Advances in Mathematics 389 (2021) 107902

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim

Sharp affine isoperimetric inequalities for the volume decomposition functionals of polytopes ☆

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ARTICLE INFO

Article history:
Received 26 January 2021
Received in revised form 4 May 2021
Accepted 28 June 2021
Available online xxxx
Communicated by Erwin Lutwak

MSC:
52A40
52B60

Keywords:
Polytope
Cone-volume
Affine isoperimetric inequality
Subspace concentration condition

ABSTRACT

The nth power of the volume functional V_n of polytopes P in $\mathbb{R}^n$, according to dimensions of the spaces spanned by any n unit outer normal vectors of P, is decomposed into n homogeneous polynomials of degree n. A set of new sharp affine isoperimetric inequalities for these volume decomposition functionals in $\mathbb{R}^3$ are established, which essentially characterize the geometric and algebraic structures of polytopes.

1. Introduction

The setting for this paper is the n-dimensional Euclidean space, $\mathbb{R}^n$. A *convex body* is a compact convex set that has a nonempty interior. Denote by $\mathcal{K}_o^n$ the set of convex

☆ Research of the authors was supported by NSFC No. 11871373.

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https://doi.org/10.1016/j.aim.2021.107902

bodies in $\mathbb{R}^n$ with the origin o in their interiors. A *polytope* in $\mathbb{R}^n$ is the convex hull of a finite set of points in $\mathbb{R}^n$ provided it has positive *volume* V_n (i.e., n-dimensional Lebesgue measure). The convex hull of a subset of these points is called a *face* of the polytope if it lies entirely on the boundary of the polytope and if it has positive $(n-1)$-dimensional Lebesgue measure. Write $\mathcal{P}_o^n$ for the set of polytopes in $\mathbb{R}^n$ with the origin in their interiors.

Let $P \in \mathcal{P}_o^n$ and u be a unit outer normal vector to a face F of P. The *cone-volume* $V_P(\{u\})$ of P associated with u is the volume of the convex hull of the origin o and face F. The simplest form of cone-volume is reduced to the area formula of triangles in ancient geometry.

However, it is striking that combining the notions of cone-volume of polytopes, linear independence of vectors and dimension of spaces, a *new* geometry of polytopes is emerged: The nth power of volume of polytopes in $\mathbb{R}^n$ is naturally decomposed into n terms, and each term is a homogeneous polynomial of degree n. See Theorem 3.1 for details. In the following, we introduce the so-called *volume decomposition functional* of polytopes.

Definition 1.1. Suppose $P \in \mathcal{P}_o^n$, and $u_1, u_2, \ldots, u_N$ are the unit outer normal vectors of its faces. We define the *kth volume decomposition functional* $X_k(P)$, $k = 1, 2, \ldots, n-1, n$, by

$$X_k(P)^n = \sum_{\dim(\mathrm{span}\{u_{i_1}, \ldots, u_{i_n}\}) = k} V_P(\{u_{i_1}\}) V_P(\{u_{i_2}\}) \cdots V_P(\{u_{i_n}\}).$$

As usual, $\mathrm{span}\{u_{i_1}, \ldots, u_{i_n}\}$ in the above definition denotes the linear subspace spanned by n normal vectors $u_{i_1}, \ldots, u_{i_n}$ of the polytope P. Obviously $X_k(P)^n$ is a homogeneous polynomial in cone-volumes of degree n, $k = 1, 2, \ldots, n$; $X_k(P)$ is *centro-affine* invariant, i.e., $X_k(TP) = X_k(P)$ for $T \in \mathrm{SL(n)}$; and $X_k(\lambda P) = \lambda^n X_k(P)$ for $\lambda > 0$.

It is interesting that if $k = n$, then

$$X_n(P) = \Big(\sum_{u_{i_1} \wedge \cdots \wedge u_{i_n} \neq 0} V_P(\{u_{i_1}\}) V_P(\{u_{i_2}\}) \cdots V_P(\{u_{i_n}\})\Big)^{\frac{1}{n}},$$

which is *identical* to the functional U introduced by E. Lutwak, D. Yang and G. Zhang (LYZ) [22] to attack the longstanding unsolved *Schneider projection problem* in convex geometry; For $k \neq n$, all the functionals $X_k(P)$ are totally *new*.

It is interesting that the nth power of the volume functional $V_n(P)$ satisfies the following tidy *identity*

$$V_n(P)^n = X_1(P)^n + X_2(P)^n + \cdots + X_n(P)^n, \tag{1.1}$$

which says that the nth power of volume $V_n(P)$ of a polytope P is decomposed into n homogeneous polynomials $X_k(P)^n$, $k = 1, 2, \ldots, n$. It seems that the investigation of these polynomials are helpful to find out the distribution of cone-volumes and outer normal vectors of the polytope P. Therefore, the geometric or algebraic structure of the polytope P is more clearly visualized.

Moreover, the identity (1.1) suggests that these new functionals X_k, $k = 1, 2, \ldots, n-1$, are *complementary* to the Lutwak-Yang-Zhang U functional. So, in some sense we trace the origin of LYZ's U functional for the first time.

For the proof of identity (1.1), please refer to Theorem 3.1 in Section 3.

At this moment, we would dwell on the story about the nth volume decomposition functional X_n, what certainly mirrors the significance of the set of new functionals X_k, $k = 1, 2, \ldots, n-1$.

In 2001, LYZ [22] conjectured that if P is a polytope in $\mathbb{R}^n$ with its centroid at the origin, then

$$\frac{X_n(P)}{V_n(P)} \geqslant \frac{(n!)^{\frac{1}{n}}}{n} \tag{1.2}$$

with equality if and only if P is a *parallelotope*.

It took more than one dozen years to completely settle this conjecture. The LYZ conjecture for origin-symmetric polytopes was firstly solved by He, Leng and Li [12], with an alternate proof given by Xiong [28]. The LYZ conjecture in $\mathbb{R}^2$ and $\mathbb{R}^3$ was also confirmed in [28]. A final solution was attributed to Henk and Linke [13]. In 2016, Böröczky and Henk [2] proved that LYZ's conjecture is also affirmative for convex bodies.

It is worth mentioning that in the process of solving the LYZ conjecture, an essential "*concentration phenomenon*" of cone-volumes of polytopes was discovered: If P is a polytope in $\mathbb{R}^n$ with its centroid at the origin and the unit outer normals of P are $u_1, u_2, \ldots, u_N$, then for each subspace $\xi \subseteq \mathbb{R}^n$,

$$\sum_{u_i \in \xi} V_P(\{u_i\}) \leq \frac{\dim\xi}{n} V_n(P) \tag{1.3}$$

with equality for a subspace ξ if and only if there exists a subspace ξ' complementary to ξ in $\mathbb{R}^n$, such that $\{u_j : u_j \notin \xi\} \subseteq \xi'$.

In 2013, Böröczky and LYZ [4] originally posed the notion of *subspace concentration condition* (see Section 2 for its definition) for finite Borel measures on the unit sphere $\mathbb{S}^{n-1}$, and proved that this condition is sufficient and necessary to guarantee the existence of solutions to even *logarithmic Minkowski problems*.

For more investigations and applications on subspace concentration condition, we refer to [3,7,5,8,17,19,27,31]. In [15], the authors pointed out that subspace concentration condition is also connected to the Yau-Tian-Donaldson conjecture in algebraic geometry.

Back to the new volume decomposition functionals X_k, $k = 1, 2, \ldots, n-1$, in light of the identity (1.1), we are tempted to raise the following problem.

Problem X. Let P be a polytope in $\mathbb{R}^n$ with its centroid at the origin. Does there exist a constant $c(n,k)$ depending on n and $k, k \in \{1, 2, \ldots, n-1\}$, such that

$$\frac{X_k(P)}{V_n(P)} \le c(n,k)?$$

In this article, the Problem X in $\mathbb{R}^3$ is satisfactorily solved.

Theorem 1.1. *If P is a polytope in $\mathbb{R}^3$ with its centroid at the origin, then*

$$\frac{X_1(P)}{V_3(P)} \le \left(\frac{1}{3}\right)^{\frac{2}{3}}, \quad \frac{X_2(P)}{V_3(P)} \le \left(\frac{2}{3}\right)^{\frac{1}{3}}, \quad \text{and} \quad \frac{X_3(P)}{V_3(P)} \ge \frac{2^{\frac{1}{3}}}{3^{\frac{2}{3}}},$$

and equality holds in each inequality if and only if P is a parallelepiped.

Theorem 1.2. *If P is a polytope in $\mathbb{R}^n$ with its centroid at the origin, then*

$$\frac{X_1(P)}{V_n(P)} \le n^{\frac{1}{n}-1}$$

with equality if and only if P a parallelotope.

Restricted to $\mathcal{P}_3^n$, i.e., the set of polytopes in $\mathbb{R}^n$ whose any *three* outer normal vectors (up to their antipodal normal vectors) are linear independent, we prove the following.

Theorem 1.3. *If P is a polytope in $\mathcal{P}_3^n$ with its centroid at the origin and $n \ge 3$, then*

$$\frac{X_2(P)}{V_n(P)} \le n^{\frac{1}{n}-1}[(2^{n-1}-1)(n-1)]^{\frac{1}{n}}$$

with equality if and only if P a parallelotope.

This article is organized as follows. In Section 3, we investigate basic properties of the functionals X_k and demonstrate why polytopes should be “positioned” with centroid at the origin in the Problem X. For polytopes P in $(k+1)$-general position, we establish the *inversion formula* for $X_k(P)$, which resembles the classical *inclusion-exclusion principle* in spirit. In the third part, we verify that X_3 does *not* attain its maximum at parallelotopes in $\mathbb{R}^4$, which suggests the complexity of the Problem X in higher dimensions.

In Section 4, we prove the Theorem 1.2 by using the *maximum principle for convex functions*. By solving two constrained optimization problems, we present the proofs of Theorem 1.1 and Theorem 1.3 in Section 5 and Section 6, respectively. Meanwhile, delicate classifications via geometry of subspaces are employed.

In the appendix, we show that the set $\mathcal{P}_k^n$, $k = 1, 2, \ldots, n$, is dense in $\mathcal{K}_o^n$ in the sense of Hausdorff metric.

2. Preliminaries

Write $\mathcal{P}_c^n$ and $\mathcal{P}_s^n$ for the set of polytopes in $\mathbb{R}^n$ with centroid at the origin and the set of polytopes in $\mathbb{R}^n$ symmetric with respect to the origin, respectively. Let $\mathrm{G}_{n,k}$ be the set of k-dimensional subspaces of $\mathbb{R}^n$.

The standard inner product of the vectors $x, y \in \mathbb{R}^n$ is denoted by $x \cdot y$. We write $\mid x \mid^2 = x \cdot x$, and $\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n : \mid x \mid = 1\}$ for the boundary of the Euclidean unit ball B in $\mathbb{R}^n$. The letter μ will be used exclusively to denote a finite Borel measure on $\mathbb{S}^{n-1}$. For such a measure μ, we denote by suppμ its *support set*.

The *support function* $h_K : \mathbb{R}^n \to \mathbb{R}^n$ of a convex body K is defined, for $x \in \mathbb{R}^n$, by

$$h_K(x) = \max\{x \cdot y : y \in K\}.$$

Observe that support functions are positively homogeneous of degree one and subadditive.

