(u_k) - x_Q \cdot u_k)^{p-1}(-u_{k,i}), \end{aligned}$$

for $i = 1,\dots,n$, where $x = (x_1,\dots,x_n)^T$ and $u_k = (u_{k,1},\dots,u_{k,n})^T$. That is,

$$\sum_{k=1}^{N} c_k (h_Q(u_k) - x_Q \cdot u_k)^{p-1} u_k = o,$$

as desired. □

From now on, we use x_Q to denote the maximal point of the functional $F_p(Q,x)$ on Q.

Lemma 3.2. *Suppose $Q_i \in P(u_1,\dots,u_N)$ and $Q_i \to Q$, as $i \to \infty$. Then*

$$x_{Q_i} \to x_Q \quad \textit{and} \quad F_p(Q_i, x_{Q_i}) \to F_p(Q, x_Q), \ \textit{as } i \to \infty.$$

Proof. Since $Q_i \to Q$, it follows that for sufficiently large i,

$$x_{Q_i} \in Q_i \subseteq Q + B.$$

So, $\{x_{Q_i}\}_i$ is a bounded sequence. Let $\{x_{Q_{i_j}}\}_j$ be a convergent subsequence of $\{x_{Q_i}\}_i$.

Assume $x_{Q_{i_j}} \to x'$, but $x' \neq x_Q$. By Lemma 2.4, it follows that $x' \in Q$. Hence,

$$F_p(Q,x') < F_p(Q,x_Q).$$

From the continuity of $F_p(Q,x)$ in Q and x, it follows that

$$\lim_{j\to\infty} F_p(Q_{i_j}, x_{Q_{i_j}}) = F_p(Q,x').$$

Meanwhile, by Lemma 2.4, for $x_Q \in Q$, there exists a $y_{i_j} \in Q_{i_j}$ such that $y_{i_j} \to x_Q$. Hence,

$$\lim_{j\to\infty} F_p(Q_{i_j}, y_{i_j}) = F_p(Q,x_Q).$$

So,

$$\lim_{j\to\infty} F_p(Q_{i_j}, x_{Q_{i_j}}) < \lim_{j\to\infty} F_p(Q_{i_j}, y_{i_j}). \tag{3.5}$$

However, for any Q_{i_j}, we have

$$F_p(Q_{i_j}, x_{Q_{i_j}}) \geqslant F_p(Q_{i_j}, y_{i_j}).$$

So,

$$\lim_{j\to\infty} F_p(Q_{i_j}, x_{Q_{i_j}}) \geqslant \lim_{j\to\infty} F_p(Q_{i_j}, y_{i_j}),$$

which contradicts (3.5). Thus, $x_{Q_{i_j}} \to x_Q$, and therefore $x_{Q_i} \to x_Q$. From the continuity of F_p, it follows that

$$F_p(Q_i, x_{Q_i}) \to F_p(Q, x_Q).$$

This completes the proof. □

Lemma 3.3. *Suppose $Q \in P(u_1, \ldots, u_N)$. Then*
(1) $F_p(Q + y, x_{Q+y}) = F_p(Q, x_Q)$, for $y \in \mathbb{R}^n$;
(2) $F_p(\lambda Q, x_{\lambda Q}) = \lambda^p F_p(Q, x_Q)$, for $\lambda > 0$.

Proof. From the definition (3.1), it follows that

$$\begin{aligned}
F_p(Q + y, x_{Q+y}) &= \max_{z\in Q+y} F_p(Q + y, z) \\
&= \max_{z-y\in Q} \sum_{k=1}^{N} c_k (h_{Q+y}(u_k) - z \cdot u_k)^p \\
&= \max_{z-y\in Q} \sum_{k=1}^{N} c_k (h_Q(u_k) - (z - y) \cdot u_k)^p \\
&= \max_{x\in Q} \sum_{k=1}^{N} c_k (h_Q(u_k) - x \cdot u_k)^p \\
&= F_p(Q, x_Q).
\end{aligned}$$

