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A New Affine Invariant Geometric Functional for Polytopes and Its Associated Affine Isoperimetric Inequalities

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A new affine invariant geometric functional for convex polytopes is introduced. Some new sharp affine isoperimetric inequalities are established for this new functional, which are extensions of Lutwak–Yang–Zhang's results on their celebrated cone-volume functional.

1 Introduction

A *convex body* (i.e., a compact convex subset with nonempty interior) K in n-dimensional Euclidean space $\mathbb{R}^n$ is uniquely determined by its *support function* $h_K : S^{n-1} \to \mathbb{R}$, which is defined for $u \in S^{n-1}$ by $h_K(u) = \max\{u \cdot x : x \in K\}$, where S^{n-1} is the unit sphere and $u \cdot x$ denotes the standard inner product of u and x. The *projection body* ΠK of K is defined as the convex body whose support function, for $u \in S^{n-1}$, is given by $h_{\Pi K}(u) = \mathrm{vol}_{n-1}(K|u^{\perp})$, where vol_{n-1} denotes $(n-1)$-dimensional volume and $K|u^{\perp}$ denotes the image of orthogonal projection of K onto the codimension 1 subspace orthogonal to u. The support function of ΠK can also be represented as

$$h_{\Pi K}(u) = \frac{1}{2}\int_{S^{n-1}} |u \cdot v| \mathrm{d}S_K(v), \qquad u \in S^{n-1}, \tag{1.1}$$

where S_K is the *surface area measure* of convex body K. Formula (1.1) follows from the Cauchy projection formula. See, for example, [28, page 569] for details.

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The projection body is one of the most important objects in convex geometry, and has been intensively investigated during the past three decades. See, for example, [1], [6], [7], [14], [16], [17], [33], etc. It is centro-affine invariant, that is, for $T \in \mathrm{SL}(n)$, $\Pi(TK) = T^{-t}(\Pi K)$, where T^{-t} denotes the inverse of the transpose of T. It is worth mentioning that there stands a celebrated unsolved problem regarding projection bodies, called Schneider's projection problem: as K ranges over the class of origin-symmetric convex bodies in $\mathbb{R}^n$, what is the least upper bound of the affine-invariant ratio

$$\left[V(\Pi K)/V(K)^{n-1}\right]^{\frac{1}{n}},$$

where $V(K)$ denotes the n-dimensional volume of K. See, for example, [27], [28], and [29]. The lower bound for the affine-invariant ratio is also unknown; Petty [24] conjectured that the minimum of this quantity is attained precisely by ellipsoids.

An effective tool to study Schneider's projection problem is the *cone-volume functional* U, which was introduced by Lutwak, Yang, and Zhang (LYZ) [19]: if P is a convex polytope in $\mathbb{R}^n$ that contains the origin o in its interior, then $U(P)$ is defined as

$$U(P)^n = \frac{1}{n^n} \sum_{u_{i_1} \wedge \cdots \wedge u_{i_n} \neq 0} h_{i_1} \cdots h_{i_n} a_{i_1} \cdots a_{i_n}, \tag{1.2}$$

where $u_1, \ldots, u_N$ are the outer normal unit vectors to the faces of P, $h_1, \ldots, h_N$ are the corresponding distances of the faces from the origin and $a_1, \ldots, a_N$ are the corresponding areas of the faces.

It is interesting that the functional U is centro-affine invariant, that is, $U(TP) = U(P)$, for $T \in \mathrm{SL}(n)$. Let $V_i = a_i h_i / n,\ i = 1, \ldots, N$, then

$$U(P)^n = \sum_{u_{i_1} \wedge \cdots \wedge u_{i_n} \neq 0} V_{i_1} \cdots V_{i_n}.$$

Since $V(P)^n = (\sum_{i=1}^{N} V_i)^n$, it follows that $U(P) < V(P)$. If P is a random polytope with a large number of faces, $U(P)$ is very close to $V(P)$. See LYZ [19, page 1772] for details. It is this important property of the functional U that makes it so powerful.

For instance, using the functional U, LYZ [19] presented an affirmative answer to the modified Schneider projection problem: if P is an origin-symmetric polytope in $\mathbb{R}^n$, then

$$\frac{V(\Pi P)}{U(P)^{\frac{n}{2}} V(P)^{\frac{n}{2}-1}} \leq 2^n \left(\frac{n^n}{n!}\right)^{\frac{1}{2}}, \tag{1.3}$$

with equality if and only if P is a parallelotope. This statement is especially interesting as for an origin-symmetric convex body K, it is known that $V(\Pi K)/V(K)^{n-1}$ is not maximized by parallelotopes on one hand and (1.3) yields an upper bound on $V(\Pi K)/V(K)^{n-1}$ of optimal order on the other hand. See, for example, Schneider [28, page 578] for details.

LYZ [19] also proved that for an origin-symmetric convex polytope P in $\mathbb{R}^n$, it holds

$$V(\Pi^* P) U(P)^{\frac{n}{2}} V(P)^{\frac{n}{2}-1} \geq \frac{2^n}{(n^n n!)^{\frac{1}{2}}}, \tag{1.4}$$

with equality if and only if P is a parallelotope. Here, $\Pi^* P$ is the polar body of ΠP.

