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Convex Bodies with Identical John and LYZ Ellipsoids

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Convex bodies with identical John and LYZ ellipsoids are characterized. This solves an important problem from convex geometry posed by G. Zhang. As applications, several sharp affine isoperimetric inequalities are established.

1 Introduction

Associated with each convex body K (compact convex set with nonempty interior) in Euclidean n-space $\mathbb{R}^n$ is a unique ellipsoid of maximal volume contained in K. This ellipsoid, denoted by JK and called the *John ellipsoid* of K, has many applications in convex geometry, functional analysis, PDEs, etc. See, for example, [1, 2, 16, 24, 25, 41]. In particular, *John's characterization theorem* of JK, as Ball commented [3, p. 19], is the starting point for a general theory that builds ellipsoids related to convex bodies by maximising determinants subject to other constraints on linear maps. This theory has played a crucial role in the development of convex geometry over the last 15 years.

John's Theorem. *The John ellipsoid JK is the standard unit ball B of $\mathbb{R}^n$ if and only if $B \subseteq K$ and for some m there are unit vectors $u_j \in S^{n-1} \cap \mathrm{bd}K$ and positive numbers λ_j,*

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$j = 1, \ldots, m$, such that

$$\sum_{j=1}^{m} \lambda_j u_j \otimes u_j = I_n \quad \text{and} \quad \sum_{j=1}^{m} \lambda_j u_j = o. \tag{1.1}$$

Here, S^{n-1} denotes the unit sphere in $\mathbb{R}^n$, that is, the boundary of B, o is the origin, I_n is the identity operator on $\mathbb{R}^n$ and for each unit vector u_j, $u_j \otimes u_j$ is the rank 1 linear operator that takes x to $(u_j \cdot x)u_j$, where $u_j \cdot x$ denotes the standard inner product of u_j and x. The necessary part of John's theorem is due to John [24], and the sufficient part is due to Ball [2]. For more information, we refer to Gruber [17, 18, 20, 21] and the references therein.

We can use the notion of *isotropy* of measures to reinterpret John's theorem. A finite positive Borel measure μ on S^{n-1} is said to be *isotropic* if

$$\frac{n}{|\mu|} \int_{S^{n-1}} u \otimes u \mathrm{d}\mu(u) = I_n,$$

where $|\mu| = \mu(S^{n-1})$. Equivalently, μ is isotropic if

$$n \int_{S^{n-1}} (u \cdot v)^2 \mathrm{d}\mu(v) = |\mu|, \quad \text{for all } u \in S^{n-1}.$$

This tells us that the inertia of μ in all directions is the same, that is, the ellipsoid of inertia of μ is a sphere. With this terminology, (1.1) says that the measure $\sum_{j=1}^{m} \lambda_j \delta_{u_j}$ is isotropic and its centroid is at the origin. Here, δ_{u_j} denotes the delta measure defined on S^{n-1} and concentrated on $u_j \in S^{n-1}$.

Isotropy is an important property of measures, which may be viewed as an extension of the Pythagorean theorem in Euclidean geometry (see, e.g., [10]). For instance, it appeared in the classical *minimal surface area theorem* proved by Petty [40]: A convex body K has minimal surface area among its $\mathrm{SL}(n)$ images if and only if its surface area measure S_K is isotropic. For more information on the role of isotropy of measures in minimization problems of convex bodies, see, for example [7, 15, 16, 45]. In 1991, Ball [1] discovered an amazing connection between the isotropy of measures and the Brascamp–Lieb inequality, and used it to establish his celebrated reverse isoperimetric inequality. Ball's work inspired much use of the notion of isotropy of measures in the study of reverse affine isoperimetric inequalities (see, e.g., [4–6, 32, 34, 35, 38, 43]). It is worth mentioning that in this article we also use *isotropy* to characterize and separate our desired class of convex bodies.

In their breakthrough article [33], Lutwak, Yang, and Zhang (LYZ) introduced a continuous family of origin-symmetric ellipsoids $\mathrm{E}_p K$ for $p > 0$, associated with a convex body K in $\mathbb{R}^n$ containing the origin in its interior. They are called the *L_p John ellipsoids* of K and generalize the John ellipsoid $\mathrm{J}K$ in the framework of the L_p Brunn–Minkowski theory (see, e.g., [28–31, 33, 35]).

It is interesting that the L_p John ellipsoids form a *spectrum* linking several fundamental objects in convex geometry: If the John point of K, that is, the center of $\mathrm{J}K$, is at the origin, then $\mathrm{E}_\infty K$ is precisely the classical John ellipsoid $\mathrm{J}K$. The L_1 John ellipsoid $\mathrm{E}_1 K$ is the so-called *Petty ellipsoid*. The volume-normalized Petty ellipsoid is obtained by minimizing the surface area of K under $\mathrm{SL}(n)$ transformations of K. The L_2 John ellipsoid $\mathrm{E}_2 K$ is the important *LYZ ellipsoid*, which was previously discovered in [29] by using the notion of L_2-curvature. LYZ's ellipsoid has similar properties to that of John's ellipsoid and is in some sense dual to the classical *Legendre ellipsoid of inertia* (see, e.g., [39, 46]).

It is worth mentioning that among the L_p John ellipsoids and even the more general Orlicz–John ellipsoids [44], *only* the LYZ ellipsoid can be represented via an explicit formula. When viewed as suitably normalized matrix-valued operators on the space of convex bodies, Ludwig [26] showed that the LYZ and Legendre ellipsoids are the *only* linearly covariant operators that satisfy the inclusion-exclusion principle. The LYZ ellipsoid is always contained in the Legendre ellipsoid [31]. This inclusion is the geometric analogue of one of the basic inequalities in information theory: the *Cramer–Rao inequality*. Using the LYZ ellipsoid, LYZ [30] studied Schneider's projection problem and established several sharp affine isoperimetric inequalities. In [35], LYZ proved that the reciprocal of the volume of the LYZ ellipsoid provides a sharp lower bound for the volume of the polar body.

Without a doubt, the John and LYZ ellipsoids, as well as the family of L_p John ellipsoids and Orlicz–John ellipsoids, have become an inherent and indispensable part of modern convex geometric analysis. Nevertheless, we still do not know for which convex bodies the John and LYZ ellipsoids coincide. This fundamental problem already goes back to the discovery of the LYZ ellipsoid [29], and was first posed by G. Zhang (through a private conversation).

Problem. *For which convex bodies do their John and LYZ ellipsoids coincide?*

The main goal of this article is to solve Zhang's problem.

