Available online at www.sciencedirect.com
ScienceDirect
Advances in Mathematics 225 (2010) 3214–3228
ADVANCES IN Mathematics
www.elsevier.com/locate/aim

Extremum problems for the cone volume functional of convex polytopes

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Received 24 August 2009; accepted 27 May 2010

Available online 3 June 2010

Communicated by Gil Kalai

Abstract

Lutwak, Yang and Zhang defined the cone volume functional U over convex polytopes in $\mathbb{R}^n$ containing the origin in their interiors, and conjectured that the greatest lower bound on the ratio of this centro-affine invariant U to volume V is attained by parallelotopes. In this paper, we give affirmative answers to the conjecture in $\mathbb{R}^2$ and $\mathbb{R}^3$. Some new sharp inequalities characterizing parallelotopes in $\mathbb{R}^n$ are established. Moreover, a simplified proof for the conjecture restricted to the class of origin-symmetric convex polytopes in $\mathbb{R}^n$ is provided.

MSC: 52A40

Keywords: Convex polytope; Parallelotope; Centro-affine invariant; Cone volume functional; Projection body

1. Introduction

A convex body K (i.e., a compact, convex subset with nonempty interior) in Euclidean n-space $\mathbb{R}^n$, is determined by its support function, $h(K,\cdot)$, on the unit sphere S^{n-1}, where $h(K,u) = \max\{u \cdot y\colon\ y \in K\}$ and where $u \cdot y$ denotes the standard inner product of u and y. The projection body, ΠK, of K is the convex body whose support function, for $u \in S^{n-1}$, is given by

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[1] Research of the author was supported by NSFC No. 11001163 and Innovation Program of Shanghai Municipal Education Commission No. 11YZ11.

0001-8708/$ – see front matter
doi:10.1016/j.aim.2010.05.016

$$h(\Pi K, u) = vol_{n-1}\big(K|u^{\perp}\big),$$

where vol_{n-1} denotes $(n-1)$-dimensional volume and $K|u^{\perp}$ denotes the image of the orthogonal projection of K onto the codimension 1 subspace orthogonal to u.

Projection bodies were introduced by Minkowski at the turn of the previous century in connection with Cauchy's surface area formula. They have been the objects of intense investigation during the past two decades. Many important results for projection bodies and their dual analogs, intersection bodies, have been obtained (see, e.g., [2–4,6,10,8,11,17,19–23,28,31–34,37,38]). In recent years, their generalizations to L_p-settings are attracted much attention and acquired remarkable advances [5,13–15,24,26,36,39].

An important unsolved problem regarding projection bodies is Schneider's projection problem (see, e.g., [7,9,18,29–31] and [35]): what is the least upper bound, as K ranges over the class of origin-symmetric convex bodies in $\mathbb{R}^n$, of the affine-invariant ratio

$$\big[V(\Pi K)/V(K)^{n-1}\big]^{\frac{1}{n}}, \tag{1}$$

where V is used to abbreviate the n-dimensional volume.

An effective tool to study Schneider's projection problem is the *cone volume functional U* introduced by Lutwak, Yang and Zhang [25]: If P is a convex polytope in $\mathbb{R}^n$ which contains the origin o in its interior, then define $U(P)$ by

$$U(P)^n = \frac{1}{n^n} \sum_{u_{i_1} \wedge \cdots \wedge u_{i_n} \neq 0} h_{i_1} \cdots h_{i_n} a_{i_1} \cdots a_{i_n}, \tag{2}$$

where $u_1, \ldots, u_N$ are the outer normal unit vectors to the corresponding facets $F_1, \ldots, F_N$ of P, and the facet with outer normal vector u_i has area (i.e. $(n-1)$-dimensional volume) a_i and distance h_i from the origin.

Let $V_i = \frac{1}{n} h_i a_i$. Then V_i is the volume of the cone $conv(o, F_i)$, and

$$U(P)^n = \sum_{u_{i_1} \wedge \cdots \wedge u_{i_n} \neq 0} V_{i_1} \cdots V_{i_n}. \tag{3}$$

Obviously the functional U is centro-affine invariant in that,

$$U(\phi P) = U(P), \quad \forall \phi \in SL(n). \tag{4}$$

Since $V(P) = \frac{1}{n} \sum_{i=1}^{N} a_i h_i$, it follows immediately that $U(P)/V(P) \leqslant 1$.

It is noted that for a random polytope with a large number of facets, $U(P)$ is very close to $V(P)$. It is this important property of the functional U which makes it so useful. For instance, with the functional U, LYZ [25] presented a modified version of Schneider's projection problem

$$\frac{V(\Pi P)}{U(P)^{\frac{n}{2}} V(P)^{\frac{n}{2}-1}} \leqslant 2^n \left(\frac{n^n}{n!}\right)^{\frac{1}{2}}, \tag{5}$$

and then gave an asymptotically optimal bound for the affine ratio (1).

As an aside, we observe that the cone volume functional U has strong connection with the cone measure: For every star-shaped body $K \subseteq \mathbb{R}^n$, the cone measure of a subset A of ∂K is the volume of $[0,1]A = \{ta\colon\ a \in A,\ 0 \leqslant t \leqslant 1\}$, i.e. the cone with base A and cusp o. The cone measure appears in the Gromov–Milman theorem [12] on the concentration of Lipschitz functions on uniformly convex bodies. In [27], Naor established the precise relation between the surface measure and cone measure on the sphere of l_p^n.