Similarly, for a convex polytope P in $\mathbb{R}^n$ with its John point at the origin, LYZ [19] established

$$\frac{V(\Pi P)}{U(P)^{\frac{n}{2}} V(P)^{\frac{n}{2}-1}} \leq \frac{n^n (n+1)^{\frac{n+1}{2}}}{(n!)^{\frac{3}{2}}}, \tag{1.5}$$

with equality if and only if P is a simplex. Recall that the John point of a convex body is precisely the center of its John ellipsoid, which is the ellipsoid contained in the body with maximal volume. See, for example, [13] and [21].

Showing the lower bound of functional U in terms of volume V makes an interesting story. LYZ [19] conjectured that for polytopes P with centroid at the origin, there holds

$$U(P) \geq \frac{(n!)^{\frac{1}{n}}}{n} V(P), \tag{1.6}$$

with equality if and only if P is a parallelotope.

It took more than a dozen years to completely settle this conjecture. In [10], He, Leng, and Li proved (1.6) for origin-symmetric polytopes, including its equality condition. In [31], the 2nd author of this article gave a simplified proof for symmetric polytopes and proved (1.6), including the equality case, for 2D and 3D polytopes with centroid at the origin. A complete solution to this conjecture was attributed to Henk and Linke [11].

In 2015, Böröczky and LYZ [4] extended the domain of cone-volume functional U to the class of convex bodies K in $\mathbb{R}^n$ with the origin in their interiors and defined

$$U(K)^n = \frac{1}{n^n} \int_{u_1 \wedge \cdots \wedge u_n \neq 0} h_K(u_1) \cdots h_K(u_n) \mathrm{d}S_K(u_1) \cdots \mathrm{d}S_K(u_n). \tag{1.7}$$

Since $V(K)^n = (\frac{1}{n} \int_{S^{n-1}} h_K \mathrm{d}S_K)^n$, it follows that $U(K) \leq V(K)$. $U(K)$ is still centro-affine invariant, that is, $U(TK) = U(K)$, for $T \in \mathrm{SL}(n)$. Recently, Böröczky and Henk [2] proved that LYZ's conjecture is also affirmative for convex bodies with centroid at the origin.

In view of its *volume* attribute of the cone-volume functional U, together with its strong applications, the main goal of this article is to further generalize the cone-volume functional $U(K)$ to the so-called *mixed cone-volume functional* $U_1(K,L)$ as

$$U_1(K,L)^n = \int_{u_1 \wedge \cdots \wedge u_n \neq 0} \mathrm{d}V_{K,L}(u_1) \cdots \mathrm{d}V_{K,L}(u_n),$$

where $V_{K,L}(\cdot)$ is the newly introduced mixed cone-volume measure. See Definition 3.1 and Definition 3.2 for details.

It is striking that as the important geometric functional *1st mixed volume* $V_1(K,L)$ (see Section 2 for its definition) generalizes the volume functional $V(K)$, the mixed cone-volume functional $U_1(K,L)$ not only generalizes the cone-volume functional $U(K)$ but also has very similar properties to $V_1(K,L)$. However, what we want to emphasize here is that $U_1(K,L)$ and $V_1(K,L)$ have different features. As an illustration, we can see later that if K and L are polytopes, then $U_1(K,L) < V_1(K,L)$. Thus, U_1 is indeed a new geometric functional for polytopes.

In this article, several sharp affine isoperimetric inequalities for $U_1(K,L)$ are established.

Theorem 1.1. If P, Q are convex polytopes in $\mathbb{R}^n$ and Q is origin-symmetric, then

$$\frac{V(\Pi P) V(Q)}{U_1(P,Q)^{\frac{n}{2}} V_1(P,Q)^{\frac{n}{2}}} \leq 2^n \left(\frac{n^n}{n!} \right)^{\frac{1}{2}}, \tag{1.8}$$

with equality if and only if P and Q are parallel parallelotopes.

Theorem 1.2. If P, Q are convex polytopes in $\mathbb{R}^n$ and Q is origin-symmetric, then

$$V(\Pi^* P) U_1(P, Q)^{\frac{n}{2}} V_1(P, Q)^{\frac{n}{2}} \geq \frac{2^n}{(n^n n!)^{\frac{1}{2}}} V(Q), \tag{1.9}$$

with equality if and only if P and Q are parallel parallelotopes.

Theorem 1.3. If P, Q are convex polytopes in $\mathbb{R}^n$ and the John point of Q is at the origin, then

$$\frac{V(\Pi P) V(Q)}{U_1(P, Q)^{\frac{n}{2}} V_1(P, Q)^{\frac{n}{2}}} \leq \frac{n^n (n+1)^{\frac{n+1}{2}}}{(n!)^{\frac{3}{2}}}, \tag{1.10}$$

with equality if and only if P and Q are parallel simplices.

It is clear that if $Q = P$ in the above theorems, then the inequalities (1.8), (1.9), and (1.10) precisely turn to the inequalities (1.3), (1.4), and (1.5), respectively.

This paper is organized as follows. After listing some basic facts on convex bodies in Section 2, we introduce the notion of *mixed cone-volume measure* $V_{K,L}$, as well as the *mixed cone-volume functional* $U_1(K, L)$, of convex bodies K and L in Section 3. Then some fundamental properties of $V_{K,L}$ and $U_1(K, L)$ are established. The proofs of Theorem 1.1, Theorem 1.2, and Theorem 1.3 are provided in Section 4.

2 Preliminaries

For quick later reference, we collect some basic facts on convex bodies. Excellent references are the books by Gardner [5], Gruber [9], Schneider [28], and Thompson [30].