To explain our solution, we need to discuss the LYZ ellipsoid in more detail. Denote by $e_1, \ldots, e_n$ the standard orthonormal basis of $\mathbb{R}^n$. Let K be a convex body in $\mathbb{R}^n$

with the origin in its interior, and $\tilde{M}_K$ be the $n \times n$ matrix with entries $\tilde{m}_{i,j}$ given by

$$\tilde{m}_{i,j} = \frac{1}{V(K)} \int\limits_{S^{n-1}} \frac{(u \cdot e_i)(u \cdot e_j)}{h_K(u)} \mathrm{d}S_K(u),$$

where $V(K)$ is the n-dimensional volume of K, h_K is the support function of K and S_K is the surface area measure of K. Then, the LYZ ellipsoid $\mathrm{E}_2 K$ is generated by the matrix $\tilde{M}_K$, that is,

$$\mathrm{E}_2 K = \left\{ x \in \mathbb{R}^n : x \cdot \tilde{M}_K x \leq 1 \right\}.$$

The LYZ ellipsoid can also be defined as the unit ball of the norm in $\mathbb{R}^n$,

$$\|x\|_{\mathrm{E}_2 K} = \left(\frac{1}{V(K)} \int\limits_{S^{n-1}} |x \cdot u|^2 h_K^{-1}(u) \mathrm{d}S_K(u) \right)^{\frac{1}{2}}, \quad x \in \mathbb{R}^n,$$

which shows clearly the L_2 characteristic of the LYZ ellipsoid.

The main result of this article can now be stated as follows.

Theorem 1.1. Suppose that K is a convex body in $\mathbb{R}^n$ with the origin in its interior. Then the following assertions are equivalent.

(1) $JK = \mathrm{E}_2 K = B$.
(2) S_K is isotropic, and $h_K(u) = 1$ for all $u \in \mathrm{supp} S_K$.
(3) $\mathrm{E}_2 K = B \subseteq K$. □

Since all the L_p John ellipsoids are affine in nature, we can readily reinterpret the equivalence of (1) and (3) in the following more simple and intuitive way.

Theorem 1.2. Suppose that K is a convex body in $\mathbb{R}^n$ with the origin in its interior. Then $JK = \mathrm{E}_2 K$, if and only if $\mathrm{E}_2 K \subseteq K$. □

Write $\mathcal{J}^n$ for the class of convex bodies in $\mathbb{R}^n$ with identical John and LYZ ellipsoids. We can verify that simplices, parallelotopes, and cross-polytopes in $\mathbb{R}^n$ with centroid at the origin, belong to $\mathcal{J}^n$. The class of origin-symmetric ellipsoids in $\mathbb{R}^n$ is the unique smooth subclass of $\mathcal{J}^n$. Except for $p = 1, 2$ and ∞, the unit ball B_p^n of the normed space l_p^n does not belong to $\mathcal{J}^n$. The characterization of parallelotopes or simplices as extremal bodies of classical functionals is a central problem in convex geometry. So it may be beneficial to consider extremum problems restricted to the class $\mathcal{J}^n$.

The proof of Theorem 1.1 is presented in Section 3. Moreover, we show that Theorem 1.1 (or Theorem 1.2) remains true if the LYZ ellipsoid is replaced by *any* member of the family of L_p John ellipsoids, or even Orlicz–John ellipsoids [44] introduced by the authors in the Orlicz–Brunn–Minkowski theory (see, e.g., Gardner, Hug, and Weil [13, 14], Ludwig [27], Haberl and LYZ [22], and LYZ [36, 37]). As applications of our main results, several sharp affine isoperimetric inequalities are established in Section 4.

2 Preliminaries

For quick later reference we collect some basic facts about convex bodies. Excellent references are the books by Gardner [12], Gruber [19], and Schneider [42].

Write $\mathcal{K}_o^n$ for the set of convex bodies in $\mathbb{R}^n$ that contain the origin in their interiors. The *support function* $h_K : \mathbb{R}^n \to \mathbb{R}$, of a convex body $K \in \mathcal{K}_o^n$, is defined by

$$h_K(x) = \max\{x \cdot y : y \in K\}, \quad x \in \mathbb{R}^n.$$

$\mathcal{K}_o^n$ is often equipped with the *Hausdorff metric* δ_H, which is defined for $K_1, K_2 \in \mathcal{K}_o^n$ by

$$\delta_H(K_1, K_2) = \max\left\{|h_{K_1}(u) - h_{K_2}(u)| : u \in S^{n-1}\right\}.$$

The classical Aleksandrov–Fenchel–Jessen *surface area measure* S_K, of a convex body K can be defined as the unique Borel measure on S^{n-1} such that

$$\int_{S^{n-1}} f(u) dS_K(u) = \int_{\partial' K} f(\nu_K(y)) \mathrm{d}\mathcal{H}^{n-1}(y)$$

for each continuous $f : S^{n-1} \to \mathbb{R}$, where $\nu_K : \partial' K \to S^{n-1}$ is the Gauss map of K, defined on $\partial' K$, the set of points of ∂K that have a unique outer unit normal, and $\mathcal{H}^{n-1}$ is $(n-1)$-dimensional Hausdorff measure. Note that $\mathcal{H}^{n-1}(\partial K \setminus \partial' K) = 0$.

For $K \in \mathcal{K}_o^n$, the *normalized cone-volume measure* $\bar{V}_K$ is defined by

$$\mathrm{d}\bar{V}_K = \frac{h_K}{nV(K)} \mathrm{d}S_K. \tag{2.1}$$

In recent years, cone-volume measures have attracted much attention. See, for example, [8–11, 23, 27].

For $K, L \in \mathcal{K}_o^n$ and $0 < p < \infty$, the *normalized L_p-mixed volume* of K and L is defined by

$$\bar{V}_p(K,L) = \left(\int_{S^{n-1}} \left(\frac{h_L}{h_K} \right)^p \mathrm{d}\bar{V}_K \right)^{\frac{1}{p}}, \tag{2.2}$$

and for $p = \infty$ define

$$\bar{V}_\infty(K,L) = \sup \left\{ \frac{h_L(u)}{h_K(u)} : u \in \operatorname{supp}\bar{V}_K \right\}. \tag{2.3}$$

Following LYZ [33], we now recall the definition of the L_p John ellipsoid. Suppose that $K \in \mathcal{K}_o^n$ and $p \in (0, \infty]$. Among all origin-symmetric ellipsoids, the *L_p John ellipsoid* E_pK of K is the unique ellipsoid that solves the constrained maximization problem

$$\max_E V(E) \qquad \text{subject to} \qquad \bar{V}_p(K,E) \leq 1. \tag{2.4}$$

Throughout this article, let Φ be the class of convex functions $\varphi : [0,\infty) \to [0,\infty)$ that are strictly increasing and satisfy $\varphi(0) = 0$ and $\lim_{t\to\infty} \varphi(t) = \infty$.