One fundamental, but still remains open extremum problem, on the ratio of U to V is posed by LYZ [25].

Conjecture. *If P is a convex polytope in $\mathbb{R}^n$ with its centroid at the origin, then*

$$\frac{U(P)}{V(P)} \geqslant \frac{(n!)^{1/n}}{n},$$

with equality if and only if P is a parallelotope.

The first progress on LYZ's conjecture was attributed to He, Leng and Li [16]. They proved that the conjecture is true when restricted to the class of origin-symmetric convex polytopes.

Theorem 1.1. *If P is an origin-symmetric convex polytope in $\mathbb{R}^n$, then*

$$\frac{U(P)}{V(P)} \geqslant \frac{(n!)^{1/n}}{n}, \tag{6}$$

with equality if and only if P is a parallelotope.

This paper is devoted to the study of LYZ's conjecture. We give affirmative answers to the conjecture in $\mathbb{R}^2$ and $\mathbb{R}^3$.

Theorem 1.2. *Let P be a convex polytope in $\mathbb{R}^n$ with its centroid at the origin. If n is equal to* 2 *or* 3*, then*

$$\frac{U(P)}{V(P)} \geqslant \frac{(n!)^{1/n}}{n},$$

with equality if and only if P is a parallelotope.

Inequality (6) is a reverse isoperimetric type inequality (see, e.g., [1]). Let $n \to \infty$, it gives

$$\frac{1}{e} \leqslant \frac{U(P)}{V(P)} \leqslant 1,$$

which is not dependent on the convex polytope and space dimension. This property will make it useful in the local theory of Banach spaces (see [4]).

Here, parallelotopes as the extremal bodies, have underlying importance in convex and discrete geometry [40–42]. The characterization of parallelotopes in the symmetric cases, or simplices in the non-symmetric cases, as extremal bodies of classical functionals is a central problem in convex geometry. In this article, some new sharp inequalities characterizing parallelotopes in $\mathbb{R}^n$ are established (Lemmas 2.3, 3.4 and 4.1), which are closely related to the functional U. At the same time, it is surprising to see that parallelotopes are the only minimizers of functional U.

It is noted that we adopt techniques of the geometric symmetrization and methods of projection to tackle the conjecture, which are readily applicable to symmetric convex polytopes. So, a mostly simplified proof for Theorem 1.1 is available. In fact, the techniques engaged in this paper are applicable to all convex bodies. However, the methods used in [16] rely on the symmetry and cannot pass to non-symmetric case.

We work in n-dimensional Euclidean space $\mathbb{R}^n$, $n \geqslant 2$, with origin o, basis $e_1, \dots, e_n$, and use coordinates $x = (x_1, \dots, x_n)^t$ for $x \in \mathbb{R}^n$. Let B_j be the centered unit ball in $\mathbb{R}^j$, whose volume is denoted by ω_j. The surface of B_j, that is the $(j-1)$-dimensional unit sphere, is denoted by S^{j-1}.

This paper, except for the introduction, is divided into three sections. The proofs of Theorem 1.2 and Theorem 1.1 are presented in Section 3 and Section 4, respectively.

2. Estimate of $\sum_{u_{i_1}\wedge\cdots\wedge u_{i_{n-1}}\wedge u_k=0} V_k$, where $\{u_{i_1}, \dots, u_{i_{n-1}}\} \subseteq \{u_1, \dots, u_N\}$ and $u_{i_1} \wedge \cdots \wedge u_{i_{n-1}} \neq 0$

From the well-known Brunn's concavity principle, it follows immediately that

Lemma 2.1. *Let $K \subseteq \mathbb{R}^n$ be a convex body and $L \subseteq \mathbb{R}^n$ be a j-dimensional subspace, $1 \leqslant j \leqslant n-1$. If*

$$f : L \mapsto \mathbb{R}, \qquad f(x) = vol_{n-j}\big(K \cap \big(L^\perp + x\big)\big),$$

then $f^{\frac{1}{n-j}}$ is concave on $K|L$.

Remark. If convex body K is origin-symmetric, then f is an even function on $K|L$, and consequently f is monotonously decreasing on any ray that starts from the origin. This property will be used in Lemma 4.1.

Lemma 2.2. *Let $K \subseteq \mathbb{R}^j \times \mathbb{R}^{n-j}$, $1 \leqslant j \leqslant n-1$, be a convex body with its centroid at the origin. Suppose $D = K|\mathbb{R}^j$ and $f(x) = vol_{n-j}(K \cap (\mathbb{R}^{n-j} + x))$, $x \in D$. Then*

$$f(0)\, vol_j(D) \geqslant V(K), \tag{7}$$

where the equality holds if and only if $f(x)$ is constant on D.