Write $\mathcal{K}^n$ and $\mathcal{K}^n_o$ for the set of convex bodies and the set of convex bodies with the origin in their interior in $\mathbb{R}^n$, respectively. Let $\mathcal{P}^n \subseteq \mathcal{K}^n$ denote the class of convex polytopes. The standard unit ball of $\mathbb{R}^n$ is denoted by $B = \{x \in \mathbb{R}^n : |x| \leq 1\}$. Its volume is $\omega_n = \pi^{n/2}/\Gamma(1 + n/2)$. For $x \in \mathbb{R}^n \setminus \{o\}$, let $\langle x \rangle = |x|^{-1} x$.

Let $K \in \mathcal{K}^n$. By the definition of support function, it follows that for $\lambda \geq 0$ and $T \in \mathrm{GL}(n)$,

$$h_K(\lambda x) = \lambda h_K(x), \quad h_{TK}(x) = h_K(T^t x), \quad x \in \mathbb{R}^n. \tag{2.1}$$

The *radial function* ρ_K of $K \in \mathcal{K}_o^n$ is defined by

$$\rho_K(x) = \max\{\lambda \geq 0 : \lambda x \in K\}, \qquad x \in \mathbb{R}^n \setminus \{o\}.$$

For $\lambda > 0$ and $T \in \mathrm{GL}(n)$, it yields that

$$\rho_K(\lambda x) = \lambda^{-1}\rho_K(x), \quad \rho_{TK}(x) = \rho_K(T^{-1}x), \qquad x \in \mathbb{R}^n \setminus \{o\}. \tag{2.2}$$

The *polar body* K^* of $K \in \mathcal{K}_o^n$ is defined by

$$K^* = \{x \in \mathbb{R}^n : \ x \cdot y \leq 1, \ y \in K\}.$$

For $K \in \mathcal{K}^n$ and $T \in \mathrm{GL}(n)$, we have

$$\rho_{K^*} = h_K^{-1} \quad \text{and} \quad (TK)^* = T^{-t}K^*. \tag{2.3}$$

The *surface area measure* S_K of $K \in \mathcal{K}^n$ is a finite Borel measure on S^{n-1}, defined for the Borel set $\omega \subseteq S^{n-1}$ by

$$S_K(\omega) = \mathcal{H}^{n-1}(\nu_K^{-1}(\omega)),$$

where $\nu_K : \partial' K \to S^{n-1}$ is the Gauss map of K, defined on $\partial' K$, the set of points of ∂K that have a unique outer unit normal. Recall that the Gauss map ν_K exists almost everywhere on ∂K with respect to the $(n-1)$-dimensional Hausdorff measure $\mathcal{H}^{n-1}$, that is, $\mathcal{H}^{n-1}(\partial K \setminus \partial' K) = 0$. Thus, for any continuous $f : S^{n-1} \to \mathbb{R}$, it holds

$$\int_{S^{n-1}} f(u)\mathrm{d}S_K(u) = \int_{\partial K} f(\nu_K(x))\mathrm{d}\mathcal{H}^{n-1}(x). \tag{2.4}$$

The *cone-volume measure* V_K of $K \in \mathcal{K}^n$ is a finite Borel measure on S^{n-1}, defined for the Borel set $\omega \subseteq S^{n-1}$ by

$$V_K(\omega) = \frac{1}{n}\int_\omega h_K(u)\mathrm{d}S_K(u). \tag{2.5}$$

Observe that V_K is $\mathrm{SL}(n)$-invariant, that is,

$$V_{TK}(\omega) = V_K(\langle T^t\omega\rangle), \quad \text{for } T \in \mathrm{SL}(n), \tag{2.6}$$

where $\langle T^t\omega\rangle = \{\langle T^t u\rangle : u \in \omega\}$.

The *Minkowski combination* of $K, L \in \mathcal{K}^n$ is defined by

$$\lambda K + \mu L = \{\lambda x + \mu y \;\; x \in K,\ y \in L\}, \quad \lambda, \mu \geq 0.$$

The *1st mixed volume* $V_1(K, L)$ of convex bodies K, L is defined by

$$V_1(K, L) = \frac{1}{n} \lim_{\epsilon \to 0^+} \frac{V(K + \epsilon L) - V(K)}{\epsilon} = \frac{1}{n} \int_{S^{n-1}} h_L(u) \mathrm{d}S_K(u). \tag{2.7}$$

From the affine invariance of volume, it follows that $V_1(TK, TL) = |\det T| V_1(K, L)$, for $T \in \mathrm{GL}(n)$. Thus, for any continuous $f : S^{n-1} \to \mathbb{R}$, we have

$$\int_{S^{n-1}} f(u) \mathrm{d}S_{TK}(u) = |\det T| \int_{S^{n-1}} f(\langle T^{-t} u \rangle) |T^{-t} u| \mathrm{d}S_K(u). \tag{2.8}$$

See, for example, Schneider [28] for its proof.