Similarly,

$$\begin{aligned}
F_p(\lambda Q, x_{\lambda Q}) &= \max_{z\in \lambda Q} F_p(\lambda Q, z) \\
&= \max_{\frac{z}{\lambda}\in Q} \sum_{k=1}^{N} c_k \left(h_{\lambda Q}(u_k) - z \cdot u_k\right)^p \\
&= \lambda^p \max_{\frac{z}{\lambda}\in Q} \sum_{k=1}^{N} c_k \left(h_Q(u_k) - \frac{z}{\lambda} \cdot u_k\right)^p
\end{aligned}$$

$$
\begin{aligned}
&= \lambda^p \max_{x\in Q} \sum_{k=1}^{N} c_k \left(h_Q(u_k) - x\cdot u_k\right)^p \\
&= \lambda^p F_p(Q, x_Q),
\end{aligned}
$$

as desired. □

4. An extremal problem related to the L_p capacitary Minkowski problem

Suppose $0 < p < 1$, $1 < \mathfrak{p} < 2$, and μ is the discrete measure on $\mathbb{S}^{n-1}$ such that

$$
\mu = \sum_{k=1}^{N} c_k \delta_{u_k}(\cdot),
$$

where $N \geqslant n+1$, $c_k > 0$, and $u_1, \dots, u_N$ are not concentrated on any closed hemisphere.

In this section, we first study the following extremal problem

$$
\inf\{\max_{x\in Q} F_p(Q, x) : Q \in P(u_1, \dots, u_N), C_{\mathfrak{p}}(Q) = 1\}, \tag{4.1}
$$

and then demonstrate that its solution is precisely the solution to our concerned L_p Minkowski problem for $\mathfrak{p}$-capacity.

Lemma 4.1. *Suppose P is an n-dimensional polytope with normal vectors $u_1, \dots, u_N$. If P is the solution to problem (4.1) and $x_P = o$, then*

$$
\lambda h_P^{1-p}\, d\mu_{\mathfrak{p}}(P, \cdot) = d\mu,
$$

where $\lambda = \frac{\mathfrak{p}-1}{n-\mathfrak{p}} \sum\limits_{k=1}^{N} c_k h_P^p(u_k)$.

Proof. For $\delta_1, \dots, \delta_N > 0$ and sufficiently small $|t| > 0$, let

$$
P_t = \{x : x\cdot u_k \leqslant h_P(u_k) + t\delta_k, k = 1, \dots, N\}
$$

and

$$
\alpha(t)P_t = C_{\mathfrak{p}}(P_t)^{-\frac{1}{n-\mathfrak{p}}} P_t.
$$

Then, $C_{\mathfrak{p}}(\alpha(t)P_t) = 1$, $\alpha(t)P_t \in P_N(u_1, \dots, u_N)$ and $\alpha(t)P_t \to P$, as $t \to 0$.

For brevity, let $x(t) = x_{\alpha(t)P_t}$. By (3.2) of Lemma 3.1, it follows that

$$
\sum_{k=1}^{N} c_k(\alpha(t)h_{P_t}(u_k) - x(t)\cdot u_k)^{p-1} u_{k,i} = 0, \quad \text{for } i = 1, \dots, n,
$$

where $u_k = (u_{k,1}, \dots, u_{k,n})^T$. Let $t = 0$. Then $P_0 = P$, $\alpha(0) = 1$, $x(0) = o$ and

$$\sum_{k=1}^{N} c_k h_P^{p-1}(u_k) u_{k,i} = 0, \quad \text{for } i = 1, \dots, n. \tag{4.2}$$

We first show $x'(t)\big|_{t=0}$ exists. Let

$$y_i(t, x_1, \dots, x_n) = \sum_{k=1}^{N} c_k [\alpha(t) h_{P_t}(u_k) - (x_1 u_{k,1} + \dots + x_n u_{k,n})]^{p-1} u_{k,i},$$

for $i = 1, \dots, n$. Then

$$\left.\frac{\partial y_i}{\partial x_j}\right|_{(0,\dots,0)} = \sum_{k=1}^{N} (1-p) c_k h_P^{p-2}(u_k) u_{k,i} u_{k,j}.$$