Proof. Since the centroid of K is at the origin, it has

$$\int_K (x, y)\, dx\, dy = 0.$$

Consequently, it gives

$$0 = \int_K x\, dx\, dy = \int_D x f(x)\, dx.$$

From Lemma 2.1 and Jessen's inequality, we have

$$f^{\frac{1}{n-j}}(0) = f^{\frac{1}{n-j}}\left(\frac{1}{V(K)}\int\limits_D xf(x)\,dx\right) \geqslant \frac{\int_D f^{\frac{n+1-j}{n-j}}(x)\,dx}{V(K)}.$$

From the well-known Hölder inequality, it follows that

$$f(0) \geqslant \left(\frac{\int_D f^{\frac{n+1-j}{n-j}}(x)\,dx}{V(K)}\right)^{n-j} \geqslant \frac{(\int_D f(x)\,dx)^{n+1-j}}{vol_j(D)V(K)^{n-j}} = \frac{V(K)}{vol_j(D)},$$

that is,

$$f(0)\,vol_j(D) \geqslant V(K).$$

If $f(x)$ is constant on D, it follows immediately that the equality of (7) holds. On the other hand, if $f(0)\,vol_j(D) = V(K)$, then all the equalities in the arguments have to be attained. From the equality condition of Hölder inequality, it follows that $f(x)$ is constant on D. This completes the proof. □

Lemma 2.3. *Let P be a convex polytope in $\mathbb{R}^n$ with its centroid at the origin. For any fixed $\{u_{i_1}, \dots, u_{i_{n-1}}\} \subseteq \{u_1, \dots, u_N\}$ with $u_{i_1} \wedge \dots \wedge u_{i_{n-1}} \neq 0$, if the normal vector u_k of F_k satisfies $u_{i_1} \wedge \dots \wedge u_{i_{n-1}} \wedge u_k = 0$, then*

$$\sum_{u_{i_1}\wedge\dots\wedge u_{i_{n-1}}\wedge u_k=0} V_k \leqslant \frac{n-1}{n} V(P). \tag{8}$$

If P is a parallelotope, then the equality of (8) *holds. Conversely, if the equalities of* (8) *hold for all subsets $\{u_{i_1}, \dots, u_{i_{n-1}}\} \subseteq \{u_1, \dots, u_N\}$ with $u_{i_1} \wedge \dots \wedge u_{i_{n-1}} \neq 0$ simultaneously, then P is a parallelotope.*

Proof. For any fixed $\{u_{i_1}, \dots, u_{i_{n-1}}\} \subseteq \{u_1, \dots, u_N\}$, $u_{i_1} \wedge \dots \wedge u_{i_{n-1}} \neq 0$, let

$$L = span\{u_{i_1}, \dots, u_{i_{n-1}}\}, \qquad f(x) = vol_1\big(P \cap \big(L^{\perp} + x\big)\big).$$

Then $f(x)$ is concave on $D = P|L$, and

$$\sum_{u_{i_1}\wedge\dots\wedge u_{i_{n-1}}\wedge u_k=0} V_k = \int\limits_{\partial D} \frac{1}{n} h\left(\frac{x}{\|x\|}\right) \cdot f(x)\,dS(x)$$
$$= \frac{n-1}{n}\int\limits_{\partial D} f(x)\left[\frac{1}{n-1} h\left(\frac{x}{\|x\|}\right) \cdot dS(x)\right]$$

where $dS(x)$ is the $(n-2)$-dimensional Lebesgue measure on ∂D.

Without loss of generality assume $L=\mathbb{R}^{n-1}$. Geometrically, it is intuitively that $\int_{\partial D} f(x)\times [\frac{1}{n-1}h(\frac{x}{\|x\|})\cdot dS(x)]$ is the volume of the set

$$\bar{P}=\{(x,y)\in\mathbb{R}^{n-1}\times\mathbb{R}^1\colon \exists x_0\in\partial D,\ (x_0,y)\in P\cap(\mathbb{R}^1+x_0),\ x\in[o,x_0]\},$$

and $\bar{P}$ has the same orthogonal projection onto $\mathbb{R}^{n-1}$ as convex polytope P.

So,

$$\sum_{u_{i_1}\wedge\cdots\wedge u_{i_{n-1}}\wedge u_k=0} V_k=\frac{n-1}{n}V(\bar{P}).$$

Now, we aim to show

$$V(\bar{P})\leqslant V(P).$$

For this aim, we make use of spherical coordinates in the subspace L. Suppose the equation of ∂D is

$$\rho=\rho_0(\theta_1,\theta_2,\ldots,\theta_{n-2}).$$

Let

$$F(\rho,\theta)=f(\rho\cos\theta_1,\ldots,\rho\sin\theta_1\cdots\sin\theta_{n-2}),\quad 0\leqslant\rho\leqslant\rho_0.$$

Then $F(\rho,\theta)$ is concave with respect to ρ, i.e.

$$F(\rho,\theta)\geqslant\frac{\rho}{\rho_0}F(\rho_0,\theta)+\frac{\rho_0-\rho}{\rho_0}F(0,0).$$

So

$$\begin{aligned} V(\bar{P})-V(P)&=\int_{S^{n-2}}d\theta\int_0^{\rho_0}\rho^{n-2}F(\rho_0,\theta)\,d\rho-\int_{S^{n-2}}d\theta\int_0^{\rho_0}\rho^{n-2}F(\rho,\theta)\,d\rho\\ &\leqslant\frac{1}{n}V(\bar{P})-\frac{1}{n(n-1)}\int_{S^{n-2}}\rho_0^{n-1}F(0,0)\,d\theta, \end{aligned}$$

that is,

$$V(P)\geqslant\frac{n-1}{n}V(\bar{P})+\frac{1}{n}F(0,0)\,vol_{n-1}(D).$$

From Lemma 2.2, it follows that $V(P)\geqslant V(\bar{P})$.