For $\varphi \in \Phi \cap C^1(0,\infty)$, that is a smooth function φ in Φ, let $S_\varphi(K,\cdot)$ be the Borel measure on S^{n-1} given by

$$\mathrm{d}S_\varphi(K,\cdot) = \frac{\varphi'\left(\frac{1}{h_K}\right)}{\varphi'(1)} \mathrm{d}S_K. \tag{2.5}$$

If $\varphi(t) = t^p$, $1 \leq p < \infty$, then $S_\varphi(K,\cdot)$ reduces to $S_p(K,\cdot)$, the *L_p surface area measure* of K, which was introduced by Lutwak [28] and is defined by

$$\mathrm{d}S_p(K,\cdot) = h_K^{1-p} \mathrm{d}S_K. \tag{2.6}$$

The *normalized Orlicz mixed volume* [14, 44] of $K, L \in \mathcal{K}_o^n$ with respect to φ is

$$\bar{V}_\varphi(K,L) = \varphi^{-1} \left(\int_{S^{n-1}} \varphi\left(\frac{h_L}{h_K} \right) \mathrm{d}\bar{V}_K \right). \tag{2.7}$$

If $\varphi(t) = t^p$, $1 \leq p < \infty$, then $\bar{V}_\varphi(K,L)$ reduces to $\bar{V}_p(K,L)$.

Suppose that $K \in \mathcal{K}_o^n$. Among all origin-symmetric ellipsoids, the *Orlicz–John ellipsoid* $\mathrm{E}_\varphi K$ of K with respect to φ is the unique ellipsoid that solves the constrained maximization problem

$$\max_E V(E) \qquad \text{subject to} \qquad \bar{V}_\varphi(K,E) \leq 1. \tag{2.8}$$

If $\varphi(t) = t^p$, $1 \le p < \infty$, then $\mathrm{E}_\varphi K$ becomes the L_p John ellipsoid $\mathrm{E}_p K$. As p tends to infinity, both $\mathrm{E}_p K$ and $\mathrm{E}_{\varphi^p} K$ approach $\mathrm{E}_\infty K$. For more information on $\mathrm{E}_\varphi K$, see [44].

3 Proof of main results

3.1 The equivalence of (1) and (2)

In this part, we prove the equivalence of (1) and (2) from Theorem 1.1. To this end, we need to make further preparations.

From the definition of the L_p John ellipsoid, we have the following Lemma 3.1.

Lemma 3.1. Suppose that $K \in \mathcal{K}_o^n$ and $0 < p \le \infty$. Then $\bar{V}_p(K, \mathrm{E}_p K) = 1$. □

Lemma 3.2. Suppose that $K \in \mathcal{K}_o^n$ and $\lambda B \subseteq K$ for some $\lambda > 0$. If $h_K(u) = \lambda$, for S_K-almost all $u \in \mathrm{supp} S_K$, then $h_K|_{\mathrm{supp} S_K} = \lambda$. □

Proof. Regard S^{n-1} as a metric space with the metric topology induced by $\mathbb{R}^n$. Let $\omega_1 = \mathrm{int}\ (\mathrm{supp} S_K)$ be the interior of $\mathrm{supp} S_K$ and $\omega_2 = S^{n-1} \setminus \omega_1$. Write μ_1 and μ_2 for the restrictions of S_K to ω_1 and ω_2, respectively. Several observations are in order.

First, if ω_1 is nonempty, then μ_1 is absolutely continuous with respect to $\mathcal{H}^{n-1}$ with a strictly positive Radon–Nikodym derivative. Hence, $\mu_1(\omega) > 0$, for each nonempty open subset $\omega \subseteq \omega_1$.

Second, if ω_2 is nonempty, then for any $u \in \omega_2$ and any nonempty open neighborhood ω of u, we have $\mu_2(\omega \cap \omega_2) > 0$.

Third, $S_K(\omega) = \mu_1(\omega \cap \omega_1) + \mu_2(\omega \cap \omega_2)$, for each open subset $\omega \subseteq S^{n-1}$.

Now, if there exists a $u_0 \in \mathrm{supp} S_K$ such that $h_K(u_0) > \lambda$, then, by the continuity of h_K, there exists a nonempty open neighborhood of u_0, say ω_0, such that $h_K|_{\omega_0} > \lambda$. Since there exists an $i_0 \in \{1, 2\}$ such that $\omega_{i_0} \cap \omega_0 \ne \emptyset$, it follows that

$$S_K(\omega_0) \ge \mu_{i_0}(\omega_0 \cap \omega_{i_0}) > 0.$$

Thus, we obtain

$$S_K\left(\left\{h_K|_{\mathrm{supp} S_K} > \lambda\right\}\right) \ge S_K(\omega_0) > 0,$$

which contradicts the assumption. ■

The following characterization of L_p John ellipsoids is implicitly contained in Lemma 2.3 of [33].

Lemma 3.3. Suppose that $K \in \mathcal{K}_o^n$ and $0 < p < \infty$. Then $S_p(K, \cdot)$ is isotropic, if and only if $\mathrm{E}_p K = \bar{V}_p(K, B)^{-1} B$. $\square$

Lemma 3.4. Suppose that the finite positive Borel measure μ on S^{n-1} is isotropic and its centroid is at the origin. Let

$$K = \bigcap_{u \in \mathrm{supp}\mu} \{x \in \mathbb{R}^n : x \cdot u \leq 1\}.$$

Then, $\mathrm{J}K = B$. $\square$

Since μ is isotropic, it follows that μ cannot be concentrated on any closed hemisphere of S^{n-1}. So K is indeed a convex body. If μ is a discrete measure, Lemma 3.4 reduces to the sufficient part of John's theorem.

Proof. Let $T \in \mathrm{GL}(n)$ be positive definite and $E = T^t B + y$, where $y \in \mathbb{R}^n$ and T^t denotes the transpose of T. Under the assumption that $E \subseteq K$, it suffices to prove that

$$V(E) \leq V(B), \tag{3.1}$$

with equality only if $E = B$.