The set $\mathcal{K}_o^n$ is often equipped with the *Hausdorff metric* δ. For $K, L \in \mathcal{K}_o^n$,

$$\delta(K, L) = \sup_{u \in \mathbb{S}^{n-1}} |h_K(u) - h_L(u)|.$$

A *hyperplane* of $\mathbb{R}^n$ can be written in the form

$$H_{u,\alpha} = \{x \in \mathbb{R}^n : x \cdot u = \alpha\}$$

with $u \in \mathbb{R}^n \backslash \{o\}$ and $\alpha \in \mathbb{R}$. The hyperplane $H_{u,\alpha}$ bounds a *closed halfspace*

$$H_{u,\alpha}^- = \{x \in \mathbb{R}^n : x \cdot u \leq \alpha\}.$$

Cone-volume measure is a natural generalization of cone-volume of polytopes to convex bodies. For $K \in \mathcal{K}_o^n$, its cone-volume measure V_K is a finite Borel measure on $\mathbb{S}^{n-1}$, defined for each Borel set $\omega \subseteq \mathbb{S}^{n-1}$ by

$$V_K(\omega) = \frac{1}{n} \int_{x \in \nu_K^{-1}(\omega)} x \cdot \nu_K(x) d\mathcal{H}^{n-1}(x), \tag{2.1}$$

where $\nu_K : \partial' K \to \mathbb{S}^{n-1}$ is the Gauss map of K, defined on $\partial' K$, the set of points of ∂K that have a unique outer unit normal, and $\mathcal{H}^{n-1}$ is $(n-1)$-dimensional Hausdorff measure.

Cone-volume measures have appeared in [1,9–11,20,21,23–26,29,30], and have been intensively investigated in recent years. See, e.g., [6,2,3,7,5,8,13,14,16,28].

Following Böröczky and LYZ [4], we present the definition of subspace concentration condition and the celebrated *Böröczky-LYZ existence theorem* of solutions to the even logarithmic Minkowski problem.

Definition 2.1. A finite Borel measure μ on $\mathbb{S}^{n-1}$ is said to satisfy the *subspace concentration inequality* if, for every subspace ξ of $\mathbb{R}^n$, such that $0 < \dim\xi < n$,

$$\mu(\xi \cap \mathbb{S}^{n-1}) \leq \frac{1}{n}\mu(\mathbb{S}^{n-1})\dim\xi. \tag{2.2}$$

The measure is said to satisfy the *subspace concentration condition* if in addition to satisfying the subspace concentration inequality (2.2), whenever

$$\mu(\xi \cap \mathbb{S}^{n-1}) = \frac{1}{n}\mu(\mathbb{S}^{n-1})\dim\xi,$$

for some subspace ξ, then there exists a subspace ξ', which is complementary to ξ in $\mathbb{R}^n$, so that also

$$\mu(\xi' \cap \mathbb{S}^{n-1}) = \frac{1}{n}\mu(\mathbb{S}^{n-1})\dim\xi',$$

or equivalently so that μ is concentrated on $\mathbb{S}^{n-1} \cap (\xi \cup \xi')$.

Lemma 2.1. (Böröczky-Lutwak-Yang-Zhang, [4]) *A non-zero finite even Borel measure on the unit sphere $\mathbb{S}^{n-1}$ is the cone-volume measure of an origin-symmetric convex body in $\mathbb{R}^n$ if and only if it satisfies the subspace concentration condition.*

3. Fundamental properties of X_k and k-general position

3.1. Fundamental properties of X_k

Theorem 3.1. *If P is a polytope in $\mathbb{R}^n$ with the origin in its interior, and the unit outer normal vectors of P are $u_1, u_2, \dots, u_N$, then $V_n(P)^n = X_1(P)^n + X_2(P)^n + \cdots + X_n(P)^n$.*

Proof. Since P contains the origin in its interior, by Definition 1.1, it follows that

$$\begin{aligned} V_n(P)^n &= (V_P(\{u_1\}) + V_P(\{u_2\}) + \cdots + V_P(\{u_N\}))^n \\ &= \sum_{k=1}^{n} \sum_{\dim(\mathrm{span}\{u_{i_1},\dots,u_{i_n}\})=k} V_P(\{u_{i_1}\})V_P(\{u_{i_2}\})\cdots V_P(\{u_{i_n}\}) \\ &= \sum_{k=1}^{n} X_k(P)^n, \end{aligned}$$

as desired. □

Remark 1. The functionals X_k, $k = 1, 2, \dots, n$, are not continuous in $\mathcal{P}_o^n$.

We see this by constructing an example. Let $P_i = \mathrm{conv}\{A_i, B_i, C_i, D_i\}$, $i \in \mathbb{N}$, where $A_i = (\frac{1}{i} - 1, 1), B_i = (-\frac{1}{i} + 1, 1), C_i = (\frac{1}{i} + 1, -1), D_i = (-\frac{1}{i} - 1, -1)$. That is, P_i is

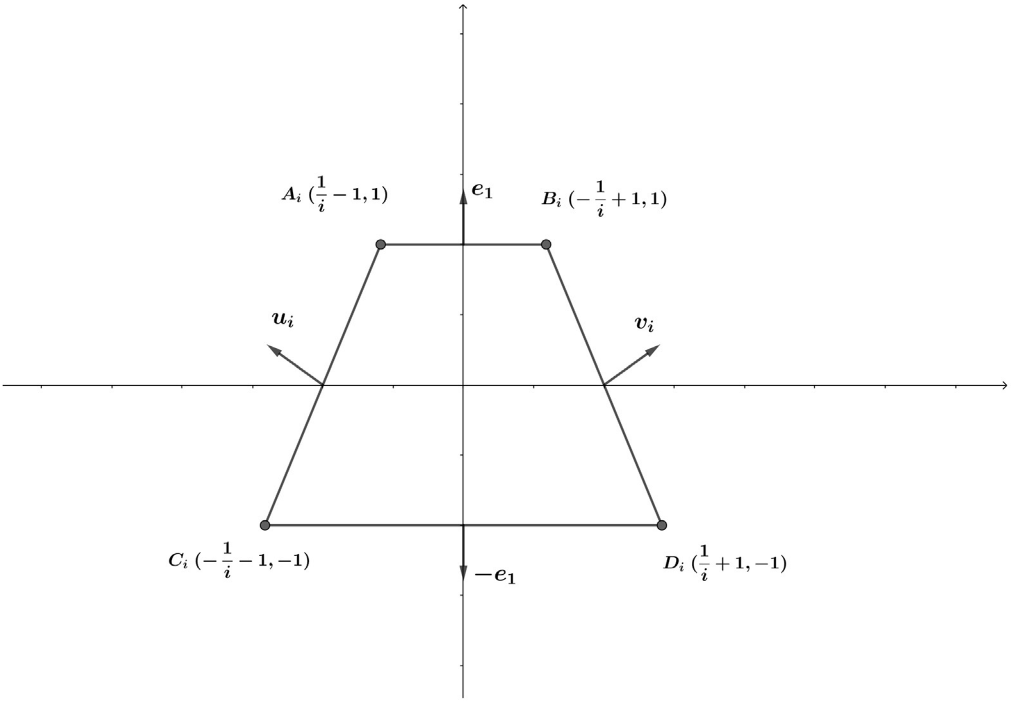

Fig. 1. P_i.

an isosceles trapezoid with symmetric to the y-axis in $\mathbb{R}^2$, whose sizes of upper bottom and lower bottom are $(2-\frac{2}{i})$ and $(2+\frac{2}{i})$, respectively; the distance from the origin o to A_iB_i and C_iD_i is 1, respectively. See Fig. 1.

Then, $\text{supp}V_{P_i}=\{e_1,-e_1,u_i,v_i\}$ and $u_i\neq -v_i$. By calculating directly, we have

$$
\begin{aligned}
&V_2(P_i)=\frac{1}{2}\cdot(2-\frac{2}{i}+2+\frac{2}{i})\cdot 2=4,\\
&V_{P_i}(\{e_1\})=\frac{1}{2}\cdot(2-\frac{2}{i})\cdot 1=1-\frac{1}{i},\\
&V_{P_i}(\{-e_1\})=\frac{1}{2}\cdot(2+\frac{2}{i})\cdot 1=1+\frac{1}{i},\\
&V_{P_i}(\{u_i\})=V_{P_i}(\{v_i\})=\frac{V_2(P_i)-V_{P_i}(\{e_1\})-V_{P_i}(\{-e_1\})}{2}=1.
\end{aligned}
$$

According to the definition of X_1, it follows that

$$
\begin{aligned}
X_1(P_i)^2=&\sum_{u\wedge v=0}V_{P_i}(\{u\})V_{P_i}(\{v\})\\
=&(1-\frac{1}{i})\cdot(1-\frac{1}{i}+1+\frac{1}{i})+(1+\frac{1}{i})\cdot(1+\frac{1}{i}+1-\frac{1}{i})+1\cdot 1+1\cdot 1\\
=&6.
\end{aligned}
$$

Note that $\lim_{i\to\infty} P_i = C$, $C = [-1,1]^2$, and $X_1(C)^2 = 4\cdot 1\cdot 2 = 8$, it follows that

$$\lim_{i\to\infty} X_1(P_i) = \sqrt{6} \neq \sqrt{8} = X_1(C) = X_1(\lim_{i\to\infty} P_i),$$

which implies that X_1 is not continuous. With the continuity of the area functional V_2 and the fact that $V_2(P_i)^2 = X_1(P_i)^2 + X_2(P_i)^2$, it follows that X_2 is not continuous.

Now, we explain the reasons why polytopes should be "positioned" with centroid at the origin, when we consider Problem X. Specifically, restricted to the set of origin-symmetric polytopes in $\mathbb{R}^n$, we prove the following theorem by *essentially* making use of Lemma 2.1.

Theorem 3.2. $\sup_{P\in\mathcal{P}_s^n} \frac{X_n(P)}{V_n(P)} = 1, \quad \inf_{P\in\mathcal{P}_s^n} \frac{X_k(P)}{V_n(P)} = 0,\ k = 1,2,\dots,n-1.$

Proof. For a natural $N, N > n$, let

$$\beta_i = (1, i, i^2, \dots, i^{n-1}), \quad i = 1,2,\dots,N.$$

By the property of Vandermonde determinant, it follows that

$$\det(\beta_{l_1}^{\mathrm{t}}, \beta_{l_2}^{\mathrm{t}}, \dots, \beta_{l_n}^{\mathrm{t}}) \neq 0, \quad \forall\ \{l_1, l_2, \dots, l_n\} \subseteq \{1,2,\dots,N\},$$

which implies that any n elements of the set $\{\beta_1, \beta_2, \dots, \beta_N\}$ are linear independent.

Constructing a finite even discrete measure μ_N on $\mathbb{S}^{n-1}$ such that

$$\begin{aligned}
&\mathrm{supp}\mu_N = \{\pm\langle\beta_1\rangle, \dots, \pm\langle\beta_N\rangle\}, \quad \text{and}\\
&\mu_N(\{\langle\beta_i\rangle\}) = \mu_N(\{-\langle\beta_i\rangle\}) = \frac{1}{2N}, \quad i = 1,2,\dots,N,
\end{aligned}$$

where $\langle\beta_i\rangle = \frac{\beta_i}{|\beta_i|}$.

For any j dimensional subspace ξ of $\mathbb{R}^n$, $0 < j < n$, since any n elements of $\{\beta_1, \beta_2, \dots, \beta_N\}$ are linear independent, it follows that any $(j+1)$ elements of $\{\beta_1, \beta_2, \dots, \beta_N\}$ are also linear independent, and therefore ξ contains at most j elements of $\{\beta_1, \beta_2, \dots, \beta_N\}$. Thus,

$$\mu_N(\xi \cap \mathbb{S}^{n-1}) \leq 2j \cdot \frac{1}{2N} < \frac{j}{n},$$

which implies that μ_N satisfies the subspace concentration condition.