So,

$$\left(\left.\frac{\partial y}{\partial x}\right|_{(0,\dots,0)}\right)_{n\times n} = \sum_{k=1}^{N} (1-p) c_k h_P^{p-2}(u_k) u_k u_k^T.$$

Since $u_1, \dots, u_N$ are not concentrated on any closed hemisphere, for $x \in \mathbb{R}^n$ with $x \neq o$, there exists a $u_{i_0} \in \{u_1, \dots, u_N\}$ such that $u_{i_0} \cdot x \neq 0$. Thus,

$$\begin{aligned} & x^T \left(\sum_{k=1}^{N} (1-p) c_k h_P^{p-2}(u_k) u_k u_k^T \right) x \\ =& \sum_{k=1}^{N} (1-p) c_k h_P^{p-2}(u_k) (x \cdot u_k)^2 \\ \geqslant& (1-p) c_{i_0} h_P^{p-2}(u_{i_0}) (x \cdot u_{i_0})^2 > 0, \end{aligned}$$

which implies that $\left(\left.\frac{\partial y}{\partial x}\right|_{(0,\dots,0)}\right)_{n\times n}$ is positively definite. By the inverse function theorem, it follows that $x'(0) = (x_1'(0), \dots, x_n'(0))$ exists.

Now, we can finish the proof. Since the functional F_p attains its minimum at the polytope P, from (2.5) and (4.2), we have

$$\begin{aligned} 0 =& \frac{1}{p} \left.\frac{dF_p(\alpha(t)P_t, x(t))}{dt}\right|_{t=0} \\ =& \sum_{j=1}^{N} c_j h_P^{p-1}(u_j) \Big[h_P(u_j) (-\frac{1}{n-\mathfrak{p}}) \left.\frac{dC_{\mathfrak{p}}(P_t)}{dt}\right|_{t=0} + \delta_j - x'(0) \cdot u_j \Big] \end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=1}^{N} c_j h_P^{p-1}(u_j)\Big[-\frac{\mathfrak{p}-1}{n-\mathfrak{p}} h_P(u_j)\big(\sum_{k=1}^{N} \delta_k \mu_{\mathfrak{p}}(P,\{u_k\})\big) + \delta_j \Big] \\
&\qquad - x'(0)\cdot\Big(\sum_{j=1}^{N} c_j h_P^{p-1}(u_j) u_j\Big) \\
&= \sum_{j=1}^{N} c_j h_P^{p-1}(u_j)\Big[-\frac{\mathfrak{p}-1}{n-\mathfrak{p}} h_P(u_j)\big(\sum_{k=1}^{N} \delta_k \mu_{\mathfrak{p}}(P,\{u_k\})\big) + \delta_j \Big] \\
&= \sum_{j=1}^{N} \delta_j \Big[c_j h_P(u_j)^{p-1} - \frac{\mathfrak{p}-1}{n-\mathfrak{p}}\big(\sum_{k=1}^{N} c_k h_P^p(u_k)\big)\mu_{\mathfrak{p}}(P,\{u_j\}) \Big].
\end{aligned}
$$

Since $\delta_1,\dots,\delta_N$ are arbitrary positive real numbers, we have

$$
\frac{\mathfrak{p}-1}{n-\mathfrak{p}}\Big(\sum_{k=1}^{N} c_k h_P^p(u_k)\Big)\mu_{\mathfrak{p}}(P,\{u_j\}) = c_j h_P^{p-1}(u_j), \quad \text{for } j=1,\dots,N.
$$

In light of P is n-dimensional and o is in its interior, it follows that $h_P(u_j) > 0$, and therefore,

$$
\frac{\mathfrak{p}-1}{n-\mathfrak{p}}\Big(\sum_{k=1}^{N} c_k h_P^p(u_k)\Big) h_P^{1-p}(u_j)\mu_{\mathfrak{p}}(P,\{u_j\}) = c_j, \quad \text{for } j=1,\dots,N.
$$

That is,

$$
\lambda h_P^{1-p}\, d\mu_{\mathfrak{p}}(P,\cdot) = d\mu,
$$

where $\lambda = \frac{\mathfrak{p}-1}{n-\mathfrak{p}} \sum_{k=1}^{N} c_k h_P^p(u_k)$. $\square$

Now we demonstrate the solution to (4.1) is precisely the scaling of the solution to our concerned L_p Minkowski problem for $\mathfrak{p}$-capacity.