From the definition of E, it follows that

$$\begin{aligned} h_E(u) &= h_{T^t B + y}(u) \\ &= h_B(Tu) + y \cdot u \\ &= |Tu| + y \cdot u. \end{aligned}$$

This, together with the assumption that $E \subseteq K$ and the definition of K, implies that for $u \in \mathrm{supp}\mu$,

$$|Tu| + y \cdot u \leq 1. \tag{3.2}$$

Since the centroid of μ is at the origin, it follows that $\int_{S^{n-1}} u \mathrm{d}\mu(u) = 0$. So,

$$\begin{aligned} \int_{S^{n-1}} |Tu| \mathrm{d}\mu(u) &= y \cdot \int_{S^{n-1}} u \mathrm{d}\mu(u) + \int_{S^{n-1}} |Tu| \mathrm{d}\mu(u) \\ &= \int_{S^{n-1}} (y \cdot u + |Tu|) \mathrm{d}\mu(u) \end{aligned}$$

$$\leq \int_{S^{n-1}} \mathrm{d}\mu(u)$$

$$= |\mu|.$$

That is,

$$\int_{S^{n-1}} |Tu| \mathrm{d}\mu(u) \leq |\mu|. \tag{3.3}$$

Let $\lambda_1, \ldots, \lambda_n$ denote the eigenvalues of T^t, with corresponding unit eigenvectors $e_1, \ldots, e_n$. Then for each $u \in S^{n-1}$,

$$\sum_{j=1}^{n} \lambda_j (u \cdot e_j)^2 = Tu \cdot u. \tag{3.4}$$

Now, from the definition of E, the AM-GM inequality, the isotropy of μ, (3.4), Cauchy's inequality and (3.3), we have

$$\left(\frac{V(E)}{V(B)}\right)^{\frac{1}{n}} = \det (T^t)^{\frac{1}{n}}$$

$$\leq \frac{1}{n} \sum_{j=1}^{n} \lambda_j$$

$$= \frac{1}{n} \sum_{j=1}^{n} \frac{n\lambda_j}{|\mu|} \int_{S^{n-1}} (u \cdot e_j)^2 \mathrm{d}\mu(u)$$

$$= \frac{1}{|\mu|} \int_{S^{n-1}} Tu \cdot u \mathrm{d}\mu(u)$$

$$\leq 1,$$

which yields inequality (3.1).

Assume that equality holds in (3.1). Then, $\det(T^t)^{\frac{1}{n}} = \frac{1}{n} \sum_{j=1}^{n} \lambda_j = 1$. This implies that $\lambda_1 = \cdots = \lambda_n = 1$, which in turn shows that T is the identity. So, by (3.2), we have $y \cdot u \leq 0$, for $u \in \operatorname{supp}\mu$. This, combined with the fact that μ is not concentrated on any closed hemisphere, gives that $y = o$. Thus, $E = B$. ■

Now, we are in the position to prove the equivalence of (1) and (2) in Theorem 1.1.

Theorem 3.5. Suppose that $K \in \mathcal{K}_o^n$ and $0 < p < \infty$. Then $\mathrm{J}K = \mathrm{E}_pK = B$, if and only if S_K is isotropic and $h_K(u) = 1$, for all $u \in \mathrm{supp}S_K$. □

Proof. Assume that $\mathrm{J}K = \mathrm{E}_pK = B$. Then $\mathrm{J}K$ is an origin-symmetric ellipsoid contained in K with maximal volume. From the Definition of $\mathrm{E}_\infty K$, it follows that

$$\mathrm{E}_\infty K = \mathrm{J}K = B.$$

By Lemma 3.1, we have

$$\begin{aligned}\bar{V}_p(K,B) &= \bar{V}_p(K,\mathrm{E}_pK)\\ &= 1\\ &= \bar{V}_\infty(K,\mathrm{E}_\infty K)\\ &= \bar{V}_\infty(K,B).\end{aligned}$$

That is,

$$\left(\int_{S^{n-1}} \left(\frac{1}{h_K}\right)^p \mathrm{d}\bar{V}_K\right)^{\frac{1}{p}} = 1 = \sup\left\{\frac{1}{h_K(u)} : u \in \mathrm{supp}S_K\right\}.$$

By the equality conditions of Jensen's inequality, it follows that for S_K-almost all $u \in \mathrm{supp}S_K$, $h_K(u) = 1$. By Lemma 3.2, it follows that $h_K|_{\mathrm{supp}S_K} = 1$. Since $\mathrm{E}_pK = B$, by Lemma 3.3, the measure $S_p(K,\cdot)$ is isotropic. From (2.6), it follows that S_K is isotropic.

Conversely, assume that S_K is isotropic and $h_K(u) = 1$ for all $u \in \mathrm{supp}S_K$. We aim to prove that $\mathrm{J}K = \mathrm{E}_pK = B$.

From the isotropy of S_K, the assumption that $h_K|_{\mathrm{supp}S_K} = 1$ and (2.6), it follows that $S_p(K,\cdot)$ is isotropic. So, $\mathrm{E}_pK = \bar{V}_p(K,B)^{-1}B$ by Lemma 3.3. Meanwhile, from (2.2) and the assumption that $h_K|_{\mathrm{supp}S_K} = 1$ again, we have $\bar{V}_p(K,B) = 1$. Thus, $\mathrm{E}_pK = B$.

Moreover, $h_K|_{\mathrm{supp}S_K} = 1$ implies that

$$K = \bigcap_{u \in \mathrm{supp}S_K} \{x \in \mathbb{R}^n : x \cdot u \le 1\}.$$

Since S_K is isotropic with its centroid at the origin, it follows from Lemma 3.4 that $\mathrm{J}K = B$.

Consequently, $\mathrm{J}K = \mathrm{E}_pK = B$. ■

Lemma 3.6. Suppose that $K \in \mathcal{K}_o^n$ and $0 < p \leq \infty$. Then $\mathrm{E}_p TK = T\mathrm{E}_p K$, for all $T \in \mathrm{GL}(n)$. □

Lemma 3.7. Suppose that $K \in \mathcal{K}_o^n$ and $0 < p < \infty$. Then modulo orthogonal transformations, K has a unique $\mathrm{SL}(n)$ image K' such that its L_p surface area measure $S_p(K', \cdot)$ is isotropic. □

For the proof of Lemmas 3.6 and 3.7, see Lemma 2.5 and Theorem 2.2 in [33], respectively. Using Lemmas 3.6 and 3.7, Theorem 3.5 can be stated as follows.