By Lemma 2.1, there exists an origin-symmetric polytope P_N, such that $V_{P_N} = \mu_N$. So,

$$\begin{aligned}
&\mathrm{supp}V_{P_N} = \{\pm\langle\beta_1\rangle, \dots, \pm\langle\beta_N\rangle\}, \quad \text{and}\\
&V_{P_N}(\{\langle\beta_i\rangle\}) = V_{P_N}(\{-\langle\beta_i\rangle\}) = \frac{1}{2N}, \quad i = 1,2,\dots,N.
\end{aligned}$$

From the definition of X_n and the fact that any n elements of the set $\{\beta_1, \beta_2, \dots, \beta_N\}$ are linear independent, it follows that

$$\begin{aligned}\frac{X_n(P_N)^n}{V_n(P_N)^n} =& \frac{\sum_{\substack{u_1,u_2,\dots,u_n \in \mathrm{supp} V_P \\ u_1 \wedge u_2 \wedge \cdots \wedge u_n \neq 0}} V_{P_N}(\{u_1\}) V_{P_N}(\{u_2\}) \cdots V_{P_N}(\{u_n\})}{1} \\ =& 2N \cdot 2(N-1) \cdots 2(N-n+1) (\frac{1}{2N})^n \\ =& \frac{N \cdot (N-1) \cdots (N-n+1)}{N^n} \to 1, \ N \to \infty.\end{aligned}$$

So,

$$0 \leq \frac{X_k(P_N)^n}{V_n(P_N)^n} \leq \frac{V_n(P_N)^n - X_n(P_N)^n}{V_n(P_N)^n} \to 0, \ N \to \infty, \quad k = 1, 2, \dots, n-1.$$

Thus,

$$\sup_{P \in \mathcal{P}_s^n} \frac{X_n(P)}{V_n(P)} = 1, \quad \inf_{P \in \mathcal{P}_s^n} \frac{X_k(P)}{V_n(P)} = 0, \ k = 1, 2, \dots, n-1,$$

as desired. □

By Theorem 3.2, it yields that $\sup_{P \in \mathcal{P}_c^n} \frac{X_n(P)}{V_n(P)} = 1, \inf_{P \in \mathcal{P}_c^n} \frac{X_k(P)}{V_n(P)} = 0, \ k = 1, 2, \dots, n-1$.

Theorem 3.3. $\sup_{P \in \mathcal{P}_o^n} \frac{X_1(P)}{V_n(P)} = 1$.

Proof. Assume that P is a regular simplex in $\mathbb{R}^n$ with $V_n(P) = 1$, and the unit outer normals of P are $u_1, u_2, \dots, u_{n+1}$. Move P to get a sequence of simplices $\{P_i\}_{i \in \mathbb{N}}$ so that $V_{P_i}(\{u_1\}) = 1 - \frac{1}{2i}$, and $V_{P_i}(\{u_j\}) = \frac{1}{2ni}$, $j = 2, 3, \dots, n+1$. Then

$$\begin{aligned}X_1(P_i)^n =& \sum_{\dim(\mathrm{span}\{u_{j_1}, u_{j_2}, \dots, u_{j_n}\}) = 1} V_{P_i}(\{u_{j_1}\}) V_{P_i}(\{u_{j_2}\}) \cdots V_{P_i}(\{u_{i_n}\}) \\ =& \sum_{j=1}^{n+1} V_{P_i}(\{u_j\})^n > V_{P_i}(\{u_1\})^n = \left(1 - \frac{1}{2i}\right)^n \to 1, \ i \to \infty.\end{aligned}$$

So, $\lim_{i \to \infty} \frac{X_1(P_i)}{V_n(P_i)} = 1$, as desired. □

Proposition 3.4. *If P is a polytope in $\mathbb{R}^n$ with the origin in its interior, then the two assertions are equivalent.*

(1) $\mathrm{supp} V_P \subseteq \{\pm u_1, \dots, \pm u_N\}$, *and* $V_P(\{\pm u_i\}) > 0$, $i = 1, 2, \dots, N$.

(2) $\mathrm{supp} V_P \cup \mathrm{supp} V_{-P} = \{\pm u_1, \dots, \pm u_N\}$.

Proof. Assume assertion (1) holds. In light of $\text{supp}V_{-P} = -\text{supp}V_P$, it yields that $\text{supp}V_{-P} \subseteq \{\pm u_1, \ldots, \pm u_N\}$. Hence, $\text{supp}V_P \cup \text{supp}V_{-P} \subseteq \{\pm u_1, \ldots, \pm u_N\}$. Meanwhile, since $V_P(\{\pm u\}) > 0$ for $u \in \{\pm u_1, \ldots, \pm u_N\}$, it follows that $u \in \text{supp}V_P$ or $u \in \text{supp}V_{-P}$. That is, $u \in \text{supp}V_P \cup \text{supp}V_{-P}$. Thus, $\text{supp}V_P \cup \text{supp}V_{-P} = \{\pm u_1, \ldots, \pm u_N\}$.

Assume assertion (2) holds. Let $u_i \in \text{supp}V_P \cup \text{supp}V_{-P}$. Then, $u_i \in \text{supp}V_P$ or $-u_i \in \text{supp}V_P$. Thus, $V_P(\{\pm u_i\}) = V_P(\{u_i\}) + V_P(\{-u_i\}) > 0$. Added that $\text{supp}V_P \subseteq \{\pm u_1, \ldots, \pm u_N\}$, then assertion (1) holds. □

Theorem 3.5. *If* $P \in \mathcal{P}_o^n$ *and* $\text{supp}V_P \cup \text{supp}V_{-P} = \{\pm u_1, \pm u_2, \ldots, \pm u_N\}$, *then*

$$X_k(P)^n = \sum_{\xi \in \mathrm{G}_{n,k}} \sum_{\substack{i_1, i_2, \ldots, i_n \in \{1,2,\ldots,N\} \\ \text{span}\{u_{i_1}, u_{i_2}, \ldots, u_{i_n}\} = \xi}} V_P(\{\pm u_{i_1}\}) V_P(\{\pm u_{i_2}\}) \cdots V_P(\{\pm u_{i_n}\}),$$

$$k = 1, 2, \ldots, n.$$

Proof. From the definition of X_k and the fact that $\text{span}\{v_1, \ldots, v_{n-1}, -v_n\} = \text{span}\{v_1, \ldots, v_{n-1}, v_n\}$, by interchange the order in the summations we have

$$\begin{aligned}
X_k(P)^n &= \sum_{\substack{v_1, \ldots, v_n \in \text{supp}V_P \\ \text{span}\{v_1, \ldots, v_n\} \in \mathrm{G}_{n,k}}} V_P(\{v_1\}) \cdots V_P(\{v_n\}) \\
&= \sum_{\xi \in \mathrm{G}_{n,k}} \sum_{\substack{v_1, \ldots, v_n \in \text{supp}V_P \\ \text{span}\{v_1, \ldots, v_n\} = \xi}} V_P(\{v_1\}) \cdots V_P(\{v_n\}) \\
&= \sum_{\xi \in \mathrm{G}_{n,k}} \sum_{v_1, \ldots, v_{n-1} \in \text{supp}V_P} V_P(\{v_1\}) \cdots V_P(\{v_{n-1}\}) \sum_{\substack{v_n \in \text{supp}V_P \\ \text{span}\{v_1, \ldots, v_n\} = \xi}} V_P(\{v_n\}) \\
&= \sum_{\xi \in \mathrm{G}_{n,k}} \sum_{v_1, \ldots, v_{n-1} \in \text{supp}V_P} V_P(\{v_1\}) \cdots V_P(\{v_{n-1}\}) \sum_{\substack{v_n \in \{u_1, u_2, \ldots, u_N\} \\ \text{span}\{v_1, \ldots, v_n\} = \xi}} V_P(\{\pm v_n\}) \\
&= \sum_{\xi \in \mathrm{G}_{n,k}} \sum_{v_n \in \{u_1, u_2, \ldots, u_N\}} V_P(\{\pm v_n\}) \sum_{\substack{v_1, \ldots, v_{n-1} \in \text{supp}V_P \\ \text{span}\{v_1, \ldots, v_n\} = \xi}} V_P(\{v_1\}) \cdots V_P(\{v_{n-1}\}) \\
&\cdots \\
&= \sum_{\xi \in \mathrm{G}_{n,k}} \sum_{\substack{v_1, \ldots, v_n \in \{u_1, u_2, \ldots, u_N\} \\ \text{span}\{v_1, \ldots, v_n\} = \xi}} V_P(\{\pm v_1\}) V_P(\{\pm v_2\}) \cdots V_P(\{\pm v_n\}) \\
&= \sum_{\xi \in \mathrm{G}_{n,k}} \sum_{\substack{i_1, i_2, \ldots, i_n \in \{1,2,\ldots,N\} \\ \text{span}\{u_{i_1}, u_{i_2}, \ldots, u_{i_n}\} = \xi}} V_P(\{\pm u_{i_1}\}) V_P(\{\pm u_{i_2}\}) \cdots V_P(\{\pm u_{i_n}\}),
\end{aligned}$$

as desired. □

Theorem 3.6. *Let P and Q be polytopes in $\mathbb{R}^n$ with the origin in their interiors. If $V_P(\{\pm u\}) = V_Q(\{\pm u\})$ for any $u \in \mathbb{S}^{n-1}$, then $X_k(P) = X_k(Q)$, $k = 1, 2, \dots, n$.*

Proof. For any $u \in \mathrm{supp}V_P \cup \mathrm{supp}V_{-P}$, it follows that $V_P\{\pm u\} > 0$ by Proposition 3.4. From the assumption that $V_Q(\{\pm u\}) = V_P(\{\pm u\})$, it yields that $V_Q\{\pm u\} > 0$, and therefore $u \in \mathrm{supp}V_Q \cup \mathrm{supp}V_{-Q}$. Hence, $\mathrm{supp}V_P \cup \mathrm{supp}V_{-P} \subseteq \mathrm{supp}V_Q \cup \mathrm{supp}V_{-Q}$.

Similarly, $\mathrm{supp}V_Q \cup \mathrm{supp}V_{-Q} \subseteq \mathrm{supp}V_P \cup \mathrm{supp}V_{-P}$. So, $\mathrm{supp}V_P \cup \mathrm{supp}V_{-P} = \mathrm{supp}V_Q \cup \mathrm{supp}V_{-Q}$. From Theorem 3.5 together with the assumption that $V_P(\{\pm u\}) = V_Q(\{\pm u\})$ for any $u \in \mathbb{S}^{n-1}$, it follows that $X_k(P) = X_k(Q)$, $k = 1, 2, \dots, n$. $\square$

Remark 2. From the equations $X_k(P) = X_k(Q), k = 1, 2, \dots, n$, we *cannot* conclude that $P = Q$. For instance, let P be an equilateral triangle in $\mathbb{R}^2$ with centroid at the origin and unit outer normals u_1, u_2, u_3; let Q be a regular hexagon with centroid at the origin and unit outer normals $\pm u_1, \pm u_2, \pm u_3$. Let P and Q have the same area. It is evidently that P and Q satisfy the conditions in Theorem 3.6.

Theorem 3.7. *If P is a polytope in $\mathbb{R}^n$ with its centroid at the origin, then there exists an origin-symmetric polytope Q in $\mathbb{R}^n$ such that $X_k(Q) = X_k(P)$, $k = 1, 2, \dots, n$.*

Proof. Let μ be a finite discrete measure on $\mathbb{S}^{n-1}$ such that for $u \in \mathbb{S}^{n-1}$,

$$\mu(\{u\}) = \mu(\{-u\}) = \frac{1}{2} V_P(\{\pm u\}).$$

Then μ is an even measure. For any subspace ξ of $\mathbb{R}^n$,

$$\begin{aligned}\mu(\xi \cap \mathbb{S}^{n-1}) = \sum_{u \in \xi \cap \mathbb{S}^{n-1}} \mu(\{u\}) &= \sum_{u \in \xi \cap \mathbb{S}^{n-1}} \frac{1}{2} \mu(\{\pm u\}) \\ &= \sum_{u \in \xi \cap \mathbb{S}^{n-1}} \frac{1}{2} V_P(\{\pm u\}) = \sum_{u \in \xi \cap \mathbb{S}^{n-1}} V_P(\{u\}) = V_P(\xi \cap \mathbb{S}^{n-1}).\end{aligned}$$

Specifically, $\mu(\mathbb{S}^{n-1}) = V_P(\mathbb{S}^{n-1})$. Hence, for every subspace ξ of $\mathbb{R}^n$ with $0 < \mathrm{dim}\xi < n$,

$$\frac{\mu(\xi \cap \mathbb{S}^{n-1})}{\mu(\mathbb{S}^{n-1})} = \frac{V_P(\xi \cap \mathbb{S}^{n-1})}{V_P(\mathbb{S}^{n-1})}.$$

Since the centroid of P is at the origin, its cone volume measure V_P satisfies the subspace concentration condition. So does the measure μ.