Finally, we prove the equality condition in (8).

Suppose that P is a parallelotope with its centroid at the origin. Since

$$V_i = V\big(conv\big(F_i \cup \{o\}\big)\big) = \frac{1}{2n} V(P), \quad i = 1, \dots, N,$$

and $u_{i_1} \wedge \cdots \wedge u_{i_{n-1}} \neq 0$, $u_{i_1} \wedge \cdots \wedge u_{i_{n-1}} \wedge u_k = 0$ if and only if $u_k = \pm u_{i_r}$, $r = 1, \dots, n-1$, it follows that

$$\sum_{u_{i_1} \wedge \cdots \wedge u_{i_{n-1}} \wedge u_k = 0} V_k = \frac{1}{2n} V(P) \cdot 2(n-1) = \frac{n-1}{n} V(P).$$

Conversely, for any $\{u_{i_1}, \dots, u_{i_{n-1}}\} \subseteq \{u_1, \dots, u_N\}$, $u_{i_1} \wedge \cdots \wedge u_{i_{n-1}} \neq 0$, suppose the equality of (8) holds simultaneously. From the above arguments, it gives $V(\bar{P}) = V(P)$ and

$$V(P) = F(0,0)\, vol_{n-1}(D).$$

From Lemma 2.2, it follows that $f(x)$ is constant on D, which thereby implies that P has one pair of opposite outer normal unit vectors $\pm v_1$, and all other outer normal unit vectors of P are in the great subsphere H_1 spanned by $u_{i_1}, u_{i_2}, \dots, u_{i_{n-1}}$.

Replace u_{i_1} by v_1. Obviously, $v_1, u_{i_2}, \dots, u_{i_{n-1}}$ are $n-1$ linearly independent outer normal unit vectors of P. Similarly, for these outer normals, the equality in (8) implies that P has another pair of opposite outer normal unit vectors $\pm v_2$, $v_2 \nparallel v_1$, and all other outer normal unit vectors are in the great subsphere H_2 spanned by v_1, $u_{i_2}, \dots, u_{i_{n-1}}$. So P has two pairs of outer normals $\pm v_1$, $\pm v_2$, and all other outer normals are in the subsphere $H_1 \cap H_2$. Using repeatedly this argument $n-2$ times, then we obtain $n-1$ pairs of linearly independent outer normals $\pm v_1, \pm v_2, \dots, \pm v_{n-1}$, of P, and all other outer normals are in the subsphere $H_1 \cap H_2 \cap \cdots \cap H_k \cap \cdots \cap H_{n-1}$, where H_k is the great subsphere spanned by $v_1, v_2, \dots, v_{k-1}, u_{i_k}, \dots, u_{i_{n-1}}$, $2 \leqslant k \leqslant n-1$. However, from the construction, we know that $H_1 \cap H_2 \cap \cdots \cap H_k \cap \cdots \cap H_{n-1}$ is a subsphere spanned only by $u_{i_{n-1}}$. For $n-1$ linearly independent outer normals, $v_1, v_2, \dots, v_{n-1}$, of P, the equality in (8) implies that $\pm u_{i_{n-1}}$ is also a pair of opposite outer normals of P. So $\pm v_1, \pm v_2, \dots, \pm v_{n-1}, \pm u_{i_{n-1}}$ are precisely the n pairs of opposite outer normals of P, which means that P is exactly a parallelotope. This completes the proof. □

3. Estimate of $\sum_{u_i \wedge u_k = 0} V_k$ and proof of Theorem 1.2

Lemma 3.1. *Suppose Q is an n-dimensional frustum of a cone in $\mathbb{R}^n$ with centroid C. Let F_1 and F_2 be the upper base and lower base of Q, respectively. Then*

$$V\big(conv(C, F_1)\big) + V\big(conv(C, F_2)\big) \leqslant \frac{1}{n} V(Q), \tag{9}$$

where the equality holds if and only if Q is a cylinder.

Proof. Without loss of generality assume $Q \subseteq \mathbb{R}^{n-1} \times \mathbb{R}^1$, whose lower base is on the $\mathbb{R}^{n-1}$-coordinate plane and centroid on the x_n-axis with coordinate z_0. Suppose the radii of upper base F_1 and lower base F_2 are r and R, respectively, and $r < R$. Let r_{x_n} denote the radius of the cross-section of Q at height x_n, $0 \leqslant x_n \leqslant h$, where h is the height of Q.