Theorem 3.8. Suppose that $K \in \mathcal{K}_o^n$ and $0 < p < \infty$. Let K' be an $\mathrm{SL}(n)$ image of K such that its surface area measure $S_{K'}$ is isotropic. Then $\mathrm{J}K = \mathrm{E}_p K$, if and only if there exists a constant $\lambda > 0$ such that $h_{K'}(u) = \lambda$, for all $u \in \mathrm{supp} S_{K'}$. □

3.2 The equivalence of (1) and (3)

In general, the L_p John ellipsoid $\mathrm{E}_p K$, $0 < p < \infty$, is not contained in its associated body K. However, if $\mathrm{E}_p K = \mathrm{J}K$, we necessarily have the inclusion $\mathrm{E}_p K \subseteq K$. In this part, we prove that the inclusion $\mathrm{E}_p K \subseteq K$, $0 < p < \infty$, is also sufficient to ensure $\mathrm{J}K = \mathrm{E}_p K$. Consequently, we can answer Zhang's problem in a more simple and intuitive way (i.e., Theorem 1.2). In this part, we complete the proof of the equivalence of (1) and (3) within the framework of the Orlicz–Brunn–Minkowski theory.

First, recall that for $K \in \mathcal{K}_o^n$, its L_∞ John ellipsoid $\mathrm{E}_\infty K$ is the unique origin-symmetric ellipsoid contained in K with maximal volume.

Lemma 3.9. Suppose that $K \in \mathcal{K}_o^n$.

(1) If there exists a $\varphi_0 \in \Phi$ such that $\mathrm{E}_{\varphi_0} K \subseteq K$, then for all $\varphi_1 \in \Phi$,

$$\mathrm{E}_{\varphi_1 \circ \varphi_0} K = \mathrm{E}_{\varphi_0} K = \mathrm{E}_\infty K.$$

(2) If there exists a $p_0 \in (0, \infty)$ such that $\mathrm{E}_{p_0} K \subseteq K$, then for all $p \in [p_0, \infty)$,

$$\mathrm{E}_p K = \mathrm{E}_\infty K.$$ □

Proof. We prove (1). Assertion (2) can be proved similarly. Assume that $\mathrm{E}_{\varphi_0} K \subseteq K$. Write $\mathcal{E}^n$ for the class of origin-symmetric ellipsoids in $\mathbb{R}^n$. First, we prove that for any $\varphi_1 \in \Phi$ and $E \in \mathcal{E}^n$,

$$\bar{V}_{\varphi_0}(K, E) \leq \bar{V}_{\varphi_1 \circ \varphi_0}(K, E) \leq \bar{V}_\infty(K, E). \tag{3.5}$$

Indeed, from the convexity of φ_1 together with Jensen's inequality and the monotonicity of ${\varphi_1}^{-1}$, and (2.3) together with the monotonicity of φ_0, φ_1 and ${\varphi_1}^{-1}$, it follows that

$$\begin{aligned}\int_{S^{n-1}} \varphi_0\left(\frac{h_E}{h_K}\right) \mathrm{d}\bar{V}_K &\leq {\varphi_1}^{-1}\left(\int_{S^{n-1}} \varphi_1\left(\varphi_0\left(\frac{h_E}{h_K}\right)\right) \mathrm{d}\bar{V}_K\right)\\ &\leq {\varphi_1}^{-1}\left(\int_{S^{n-1}} \varphi_1\left(\varphi_0\left(\bar{V}_\infty(K,E)\right)\right) \mathrm{d}\bar{V}_K\right)\\ &= \varphi_0\left(\bar{V}_\infty(K,E)\right).\end{aligned}$$

By the monotonicity of φ_0 and (2.7), this yields (3.5).

Let $\varphi_1 \in \Phi$. By (3.5), we have

$$\begin{aligned}&\left\{E \in \mathcal{E}^n : \bar{V}_{\varphi_0}(K,E) \leq 1\right\}\\ &\quad\supseteq \left\{E \in \mathcal{E}^n : \bar{V}_{\varphi_1\circ\varphi_0}(K,E) \leq 1\right\}\\ &\quad\supseteq \left\{E \in \mathcal{E}^n : \bar{V}_{\infty}(K,E) \leq 1\right\}.\end{aligned}$$

Thus, we have

$$\begin{aligned}&\max_{E\in\left\{E\in\mathcal{E}^n:\bar{V}_{\varphi_0}(K,E)\leq 1\right\}} V(E)\\ &\quad\geq \max_{E\in\left\{E\in\mathcal{E}^n:\bar{V}_{\varphi_1\circ\varphi_0}(K,E)\leq 1\right\}} V(E)\\ &\quad\geq \max_{E\in\left\{E\in\mathcal{E}^n:\bar{V}_{\infty}(K,E)\leq 1\right\}} V(E).\end{aligned}$$

From the definition of $\mathrm{E}_{\varphi_0}K$, $\mathrm{E}_{\varphi_1\circ\varphi_0}K$ and $\mathrm{E}_\infty K$, we obtain

$$V(\mathrm{E}_{\varphi_0}K) \geq V(\mathrm{E}_{\varphi_1\circ\varphi_0}K) \geq V(\mathrm{E}_\infty K). \tag{3.6}$$

Meanwhile, since $\mathrm{E}_{\varphi_0}K \subseteq K$, by (2.3) we have $\bar{V}_\infty(K, \mathrm{E}_{\varphi_0}K) \leq 1$. By (3.5) again, we obtain

$$\mathrm{E}_{\varphi_0}K \in \left\{E \in \mathcal{E}^n : \bar{V}_{\varphi_1\circ\varphi_0}(K,E) \leq 1\right\} \quad \text{and} \quad \mathrm{E}_{\varphi_0}K \in \left\{E \in \mathcal{E}^n : \bar{V}_{\infty}(K,E) \leq 1\right\}. \tag{3.7}$$

From the uniqueness of $\mathrm{E}_{\varphi_1\circ\varphi_0}K$ and $\mathrm{E}_\infty K$, (3.6) and (3.7), we conclude that

$$\mathrm{E}_{\varphi_1\circ\varphi_0}K = \mathrm{E}_{\varphi_0}K = \mathrm{E}_\infty K,$$

as desired. ∎

Lemma 3.9 reveals an interesting phenomenon: If there is an Orlicz–John ellipsoid contained in the convex body K, then the Orlicz–John ellipsoids "*behind*" it are all contained in K.