According to Lemma 2.1, there exists an origin-symmetric polytope Q so that $V_Q = \mu$. Thus, $V_Q(\{\pm u\}) = \mu(\{\pm u\}) = V_P(\{\pm u\})$. By Theorem 3.6, it yields that $X_k(Q) = X_k(P)$, $k = 1, 2, \dots, n$, as desired. $\square$

By Theorem 3.7, it yields that $\sup_{P \in \mathcal{P}_c^n} \frac{X_k(P)}{V_n(P)} = \sup_{P \in \mathcal{P}_s^n} \frac{X_k(P)}{V_n(P)}, k = 1, 2, \dots, n-1$.

3.2. k-general position

In this part, we pose the notion of *k-general position*, $k \in \{1, 2, \dots, n\}$, which generalizes the essential geometric notion: general position. Restricted to the class of polytopes in $(k+1)$-general position in $\mathbb{R}^n$, we establish the *inversion formula* of the kth volume decomposition X_k. The fact that polytopes in k-general position in $\mathbb{R}^n$ is dense in the set $\mathcal{K}_o^n$ is proved in Appendix A.

Definition 3.1. Let Z be a finite set of unit vectors in $\mathbb{R}^n$, and $Z \cup (-Z) = \{\pm u_1, \pm u_2, \dots, \pm u_N\}$. Z is said to be in *k-general position*, $k \in \{1, 2, \dots, n\}$, if Z is not contained in a closed hemisphere of $\mathbb{S}^{n-1}$ and any k elements of $\{u_1, u_2, \dots, u_N\}$ are linearly independent.

A polytope P in $\mathbb{R}^n$ is said to be in *k-general position*, if the set of unit outer normals of P is in k-general position.

Write $\mathcal{P}_k^n$ for the set of polytopes in $\mathbb{R}^n$ which are in k-general position and contain the origin in their interiors. The inclusion relation

$$\mathcal{P}_n^n \subseteq \mathcal{P}_{n-1}^n \subseteq \cdots \subseteq \mathcal{P}_3^n \subseteq \mathcal{P}_2^n = \mathcal{P}_1^n = \mathcal{P}_o^n$$

is evident.

Recall that Károlyi and Lovász [18] first posed the notion of *general position*: A finite set Z of unit vectors in $\mathbb{R}^n$ is said to be in general position, if Z is not contained in a closed hemisphere of $\mathbb{S}^{n-1}$ and any n elements of Z are linearly independent. So, the n-general position is indeed the general position in the sense of Károlyi and Lovász, up to antipodal unit outer normals.

Write $\mathcal{L}^n \subseteq \mathcal{P}_o^n$ for the set of polytopes in $\mathbb{R}^n$ whose unit outer normals are in general position. Then, $\mathcal{L}^n \subseteq \mathcal{P}_n^n$. However, a cube $C = [-1, 1]^n \in \mathcal{P}_n^n$, but $C \notin \mathcal{L}^n$.

Restricted in $\mathcal{P}_{k+1}^n$, $k \in \{1, 2, \dots, n-1\}$, we establish the inversion formula of the kth volume decomposition X_k.

Theorem 3.8. *If $P \in \mathcal{P}_{k+1}^n$ and $\operatorname{supp} V_P \cup \operatorname{supp} V_{-P} = \{\pm u_1, \dots, \pm u_N\}$, then*

$$X_k(P)^n = \sum_{1 \le i_1 < i_2 < \cdots < i_k \le N} \sum_{l=1}^{k} (-1)^{k-l} \sum_{\{j_1, \dots, j_l\} \subseteq \{i_1, \dots, i_k\}} (a_{j_1} + \cdots + a_{j_l})^n,$$

where $a_j = V_P(\{\pm u_j\})$, $j = 1, \dots, N$.

Proof. From Theorem 3.5, and the assumption that P is in $(k+1)$-general position (therefore, any k-dimensional subspace spanned by unit outer normals of P precisely contains k normals of P, up to their antipodal normals), it follows that

$$\begin{aligned}
X_k(P)^n &= \sum_{\xi \in \mathrm{G}_{n,k}} \sum_{\substack{j_1,\ldots,j_n \in \{1,\ldots,N\} \\ \mathrm{span}\{u_{j_1},\ldots,u_{j_n}\}=\xi}} a_{j_1} \cdots a_{j_n} \\
&= \sum_{1 \le i_1 < i_2 < \cdots < i_k \le N} \sum_{\cup_{m=1}^{n}\{j_m\}=\{i_1,\ldots,i_k\}} a_{j_1} \cdots a_{j_n} \\
&= \sum_{1 \le i_1 < \cdots < i_k \le N} \sum_{\substack{l_1,\ldots,l_k \in \mathbb{N}^+ \\ l_1+\cdots+l_k=n}} a_{i_1}^{l_1} a_{i_2}^{l_2} \cdots a_{i_k}^{l_k} \binom{n}{l_1} \\
&\quad \times \binom{n-l_1}{l_2} \cdots \binom{n-l_1-l_2-\cdots-l_{k-1}}{l_k} \\
&= \sum_{1 \le i_1 < \cdots < i_k \le N} \sum_{\substack{l_1,\ldots,l_k \in \mathbb{N}^+ \\ l_1+\cdots+l_k=n}} \frac{a_{i_1}^{l_1} \cdots a_{i_k}^{l_k}}{l_1! \cdots l_k!} n!.
\end{aligned}$$

Fixing $i_1, \ldots, i_k$, let

$$\sum_{l=1}^{k} (-1)^{k-l} \sum_{\{j_1,\ldots,j_l\} \subseteq \{i_1,\ldots,i_k\}} (a_{j_1} + \cdots + a_{j_l})^n = \sum_{\substack{l_1,\ldots,l_k \in \mathbb{N} \\ l_1+\cdots+l_k=n}} c(l_1,\ldots,l_k) a_{i_1}^{l_1} \cdots a_{i_k}^{l_k}, \quad (3.1)$$

where $c(l_1, \ldots, l_k)$ is the coefficient of $a_{i_1}^{l_1} \cdots a_{i_k}^{l_k}$.

Observe that when $l_1, \ldots, l_k \in \mathbb{N}^+$, $a_{i_1}^{l_1} \cdots a_{i_k}^{l_k}$ appears only if $l = k$ on the left side. So,

$$c(l_1, \ldots, l_k) = \binom{n}{l_1} \binom{n-l_1}{l_2} \cdots \binom{n-l_1-l_2-\cdots-l_{k-1}}{l_k} = \frac{n!}{l_1! \cdots l_k!}.$$

When $l_1 \cdot l_2 \cdots l_k = 0$, assume $l_{p_1}, \ldots, l_{p_m} > 0$ and $l_{p_1} + \cdots + l_{p_m} = n$, $m \le k-1$. By comparing the coefficient of $a_{i_{p_1}}^{l_{p_1}} \cdots a_{i_{p_m}}^{l_{p_m}}$ of both sides of the equation (3.1), we have

$$\begin{aligned}
c(l_1, \ldots, l_k) &= \sum_{l=m}^{k} (-1)^{k-l} \sum_{\{p_1,\ldots,p_m\} \subseteq \{j_1,\ldots,j_l\} \subseteq \{i_1,\ldots,i_k\}} \bar{c} \\
&= \bar{c} \sum_{l=m}^{k} (-1)^{k-l} \binom{k-m}{l-m} = \bar{c} \sum_{l=0}^{k-m} (-1)^{k-m-l} \binom{k-m}{l} \\
&= \bar{c}(1-1)^{k-m} = 0,
\end{aligned}$$

where $\bar{c}$ is the coefficient of $a_{p_1}^{l_{p_1}} \cdots a_{p_m}^{l_{p_m}}$ in $(a_{p_1} + \cdots + a_{p_m})^n$. So,

$$\sum_{l=1}^{k} (-1)^{k-l} \sum_{\{j_1,\ldots,j_l\} \subseteq \{i_1,\ldots,i_k\}} (a_{j_1} + \cdots + a_{j_l})^n = \sum_{\substack{l_1,\ldots,l_k \in \mathbb{N}^+ \\ l_1+\cdots+l_k=n}} \frac{a_{i_1}^{l_1} \cdots a_{i_k}^{l_k}}{l_1! \cdots l_k!} n!.$$

Hence,

$$X_k(P)^n = \sum_{1\le i_1<i_2<\cdots<i_k\le N} \sum_{l=1}^{k}(-1)^{k-l} \sum_{\{j_1,\dots,j_l\}\subseteq\{i_1,\dots,i_k\}} (a_{j_1}+\cdots+a_{j_l})^n. \quad \square$$

3.3. The complexity of the Problem X

In this part, we construct two examples to show that in $\mathbb{R}^4$, the volume decomposition functional $X_3(P)$ does *not* attain its maximum at parallelotopes, which indicates the complexity of the Problem X in higher dimensions.

Example 1. Let Q be an origin-symmetric parallelotope in $\mathbb{R}^n$. Suppose that $\mathrm{supp}V_Q = \{\pm u_1,\dots,\pm u_n\}$ and $V_Q(\{\pm u_i\}) = 1$, $i = 1,2,\dots,n$. Then, $Q \in \mathcal{P}_n^n$ and $V_n(Q) = n$.

By Theorem 3.8, it follows that

$$X_k(Q)^n = \binom{n}{k}\sum_{l=1}^{k}\binom{k}{l}(-1)^{k-l}l^n.$$

In particular,

$$\begin{aligned}
X_1(Q)^n &= \binom{n}{1} = n,\\
X_2(Q)^n &= \binom{n}{2}\big(2^n - \binom{2}{1}\big) = n(n-1)(2^{n-1}-1),\\
X_3(Q)^n &= \binom{n}{3}\big(3^n - \binom{3}{2}\cdot 2^n + \binom{1}{1}\cdot 1^n\big) = \frac{n(n-1)(n-2)}{2}(3^{n-1}-2^n+1).
\end{aligned}$$

Example 2. Let P be a simplex in $\mathbb{R}^4$ with its centroid at the origin. Suppose that $\mathrm{supp}V_P = \{u_1,u_2,u_3,u_4,u_5\}$ and $V_P(\{u_i\}) = 1$, $i = 1,2,3,4,5$. Then, $P \in \mathcal{P}_4^4$, $V_4(P) = 5$.

By Theorem 3.8, it follows that

$$X_3(P)^4 = \binom{5}{3}\big[\binom{3}{1}1^4 - \binom{3}{2}2^4 + \binom{3}{3}3^4\big] = 360.$$

Thus,

$$\frac{X_3(P)^4}{V_4(P)^4} = \frac{360}{5^4} = \frac{72}{125}.$$

However, taking $n = 4$ in Example 1, we have

$$\frac{X_3(Q)^4}{V_4(Q)^4} = \frac{4(4-1)(4-2)}{2}(3^{4-1}-2^4+1)\frac{1}{4^4} = \frac{9}{16} = \frac{1125}{2000} < \frac{1152}{2000} = \frac{72}{125} = \frac{X_3(P)^4}{V_4(P)^4},$$

which implies that a parallelotope is *not* the extremal body for X_3 in $\mathbb{R}^4$.