Lemma 4.2. *Suppose the n-dimensional polytope P solves problem (4.1) and $x_P = o$.*
(1) If $p + \mathfrak{p} \neq n$, then for

$$
\lambda_0 = \Big(\frac{\mathfrak{p}-1}{n-\mathfrak{p}} \sum_{k=1}^{N} c_k h_P^p(u_k) \Big)^{\frac{1}{n-p-\mathfrak{p}}},
$$

we have

$$
d\mu_{p,\mathfrak{p}}(\lambda_0 P,\cdot) = d\mu.
$$

(2) If $p + \mathfrak{p} = n$, then for any $\alpha > 0$,

$$d\mu_{p,\mathfrak{p}}(\alpha P, \cdot) = d\mu_{p,\mathfrak{p}}(P, \cdot) = \frac{d\mu}{\frac{\mathfrak{p}-1}{n-\mathfrak{p}} \sum_{k=1}^{N} c_k h_P^p(u_k)}.$$

Proof. Let $\alpha > 0$ and $P \in P(u_1, \dots, u_N)$. Then

$$d\mu_{p,\mathfrak{p}}(\alpha P, \cdot) = \alpha^{n-p-\mathfrak{p}} h_P^{1-p}\, d\mu_{\mathfrak{p}}(P, \cdot) = \alpha^{n-p-\mathfrak{p}}\, d\mu_{p,\mathfrak{p}}(P, \cdot). \tag{4.3}$$

Assume $p + \mathfrak{p} \neq n$. If P is the solution to (4.1), by Lemma 4.1, we have

$$\lambda\, d\mu_{p,\mathfrak{p}}(P, \cdot) = \lambda h_P^{1-p}\, d\mu_{\mathfrak{p}}(P, \cdot) = d\mu,$$

where $\lambda = \frac{\mathfrak{p}-1}{n-\mathfrak{p}} \sum_{k=1}^{N} c_k h_P^p(u_k)$. Combining with (4.3), it follows that

$$d\mu_{p,\mathfrak{p}}(\lambda_0 P, \cdot) = d\mu,$$

where $\lambda_0 = \lambda^{\frac{1}{n-p-\mathfrak{p}}}$.

Assume $p + \mathfrak{p} = n$. (4.3) implies that $d\mu_{p,\mathfrak{p}}(\alpha P, \cdot) = d\mu_{p,\mathfrak{p}}(P, \cdot)$, for any $\alpha > 0$. If P is the solution to (4.1), then

$$d\mu_{p,\mathfrak{p}}(\alpha P, \cdot) = d\mu_{p,\mathfrak{p}}(P, \cdot) = \frac{d\mu}{\frac{\mathfrak{p}-1}{n-\mathfrak{p}} \sum_{k=1}^{N} c_k h_P^p(u_k)}.$$

This completes the proof. □

5. Existence of solutions to the L_p Minkowski problem for $\mathfrak{p}$-capacity

Recall that the extremal problem (4.1) is the following

$$\inf\{\max_{x \in Q} F_p(Q, x) : Q \in P(u_1, \dots, u_N), C_{\mathfrak{p}}(Q) = 1\}.$$

To finish the proof of the existence of the solution to the L_p Minkowski problem for $\mathfrak{p}$-capacity, we need to prove the following two lemmas.

Lemma 5.1. *Suppose the n-dimensional polytope P solves problem (4.1) and $x_P = o$. Then P has exactly N facets whose normal vectors are $u_1, \dots, u_N$.*

Proof. We argue by contradiction. Assume that $u_{i_0} \in \{u_1, \dots, u_N\}$, but the support set $F(P, u_{i_0}) = P \cap H(P, u_{i_0})$ is not a facet of P.