Lemma 3.10. Suppose that $K \in \mathcal{K}_o^n$ and let K' be an $\mathrm{SL}(n)$ image of K such that its surface area measure $S_{K'}$ is isotropic and $h_{K'}|_{\mathrm{supp}S_{K'}}$ is a positive constant. Then for all $\varphi \in \Phi$,

$$\mathrm{E}_\varphi K = \mathrm{E}_\infty K = JK. \qquad \square$$

Proof. From Theorem 3.8, it follows that $\mathrm{E}_1 K = JK$. So, JK is origin-symmetric and $\mathrm{E}_\infty K = JK$. From Lemma 3.9 (1), it follows that $\mathrm{E}_\varphi K = \mathrm{E}_\infty K = JK$, for all $\varphi \in \Phi$. ■

To prove Lemma 3.12, we need the following result. For its proof, see Lemma 4.3, Lemma 4.6 and Theorem 8.4 in [44], respectively.

Lemma 3.11. Suppose that $K \in \mathcal{K}_o^n$ and $\varphi \in \Phi$. Then the following assertions hold.

(1) $\bar{V}_\varphi(K, \mathrm{E}_\varphi K) = 1$.
(2) $\mathrm{E}_\varphi TK = T\mathrm{E}_\varphi K$, for all $T \in \mathrm{GL}(n)$.
(3) If $\varphi \in \Phi \cap C^1(0,\infty)$ and $\mathrm{E}_\varphi K = B$, then $S_\varphi(K,\cdot)$ is isotropic. □

Lemma 3.12. Suppose that $K \in \mathcal{K}_o^n$ and $\varphi \in \Phi$. If $\mathrm{E}_\varphi K \subseteq K$, then there exists an $\mathrm{SL}(n)$ image K' of K such that

$$h_{K'}|_{\mathrm{supp}S_{K'}} = \left(\frac{V(\mathrm{E}_\infty K)}{\omega_n}\right)^{1/n}.$$

Moreover, if $\varphi \in \Phi \cap C^1(0,\infty)$, then $S_{K'}$ is isotropic. □

Proof. Let $\lambda = (V(\mathrm{E}_\infty K)/\omega_n)^{1/n}$. Take a $T \in \mathrm{SL}(n)$ such that $\mathrm{E}_\infty K = \lambda T^{-1} B$.

Since $\mathrm{E}_\varphi K \subseteq K$, it follows that $\mathrm{E}_\varphi K = \mathrm{E}_\infty K$ by Lemma 3.9 (1). Let $K' = TK$. By Lemma 3.11 (2), we have $\mathrm{E}_\varphi K' = \mathrm{E}_\infty K' = \lambda B$. By Lemma 3.9 (1) again, for $\varphi_1 \in \Phi$, we have

$$\mathrm{E}_{\varphi_1\circ\varphi} K' = \mathrm{E}_\varphi K' = \mathrm{E}_\infty K' = \lambda B.$$

So, by Lemma 3.11 (1), we have

$$\bar{V}_{\varphi_1\circ\varphi}(K', \lambda B) = \bar{V}_\varphi(K', \lambda B) = 1.$$

That is,

$$\varphi^{-1}\left(\varphi_1^{-1}\left(\int\limits_{\mathrm{supp}\bar{V}_{K'}}\varphi_1\left(\varphi\left(\frac{\lambda}{h_{K'}}\right)\right)\mathrm{d}\bar{V}_{K'}\right)\right)=\varphi^{-1}\left(\int\limits_{\mathrm{supp}\bar{V}_{K'}}\varphi\left(\frac{\lambda}{h_{K'}}\right)\mathrm{d}\bar{V}_{K'}\right)=1.$$

By the injectivity of φ and φ_1, we have

$$\int\limits_{\mathrm{supp}\bar{V}_{K'}}\varphi_1\left(\varphi\left(\frac{\lambda}{h_{K'}}\right)\right)\mathrm{d}\bar{V}_{K'}=\varphi_1\left(\int\limits_{\mathrm{supp}\bar{V}_{K'}}\varphi\left(\frac{\lambda}{h_{K'}}\right)\mathrm{d}\bar{V}_{K'}\right)=\varphi_1(\varphi(1)).$$

These equalities hold for all $\varphi_1\in\Phi$. So, we can choose a strictly convex φ_1. From the equality conditions of Jensen's inequality, it follows that

$$\varphi_1\left(\varphi\left(\frac{\lambda}{h_{K'}(u)}\right)\right)=\varphi_1(\varphi(1)),$$

for $\bar{V}_{K'}$-almost all $u\in\mathrm{supp}\bar{V}_{K'}$. This, combined with (2.1), implies that $h_{K'}(u)=\lambda$, for $S_{K'}$-almost all $u\in\mathrm{supp}S_{K'}$. From Lemma 3.2, it follows that $h_{K'}|_{\mathrm{supp}\,S_{K'}}=\lambda$.

It remains to show that $S_{K'}$ is isotropic under the assumption that $\varphi\in\Phi\cap C^1(0,\infty)$.

By (2.5) and the previous result, for any Borel set $\omega\subseteq S^{n-1}$, we have

$$\begin{aligned}S_\varphi(\lambda^{-1}K',\omega)&=\frac{1}{\varphi'(1)}\int\limits_{\omega\cap\mathrm{supp}S_{\lambda^{-1}K'}}\varphi'\left(\frac{1}{h_{\lambda^{-1}K'}}\right)\mathrm{d}S_{\lambda^{-1}K'}\\&=\frac{\lambda^{-(n-1)}}{\varphi'(1)}\int\limits_{\omega\cap\mathrm{supp}S_{K'}}\varphi'\left(\frac{1}{h_{\lambda^{-1}K'}}\right)\mathrm{d}S_{K'}\\&=\frac{\lambda^{-(n-1)}}{\varphi'(1)}\int\limits_{\omega\cap\mathrm{supp}S_{K'}}\varphi'(1)\mathrm{d}S_{K'}\\&=\lambda^{-(n-1)}S_{K'}(\omega).\end{aligned}$$

Thus,

$$S_\varphi(\lambda^{-1}K',\cdot)=\lambda^{-(n-1)}S_{K'}.$$

Since $\mathrm{E}_\varphi K'=\lambda B$, it follows that $\mathrm{E}_\varphi(\lambda^{-1}K')=B$ by Lemma 3.11 (2). Moreover, since $h_{\lambda^{-1}K'}|_{\mathrm{supp}S_{\lambda^{-1}K'}}=1$, this implies that $\bar{V}_\varphi(\lambda^{-1}K',B)=1$. By Lemma 3.11 (3), $S_\varphi\left(\lambda^{-1}K',\cdot\right)$ is isotropic. Consequently, $S_{K'}$ is isotropic. ∎

With these results in hand, we can now prove the main result of this part.