4. The Problem X for X_1 in $\mathbb{R}^n$

Lemma 4.1. *If P is a polytope in $\mathbb{R}^n$ with its centroid at the origin, and $u_1, u_2, \ldots, u_N$ are the unit outer normal vectors to the faces of P, then*

$$V_P(\{\pm u_i\}) \leq \frac{1}{n} V(P), \quad i = 1, 2, \ldots, N. \tag{4.1}$$

If P is a parallelotope, then equality holds in (4.1). Conversely, if equality holds in (4.1) for each $i \in \{1, 2, \ldots, N\}$, then P is a parallelotope.

One can refer to [28, p. 3222] for its proof. In the following, we prove Theorem 1.2.

Theorem 4.2. *If P is a polytope in $\mathbb{R}^n$ with its centroid at the origin, then*

$$\frac{X_1(P)}{V_n(P)} \leq n^{\frac{1}{n}-1}$$

with equality if and only if P is a parallelotope.

Proof. Let $\mathrm{supp}V_P \cup \mathrm{supp}V_{-P} = \{\pm u_1, \pm u_2, \ldots, \pm u_N\}$ with $N \geq n \geq 2$, and assume $V_n(P) = 1$. By the definition of X_1 and Theorem 3.5, it follows that

$$X_1(P)^n = \sum_{i=1}^{N} a_i^n,$$

where $a_i = V_P(\{\pm u_i\})$, $i = 1, \ldots, N$.

In the following, we investigate the maximum of the function

$$f(a_1, a_2, \ldots, a_N) = \sum_{i=1}^{N} a_i^n$$

on the compact convex set

$$\Omega = \{(a_1, a_2, \ldots, a_N) \in [0, \frac{1}{n}]^N : \sum_{i=1}^{N} a_i = 1\}.$$

For this purpose, two claims are needed.

Claim 1. The function f is *strictly* convex on Ω.

For any two distinct points $Y_1, Y_2 \in \Omega$, assume $Y_1 = (x_1, x_2, \ldots, x_N)$ and $Y_2 = (y_1, y_2, \ldots, y_N)$. Since $Y_1 \neq Y_2$, and $x_i, y_i \geq 0$ for each $i \in \{1, 2, \ldots, N\}$, it follows that

$$\begin{aligned}\frac{d^2 f(tY_1+(1-t)Y_2)}{dt^2} =&(Y_1-Y_2)\mathrm{Hess}(f)(tY_1+(1-t)Y_2)(Y_1-Y_2)^t\\ =&\sum_{i=1}^{N} n(n-1)(tx_i+(1-t)y_i)^{n-2}(x_i-y_i)^2>0,\quad t\in(0,1),\end{aligned}$$

which implies that f is strictly convex.

Claim 2. The extreme points of Ω (which is a polytope in a hyperplane in $\mathbb{R}^N$) are exactly the vertices with n coordinates $\frac{1}{n}$ and $N-n$ coordinates 0.

Assume $Y_0=(\bar{a}_1,\dots,\bar{a}_N)\in\Omega\backslash\{0,\frac{1}{n}\}^N$. Then there exist distinct $i,j\in\{1,2,\dots,N\}$, such that $\bar{a}_i,\bar{a}_j\in(0,\frac{1}{n})$. Let $\varepsilon=\min\{\bar{a}_i,\bar{a}_j,\frac{1}{n}-\bar{a}_i,\frac{1}{n}-\bar{a}_j\}$. Then

$$(\bar{a}_1,\dots,\bar{a}_i+\varepsilon,\dots,\bar{a}_j-\varepsilon,\dots,\bar{a}_N)\in\Omega,\quad(\bar{a}_1,\dots,\bar{a}_i-\varepsilon,\dots,\bar{a}_j+\varepsilon,\dots,\bar{a}_N)\in\Omega,$$

and Y_0 is precisely the *midpoint* of the line segment joining the above two points. Thus, Y_0 is *not* an extreme point of the polytope Ω, and therefore all the extreme points of Ω are contained in $\Omega\cap\{0,\frac{1}{n}\}^N$. With the definition of Ω, the Claim 2 is confirmed.

Combining the Claim 1 and Claim 2, by the *maximum principle for convex functions*, it follows that f can only achieve its maximum on the extreme points of Ω. That is,

$$\max\{f(a_1,\dots,a_N):(a_1,\dots,a_N)\in\Omega\}=\sum_{i=1}^{n}\Big(\frac{1}{n}\Big)^n=n^{1-n}.$$

Therefore,

$$\max\{X_1(P)^n:P\in\mathcal{P}_c^n\}=\max\{f(a_1,\dots,a_N):(a_1,\dots,a_N)\in\Omega\}=n^{1-n}.$$

From Lemma 4.1, the maximum of X_1 is achieved only when P is a parallelotope. □

5. A complete solution to the Problem X in $\mathbb{R}^3$

With Theorem 4.2 in hand, to prove Theorem 1.1 it suffices to prove the following.

Theorem 5.1. *If P is a polytope in $\mathbb{R}^3$ with its centroid at the origin, then*

$$\frac{X_2(P)}{V_3(P)}\le\Big(\frac{2}{3}\Big)^{\frac{1}{3}}$$

with equality if and only if P is a parallelepiped.

Proof. Assume that $\mathrm{supp}V_P\cup\mathrm{supp}V_{-P}=\{\pm u_1,\pm u_2,\dots,\pm u_N\}$ and $V_3(P)=1$.

If P is a parallelepiped, taking $n=3$ and $k=2$ in Theorem 3.8, then

$$\frac{X_2(P)}{V_3(P)}=\Big[\binom{3}{2}\Big(\big(\frac{1}{3}+\frac{1}{3}\big)^3-2\big(\frac{1}{3}\big)^3\Big)\Big]^{\frac{1}{3}}=\Big(\frac{2}{3}\Big)^{\frac{1}{3}}.$$

In the following, we aim to show that

$$\frac{X_2(P)}{V_3(P)} < \left(\frac{2}{3}\right)^{\frac{1}{3}},$$

as long as P is not a parallelepiped.

Suppose $\{\xi_1, \xi_2, \ldots, \xi_m\} = \{\text{span}\{u_i, u_j\} : i, j = 1, 2, \ldots, N, i \neq j\}$, $i = 1, 2, \ldots, N$. That is to say, $\{\xi_1, \xi_2, \ldots, \xi_m\}$ consists of *all* the 2-dimensional subspaces spanned by unit outer normals of the polytope P.

Let $a_i = V_P(\{\pm u_i\})$. From the assumption that $V_3(P) = 1$ and Lemma 2.1, it follows that $\sum_{i=1}^{N} a_i = 1$; $(a_1, a_2, \ldots, a_N) \in (0, \frac{1}{3}]^N$; and $\sum_{\{j:u_j\in\xi_i\}} a_j \leq \frac{2}{3}$, $i = 1, 2, \ldots, m$.

Observe that for $u_{j_1}, u_{j_2}, u_{j_3} \in \xi_i$, $\text{span}\{u_{j_1}, u_{j_2}, u_{j_3}\} \neq \xi_i$ if and only if $j_1 = j_2 = j_3$. By Theorem 3.5, it follows that

$$\begin{aligned}
X_2(P)^3 &= \sum_{\xi\in G_{3,2}} \sum_{\substack{j_1,j_2,j_3\in\{1,2,\ldots,N\}\\ \text{span}\{u_{j_1},u_{j_2},u_{j_3}\}=\xi}} V_P(\{\pm u_{j_1}\})V_P(\{\pm u_{j_2}\})V_P(\{\pm u_{j_3}\}) \\
&= \sum_{i=1}^{m} \sum_{\substack{j_1,j_2,j_3\in\{j:u_j\in\xi_i\}\\ \text{span}\{u_{j_1},u_{j_2},u_{j_3}\}=\xi_i}} a_{j_1}a_{j_2}a_{j_3} \\
&= \sum_{i=1}^{m} \Big(\big(\sum_{j_1,j_2,j_3\in\{j:u_j\in\xi_i\}} a_{j_1}a_{j_2}a_{j_3} \big) - \big(\sum_{\substack{j_1,j_2,j_3\in\{j:u_j\in\xi_i\}\\ \text{span}\{u_{j_1},u_{j_2},u_{j_3}\}\neq\xi_i}} a_{j_1}a_{j_2}a_{j_3} \big) \Big) \\
&= \sum_{i=1}^{m} \Big(\big(\sum_{j_1,j_2,j_3\in\{j:u_j\in\xi_i\}} a_{j_1}a_{j_2}a_{j_3} \big) - \sum_{j_1\in\{j:u_j\in\xi_i\}} a_{j_1}^3 \Big) \\
&= \sum_{i=1}^{m} \Big(\big(\sum_{\{j:u_j\in\xi_i\}} a_j \big)^3 - \sum_{\{j:u_j\in\xi_i\}} a_j^3 \Big).
\end{aligned}$$

In the rest, we investigate the maximum of the function

$$f(a_1, a_2, \ldots, a_n) = \sum_{i=1}^{m} \Big(\big(\sum_{\{j:u_j\in\xi_i\}} a_j \big)^3 - \sum_{\{j:u_j\in\xi_i\}} a_j^3 \Big) \tag{5.1}$$

under the constrained conditions (A1)–(A3):

$$\sum_{i=1}^{N} a_i = 1, \tag{A1}$$

$$(a_1, a_2, \ldots, a_N) \in [0, \frac{1}{3}]^N, \tag{A2}$$

$$\sum_{\{j:u_j\in\xi_i\}} a_j \leq \frac{2}{3}, \; i = 1, 2, \ldots, m. \tag{A3}$$

Since the set of points $(a_1, a_2, \dots, a_N)$ satisfying conditions (A1)–(A3) is compact and f is a polynomial, it follows that f attains its maximum on this constrained domain. Suppose that f attains its maximum at the point $Y_0 = (\bar{a}_1, \bar{a}_2, \dots, \bar{a}_N)$. What follows aims to show that $\bar{a}_1, \bar{a}_2, \dots, \bar{a}_N \in \{0, \frac{1}{3}\}$. We show this by contradiction.

Assume that there exists at least one member among $\bar{a}_1, \bar{a}_2, \dots, \bar{a}_N$ lying in $(0, \frac{1}{3})$.

Step 1. We prove that there exist $\bar{a}_k, \bar{a}_l \in (0, \frac{1}{3}),\ k \neq l$, and sufficiently small $\varepsilon > 0$ such that

$$Y(t) = Y_0 + te_k - te_l, \quad \forall t \in (-\varepsilon, \varepsilon)$$

still satisfies the conditions (A1)–(A3), where $e_k = (\underbrace{0, \dots, 0}_{k-1}, 1, 0, \dots, 0)$.

Case 1. Assume that $Y_0 = (\bar{a}_1, \bar{a}_2, \dots, \bar{a}_N)$ satisfies all the *strict* inequalities in the condition (A3), i.e., for any $i \in \{1, 2, \dots, m\}$, $\sum_{\{j:u_j \in \xi_i\}} \bar{a}_j < \frac{2}{3}$.

Since there exists at least one member among $\bar{a}_1, \bar{a}_2, \dots, \bar{a}_N$ lying in $(0, \frac{1}{3})$, by conditions (A1) and (A2), it follows that there exist at least *two* members among $\bar{a}_1, \bar{a}_2, \dots, \bar{a}_N$, say $\bar{a}_k, \bar{a}_l$, such that $\bar{a}_k, \bar{a}_l \in (0, \frac{1}{3})$. Hence, there exists a sufficiently small $\varepsilon > 0$ such that $Y(t)$ still satisfies the conditions (A1)–(A3).

Case 2. Assume that $Y_0 = (\bar{a}_1, \bar{a}_2, \dots, \bar{a}_N)$ satisfies a certain *equality* in the condition (A3), i.e., there exists an $i \in \{1, 2, \dots, m\}$, say $i = 1$, such that $\sum_{\{j:u_j \in \xi_1\}} \bar{a}_j = \frac{2}{3}$.