Suppose the frustum of a cone is formed from a cone M with height b. Then

$$r_{x_n} = (b - x_n)\tan\theta, \qquad b - h = \frac{r}{R - r}h,$$

where θ is the half angle of the cone M. The x_n-coordinate z_0 of centroid C is

$$z_0 = \frac{\int_0^h x_n \omega_{n-1} r_{x_n}^{n-1}\,dx_n}{\int_0^h \omega_{n-1} r_{x_n}^{n-1}\,dx_n} = \frac{\frac{1}{n+1}\cdot\frac{R^{n+1}-r^{n+1}}{R-r} - r^n}{R^n - r^n}h.$$

Since

$$V\big(conv(C, F_1)\big) + V\big(conv(C, F_2)\big) - \frac{1}{n}V(Q)$$
$$= \frac{\omega_{n-1}}{n}\left[z_0\big(R^{n-1} - r^{n-1}\big) + r^{n-1}h - \frac{1}{n}\frac{R^n - r^n}{R - r}h\right],$$

substituting z_0 into the formula, then it is sufficient to prove

$$\frac{\frac{1}{n+1}\frac{R^{n+1}-r^{n+1}}{R-r} - r^n}{R^n - r^n} \leqslant \frac{\frac{1}{n}\frac{R^n-r^n}{R-r} - r^{n-1}}{R^{n-1} - r^{n-1}}. \tag{10}$$

Since function

$$g(n) = \frac{\frac{1}{n+1}\frac{a^{n+1}-1}{a-1} - 1}{a^n - 1} = \frac{1}{a-1}\int_1^a \frac{t^n - 1}{a^n - 1}\,dt, \quad a > 1,$$

is strictly monotonously decreasing on n, it follows that inequality (10) holds.

If Q is a cylinder, it follows immediately that the equality of (9) holds. On the contrary, if Q is not a cylinder, since $g(n)$ is strictly decreasing, then the equality in (10) will not hold, and consequently the equality in (9) will not hold. This completes the proof. □

Lemma 3.2. *Let P be a convex polytope in $\mathbb{R}^n$ with its centroid at the origin. For any $\{u_i\} \subseteq \{u_1, \ldots, u_N\}$, let S be the Schwartz symmetrization of P with respect to u_i. Suppose the centroid of S is C_S, and the upper base and lower base of S are F_1 and F_2, respectively. Then*

$$V\big(conv(o, F_1)\big) + V\big(conv(o, F_2)\big) = V\big(conv(C_S, F_1)\big) + V\big(conv(C_S, F_2)\big). \tag{11}$$

Proof. Without loss of generality, for any given $\{u_i\} \subseteq \{u_1, \ldots, u_N\}$, we assume that u_i is parallel to the x_n-axis. Let $A_P(x_n)$ and $A_S(x_n)$ be the areas of cross-sections of P and S at height x_n, $h_1 \leqslant x_n \leqslant h_2$, respectively. Then the x_n coordinate of C_S is

$$x_n(C_S) = \int_{h_1}^{h_2} x_n A_S(x_n)\,dx_n = \int_{h_1}^{h_2} x_n A_P(x_n)\,dx_n = 0.$$

Hence, the equality (11) can be derived immediately. This completes the proof. □

We will need the following elementary geometrical fact:

Lemma 3.3. *Let Q be an n-dimensional frustum of a cone in $\mathbb{R}^n$ with upper base F_1 and lower base F_2, respectively. Suppose $vol_{n-1}(F_1) \leqslant vol_{n-1}(F_2)$. For any two points $C_1, C_2 \in int\, Q$, if the distance from C_1 to F_1 is not greater than the distance from C_2 to F_1, then*

$$V\big(conv(C_1, F_1)\big) + V\big(conv(C_1, F_2)\big) \geqslant V\big(conv(C_2, F_1)\big) + V\big(conv(C_2, F_2)\big). \tag{12}$$

Proof. For any point $C \in int\, Q$, suppose the distances from C to F_1 and F_2 are h_1 and h_2, respectively. Assume the height of Q is h. Then

$$V\big(conv(C, F_1)\big) + V\big(conv(C, F_2)\big) = \frac{1}{n} h\, vol_{n-1}(F_2) - \frac{1}{n} h_1 \big[vol_{n-1}(F_2) - vol_{n-1}(F_1)\big].$$

It implies that

$$V\big(conv(C, F_1)\big) + V\big(conv(C, F_2)\big)$$

is strictly decreasing on h_1. This completes the proof. □

Lemma 3.4. *Let P be a convex polytope in $\mathbb{R}^n$ with its centroid at the origin. For any fixed $\{u_i\} \subseteq \{u_1, \ldots, u_N\}$, if the normal vector u_k of F_k satisfies $u_k \wedge u_i = 0$, then*

$$\sum_{u_k \wedge u_i = 0,\ u_k \in \{u_1, \ldots, u_N\}} V_k \leqslant \frac{1}{n} V(P). \tag{13}$$

If P is a parallelotope, then the equality of (13) *holds. Conversely, if the equalities of* (13) *hold for all u_i's simultaneously, then P is a parallelotope.*

Proof. Without loss of generality, for any fixed $\{u_i\} \subseteq \{u_1, \ldots, u_N\}$, we assume that u_i is parallel to the x_n-axis. Suppose S is the Schwartz symmetrization of convex polytope P with respect to the x_n-axis, $h_1 \leqslant x_n \leqslant h_2$, and T is the frustum of a cone inscribed inside S with x_n-axis as its rotation axis. Suppose the upper base and lower base of T are F_1 and F_2, respectively, and $vol_{n-1}(F_1) \leqslant vol_{n-1}(F_2)$.