Theorem 3.13. Suppose that $K \in \mathcal{K}_o^n$ and $\varphi_0 \in \Phi \cap C^1(0,\infty)$. Then the following assertions are equivalent.

(1) $\mathrm{E}_{\varphi_0} K \subseteq K$.
(2) $\mathrm{E}_{\varphi} K \subseteq K$, for all $\varphi \in \Phi$.
(3) $\mathrm{E}_{\varphi_0} K = JK$.
(4) $\mathrm{E}_{\varphi} K = JK$, for all $\varphi \in \Phi$. □

Proof. From Lemmas 3.12 and 3.10, the implication "(1) $\Longrightarrow$ (4)" follows. The implications "(4) $\Longrightarrow$ (2)", "(4) $\Longrightarrow$ (3)", "(2) $\Longrightarrow$ (1)", and "(3) $\Longrightarrow$ (1)" are obvious. ■

It is obvious that if $\varphi_0 = t^2$, then the equivalence of (1) and (3) from Theorem 3.13 implies the equivalence of (1) and (3) from Theorem 1.1.

Similar to the proof of Lemma 3.12, one can show the following result.

Lemma 3.14. Suppose that $K \in \mathcal{K}_o^n$ and $0 < p < \infty$. If $\mathrm{E}_p K \subseteq K$, then there exists an $\mathrm{SL}(n)$ image K' of K such that

$$h_{K'}|_{\mathrm{supp} S_{K'}} = \left(\frac{V(\mathrm{E}_\infty K)}{\omega_n}\right)^{1/n}$$

and $S_{K'}$ is isotropic. □

From Theorems 3.8 and 3.13, Lemma 3.14, we obtain the following result, which naturally encompasses Theorem 1.2 as a special case.

Theorem 3.15. Suppose that $K \in \mathcal{K}_o^n$. Then the following assertions are equivalent.

(1) $\mathrm{E}_2 K = JK$.
(2) $\mathrm{E}_p K = JK$, for all $p \in (0,\infty)$.
(3) $\mathrm{E}_\varphi K = JK$, for all $\varphi \in \Phi$.
(4) $\mathrm{E}_2 K \subseteq K$. □

3.3 Convex bodies containing their LYZ ellipsoids

In view of the special role of those convex bodies which contain their LYZ ellipsoids, we introduce the following notation.

Let $\mathcal{J}^n$ denote the class of convex bodies in $\mathbb{R}^n$ with identical John and LYZ ellipsoids. Write $\mathcal{J}_s^n$ for the subset of $\mathcal{J}^n$ whose elements are origin-symmetric, and let $\mathcal{J}_e^n \subseteq \mathcal{J}^n$ be the set of convex bodies whose LYZ ellipsoids are the unit ball B in $\mathbb{R}^n$.

Lemma 3.16. Let $\{K_j\}_j$ be a sequence in $\mathcal{J}^n$ and E be an origin-symmetric ellipsoid such that $\mathrm{E}_2K_j = E$, for all j. If $\{K_j\}_j$ converges to K with respect to δ_H, then $\mathrm{E}_2K = E$ and $K \in \mathcal{J}^n$. □

Proof. By compactness and convexity of each K_j, K is also compact and convex. Since $E \subseteq K_j$ for all j and the volume functional V is continuous with respect to δ_H, it follows that

$$V(K) = \lim_{j\to\infty} V(K_j) \geq V(E) > 0.$$

Thus the interior of K is nonempty and K is a convex body. By Theorem 5.1 in [44],

$$\mathrm{E}_2K = \lim_{j\to\infty} \mathrm{E}_2K_j = E.$$

Since $\mathrm{E}_2K_j = E \subseteq K_j$ for all j, it follows that $\mathrm{E}_2K \subseteq K$. By Theorem 1.2, this implies that $\mathrm{E}_2K = \mathrm{J}K$. Hence, $K \in \mathcal{J}^n$. ■

Proposition 3.17. $\mathcal{J}_e^n$ is closed with respect to the Hausdorff metric δ_H. □

John's inclusion states that $\mathrm{J}K \subseteq K \subseteq n\mathrm{J}K$. Hence, $\mathcal{J}_e^n$ is bounded with respect to δ_H. Therefore, we have the following result.

Proposition 3.18. $\mathcal{J}_e^n$ is compact with respect to the Hausdorff metric δ_H. □

4 Applications

Let $\mathcal{K}_{o,j}^n$ be the class of convex bodies in $\mathbb{R}^n$ whose John point is at the origin. In this section, we provide some applications of our extracted class $\mathcal{J}^n$. We begin with the following celebrated reverse isoperimetric inequality established by Ball [1].

1. Ball's volume ratio inequality. *If K is a convex body in $\mathbb{R}^n$, then*

$$\frac{V(K)}{V(\mathrm{J}K)} \leq \frac{(n+1)^{\frac{n+1}{2}} n^{\frac{n}{2}}}{n!\omega_n}, \tag{4.1}$$

with equality if and only if K is a simplex. If in addition K is centrally symmetric, then

$$\frac{V(K)}{V(\mathrm{J}K)} \leq \frac{2^n}{\omega_n}, \tag{4.2}$$

with equality if and only if K is a parallelotope.

The necessary parts of the equality conditions in Ball's volume-ratio inequalities were proved by Barthe [4]. In [29], LYZ showed that if the John point of K is at the origin, then (4.1) and (4.2) still hold for the LYZ ellipsoid E_2K. Later, LYZ [33] showed that if the convex body K is origin-symmetric, (4.2) holds for all L_p John ellipsoids E_pK. Schuster and Weberndorfer [43] proved that when restricted to the class $\mathcal{K}^n_{o,j}$, (4.1) holds for all E_pK. Recently, the authors of this article showed [44] that if K is origin-symmetric, then (4.2) holds for all Orlicz–John ellipsoids $\mathrm{E}_\varphi K$.

Now, we show that when restricted to the class $\mathcal{K}^n_{o,j}$, (4.1) also holds for all $\mathrm{E}_\varphi K$. The following Lemma 4.1 was previously proved by the authors in [45] (See Lemma 5.4).