For convenience, we introduce the U-functional for measures. Let μ be a finite Borel measure on S^{n-1}. Then the *U-functional* $U(\mu)$ of μ is defined by

$$U(\mu)^n = \int_{u_1 \wedge \cdots \wedge u_n \neq 0} \mathrm{d}\mu(u_1) \cdots \mathrm{d}\mu(u_n). \tag{2.9}$$

$U(\mu)$ is centro-affine invariant in the sense that

$$U(T\mu) = U(\mu), \qquad T \in \mathrm{SL}(n), \tag{2.10}$$

where $T\mu$ is the affine image of measure μ, that is,

$$T\mu(\omega) = \mu(\langle T^{-1} \omega \rangle), \qquad \text{for Borel set } \omega \subseteq S^{n-1}. \tag{2.11}$$

A finite positive Borel measure μ on S^{n-1} is said to be *isotropic* if

$$\int_{S^{n-1}} |u \cdot v|^2 \mathrm{d}\mu(v) = 1, \quad \text{for all } u \in S^{n-1}.$$

Specially, the discrete measure μ is isotropic if

$$\sum_{v \in \mathrm{supp}\mu} |u \cdot v|^2 \mu(\{v\}) = 1, \quad \text{for all } u \in S^{n-1}.$$

For $x_1, \dots, x_n \in \mathbb{R}^n$, $[o, x_1] + \cdots + [o, x_n]$ is a parallelotope. Write $[x_1, \dots, x_n]$ for its n-dimension volume. Suppose μ is an isotropic measure on S^{n-1}. Then

$$\frac{1}{n!}\int_{S^{n-1}} \cdots \int_{S^{n-1}} [u_1, \dots, u_n]^2 \mathrm{d}\mu(u_1) \cdots \mathrm{d}\mu(u_n) = 1. \tag{2.12}$$

See, for example, LYZ [20] or [22] for its proof.

Suppose Z is a convex body in $\mathbb{R}^n$ with its support function

$$h_Z(u) = \int_{S^{n-1}} |u \cdot v| \mathrm{d}\mu(v), \qquad u \in S^{n-1},$$

where μ is a finite Borel measure on S^{n-1}. The McMullen–Matheron–Weil formula reads

$$V(Z) = \frac{2^n}{n!}\int_{S^{n-1}} \cdots \int_{S^{n-1}} [u_1, \dots, u_n] \mathrm{d}\mu(u_1) \cdots \mathrm{d}\mu(u_n). \tag{2.13}$$

See, e.g., Matheron [23] or Weil [31] for its proof.

3 The Mixed Cone-Volume Functional and the Mixed LYZ Ellipsoid

In this section, we introduce a new notion: the mixed cone-volume functional $U_1(K, L)$. To define this functional, a new measure, called the mixed cone-volume measure $V_{K,L}$, is involved. It is pointed out that the mixed cone-volume measure $V_{K,L}$ is a natural extension of the important cone-volume measure V_K.

Definition 3.1. Let $K, L \in \mathcal{K}^n$ and L contains the origin. The *mixed cone-volume measure* $V_{K,L}$ of K and L is defined by

$$V_{K,L}(\omega) = \frac{1}{n}\int_{\omega} h_L(u) \mathrm{d}S_K(u), \quad \text{for Borel set } \omega \subseteq S^{n-1}.$$

Note that the total mass of $V_{K,L}$ is exactly the 1st mixed volume $V_1(K, L)$, that is, $V_{K,L}(S^{n-1}) = V_1(K, L)$. In additional, if L is the unit ball B, then $V_{K,B} = S_K/n$; if $L = K$, then $V_{K,K} = V_K$. This means that the mixed cone-volume measure $V_{K,L}$ contains two fundamental measures in geometry: the surface area measure S_K and the cone volume measure V_K. It is well known that the surface area measure is characterized by the classical *Minkowski problem*, which is one of the cornerstones of the Brunn–Minkowski theory of convex bodies. In recent years, cone-volume measures have appeared and were studied in various contexts, see, for example, [2], [3], [4], [10], [11], and [32].

If $P \in \mathcal{P}^n$ and $Q \in \mathcal{K}^n$ with $o \in Q$, then $V_{P,Q}$ can be represented as

$$V_{P,Q}(\omega) = \frac{1}{n} \sum_{u \in \mathrm{supp} S_P} h_Q(u) S_P(u) \delta_u(\omega), \quad \text{for Borel set } \omega \subseteq S^{n-1}, \tag{3.1}$$

where $\delta_u(\cdot)$ is the Delta measure on S^{n-1} concentrated on u.

Definition 3.2. Let $K, L \in \mathcal{K}^n$ and L contains the origin. The *mixed cone-volume functional* $U_1(K, L)$ of K and L, is defined by

$$U_1(K, L)^n = \int_{u_1 \wedge \cdots \wedge u_n \neq 0} \mathrm{d}V_{K,L}(u_1) \cdots \mathrm{d}V_{K,L}(u_n).$$

Observe that if $L = K$, it immediately yields $U_1(K, L) = U_1(K, K) = U(K)$, which turns to the U functional introduced by Böröczky and LYZ [4]. By (2.9), it follows that $U_1(K, L) = U(V_{K,L})$.

Specifically, if $P \in \mathcal{P}^n$ and $Q \in \mathcal{K}^n$ with $o \in Q$, then $U_1(P, Q)$ can be represented as

$$U_1(P, Q)^n = \frac{1}{n^n} \sum_{u_1 \wedge \cdots \wedge u_n \neq 0} h_Q(u_1) \cdots h_Q(u_n) S_P(u_1) \cdots S_P(u_n), \tag{3.2}$$

where $u_i \in \mathrm{supp} S_P,\ i = 1 \ldots, n$. Moreover, if $Q = P$, then $U_1(P, Q) = U(P)$, which goes to LYZ's original definition (1.2).