Let Q be the frustum of a cone with the same volume, the same height, and the same lower base F_2 of S. The frustum of a cone Q is also a body of revolution about the x_n-axis. Suppose the upper base of Q is F_3. Then $vol_{n-1}(F_1) \leqslant vol_{n-1}(F_3)$.

If $S = Q$, it means that the Schwartz symmetrization itself is a frustum of a cone, then the inequality (13) can be derived from Lemmas 3.2 and 3.1 immediately.

If $S \neq Q$, we will show that there exists a unique number l, $h_1 < l < h_2$, such that the lateral boundary of S (the boundary of S taking away the top and the base) and the lateral boundary of Q intersect at the height l. For this aim, let H be a 2-plane that contains the x_n-axis. Then the intersection of H with the lateral boundary of Q is a line segment N, and the intersection of H with the lateral boundary of S is a convex curve C. The line segment N and the convex curve C intersect at a boundary point on the lower base F_2 of Q and S. If N and C intersect at only one point or at more than two points, then the convex curve C must be on one side of the line segment N. This implies that either $S \subset Q$ or $Q \subseteq S$ because both S and Q are bodies of revolution. In view of $V(S) = V(Q)$, the case $S \subset Q$ gives $V(S) < V(Q)$ that is impossible, and the case $Q \subseteq S$ gives $S = Q$. Therefore, the line segment N and the convex curve C intersect at exactly two points if $S \neq Q$. This shows the uniqueness of the height l if $S \neq Q$.

Next, we will show that the distance from the centroid C_Q of Q to F_3 is not greater than the distance from the centroid C_S of S to F_3. For this aim, we only need to compare the x_n coordinates of C_Q and C_S. Suppose $V(P) = V(S) = V(Q) = V$. Let $A_Q(x_n)$, $A_S(x_n)$ be the area of cross-section of Q and S at height x_n, $h_1 \leqslant x_n \leqslant h_2$, respectively. Then

$$\begin{aligned} x_n(C_Q) - x_n(C_S) &= \frac{1}{V}\left[\int_{h_1}^{h_2} x_n A_Q(x_n)\,dx_n - \int_{h_1}^{h_2} x_n A_S(x_n)\,dx_n\right] \\ &= \frac{1}{V}\left[\int_{h_1}^{l} x_n\big(A_Q(x_n) - A_S(x_n)\big)\,dx_n + \int_{l}^{h_2} x_n\big(A_Q(x_n) - A_S(x_n)\big)\,dx_n\right] \\ &\geqslant \frac{1}{V}\left[l\int_{h_1}^{l} \big(A_Q(x_n) - A_S(x_n)\big)\,dx_n + l\int_{l}^{h_2} \big(A_Q(x_n) - A_S(x_n)\big)\,dx_n\right] \\ &= 0, \end{aligned}$$

that is, $x_n(C_Q) \geqslant x_n(C_S)$.

Then, from Lemma 3.2, followed by Lemmas 3.3 and 3.1, we have

$$\begin{aligned} \sum_{u_k \wedge u_i = 0,\ u_k \in \{u_1,\dots,u_N\}} V_k &= V\big(conv(o, F_1)\big) + V\big(conv(o, F_2)\big) \\ &= V\big(conv(C_S, F_1)\big) + V\big(conv(C_S, F_2)\big) \\ &\leqslant V\big(conv(C_Q, F_1)\big) + V\big(conv(C_Q, F_2)\big) \\ &\leqslant V\big(conv(C_Q, F_3)\big) + V\big(conv(C_Q, F_2)\big) \\ &\leqslant \frac{1}{n}V(Q) = \frac{1}{n}V(P), \end{aligned} \tag{14}$$

that is,

$$\sum_{u_k \wedge u_i = 0,\ u_k \in \{u_1,\dots,u_N\}} V_k \leqslant \frac{1}{n}V(P).$$

Suppose that P is a parallelotope centered at the origin. Since

$$V_i = V\big(conv\big(F_i \cup \{o\}\big)\big) = \frac{1}{2n}V(P), \quad i = 1, \dots, N,$$

and $u_k \wedge u_i = 0$ if and only if $u_k = \pm u_i$, we have

$$\sum_{u_k \wedge u_i = 0,\ u_k \in \{u_1,\dots,u_N\}} V_k = 2 \cdot \frac{1}{2n}V(P) = \frac{1}{n}V(P).$$

Conversely, suppose the equality

$$\sum_{u_k \wedge u_i = 0,\ u_k \in \{u_1, \ldots, u_N\}} V_k = \frac{1}{n} V(P)$$

holds for any $u_i \in \{u_1, \ldots, u_N\}$ simultaneously. Then all the equalities in (14) have to hold. From Lemma 3.1, it follows that the Schwartz symmetrization of P with respect to each u_i is a cylinder. So u_i and $-u_i$ both are outer normal unit vectors to the facets of P. With the condition that the centroid of P is at the origin, it follows that $V_i = \frac{1}{2n} V(P)$, $i = 1, \ldots, N$. So P has exactly $2n$ facets and $\pm u_1, \ldots, \pm u_n$ are its outer normal unit vectors, which implies that P is a parallelotope. This completes the proof. □

With these lemmas in hand, we can complete the proof of Theorem 1.2.