Fix $\delta > 0$, let

$$P_\delta = P \cap \{x : x \cdot u_{i_0} \leqslant h_P(u_{i_0}) - \delta\}$$

and

$$\tau P_\delta = \tau(\delta) P_\delta = C_\mathfrak{p}(P_\delta)^{-\frac{1}{n-\mathfrak{p}}} P_\delta.$$

Then, $C_\mathfrak{p}(\tau P_\delta) = 1$ and $\tau P_\delta \to P$, as $\delta \to 0^+$. By Lemma 3.2, it follows that $x_{P_\delta} \to x_P = o \in \operatorname{int} P$, as $\delta \to 0^+$. Thus, for sufficiently small $\delta > 0$, we can assume that $x_{P_\delta} \in \operatorname{int} P$ and

$$h_P(u_k) - x_{P_\delta} \cdot u_k > \delta > 0, \quad \text{for } k = 1, \dots, N.$$

In the following, we show $F_p(\tau P_\delta, x_{\tau P_\delta}) < F_p(P, o)$, which contradicts the fact that $F_p(P, o)$ is the minimum. Since

$$\begin{aligned}
F_p(\tau P_\delta, x_{\tau P_\delta}) &= \tau^p \sum_{k=1}^N c_k (h_{P_\delta}(u_k) - x_{P_\delta} \cdot u_k)^p \\
&= \tau^p \left(\sum_{k=1}^N c_k (h_P(u_k) - x_{P_\delta} \cdot u_k)^p \right) - \tau^p c_{i_0} (h_P(u_{i_0}) - x_{P_\delta} \cdot u_{i_0})^p \\
&\quad + \tau^p c_{i_0} (h_{P_\delta}(u_{i_0}) - x_{P_\delta} \cdot u_{i_0})^p \\
&= F_p(P, x_{P_\delta}) + G(\delta),
\end{aligned}$$

where

$$\begin{aligned}
G(\delta) = (\tau^p - 1) &\left(\sum_{k=1}^N c_k (h_P(u_k) - x_{P_\delta} \cdot u_k)^p \right) \\
&+ c_{i_0} \tau^p [(h_P(u_{i_0}) - x_{P_\delta} \cdot u_{i_0} - \delta)^p - (h_P(u_{i_0}) - x_{P_\delta} \cdot u_{i_0})^p].
\end{aligned}$$

If we can show $G(\delta) < 0$, then $F_p(\tau P_\delta, x_{\tau P_\delta}) < F_p(P, x_{P_\delta}) \leqslant F_p(P, o)$, as desired.

Since $0 < h_P(u_{i_0}) - x_{P_\delta} \cdot u_{i_0} - \delta < h_P(u_{i_0}) - x_{P_\delta} \cdot u_{i_0} < d_0$, where d_0 is the diameter of P, by the concavity of t^p on $[0, \infty)$ for $0 < p < 1$, it follows that

$$(h_P(u_{i_0}) - x_{P_\delta} \cdot u_{i_0} - \delta)^p - (h_P(u_{i_0}) - x_{P_\delta} \cdot u_{i_0})^p < (d_0 - \delta)^p - d_0^p.$$

Hence,

$$G(\delta) < (\tau^p - 1) \left(\sum_{k=1}^N c_k (h_P(u_k) - x_{P_\delta} \cdot u_k)^p \right) + c_{i_0} \tau^p [(d_0 - \delta)^p - d_0^p]$$

$$= \tau^p \left[(d_0-\delta)^p - d_0^p\right] \left(c_{i_0} + \frac{\tau^p - 1}{(d_0-\delta)^p - d_0^p} \frac{1}{\tau^p} \sum_{k=1}^{N} c_k (h_P(u_k) - x_{P_\delta} \cdot u_k)^p \right).$$