Lemma 4.1. If $K \in \mathcal{K}^n_{o,j}$ is a regular simplex, then S_K is isotropic and $h_K|_{\mathrm{supp} S_K}$ is a positive constant. □

Theorem 4.2. Suppose that $K \in \mathcal{K}^n_{o,j}$ and $\varphi \in \Phi$. Then

$$\frac{V(K)}{V(\mathrm{E}_\varphi K)} \leq \frac{(n+1)^{\frac{n+1}{2}} n^{\frac{n}{2}}}{n!\omega_n},$$

with equality if and only if K is a simplex. □

Proof. Since $\mathrm{E}_\infty K = \mathrm{J}K$ for $K \in \mathcal{K}^n_{o,j}$, using (4.1) and the fact that $V(\mathrm{E}_\varphi K) \geq V(\mathrm{E}_\infty K)$, we have

$$\frac{V(K)}{V(\mathrm{E}_\varphi K)} \leq \frac{V(K)}{V(\mathrm{J}K)} \leq \frac{(n+1)^{\frac{n+1}{2}} n^{\frac{n}{2}}}{n!\omega_n}. \tag{4.3}$$

If K is a simplex, we have $\mathrm{E}_\varphi K = \mathrm{J}K$ by Lemmas 4.1 and 3.10. Thus, equalities hold in (4.3). Conversely, if equalities hold in (4.3), the equality condition of (4.1) guarantees that K is a simplex. ■

A dual result to Ball's volume ratio inequality is called *Barthe's volume ratio inequality*, which concerns the *Löwner ellipsoid* $\mathrm{L}K$, the unique ellipsoid of minimal volume containing K.

2. Barthe's volume ratio inequality. *If K is a convex body in $\mathbb{R}^n$, then*

$$\frac{V(\mathrm{L}K)}{V(K)} \leq \frac{\omega_n n! n^{\frac{n}{2}}}{(n+1)^{\frac{n+1}{2}}}, \tag{4.4}$$

with equality if and only if K is a simplex.

Recall that for a convex body $K \in \mathcal{K}_o^n$, its *polar body* is

$$K^* = \{x \in \mathbb{R}^n : x \cdot y \leq 1, y \in K\}.$$

An immediate consequence of (4.4) reads as follows: If $K \in \mathcal{K}_o^n$, then

$$V(K^*) \geq \frac{(n+1)^{\frac{n+1}{2}} \omega_n}{n! n^{\frac{n}{2}}} \frac{1}{V(\mathrm{J}K)}, \tag{4.5}$$

with equality if and only if K is a simplex with its John point at the origin. Schuster and Weberndorfer [43] proved that when restricted to the class $\mathcal{K}_{o,j}^n$, (4.5) still holds for all $\mathrm{E}_p K$, $1 \leq p \leq \infty$. We now want to prove the following natural extension.

Theorem 4.3. Suppose that $K \in \mathcal{K}_{o,j}^n$ and $\varphi \in \Phi$. Then

$$V(K^*) \geq \frac{(n+1)^{\frac{n+1}{2}} \omega_n}{n! n^{\frac{n}{2}}} \frac{1}{V(\mathrm{E}_\varphi K)},$$

with equality if and only if K is a simplex. □

Proof. Since $\mathrm{E}_\infty K = \mathrm{J}K$ for $K \in \mathcal{K}_{o,j}^n$, using (4.5) and the fact that $V(\mathrm{E}_\varphi K) \geq V(\mathrm{E}_\infty K)$, we have

$$\begin{aligned} V(K^*)V(\mathrm{E}_\varphi K) &\geq V(K^*)V(\mathrm{E}_\infty K) \\ &= V(K^*)V(\mathrm{J}K) \\ &\geq \frac{(n+1)^{\frac{n+1}{2}} \omega_n}{n! n^{\frac{n}{2}}}. \end{aligned}$$

If K is a simplex, we have $\mathrm{E}_\varphi K = \mathrm{E}_\infty K = \mathrm{J}K$ by Lemmas 4.1 and 3.10. Thus, equalities hold in all the above inequalities. Conversely, if equality holds in the third line, the equality condition of (4.5) guarantees that K is a simplex. ■

Similar to the proof of Theorem 4.3, one can obtain the following result.

Theorem 4.4. Suppose that $K \in \mathcal{K}_{o,j}^n$ and $0 < p \leq \infty$. Then

$$V(K^*) \geq \frac{(n+1)^{\frac{n+1}{2}}\omega_n}{n!n^{\frac{n}{2}}} \frac{1}{V(\mathrm{E}_p K)},$$

with equality if and only if K is a simplex. □

In [35], LYZ proved that for $K \in \mathcal{K}_o^n$ whose John point is not necessarily at the origin, (4.5) still holds for $\mathrm{E}_2 K$. This is the following LYZ's volume inequality.

3. LYZ's volume inequality. *If $K \in \mathcal{K}_o^n$, then*

$$V(K^*) \geq \frac{(n+1)^{\frac{n+1}{2}}\omega_n}{n!n^{\frac{n}{2}}} \frac{1}{V(\mathrm{E}_2 K)}, \tag{4.6}$$

with equality if and only if K is a simplex with its John point at the origin.

Using the inequality

$$V(\mathrm{E}_q K) \leq V(\mathrm{E}_p K), \quad 0 < p < q \leq \infty,$$

Schuster and Weberndorfer [43] proved that if $K \in \mathcal{K}_o^n$ and $1 \leq p < 2$, then (4.6) remains true if $\mathrm{E}_2 K$ is replaced by $\mathrm{E}_p K$, $1 \leq p \leq 2$. We now show that LYZ's volume inequality, as well as its equality conditions, remains true when the range of p is extended to $(0, 2]$.

Theorem 4.5. Suppose that $K \in \mathcal{K}_o^n$ and $0 < p < 2$. Then

$$V(K^*) \geq \frac{(n+1)^{\frac{n+1}{2}}\omega_n}{n!n^{\frac{n}{2}}} \frac{1}{V(\mathrm{E}_p K)},$$

with equality if and only if K is a simplex with its John point at the origin. □

Proof. Note that

$$V(K^*)V(\mathrm{E}_p K) \geq V(K^*)V(\mathrm{E}_2 K) \geq \frac{(n+1)^{\frac{n+1}{2}}\omega_n}{n!n^{\frac{n}{2}}}.$$

If K is a simplex with its John point at the origin, then $\mathrm{E}_p K = \mathrm{E}_2 K$ for all $p \in (0, 2)$ by Lemma 4.1 and Theorem 3.15. Therefore, equality holds in all the above inequalities. ■

We conclude this article with an open problem, which is a further extension of Zhang's problem.

Problem 4.6. For which convex bodies K in $\mathbb{R}^n$ do we have $V(\mathrm{E}_2K) \leq V(\mathrm{J}K)$? In particular, for which convex bodies K in $\mathbb{R}^n$ holds $V(\mathrm{E}_2K) = V(\mathrm{J}K)$? □

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