Proof of Theorem 1.2. Firstly, suppose that n is equal to 2.

From the definition of $U(P)$, followed by Lemma 3.4 (or Lemma 2.3), we have

$$\begin{aligned} U(P)^2 &= \sum_{u_{i_1} \wedge u_{i_2} \neq 0} V_{i_1} V_{i_2} = \sum_{u_{i_1} \neq 0} V_{i_1} \left(V - \sum_{u_{i_1} \wedge u_k = 0} V_k \right) \\ &\geqslant \sum_{u_{i_1} \neq 0} V_{i_1} \left(V - \frac{1}{2} V \right) = \frac{1}{2} V^2, \end{aligned}$$

that is,

$$\frac{U(P)}{V(P)} \geqslant \frac{\sqrt{2}}{2}.$$

Then, suppose that n is equal to 3.

From the definition of $U(P)$, followed by Lemma 2.3, then Lemma 3.4, we have

$$\begin{aligned} U(P)^3 &= \sum_{u_{i_1} \wedge u_{i_2} \wedge u_{i_3} \neq 0} V_{i_1} V_{i_2} V_{i_3} = \sum_{u_{i_1} \wedge u_{i_2} \neq 0} V_{i_1} V_{i_2} \left(V - \sum_{u_{i_1} \wedge u_{i_2} \wedge u_k = 0} V_k \right) \\ &\geqslant \sum_{u_{i_1} \wedge u_{i_2} \neq 0} V_{i_1} V_{i_2} \left(V - \frac{2}{3} V \right) = \frac{1}{3} V \sum_{u_{i_1} \wedge u_{i_2} \neq 0} V_{i_1} V_{i_2} \\ &= \frac{1}{3} V \sum_{u_{i_1} \neq 0} V_{i_1} \left(V - \sum_{u_{i_1} \wedge u_k = 0} V_k \right) \geqslant \frac{1}{3} V \sum_{u_{i_1} \neq 0} V_{i_1} \left(V - \frac{1}{3} V \right) \\ &= \frac{2!}{3^2} V^2 \sum_{u_{i_1} \neq 0} V_{i_1} = \frac{3!}{3^3} V^3, \end{aligned}$$

that is

$$\frac{U(P)}{V(P)} \geqslant \frac{(3!)^{1/3}}{3}.$$

The equality condition can be derived from Lemma 3.4 immediately. This completes the proof. □

4. Estimate of $\sum_{u_{i_1}\wedge\cdots\wedge u_{i_j}\wedge u_k=0} V_k$ for symmetrical convex polytopes, where $\{u_{i_1},\ldots,u_{i_j}\}\subseteq\{u_1,\ldots,u_N\}$, $u_{i_1}\wedge\cdots\wedge u_{i_j}\neq 0$, $2\leqslant j\leqslant n-1$

If P is an origin-symmetric convex polytope in $\mathbb{R}^n$, we can give a unified estimate for the sums of cone volumes on any finite ($2\leqslant j\leqslant n-1$) outer normal unit vectors, which is also obtained in [16] but through long and complicated arguments.

Lemma 4.1. *Let P be an origin-symmetric polytope in $\mathbb{R}^n$ with interior points. For any fixed $\{u_{i_1},\ldots,u_{i_j}\}\subseteq\{u_1,\ldots,u_N\}$, such that $u_{i_1}\wedge\cdots\wedge u_{i_j}\neq 0$, $2\leqslant j\leqslant n-1$, if the normal vector u_k of F_k satisfies $u_{i_1}\wedge\cdots\wedge u_{i_j}\wedge u_k=0$, then*

$$\sum_{u_{i_1}\wedge\cdots\wedge u_{i_j}\wedge u_k=0} V_k \leqslant \frac{j}{n}V(P). \tag{15}$$

If P is a parallelotope, then the equality of (15) *holds. Conversely, if the equalities of* (15) *hold for all subsets $\{u_{i_1},\ldots,u_{i_j}\}\subseteq\{u_1,\ldots,u_N\}$ with $u_{i_1}\wedge\cdots\wedge u_{i_j}\neq 0$ simultaneously, $2\leqslant j\leqslant n-1$, then P is a parallelotope.*

Proof. For any fixed $\{u_{i_1},\ldots,u_{i_j}\}\subseteq\{u_1,\ldots,u_N\}$, $u_{i_1}\wedge\cdots\wedge u_{i_j}\neq 0$, let

$$L=span\{u_{i_1},\ldots,u_{i_j}\},\qquad f(x)=vol_{n-j}\big(P\cap\big(L^{\perp}+x\big)\big).$$

From Lemma 2.1, $f^{\frac{1}{n-j}}(x)$ is concave on $D=P|L$. Then

$$\sum_{u_{i_1}\wedge\cdots\wedge u_{i_j}\wedge u_k=0} V_k=\int_{\partial D}\frac{1}{n}h\left(\frac{x}{\|x\|}\right)\cdot f(x)\,dS(x)$$
$$=\frac{j}{n}\int_{\partial D} f(x)\left[\frac{1}{j}h\left(\frac{x}{\|x\|}\right)\cdot dS(x)\right]$$

where $dS(x)$ is the $(j-1)$-dimensional Lebesgue area measure on ∂D.