Several observations for the relations between $U_1(K, L)$ and $V_1(K, L)$ are in order. First, for general convex bodies, it always holds $U_1(K, L) \leq V_1(K, L)$, which can be seen from the fact that $V_1(K, L)^n = (\int_{S^{n-1}} dV_{K,L}(u))^n$. Second, if convex body K is smooth, then $U_1(K, L) = V_1(K, L)$. See Böröczky and LYZ [4] for details. Third, if K is a polytope, in light of

$$V_1(K, L)^n = \frac{1}{n^n} \sum_{u_1, \ldots, u_n \in \mathrm{supp} S_K} h_L(u_1) \cdots h_L(u_n) S_K(u_1) \cdots S_K(u_n),$$

it follows that $U_1(K, L) < V_1(K, L)$. Since $U_1(K, L)$ is affine invariant by Proposition 3.6, U_1 is indeed a new affine invariant geometric functional for polytopes.

To prove the main results, we need to use the mixed LYZ ellipsoid introduced by Hu–Xiong–Zou. For more information on the mixed LYZ ellipsoid and its associated affine isoperimetric inequalities, refer to [12].

Definition 3.3. Let $K, L \in \mathcal{K}^n$ and L contains the origin in its interior. The *mixed LYZ ellipsoid* $\Gamma_{-2}(K, L)$ of K and L is defined by

$$\rho_{\Gamma_{-2}(K,L)}^{-2}(u) = \frac{n}{V_1(K,L)} \int_{S^{n-1}} \left(\frac{|u \cdot v|}{h_L(v)} \right)^2 \mathrm{d}V_{K,L}(v), \quad \text{for } u \in S^{n-1}.$$

It is clear that when $L = K$, then $\Gamma_{-2}(K, K) = \Gamma_{-2}K$, which is precisely the celebrated LYZ ellipsoids defined in [18]. The LYZ ellipsoid is in some sense dual to the classical Legendre ellipsoid of inertia in mechanics. When viewed as suitably normalized matrix-valued operators on the space of convex bodies, it was proved by Ludwig [15] that the Legendre ellipsoid and the LYZ ellipsoid are the only linearly invariant operators that satisfy the inclusion–exclusion principle.

Let $M = (m_{ij})_{n \times n}$ be the symmetric positive definite matrix with entries

$$m_{ij} = \frac{1}{V_1(K,L)} \int_{S^{n-1}} (v \cdot e_i)(v \cdot e_j) h_L^{-1}(v) \mathrm{d}S_K(v),$$

where $e_i \ldots, e_n$ are the orthonormal basis for $\mathbb{R}^n$. Then

$$\Gamma_{-2}(K, L) = \{x \in \mathbb{R}^n \ \ x \cdot Mx \le 1\}.$$

If $P \in \mathcal{P}^n$ and $Q \in \mathcal{K}^n$ with the origin in its interior, then $\Gamma_{-2}(P, Q)$ is defined by

$$\rho_{\Gamma_{-2}(P,Q)}^{-2}(u) = \frac{1}{V_1(K,L)} \sum_{v \in \operatorname{supp} S_P} |u \cdot v|^2 \frac{S_P(v)}{h_Q(v)}, \quad \text{for } u \in S^{n-1}. \tag{3.3}$$

Recall that $\nu_K : \partial' K \to S^{n-1}$ is the Gauss map of K, defined on $\partial' K$, the set of points of ∂K that have a unique outer unit normal.

Proposition 3.4. Suppose that $K \in \mathcal{K}^n$ and $T \in \mathrm{GL}(n)$. For $x \in \partial' K$, then

$$\nu_{TK}(y) = \langle T^{-t} \nu_K(x) \rangle, \quad \text{for } y = Tx \in \partial'(TK).$$

Proof. By the definition of support function, it follows that $h_K(\nu_K(x)) = x \cdot \nu_K(x)$, for $x \in \partial' K$. According to (2.1), we have

$$h_{TK}(T^{-t} \nu_K(x)) = h_K(\nu_K(x)) = x \cdot \nu_K(x) = Tx \cdot T^{-t} \nu_K(x).$$

Thus,

$$h_{TK}(\langle T^{-t}\nu_K(x)\rangle) = y \cdot \langle T^{-t}\nu_K(x)\rangle.$$

From $y = Tx \in \partial'(TK)$, it follows that $\nu_{TK}(y) = \langle T^{-t}\nu_K(x)\rangle$. ■

Proposition 3.5. The mixed cone-volume measure $V_{K,L}$ is $\mathrm{SL}(n)$-invariant. That is, for $T \in \mathrm{SL}(n)$ and Borel $\omega \subseteq S^{n-1}$, then $V_{TK,TL}(\omega) = V_{K,L}(\langle T^t\omega\rangle)$.

Proof. Let $x \in \partial' K$. Then $y = Tx \in \partial'(TK)$. Since $T \in \mathrm{SL}(n)$, from (2.6) and Proposition 3.4, it follows that

$$\mathrm{d}V_{TK}(\nu_{TK}(y)) = \mathrm{d}V_K(\langle T^t\nu_{TK}(y)\rangle) = \mathrm{d}V_K(\nu_K(x)).$$

Thus,

$$\frac{1}{n} y \cdot \nu_{TK}(y)\mathrm{d}\mathcal{H}^{n-1}(y) = \frac{1}{n} x \cdot \nu_K(x)\mathrm{d}\mathcal{H}^{n-1}(x).$$

By Proposition 3.4, we have

$$\mathrm{d}\mathcal{H}^{n-1}(y) = |T^{-t}\nu_K(x)|\mathrm{d}\mathcal{H}^{n-1}(x). \tag{3.4}$$