Case 2.1. If the set $\{i : u_i \in \xi_1 \text{ and } \bar{a}_i > 0\}$ contains only *two* elements, say i_1, i_2, then $\bar{a}_{i_1} = \bar{a}_{i_2} = \frac{1}{3}$.

Since there exists at least one member among $\bar{a}_1, \bar{a}_2, \dots, \bar{a}_N$ lying in $(0, \frac{1}{3})$, by conditions (A1) and (A2), it follows that there exist at least two members among $\bar{a}_1, \bar{a}_2, \dots, \bar{a}_N$, say $\bar{a}_k, \bar{a}_l$, such that $\bar{a}_k, \bar{a}_l \in (0, \frac{1}{3})$. Let $\varepsilon = \min\{\bar{a}_k, \bar{a}_l, \frac{1}{3} - \bar{a}_k, \frac{1}{3} - \bar{a}_l\}$. Then $Y(t)$ satisfies the condition (A2). That $Y(t)$ satisfies the condition (A1) is evident by its definition. It remains to show that $Y(t)$ also satisfies the condition (A3). For this aim, it suffices to show that any index i in the set $\{i : u_k \in \xi_i, \text{ or } u_l \in \xi_i\}$ satisfies the condition (A3).

Let $i_0 \in \{i : u_k \in \xi_i, \text{ or } u_l \in \xi_i\}$. Since $u_k, u_l \notin \xi_1 = \operatorname{span}\{u_{i_1}, u_{i_2}\}$, it yields that $\xi_{i_0} \neq \xi_1$. So, $u_{i_1} \notin \xi_{i_0}$ or $u_{i_2} \notin \xi_{i_0}$. Hence, if a point $(a_1, a_2, \dots, a_N)$ satisfies the conditions (A1)–(A2) and $a_{i_1} = a_{i_2} = \frac{1}{3}$, then

$$\sum_{\{j:u_j \in \xi_{i_0}\}} a_j \leq \sum_{\{j:u_j \neq u_{i_1}\}} a_j = 1 - a_{i_1} = 1 - \frac{1}{3} = \frac{2}{3},$$

or

$$\sum_{\{j:u_j \in \xi_{i_0}\}} a_j \leq \sum_{\{j:u_j \neq u_{i_2}\}} a_j = 1 - a_{i_2} = 1 - \frac{1}{3} = \frac{2}{3}.$$

In particular, $Y(t)$ satisfies the condition (A3).

Case 2.2. If the set $\{i : u_i \in \xi_1 \text{ and } \bar{a}_i > 0\}$ contains at least *three* elements, say $i_1, i_2, \ldots, i_M$, $2 < M < N$, then $\bar{a}_{i_1}, \bar{a}_{i_2}, \ldots, \bar{a}_{i_M} \in (0, \frac{1}{3}]$ and $\sum_{j=1}^{M} \bar{a}_{i_j} = \frac{2}{3}$. So, there exist at least two members among $\bar{a}_{i_1}, \bar{a}_{i_2}, \ldots, \bar{a}_{i_M}$, say $\bar{a}_k, \bar{a}_l$, such that $\bar{a}_k, \bar{a}_l \in (0, \frac{1}{3})$. Let $\varepsilon = \min\{\bar{a}_k, \bar{a}_l, \frac{1}{3} - \bar{a}_k, \frac{1}{3} - \bar{a}_l\}$. Then $Y(t)$ satisfies the condition (A2). That $Y(t)$ satisfies the condition (A1) is evident by its definition. It remains to show that $Y(t)$ also satisfies the condition (A3). For this aim, it suffices to show that any index i in the set $\{i : u_k \in \xi_i, \text{ or } u_l \in \xi_i\}$ satisfies the condition (A3).

Let $i_0 \in \{i : u_k \in \xi_i, \text{ or } u_l \in \xi_i\}$. If $\xi_{i_0} = \xi_1$, then the condition (A3) for i_0 is naturally satisfied; If $\xi_{i_0} \neq \xi_1$, then

$$\xi_{i_0} \cap \xi_1 \cap \mathbb{S}^2 = \{\pm u_k\}, \text{ or } \xi_{i_0} \cap \xi_1 \cap \mathbb{S}^2 = \{\pm u_l\}.$$

Hence, if a point $(a_1, a_2, \ldots, a_N)$ satisfies conditions (A1)-(A2) and $\sum_{\{j:u_j \in \xi_1\}} a_j = \frac{2}{3}$, then

$$\sum_{\{j:u_j \in \xi_{i_0}\}} a_j = \sum_{\{j:u_j \in \xi_{i_0} \cap \xi_1\}} a_j + \sum_{\{j:u_j \in \xi_{i_0} \setminus \xi_1\}} a_j \leq a_k + \sum_{\{j:u_j \notin \xi_1\}} a_j = a_k + (1 - \frac{2}{3}) \leq \frac{2}{3},$$

or

$$\sum_{\{j:u_j \in \xi_{i_0}\}} a_j = \sum_{\{j:u_j \in \xi_{i_0} \cap \xi_1\}} a_j + \sum_{\{j:u_j \in \xi_{i_0} \setminus \xi_1\}} a_j \leq a_l + \sum_{\{j:u_j \notin \xi_1\}} a_j = a_l + (1 - \frac{2}{3}) \leq \frac{2}{3}.$$

In particular, $Y(t)$ satisfies the condition (A3).

Step 2. Let $F(t) = f(Y(t))$, $t \in (-\varepsilon, \varepsilon)$. By the assumption that f attains its maximum at the point Y_0, it implies that $F(t)$ attains its maximum at $t = 0$. Hence, it is necessary that

$$F'(0) = 0, \quad \text{and} \quad F''(0) \leq 0.$$

In the following, it suffices to show that $F''(0) > 0$ to get the contradiction.

Assume that $k = 1$, $l = 2$, here k, l are found in Step 1. Then

$$F(t) = f(Y(t)) = f(Y_0 + te_1 - te_2).$$

So,

$$F''(0) = (\frac{\partial^2 f}{\partial a_1^2} + \frac{\partial^2 f}{\partial a_2^2} - 2\frac{\partial^2 f}{\partial a_1 \partial a_2})(Y_0).$$

In the following, we calculate $\frac{\partial^2 f}{\partial a_1^2}$, $\frac{\partial^2 f}{\partial a_2^2}$ and $\frac{\partial^2 f}{\partial a_1 \partial a_2}$.

From (5.1), and the fact that the polynomial $\sum_{\{i:u_1 \notin \xi_i\}} \Big((\sum_{\{j:u_j \in \xi_i\}} a_j)^3 - \sum_{\{j:u_j \in \xi_i\}} a_j^3 \Big)$ does not contain the variable a_1, it follows that

$$\begin{aligned}\frac{\partial f}{\partial a_1} &= \frac{\partial}{\partial a_1}\sum_{i=1}^{m}\Big((\sum_{\{j:u_j\in\xi_i\}} a_j)^3 - \sum_{\{j:u_j\in\xi_i\}} a_j^3\Big)\\ &= \frac{\partial}{\partial a_1}\sum_{\{i:u_1\in\xi_i\}}\Big((\sum_{\{j:u_j\in\xi_i\}} a_j)^3 - \sum_{\{j:u_j\in\xi_i\}} a_j^3\Big)\\ &= 3\sum_{\{i:u_1\in\xi_i\}}\Big((\sum_{\{j:u_j\in\xi_i\}} a_j)^2 - a_1^2\Big).\end{aligned}$$

So,

$$\frac{\partial^2 f}{\partial a_1^2} = 6\sum_{\{i:u_1\in\xi_i\}}\Big((\sum_{\{j:u_j\in\xi_i\}} a_j) - a_1\Big) = 6\sum_{\{i:u_1\in\xi_i\}}\sum_{\{j:u_j\in\xi_i\setminus\{u_1\}\}} a_j.$$

Observe that the intersection of two dimensional subspaces containing u_1 with $\{u_2, u_3, \ldots, u_N\}$ is indeed a disjoint partition of $\{u_2, u_3, \ldots, u_N\}$. Thus,

$$\frac{\partial^2 f}{\partial a_1^2} = 6\sum_{j=2}^{N} a_j = 6(1 - a_1).$$

Similarly, we obtain

$$\frac{\partial^2 f}{\partial a_2^2} = 6(1 - a_2).$$

Now, we calculate $\frac{\partial^2 f}{\partial a_1 \partial a_2}$.

$$\begin{aligned}\frac{\partial^2 f}{\partial a_1\partial a_2} &= 3\frac{\partial}{\partial a_2}\sum_{\{i:u_1\in\xi_i\}}((\sum_{\{j:u_j\in\xi_i\}} a_j)^2 - a_1^2)\\ &= 3\frac{\partial}{\partial a_2}\big(\sum_{\{j:u_j\in\mathrm{span}\{u_1,u_2\}\}} a_j\big)^2\\ &= 6\sum_{\{j:u_j\in\mathrm{span}\{u_1,u_2\}\}} a_j.\end{aligned}$$

According to the condition (A3), it follows that

$$\frac{\partial^2 f}{\partial a_1 \partial a_2} \le 6\cdot\frac{2}{3} = 4.$$

Added with $\bar{a}_1, \bar{a}_2 \in (0, \frac{1}{3})$, it follows that

$$F''(0) \ge 6(1 - \bar{a}_1) + 6(1 - \bar{a}_2) - 2\cdot 4 > 6(1 - \frac{1}{3}) + 6(1 - \frac{1}{3}) - 8 = 0.$$

Step 3. The above two steps indicate that $\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_N \in \{0, \frac{1}{3}\}$.

Suppose that $\bar{a}_1 = \bar{a}_2 = \bar{a}_3 = \frac{1}{3}$, $\bar{a}_4 = \bar{a}_5 = \cdots = \bar{a}_N = 0$. Since $(\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_N)$ satisfies the condition (A3), it follows that $u_1 \wedge u_2 \wedge u_3 \neq 0$. Thus,

$$f(Y_0) = \sum_{i=1}^{m} \Big(\big(\sum_{\{j: u_j \in \xi_i\}} \bar{a}_j \big)^3 - \sum_{\{j: u_j \in \xi_i\}} \bar{a}_j^3 \Big) = 3 \cdot \Big(\big(\frac{2}{3}\big)^3 - 2 \cdot \big(\frac{1}{3}\big)^3 \Big) = \frac{2}{3}.$$

So, for any point $(a_1, a_2, \ldots, a_N)$ satisfying the conditions (A1), (A2) and (A3), we have

$$f(a_1, a_2, \ldots, a_N) \leq \frac{2}{3},$$

the equality holds if and only if there exist $\{i, j, k\} \subseteq \{1, 2, \ldots, N\}$ such that $a_i = a_j = a_k = \frac{1}{3}$ and $u_i \wedge u_j \wedge u_k \neq 0$. Consequently, if P is not a parallelepiped, then

$$\frac{X_2(P)^3}{V_3(P)^3} < \max f = \frac{2}{3}.$$

To sum up, for any polytope P with its centroid at the origin, it follows that

$$\frac{X_2(P)}{V_3(P)} \leq \left(\frac{2}{3}\right)^{\frac{1}{3}}$$

with equality if and only if P is a parallelepiped. □

In light of that $V_3^3 = X_1^3 + X_2^3 + X_3^3$, combining Theorem 4.2 with Theorem 5.1, we obtain the following immediately.

Corollary 5.2. *If P is a polytope in $\mathbb{R}^3$ with its centroid at the origin, then*

$$\frac{X_3(P)}{V_3(P)} \geq \frac{2^{\frac{1}{3}}}{3^{\frac{2}{3}}}$$

with equality if and only if P is a parallelepiped.

By Theorem 4.2, Theorem 5.1 and Corollary 5.2, we prove the main results.