Geometrically, it is intuitively that $\int_{\partial D} f(x)[\frac{1}{j}h(\frac{x}{\|x\|})\cdot dS(x)]$ is the volume of the set

$$\bar{P}=\big\{(x,y)\in L\times L^{\perp}\colon\ \exists x_0\in\partial D,\ (x_0,y)\in P\cap\big(L^{\perp}+x_0\big),\ x\in[o,x_0]\big\},$$

and $\bar{P}$ has the same orthogonal projection onto L as convex polytope P.

So,

$$\sum_{u_{i_1}\wedge\cdots\wedge u_{i_j}\wedge u_k=0} V_k = \frac{j}{n} V(\bar{P}).$$

Now, we aim to show

$$V(\bar{P}) \leqslant V(P).$$

For this aim, we make use of spherical coordinate in the subspace L. Suppose the equation of ∂D is:

$$\rho = \rho_0(\theta_1, \theta_2, \ldots, \theta_{j-1}).$$

Let

$$F(\rho, \theta) = f(\rho \cos\theta_1, \ldots, \rho \sin\theta_1 \cdots \sin\theta_{j-1}).$$

Then

$$V(\bar{P}) - V(P) = \int_{S^{j-1}} d\theta \int_0^{\rho_0} \rho^{j-1}\big[F(\rho_0, \theta) - F(\rho, \theta)\big]\, d\rho \leqslant 0,$$

the inequality holds since P is origin-symmetric.

If P is a parallelotope centered at the origin, it follows that

$$\sum_{u_{i_1}\wedge\cdots\wedge u_{i_j}\wedge u_k=0} V_k = \frac{1}{2n} V(P) \cdot 2j = \frac{j}{n} V(P).$$

Conversely, for any $\{u_{i_1}, \ldots, u_{i_j}\} \subseteq \{u_1, \ldots, u_N\}$, with $u_{i_1} \wedge \cdots \wedge u_{i_j} \neq 0$, $2 \leqslant j \leqslant n-1$, suppose that

$$\sum_{u_{i_1}\wedge\cdots\wedge u_{i_j}\wedge u_k=0} V_k = \frac{j}{n} V(P).$$

Set $j = n-1$. From the equality condition in Lemma 2.3, it follows that P is a parallelotope. This completes the proof. □

The proof of Theorem 1.1 is basically similar to the proof of He–Leng–Li Theorem [16]. To make the paper self-contained, we present it here.

Proof of Theorem 1.1. From the definition of $U(P)$, followed by Lemma 4.1, then Lemma 3.4, we have

$$
\begin{aligned}
U(P)^n &= \sum_{u_{i_1}\wedge\cdots\wedge u_{i_n}\neq 0} V_{i_1}\cdots V_{i_n} \\
&= \sum_{u_{i_1}\wedge\cdots\wedge u_{i_{n-1}}\neq 0} V_{i_1}\cdots V_{i_{n-1}}\Bigl(V - \sum_{u_{i_1}\wedge\cdots\wedge u_{i_{n-1}}\wedge u_k = 0} V_k\Bigr) \\
&\geqslant \sum_{u_{i_1}\wedge\cdots\wedge u_{i_{n-1}}\neq 0} V_{i_1}\cdots V_{i_{n-1}}\Bigl(V - \frac{n-1}{n}V\Bigr) \\
&= \frac{1}{n}V \sum_{u_{i_1}\wedge\cdots\wedge u_{i_{n-1}}\neq 0} V_{i_1}\cdots V_{i_{n-1}} \\
&\ \vdots \\
&= \frac{(n-2)!}{n^{n-2}}V^{n-2}\sum_{u_{i_1}\wedge u_{i_2}\neq 0} V_{i_1}V_{i_2} \\
&= \frac{(n-2)!}{n^{n-2}}V^{n-2}\sum_{u_{i_1}\neq 0} V_{i_1}\Bigl(V - \sum_{u_{i_1}\wedge u_k = 0} V_k\Bigr) \\
&\geqslant \frac{(n-1)!}{n^{n-1}}V^{n-1}\sum_{u_{i_1}\neq 0} V_{i_1} = \frac{n!}{n^n}V^n,
\end{aligned}
$$

that is

$$U(P) \geqslant \frac{(n!)^{1/n}}{n} V(P).$$

The condition of equality can be derived from Lemma 4.1 immediately. This completes the proof. □

Combined (6) with (5), it gives

$$\frac{V(\Pi K)}{U(K)^{n-1}} \leqslant 2^n \frac{n^{n-1}}{(n!)^{\frac{n-1}{n}}}, \tag{16}$$

where the equality holds if and only if K is a parallelotope. It also can be regarded as a modified version of Schneider's projection problem.

Acknowledgments

This work was done when I was visiting Polytechnic Institute of NYU during 2008–09. I was partially supported by CSC (China Scholarship Council) and Polytechnic Institute of NYU. I am very grateful for academic guidance and hospitality of professors Monika Ludwig, Erwin Lutwak, Deane Yang and Gaoyong Zhang. I thank professor Deyi Li for many discussions. I also thank the referee for careful reading and helpful comments on the paper.

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