Meanwhile, if $Tx = y \in \nu_{TK}^{-1}(\omega)$, then $\nu_{TK}(y) \in \omega$. By Proposition 3.4, it follows that

$$\nu_K(x) \in \langle T^t\omega\rangle,$$

which is equivalent to $x \in \nu_K^{-1}(\langle T^t\omega\rangle)$. Thus,

$$\nu_{TK}^{-1}(\omega) = T(\nu_K^{-1}(\langle T^t\omega\rangle)). \tag{3.5}$$

From Definition 3.1, (2.4), (3.5), Proposition 3.4, (3.4), (2.4), and Definition 3.1 again, it follows that

$$\begin{aligned}
V_{TK,TL}(\omega) &= \frac{1}{n}\int_{\omega} h_{TL}(u)\mathrm{d}S_{TK}(u) \\
&= \frac{1}{n}\int_{\nu_{TK}^{-1}(\omega)} h_{TL}(\nu_{TK}(y))\mathrm{d}\mathcal{H}^{n-1}(y) \\
&= \frac{1}{n}\int_{T(\nu_{K}^{-1}(\langle T^{t}\omega\rangle))} h_{TL}(\nu_{TK}(y))\mathrm{d}\mathcal{H}^{n-1}(y) \\
&= \frac{1}{n}\int_{\nu_{K}^{-1}(\langle T^{t}\omega\rangle)} h_{TL}(\langle T^{-t}\nu_{K}(x)\rangle)|T^{-t}\nu_{K}(x)|\mathrm{d}\mathcal{H}^{n-1}(x) \\
&= \frac{1}{n}\int_{\nu_{K}^{-1}(\langle T^{t}\omega\rangle)} h_{L}(\nu_{K}(x))\mathrm{d}\mathcal{H}^{n-1}(x) \\
&= \frac{1}{n}\int_{\langle T^{t}\omega\rangle} h_{L}(u)\mathrm{d}S_{K}(u) \\
&= V_{K,L}(\langle T^{t}\omega\rangle).
\end{aligned}$$

This completes the proof. ■

Proposition 3.6. The mixed cone-volume functional $U_1(K,L)$ is SL(n)-invariant. That is, for $T \in \mathrm{SL}(n)$, then $U_1(TK,TL) = U_1(K,L)$.

Proof. From Proposition 3.5 and (2.11), it follows that for Borel $\omega \subseteq S^{n-1}$,

$$V_{TK,TL}(\omega) = V_{K,L}(\langle T^{t}\omega\rangle) = T^{-t}V_{K,L}(\omega). \tag{3.6}$$

So, from Definition 3.2 combining with (2.9), (3.6), and (2.10), it follows that

$$U_1(TK,TL) = U(V_{TK,TL}) = U(T^{-t}V_{K,L}) = U(V_{K,L}) = U_1(K,L).$$

This completes the proof. ■

Proposition 3.7. The mixed LYZ ellipsoid $\Gamma_{-2}(K,L)$ is affine invariant. That is, for $T \in \mathrm{GL}(n)$, then $\Gamma_{-2}(TK,TL) = T(\Gamma_{-2}(K,L))$.

Proof. For $u \in S^{n-1}$, from Definition 3.3, (2.8), (2.1), and (2.2), it follows that

$$\begin{aligned}
&\rho^{-2}_{\Gamma_{-2}(TK,TL)}(u)\\
&= \frac{n}{V_1(TK,TL)} \int_{S^{n-1}} \left(\frac{|u\cdot v|}{h_{TL}(v)}\right)^2 \mathrm{d}V_{TK,TL}(v)\\
&= \frac{|\det T|}{|\det T| V_1(K,L)} \int_{S^{n-1}} |u\cdot \langle T^{-t}v\rangle|^2 h^{-1}_{TL}(\langle T^{-t}v\rangle)|T^{-t}v|\mathrm{d}S_K(v)\\
&= \frac{1}{V_1(K,L)} \int_{S^{n-1}} |u\cdot T^{-t}v|^2 h^{-1}_{TL}(T^{-t}v)\mathrm{d}S_K(v)\\
&= \frac{1}{V_1(K,L)} \int_{S^{n-1}} |T^{-1}u\cdot v|^2 h^{-1}_{L}(v)\mathrm{d}S_K(v)\\
&= \frac{n}{V_1(K,L)} \int_{S^{n-1}} \left(\frac{|T^{-1}u\cdot v|}{h_{L}(v)}\right)^2 \mathrm{d}V_{K,L}(v)\\
&= \rho^{-2}_{\Gamma_{-2}(K,L)}(T^{-1}u) = \rho^{-2}_{T(\Gamma_{-2}(K,L))}(u).
\end{aligned}$$

This completes the proof. ■

4 Affine Isoperimetric Inequalities for the Mixed Cone-Volume Functional

The following inequality relates the mixed cone-volume functional $U_1(K,L)$ with the volume of the mixed LYZ ellipsoid $\Gamma_{-2}(K,L)$.

Theorem 4.1. Suppose that $P \in \mathcal{P}^n$ and $Q \in \mathcal{K}^n$ with the origin in its interior. Then

$$\frac{V(\Gamma_{-2}(P,Q))V(\Pi P)}{U_1(P,Q)^{\frac{n}{2}} V_1(P,Q)^{\frac{n}{2}}} \le \left(\frac{n^n}{n!}\right)^{\frac{1}{2}} \omega_n,$$

with equality if and only if $[u_1,\dots,u_n]/h_Q(u_1)\cdots h_Q(u_n)$ is independent of the choice of $u_1,\dots,u_n$ on $\mathrm{supp}S_P$, whenever $u_1 \wedge \cdots \wedge u_n \neq 0$.