From the variational formula for $\mathfrak{p}$-capacity (2.5), it follows that

$$\begin{aligned}
\lim_{\delta\to 0^+} \frac{\tau^p - 1}{(d_0-\delta)^p - d_0^p} &= \lim_{\delta\to 0^+} \frac{(C_\mathfrak{p}(P_\delta))^{-\frac{p}{n-\mathfrak{p}}} - 1}{(d_0-\delta)^p - d_0^p} \\
&= \frac{-\frac{p(\mathfrak{p}-1)}{n-\mathfrak{p}} \sum\limits_{k=1}^{N} \mu_\mathfrak{p}(P,\{u_k\}) h'(u_k,0)}{-p d_0^{p-1}} \\
&= \frac{\mathfrak{p}-1}{n-\mathfrak{p}} \frac{\sum\limits_{k=1}^{N} \mu_\mathfrak{p}(P,\{u_k\}) h'(u_k,0)}{p d_0^{p-1}}.
\end{aligned}$$

Here, $h'(u_k,0) = \lim\limits_{\delta\to 0^+} \frac{h_{P_\delta}(u_k) - h_P(u_k)}{\delta}$.

Assume $\mu_\mathfrak{p}(P,\{u_k\}) \neq 0$, for some k. Since $\mu_\mathfrak{p}(P,\cdot)$ is absolutely continuous with respect to the surface measure $S(P,\cdot)$, it follows that P has a facet with normal vector u_k. By the definition of P_δ, it implies $h_{P_\delta}(u_k) = h_P(u_k)$, for sufficiently small $\delta > 0$. Thus, $h'(u_k,0) = 0$ and

$$\sum_{k=1}^{N} \mu_\mathfrak{p}(P,\{u_k\}) h'(u_k,0) = 0.$$

Therefore,

$$\lim_{\delta\to 0^+} \frac{\tau^p - 1}{(d_0-\delta)^p - d_0^p} = 0.$$

Combining $(d_0-\delta)^p - d_0^p < 0$, $c_{i_0} > 0$ and

$$\frac{1}{\tau^p} \sum_{k=1}^{N} c_k (h_P(u_k) - x_{P_\delta} \cdot u_k)^p \to \sum_{k=1}^{N} c_k h_P^p(u_k) > 0, \quad \text{as } \delta \to 0^+,$$

it follows that for sufficiently small $\delta > 0$, $G(\delta) < 0$.

Consequently, P has exactly N facets. This completes the proof. $\square$

Lemma 5.2. *Let μ be a finite positive Borel measure on $\mathbb{S}^{n-1}$ which is not concentrated on any closed hemisphere, $0 < p < 1$ and $1 < \mathfrak{p} < 2$. If μ is discrete and satisfies $\mu(\{-u\}) = 0$ whenever $\mu(\{u\}) > 0$, for $u \in \mathbb{S}^{n-1}$, then there exists an n-dimensional polytope P solving problem (4.1).*

Proof. Let

$$\beta = \inf\{\max_{x\in Q} F_p(Q,x) : Q \in P(u_1,\dots,u_N), C_{\mathfrak{p}}(Q) = 1\}.$$

Take a minimizing sequence $\{P_i\}_i$ such that $P_i \in P(u_1,\dots,u_N)$, $x_{P_i} = o$, $C_{\mathfrak{p}}(P_i) = 1$ and $\lim_{i\to\infty} F_p(P_i,o) = \beta$.

First, we claim $\{P_i\}_i$ is bounded. Since $x_P = o$, by the definition of F_p, it follows that

$$\begin{aligned}\sum_{k=1}^{N} c_k h_{P_i}(u_k)^p &= \max_{x\in P_i} \sum_{k=1}^{N} c_k (h_{P_i}(u_k) - x\cdot u_k)^p \\ &\leqslant \max_{x\in \tau Q} \sum_{k=1}^{N} c_k (h_{\tau Q}(u_k) - x\cdot u_k)^p + 1,\end{aligned}$$

where $Q = \{x : x\cdot u_k \leqslant 1, k = 1,\dots,N\}$ and τ satisfies $C_{\mathfrak{p}}(\tau Q) = 1$. Let

$$M = \max_{x\in \tau Q} \sum_{k=1}^{N} c_k (h_{\tau Q}(u_k) - x\cdot u_k)^p + 1.$$

Then $M > 0$ is independent of i. Hence, for any i,

$$h_{P_i}(u_k) \leqslant \left(\frac{M}{\min\limits_{1\leqslant k\leqslant N} c_k}\right)^{\frac{1}{p}} < \infty, \quad \text{for } k = 1,\dots,N,$$

which implies that $\{P_i\}_i$ is bounded.