Theorem 5.3. *If P is a polytope in $\mathbb{R}^3$ with its centroid at the origin, then*

$$\frac{X_1(P)}{V_3(P)} \leq \left(\frac{1}{3}\right)^{\frac{2}{3}}, \quad \frac{X_2(P)}{V_3(P)} \leq \left(\frac{2}{3}\right)^{\frac{1}{3}}, \quad \text{and} \quad \frac{X_3(P)}{V_3(P)} \geq \frac{2^{\frac{1}{3}}}{3^{\frac{2}{3}}},$$

and equality holds in each inequality if and only if P is a parallelepiped.

Theorem 5.4. *If P is a polytope in $\mathbb{R}^3$ with its centroid at the origin, then*

$$\frac{X_1(P)}{X_3(P)} \leq \left(\frac{1}{2}\right)^{\frac{1}{3}}, \text{ and } \frac{X_2(P)}{X_3(P)} \leq \left(\frac{1}{3}\right)^{\frac{1}{3}},$$

and equality holds in each inequality if and only if P is a parallelepiped.

6. The solution to the Problem X for X_2 in $\mathcal{P}_3^n$

Recall that $\mathcal{P}_3^n$ is the set of polytopes in $\mathbb{R}^n$ whose any *three* unit outer normals, up to their antipodal unit outer normals, are linear independent.

Theorem 6.1. *If P is a polytope in $\mathcal{P}_3^n$ with its centroid at the origin and $n \geq 3$, then*

$$\frac{X_2(P)}{V_n(P)} \leq n^{\frac{1}{n}-1}[(2^{n-1}-1)(n-1)]^{\frac{1}{n}}$$

with equality if and only if P is a parallelotope.

Proof. Assume that $\mathrm{supp}V_P \cup \mathrm{supp}V_{-P} = \{\pm u_1, \pm u_2, \ldots, \pm u_N\}$ and $V_n(P) = 1$. From Lemma 4.1 and the assumption that $V_n(P) = 1$, it follows that

$$0 < V_P(\{\pm u_i\}) \leq \frac{1}{n}, \ i = 1, \ldots, N; \quad \text{and} \quad \sum_{i=1}^{N} V_P(\{\pm u_i\}) = 1. \tag{6.1}$$

Since $P \in \mathcal{P}_3^n$, by Theorem 3.8, it follows that

$$\begin{aligned} X_2(P)^n &= \sum_{1\leq i<j\leq N} \sum_{l=1}^{2} (-1)^{2-l} \sum_{\{j_1,\ldots,j_l\}\subseteq\{i,j\}} (V_P(\{\pm u_{j_1}\}) + \cdots + V_P(\{\pm u_{j_l}\}))^n \\ &= \frac{1}{2}\sum_{i=1}^{N}\sum_{j\neq i}[(V_P(\{\pm u_i\}) + V_P(\{\pm u_j\}))^n - (V_P(\{\pm u_i\})^n + V_P(\{\pm u_j\})^n)] \\ &\triangleq f(V_P(\{\pm u_1\}), V_P(\{\pm u_2\}), \ldots, V_P(\{\pm u_N\})). \end{aligned}$$

Let $x_i = V_P(\{\pm u_i\})$. To prove the theorem, we consider the following constrained maximizing problem of the function

$$f(x_1, x_2, \ldots, x_N) = \frac{1}{2}\sum_{i=1}^{N}\sum_{j\neq i}[(x_i + x_j)^n - (x_i^n + x_j^n)], \tag{6.2}$$

$$\text{subject to} \quad \begin{cases} (x_1, x_2, \ldots, x_N) \in [0, \frac{1}{n}]^N, \\ \sum_{i=1}^{N} x_i = 1. \end{cases} \tag{6.3}$$

In light of

$$f(x_1,\ldots,x_{i-1},x_i,\ldots,x_j,x_{j+1},\ldots,x_N)=f(x_1,\ldots,x_{i-1},x_j,\ldots,x_i,x_{j+1},\ldots,x_N),$$

we assume that f attains its maximum at the point $Y_0=(\bar{x}_1,\ldots,\bar{x}_M,0,\ldots,0)$, with

$$0<\bar{x}_1\le\bar{x}_2\cdots\le\bar{x}_M\le\frac{1}{n},\text{ and } n\le M\le N.$$

What follows aims to show that $\bar{x}_i=\frac{1}{n}, i=1,2,\ldots,M$. We show this by contradiction.

Assume that there is some $\bar{x}_i$ lying in $(0,\frac{1}{n})$, $1\le i\le M$.

Step 1. We prove that there must be the other $\bar{x}_{i'}$ lying in $(0,\frac{1}{n})$, too. Then, we conclude that $\bar{x}_1,\bar{x}_2\in(0,\frac{1}{n})$ are the *smallest two members* among $\bar{x}_1,\ldots,\bar{x}_M$.

Otherwise, $i=1$ and $\bar{x}_2=\cdots=\bar{x}_M=\frac{1}{n}$. So,

$$0<\bar{x}_1=1-\frac{M-1}{n}<\frac{1}{n}.$$

That is,

$$0<n-M+1<1,$$

which implies that $n-M+1$ is an integer greater than 0 but less than 1, which is a contradiction.

Step 2. For $\varepsilon=\min\{\bar{x}_1,\bar{x}_2,\frac{1}{n}-\bar{x}_1,\frac{1}{n}-\bar{x}_2\}>0$, $Y(t)=Y_0+te_1-te_2$, $t\in(-\varepsilon,\varepsilon)$, still satisfies condition (6.3). Let

$$F(t)=f(Y(t)),\ t\in(-\varepsilon,\varepsilon).$$

By the assumption that f attains its maximum at the point Y_0, it yields that $F(t)$ attains its maximum at $t=0$. Hence, it is necessary that

$$F'(0)=0,\qquad\text{and}\qquad F''(0)\le 0.$$

In the following, it suffices to show that $F''(0)>0$ to get the contradiction.

Since $F(t)=f(Y(t))=f(Y_0+te_1-te_2)$, it follows that

$$F''(0)=(\frac{\partial^2 f}{\partial x_1^2}+\frac{\partial^2 f}{\partial x_2^2}-2\frac{\partial^2 f}{\partial x_1\partial x_2})(Y_0).$$

In the following, we calculate $\frac{\partial^2 f}{\partial x_1^2}$, $\frac{\partial^2 f}{\partial x_2^2}$ and $\frac{\partial^2 f}{\partial x_1\partial x_2}$.

From the equation (6.2), we have

$$\frac{\partial f}{\partial x_i}=\frac{\partial}{\partial x_i}\Big(\frac{1}{2}\sum_{l=1}^{N}\sum_{\substack{j=1\\ j\ne l}}^{N}\big((x_l+x_j)^n-(x_l^n+x_j^n)\big)\Big)$$

$$
\begin{aligned}
&= \frac{\partial}{\partial x_i}\Big(\frac{1}{2}\sum_{l\neq i}\sum_{j\neq l}\big((x_l+x_j)^n-(x_l^n+x_j^n)\big)+\frac{1}{2}\sum_{j\neq i}\big((x_i+x_j)^n-(x_i^n+x_j^n)\big)\Big)\\
&= \frac{\partial}{\partial x_i}\Big(\frac{1}{2}\sum_{l\neq i}\big((x_l+x_i)^n-(x_l^n+x_i^n)\big)+\frac{1}{2}\sum_{j\neq i}\big((x_i+x_j)^n-(x_i^n+x_j^n)\big)\Big)\\
&= \frac{\partial}{\partial x_i}\Big(\sum_{j\neq i}\big((x_i+x_j)^n-x_i^n\big)\Big)\\
&= n\sum_{j\neq i}\big(\,(x_i+x_j)^{n-1}-x_i^{n-1}\big)\\
&= n\sum_{j\neq i}(x_i+x_j)^{n-1}-n(N-1)x_i^{n-1}, \qquad i=1,2.
\end{aligned}
$$

So,

$$
\begin{aligned}
&\frac{\partial^2 f}{\partial x_1^2}=n(n-1)\Big(\,(x_1+x_2)^{n-2}+\sum_{j=3}^{N}(x_1+x_j)^{n-2}-(N-1)x_1^{n-2}\Big);\\
&\frac{\partial^2 f}{\partial x_2^2}=n(n-1)\Big(\,(x_1+x_2)^{n-2}+\sum_{j=3}^{N}(x_2+x_j)^{n-2}-(N-1)x_2^{n-2}\Big);\\
&\frac{\partial^2 f}{\partial x_1\partial x_2}=n(n-1)\,(x_1+x_2)^{n-2}.
\end{aligned}
$$

By $\bar{x}_{M+1}=\cdots=\bar{x}_N=0$, it follows that

$$
\begin{aligned}
F''(0) =&(\frac{\partial^2 f}{\partial x_1^2}+\frac{\partial^2 f}{\partial x_2^2}-2\frac{\partial^2 f}{\partial x_1\partial x_2})(Y_0)\\
=&n(n-1)\Big(\sum_{j=3}^{N}\Big((\bar{x}_1+\bar{x}_j)^{n-2}+(\bar{x}_2+\bar{x}_j)^{n-2}\Big)-(N-1)\left(\bar{x}_1^{n-2}+\bar{x}_2^{n-2}\right)\Big)\\
=&n(n-1)\Big(\sum_{j=3}^{M}\Big((\bar{x}_1+\bar{x}_j)^{n-2}+(\bar{x}_2+\bar{x}_j)^{n-2}\Big)-(M-1)\left(\bar{x}_1^{n-2}+\bar{x}_2^{n-2}\right)\Big).
\end{aligned}
$$

Since $\bar{x}_1+\bar{x}_2+\cdots+\bar{x}_M=1, \bar{x}_i\leq\frac{1}{n}$ and $n\geq 3$, we must have $M\geq 4$. It follows from the facts $0<\bar{x}_1\leq\bar{x}_2\leq\bar{x}_3\leq\bar{x}_4$ and $\bar{x}_j\geq 0$ that

$$
\begin{aligned}
\frac{1}{n(n-1)}F''(0) \geq\ & (\bar{x}_1+\bar{x}_3)^{n-2}+(\bar{x}_2+\bar{x}_3)^{n-2}+(\bar{x}_1+\bar{x}_4)^{n-2}+(\bar{x}_2+\bar{x}_4)^{n-2}\\
&-3(\bar{x}_1^{n-2}+\bar{x}_2^{n-2})\\
\geq\ & 2(2\bar{x}_1)^{n-2}+2(2\bar{x}_2)^{n-2}-3\bar{x}_1^{n-2}-3\bar{x}_2^{n-2}\\
>\ & 0.
\end{aligned}
$$

Step 3. The above two steps indicate that $\bar{x}_1 = \cdots = \bar{x}_n = \frac{1}{n}$, $\bar{x}_{n+1} = \cdots = \bar{x}_N = 0$. So,

$$f(Y_0) = \frac{1}{2}\sum_{i=1}^{n}\sum_{j\neq i}^{n}[(\frac{1}{n}+\frac{1}{n})^n - ((\frac{1}{n})^n + (\frac{1}{n})^n)] = n^{1-n}(2^{n-1}-1)(n-1).$$

That is to say, $f(x_1, x_2, \dots, x_N) \leq n^{1-n}(2^{n-1}-1)(n-1)$, with equality if and only if there are precisely n members of $x_1, x_2, \dots, x_N$ equal to $\frac{1}{n}$. Hence,

$$\frac{X_2(P)}{V_n(P)} \leq n^{\frac{1}{n}-1}[(2^{n-1}-1)(n-1)]^{\frac{1}{n}}$$

with equality if and only if P is a parallelotope. □

7. Open problems

The domain of functionals X_k, $k = 1, 2, \dots, n-1$, can be extended to general convex bodies and even to general measures in a natural manner.