Proof. Let $P \in \mathcal{P}^n$. From (1.1), the support function of ΠP is given by

$$h_{\Pi P}(u) = \frac{1}{2} \sum_{v \in \mathrm{supp} S_P} |u \cdot v| S_P(v), \qquad u \in S^{n-1}.$$

According to the McMullen–Matheron–Weil formula (2.13), we have

$$V(\Pi P) = \frac{1}{n!} \sum_{u_1,\ldots,u_n \in \mathrm{supp} S_P} [u_1, \ldots, u_n] S_P(u_1) \cdots S_P(u_n). \tag{4.1}$$

From (1.2), Propositions 3.6 and 3.7, it follows that the left side of the desired inequality is $\mathrm{SL}(n)$-invariant. In order to prove the theorem we may assume, w.l.o.g.,

$$\Gamma_{-2}(P, Q) = \left(\frac{V(\Gamma_{-2}(P, Q))}{\omega_n} \right)^{\frac{1}{n}} B.$$

Then from (3.3), for $u \in S^{n-1}$, it follows that

$$\left(\frac{\omega_n}{V(\Gamma_{-2}(P, Q))} \right)^{\frac{2}{n}} = \frac{1}{V_1(P, Q)} \sum_{v \in \mathrm{supp} S_P} |u \cdot v|^2 \frac{S_P(v)}{h_Q(v)},$$

which implies the discrete measure

$$\mu := \frac{1}{V_1(P, Q)} \left(\frac{V(\Gamma_{-2}(P, Q))}{\omega_n} \right)^{\frac{2}{n}} \frac{S_P}{h_Q}$$

is isotropic on S^{n-1}. Thus, by (2.12), we have

$$\left(\frac{\omega_n}{V(\Gamma_{-2}(P, Q))} \right)^2 = \frac{1}{n!\, V_1(P, Q)^n} \sum_{u_1,\ldots,u_n \in \mathrm{supp} S_P} [u_1, \ldots, u_n]^2 \frac{S_P(u_1) \cdots S_P(u_n)}{h_Q(u_1) \cdots h_Q(u_n)}. \tag{4.2}$$

From (4.2), the Jensen inequality and (4.1), it follows that

$$\begin{aligned}
&\left(\frac{\omega_n}{V(\Gamma_{-2}(P,Q))}\right)^2 \frac{n!\,V_1(P,Q)^n}{n^n U_1(P,Q)^n} \\
&= \frac{1}{n^n U_1(P,Q)^n} \sum_{u_1,\dots,u_n\in \mathrm{supp} S_P} \left(\frac{[u_1,\dots,u_n]}{h_Q(u_1)\cdots h_Q(u_n)}\right)^2 h_Q(u_1)\cdots h_Q(u_n) S_P(u_1)\cdots S_P(u_n) \\
&= \frac{1}{n^n U_1(P,Q)^n} \sum_{u_1\wedge\cdots\wedge u_n\neq 0} \left(\frac{[u_1,\dots,u_n]}{h_Q(u_1)\cdots h_Q(u_n)}\right)^2 h_Q(u_1)\cdots h_Q(u_n) S_P(u_1)\cdots S_P(u_n) \\
&\geq \left(\frac{1}{n^n U_1(P,Q)^n} \sum_{u_1\wedge\cdots\wedge u_n\neq 0} \frac{[u_1,\dots,u_n]}{h_Q(u_1)\cdots h_Q(u_n)} h_Q(u_1)\cdots h_Q(u_n) S_P(u_1)\cdots S_P(u_n)\right)^2 \\
&= \left(\frac{1}{n^n U_1(P,Q)^n} \sum_{u_1,\dots,u_n\in \mathrm{supp} S_P} [u_1,\dots,u_n] S_P(u_1)\cdots S_P(u_n)\right)^2 = \left(\frac{n!\,V(\Pi P)}{n^n U_1(P,Q)^n}\right)^2,
\end{aligned}$$

with equality if and only if

$$\frac{[u_1,\dots,u_n]}{h_Q(u_1)\cdots h_Q(u_n)}$$

is independent of the choice of $u_1,\dots,u_n$ on $\mathrm{supp} S_P$, whenever $u_1\wedge\cdots\wedge u_n\neq 0$. ■

To prove the main results, the following volume ratio inequality for $\Gamma_{-2}(K,L)$, which was previously established by Hu–Xiong–Zou [12], is needed.

Lemma 4.2. Suppose $P, Q \in \mathcal{P}^n$.

(1). If Q is origin symmetric, then

$$V(\Gamma_{-2}(P,Q)) \geq \frac{\omega_n}{2^n} V(Q),$$

with equality if and only if P and Q are parallel parallelotopes.

(2). If the John point of Q is at the origin, then

$$V(\Gamma_{-2}(P,Q)) \geq \frac{n!\,\omega_n}{n^{\frac{n}{2}}(n+1)^{\frac{n+1}{2}}} V(Q),$$

with equality if and only if P and Q are parallel simplices.

Theorem 4.3. Suppose that $P, Q \in \mathcal{P}^n$ and Q is origin symmetric. Then

$$\frac{V(\Pi P)V(Q)}{U_1(P,Q)^{\frac{n}{2}}V_1(P,Q)^{\frac{n}{2}}} \le 2^n \left(\frac{n^n}{n!}\right)^{\frac{1}{2}},$$

with equality if and only if P and Q are parallel parallelotopes.