By the Blaschke Selection theorem, there exists a convergent subsequence $\{P_{i_j}\}_j$ of $\{P_i\}_i$ such that $P_{i_j} \to P$.

In the following, we prove P is an n-dimensional polytope.

Case 1. If $\dim P \leqslant n-2 < n-\mathfrak{p}$, then $C_{\mathfrak{p}}(P) = 0$ by [26, p. 179], which contradicts that $C_{\mathfrak{p}}(P) = 1$.

Case 2. If $\dim P = n-1$, there exists a unit vector $u \in \mathbb{S}^{n-1}$ such that $P \subset u^{\perp}$. Then $u, -u \in \{u_1,\dots,u_N\} = \operatorname{supp}\mu$. But μ satisfies $\mu(\{-u\}) = 0$ whenever $\mu(\{u\}) > 0$ for any $u \in \mathbb{S}^{n-1}$, which is a contradiction.

So, $\dim P = n$, as desired. $\square$

Now, we can conclude the proof of Theorem 1.1 in this article.

Theorem 5.3. *Let μ be a finite positive Borel measure on $\mathbb{S}^{n-1}$ which is not concentrated on any closed hemisphere, $0 < p < 1$ and $1 < \mathfrak{p} < 2$. Suppose μ is discrete and satisfies $\mu(\{-u\}) = 0$ whenever $\mu(\{u\}) > 0$, for $u \in \mathbb{S}^{n-1}$.*

(1) If $p+\mathfrak{p} \neq n$, then there exists an n-dimensional polytope P such that $\mu_{p,\mathfrak{p}}(P,\cdot) = \mu$;

(2) If $p+\mathfrak{p}=n$, then there exists an n-dimensional polytope P and a constant $\lambda>0$ such that $\lambda\mu_{p,\mathfrak{p}}(P,\cdot)=\mu$.

Proof. For the discrete measure μ, by Lemma 5.2, there exists an n-dimensional polytope Q_0 which solves problem (4.1). That is, $C_{\mathfrak{p}}(Q_0)=1$ and

$$F_p(Q_0,x_{Q_0})=\inf\{\max_{x\in Q}F_p(Q,x):Q\in P(u_1,\dots,u_N),C_{\mathfrak{p}}(Q)=1\}.$$

By Lemma 3.3 (1), it implies that $P_0=Q_0-x_{Q_0}$ is still the solution to (4.1) and $x_{P_0}=o$. Combining this with Lemma 5.1, Lemma 4.1 and Lemma 4.2 (1), if $p+\mathfrak{p}\neq n$, we have

$$\mu_{p,\mathfrak{p}}(\lambda_0P_0,\cdot)=\mu,$$

where $\lambda_0=\big(\frac{\mathfrak{p}-1}{n-\mathfrak{p}}\sum_{k=1}^{N}c_kh_P^p(u_k)\big)^{\frac{1}{n-p-\mathfrak{p}}}$. That is, $P=\lambda_0P_0$ is the desired solution.

By Lemma 5.1, Lemma 4.1 and Lemma 4.2 (2), if $p+\mathfrak{p}=n$, we have

$$\lambda\mu_{p,\mathfrak{p}}(Q_0,\cdot)=\mu,$$

where $\lambda=\frac{\mathfrak{p}-1}{n-\mathfrak{p}}\sum_{k=1}^{N}c_kh_P^p(u_k)$. That is, $P=Q_0$ is the desired solution. □

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