Let μ be a finite Borel measure on the unit sphere $\mathbb{S}^{n-1}$. Define $X_k(\mu)$ as an integral over a subset of $\mathbb{S}^{n-1} \times \cdots \times \mathbb{S}^{n-1}$:

$$X_k(\mu)^n = \int_{\dim(\mathrm{span}\{u_1,\dots,u_n\})=k} d\mu(u_1)\cdots d\mu(u_n),\ \ k = 1, 2, \dots, n-1, \tag{7.1}$$

where $\mathrm{span}\{u_1, \dots, u_n\}$ denotes the linear subspace spanned by vectors $u_1, \dots, u_n$.

Specifically, for $K \in \mathcal{K}_o^n$, define

$$X_k(K)^n = \int_{\dim(\mathrm{span}\{u_1,\dots,u_n\})=k} dV_K(u_1)\cdots dV_K(u_n),\ \ k = 1, 2, \dots, n-1. \tag{7.2}$$

While we have chosen to present our inequalities only for convex polytopes, all of the inequalities presented in this article hold for arbitrary convex bodies.

Two obvious questions regarding functionals X_k beg to be asked.

Problem 7.1. Let K be a convex body in $\mathbb{R}^n$ with its centroid at the origin. Does there exist a constant $c(n, k)$ depending on n and $k, k \in \{1, 2, \dots, n-1\}$, such that

$$\frac{X_k(K)}{V_n(K)} \leq c(n, k)?$$

Problem 7.2. Let K be a convex body in $\mathbb{R}^n$ with its centroid at the origin. Do there exist constants c_1 and c_2 depending on n and $k, k \in \{1, 2, \dots, n-1\}$, such that

$$c_1 X_k(K) \leq X_{k+1}(K) \leq c_2 X_k(K)?$$

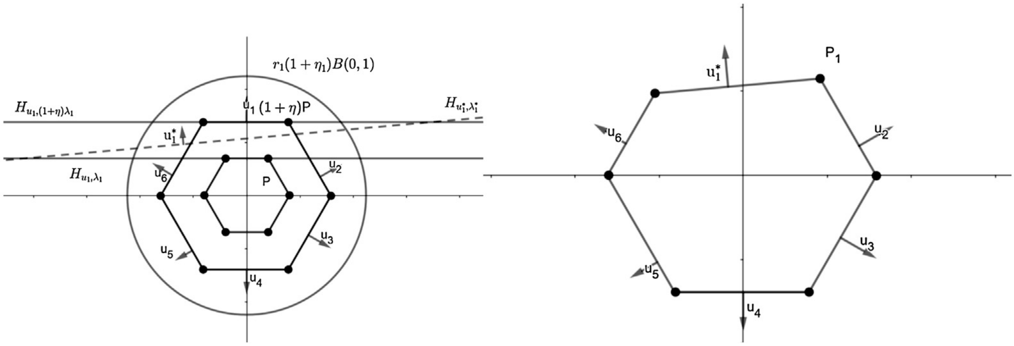

Fig. 2. $P \to P_1$.

Acknowledgments

We are *very* grateful to the reviewers for the *very* thoughtful and careful reading given to the original draft of this paper, and for the *very* nice suggested improvements.

Appendix A

Lemma A.1. *$\mathcal{P}_k^n$, $k = 1, 2, \dots, n$, is dense in $\mathcal{K}_o^n$ in the sense of Hausdorff metric δ.*

Proof. Since $\mathcal{P}_o^n$ is dense in $\mathcal{K}_o^n$ and $\mathcal{L}^n \subseteq \mathcal{P}_k^n$ for any $k \in \{1, 2, \dots, n\}$, it suffices to prove that $\mathcal{L}^n$ is dense in $\mathcal{P}_o^n$ in the sense of Hausdorff metric δ.

Let $P \in \mathcal{P}_o^n$. Assume that $\text{supp}V_P = \{u_1, u_2, \dots, u_N\}$ and $h_P(u_i) = \lambda_i$, $i = 1, 2, \dots, N$. Then, $P = \cap_{i=1}^N H^-_{u_i,\lambda_i}$ (Fig. 2).

Step 1. Assume $P \subseteq r_1 B(0, 1)$, $r_1 > 0$, here $B(0, 1) = \{x \in \mathbb{R}^n : |x| \leq 1\}$.

For any $\epsilon > 0$, let $\eta_1 = \frac{\varepsilon}{r_1}$. Then,

$$\delta((1 + \eta_1)P, P) \leq \max \mid h_{(1+\eta_1)P} - h_P \mid = \eta_1 \max h_P \leq \eta_1 r_1 = \varepsilon.$$

Let

$$Q_1 = H^+_{u_1,\lambda_1} \cap H^-_{u_1,\lambda_1(1+\eta_1)} \cap r_1(1 + \eta_1)B(0, 1),$$

$$A_1 = \{u \in \mathbb{S}^{n-1} : \exists \lambda \in \mathbb{R}, \text{ s.t., } H^-_{u_1,\lambda_1} \cap Q_1 \subseteq H^-_{u,\lambda} \cap Q_1 \subseteq H^-_{u_1,\lambda_1(1+\eta_1)} \cap Q_1\},$$

$$B_1 = \bigcup_{\substack{v_j \in \text{supp}V_P \setminus \{u_1\} \\ j=1,2,\dots,n-1}} \text{span}\{v_1, v_2, \dots, v_{n-1}\} \bigcap \mathbb{S}^{n-1}.$$

Then, $\mathcal{H}^{n-1}(A_1) > 0$ and $\mathcal{H}^{n-1}(B_1) = 0$. So, $A_1 \backslash B_1 \neq \emptyset$.

Pick up a $u_1^* \in A_1 \backslash B_1$, then

$$u_1^* \notin \bigcup_{\{v_1,v_2,\dots,v_{n-1}\} \subseteq \{u_2,u_3,\dots,u_N\}} \text{span}\{v_1, v_2, \dots, v_{n-1}\} \bigcap \mathbb{S}^{n-1}.$$

By the construction of A_1, there exists a $\lambda_1^* \in \mathbb{R}$ such that

$$H^-_{u_1,\lambda_1} \cap Q_1 \subseteq H^-_{u_1^*,\lambda_1^*} \cap Q_1 \subseteq H^-_{u_1,\lambda_1(1+\eta_1)} \cap Q_1.$$

Since

$$\begin{aligned} P &\subseteq H^-_{u_1,\lambda_1} \cap r_1(1+\eta_1)B(0,1) \subseteq H^-_{u_1^*,\lambda_1^*} \cap r_1(1+\eta_1)B(0,1) \\ &\subseteq H^-_{u_1,\lambda_1(1+\eta_1)} \cap r_1(1+\eta_1)B(0,1), \end{aligned}$$

it follows that

$$P \cap (1+\eta_1)P = P \subseteq H^-_{u_1^*,\lambda_1^*} \cap r_1(1+\eta_1)B(0,1) \cap (1+\eta_1)P \subseteq (1+\eta_1)P.$$

Let $P_1 = H^-_{u_1^*,\lambda_1^*} \cap (1+\eta_1)P$. Then, $P_1 = H^-_{u_1^*,\lambda_1^*} \bigcap_{i=2}^N H^-_{u_i,\lambda_i(1+\eta_1)}$ and $P \subseteq P_1 \subseteq (1+\eta_1)P$. Thus,

$$\delta(P_1, P) \leq \delta((1+\eta_1)P, P) \leq \varepsilon.$$

Step 2. Assume that $P_1 \subseteq r_2 B(0,1), r_2 > 0$. Let $\eta_2 = \frac{\varepsilon}{r_2}$. Then,

$$\delta((1+\eta_2)P_1, P_1) \leq \max \mid h_{(1+\eta_2)P_1} - h_{P_1} \mid = \eta_2 \max h_{P_1} \leq \eta_2 r_2 = \varepsilon.$$

Let

$$\begin{aligned} &\widetilde{\lambda_i} = \lambda_i(1+\eta_1),\ i = 2, 3, \ldots, N, \\ &Q_2 = H^+_{u_2,\widetilde{\lambda_2}} \cap H^-_{u_2,\widetilde{\lambda_2}(1+\eta_2)} \cap r_2(1+\eta_2)B(0,1), \\ &A_2 = \{u \in \mathbb{S}^{n-1} : \exists \lambda \in \mathbb{R},\ \text{s.t.,}\ H^-_{u_2,\widetilde{\lambda_2}} \cap Q_2 \subseteq H^-_{u,\lambda} \cap Q_2 \subseteq H^-_{u_2,\widetilde{\lambda_2}(1+\eta_2)} \cap Q_2\}, \\ &B_2 = \bigcup_{\{v_1,v_2,\ldots,v_{n-1}\} \subseteq \{u_1^*,u_3,u_4,\ldots,u_N\}} \text{span}\{v_1, v_2, \ldots, v_{n-1}\} \bigcap \mathbb{S}^{n-1}. \end{aligned}$$

Similarly, we can find $u_2^* \in A_2 \backslash B_2$ and $\lambda_2^* \in \mathbb{R}$ such that

$$u_2^* \notin \bigcup_{\{v_1,v_2,\ldots,v_{n-1}\} \subseteq \{u_1^*,u_3,u_4,\ldots,u_N\}} \text{span}\{v_1, v_2, \ldots, v_{n-1}\} \bigcap \mathbb{S}^{n-1}$$

and

$$H^-_{u_2,\widetilde{\lambda_2}} \cap Q_2 \subseteq H^-_{u_2^*,\lambda_2^*} \cap Q_2 \subseteq H^-_{u_2,\widetilde{\lambda_2}(1+\eta_2)} \cap Q_2.$$

Let $P_2 = H^-_{u_2^*,\lambda_2^*} \cap (1+\eta_2)P_1$. Then, $P_2 = \bigcap_{i=1}^2 H^-_{u_i^*,\lambda_i^*} \bigcap_{i=3}^N H^-_{u_i,\widetilde{\lambda_i}(1+\eta_2)}$ and $P_1 \subseteq P_2 \subseteq (1+\eta_2)P_1$. Thus,

$$\delta(P_2, P_1) \leq \delta((1+\eta_2)P_1, P_1) \leq \varepsilon.$$

Step N. Repeating the above process N times. We can find $u_N^* \in A_N \backslash B_N$ and $\lambda_N^* \in \mathbb{R}$ such that

$$u_N^* \notin \bigcup_{\{v_1,v_2,\ldots,v_{n-1}\}\subseteq\{u_1^*,u_2^*,\ldots,u_{N-1}^*\}} \operatorname{span}\{v_1, v_2, \ldots, v_{n-1}\} \bigcap \mathbb{S}^{n-1}.$$

Let $P_N = H^-_{u_N^*,\lambda_N^*} \cap (1+\eta_N)P_{N-1}$. Then, $P_N = \cap_{i=1}^N H^-_{u_i^*,\lambda_i^*}$ and

$$\delta(P_N, P_{N-1}) \leq \delta((1+\eta_N)P_{N-1}, P_{N-1}) \leq \varepsilon.$$

By the construction of u_i^*, it follows that $\{u_1^*, u_2^*, \ldots, u_N^*\}$ are in *general position*. Added that $\operatorname{supp}V_{P_N} \subseteq \{u_1^*, u_2^*, \ldots, u_N^*\}$, it implies that any n elements of $\operatorname{supp}V_P$ are linear independent. Since $P \subseteq P_N$, it follows that P_N is an n-dimensional polytope, and therefore $\operatorname{supp}V_{P_N}$ does not concentrate on any closed hemisphere of $\mathbb{S}^{n-1}$. Hence, $\operatorname{supp}V_{P_N}$ is in *general position*.

Thus, for any $\varepsilon > 0$, we find a polytope P_N in general position such that

$$\begin{aligned}\delta(P_N, P) &\leq \delta(P_N, P_{N-1}) + \delta(P_{N-1}, P) \leq \delta(P_N, P_{N-1}) + \varepsilon \leq \cdots \leq \delta(P_2, P_1) + (N-1)\epsilon \\ &\leq N\varepsilon,\end{aligned}$$

which implies that $\mathcal{L}^n$ is dense in $\mathcal{P}_o^n$. □

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