Proof. From Theorem 4.1 and Lemma 4.2 (1), we have

$$\frac{\omega_n}{2^n}\frac{V(\Pi P)V(Q)}{U_1(P,Q)^{\frac{n}{2}}V_1(P,Q)^{\frac{n}{2}}} \le \frac{V(\Gamma_{-2}(P,Q))V(\Pi P)}{U_1(P,Q)^{\frac{n}{2}}V_1(P,Q)^{\frac{n}{2}}} \le \left(\frac{n^n}{n!}\right)^{\frac{1}{2}}\omega_n,$$

which yields the desired inequality.

If the equality holds in the inequality, by the equality condition of Lemma 4.2 (1), it follows that P and Q are parallel parallelotopes.

Conversely, assume $Q = [-1, 1]^n$, then the polar body Q^* is a cross-polytope with vertices $\{u\rho_{Q^*}(u) : u \in \mathrm{supp}S_Q\}$. Since $\mathrm{supp}S_Q = \mathrm{supp}S_P$, by the identity

$$\frac{[u_1,\ldots,u_n]}{h_Q(u_1)\cdots h_Q(u_n)} = [u_1\rho_{Q^*}(u_1),\ldots,u_n\rho_{Q^*}(u_n)],$$

it follows that it is constant for all $u_1,\ldots,u_n$ on $\mathrm{supp}S_P$, whenever $u_1 \wedge \cdots \wedge u_n \neq 0$. By the equality conditions of Theorem 4.1 and Lemma 4.2 (1), the equality holds. This completes the proof. ∎

Recall that the Reisner inequality reads: If K is a projection body in $\mathbb{R}^n$, then

$$V(K)V(K^*) \ge \frac{4^n}{n!},$$

with equality if and only if K is a parallelotope. See, for example, Reisner [25, 26] and Gordon–Meyer–Reisner [8].

Theorem 4.4. Suppose that $P, Q \in \mathcal{P}^n$ and Q is origin symmetric. Then

$$V(\Pi^* P)U_1(P,Q)^{\frac{n}{2}}V_1(P,Q)^{\frac{n}{2}} \ge \frac{2^n}{(n^n n!)^{\frac{1}{2}}}V(Q),$$

with equality if and only if P and Q are parallel parallelotopes.

Proof. From Theorem 4.3 and Reisner's inequality, it follows that

$$\frac{4^n}{n!}\frac{V(Q)}{V(\Pi^*P)U_1(P,Q)^{\frac{n}{2}}V_1(P,Q)^{\frac{n}{2}}} \le \frac{V(\Pi P)V(Q)}{U_1(P,Q)^{\frac{n}{2}}V_1(P,Q)^{\frac{n}{2}}} \le 2^n\left(\frac{n^n}{n!}\right)^{\frac{1}{2}},$$

which yields the desired inequality.

If P and Q are parallel parallelotopes, then ΠP is still a parallelotope. By the equality conditions of Theorem 4.3 and Reisner's inequality, the equality holds. This completes the proof. ∎

Theorem 4.5. Suppose that $P, Q \in \mathcal{P}^n$ and the John point of Q is at the origin. Then

$$\frac{V(\Pi P)V(Q)}{U_1(P,Q)^{\frac{n}{2}}V_1(P,Q)^{\frac{n}{2}}} \le \frac{n^n(n+1)^{\frac{n+1}{2}}}{(n!)^{\frac{3}{2}}},$$

with equality if and only if P and Q are parallel simplices.

Proof. From Theorem 4.1 and Lemma 4.2 (2), we have

$$\frac{n!\,\omega_n}{n^{\frac{n}{2}}(n+1)^{\frac{n+1}{2}}}\frac{V(\Pi P)V(Q)}{U_1(P,Q)^{\frac{n}{2}}V_1(P,Q)^{\frac{n}{2}}} \le \frac{V(\Gamma_{-2}(P,Q))V(\Pi P)}{U_1(P,Q)^{\frac{n}{2}}V_1(P,Q)^{\frac{n}{2}}} \le \left(\frac{n^n}{n!}\right)^{\frac{1}{2}}\omega_n,$$

which yields the desired inequality.

If the equality holds in the inequality, by the equality condition of Lemma 4.2 (2), it follows that P and Q are parallel simplices.

Conversely, assume Q is a regular simplex in $\mathbb{R}^n$, then the polar body Q^* is also a regular simplex with vertices $\{u\rho_{Q^*}(u) : u \in \mathrm{supp}S_Q\}$. Since $\mathrm{supp}S_Q = \mathrm{supp}S_P$, by the identity

$$\frac{[u_1,\ldots,u_n]}{h_Q(u_1)\cdots h_Q(u_n)} = [u_1\rho_{Q^*}(u_1),\ldots,u_n\rho_{Q^*}(u_n)],$$

it follows that it is constant for all $u_1,\ldots,u_n$ on $\mathrm{supp}S_P$, whenever $u_1\wedge\cdots\wedge u_n \neq 0$. By the equality conditions of Theorem 4.1 and Lemma 4.2 (2), the equality holds. This completes the proof. ∎

One obvious question regarding the functionals U_1 and V_1 beg to be asked.

Problem 4.6. Suppose K and L are convex bodies in $\mathbb{R}^n$. Is there an absolute constant c only depending on the dimension n, such that

$$\frac{U_1(K,L)}{V_1(K,L)} \geq c\ ?$$

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