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Bounds for inclusion measures of convex bodies ☆

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Abstract

By using the method of mixed volumes, we give sharp bounds for inclusion measures of convex bodies in n-dimensional Euclidean space. In the special cases where the random convex body is the unit ball or when $n = 3$, neater and simpler bounds are obtained. All the associated inequalities proved are new isoperimetric-type inequalities.

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1. Introduction

The setting for this paper is in the n-dimensional Euclidean space $\mathbb{R}^n$. A *convex figure* is a compact convex subset of $\mathbb{R}^n$, and a *convex body* is a convex figure with nonempty interior. The *principal kinematic formula* in integral geometry gives the measure of the set of congruent convex bodies intersecting with a fixed convex body. These formulas can be viewed as integral formulas for various *intersection measures*. They are useful for solving problems in geometric probability and stochastic geometry. Some problems in geometric probability require more tools

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than merely intersection measures. For instance, solutions to the Buffon needle problem of lattices need to compute the measure of the needle that is contained in a fundamental region of the lattice. The kinematic measure of a moving geometric figure that is contained in a fixed geometric figure is called the *inclusion measure*. Specifically, let K, L be two convex bodies in $\mathbb{R}^n$ and $G(n)$ be the group of special motions in $\mathbb{R}^n$. Each element $g : \mathbb{R}^n \to \mathbb{R}^n$ of $G(n)$ can be represented by

$$x \mapsto g(x) = ex + b,$$

where $b \in \mathbb{R}^n$ and e is an orthogonal matrix of determinant 1. Let μ be the Haar measure on $G(n)$ normalized as follows. Let $\varphi : \mathbb{R}^n \times SO(n) \to G(n)$ be defined by $\varphi(t, e)x = ex + t$, $x \in \mathbb{R}^n$, where $SO(n)$ is the rotation group of $\mathbb{R}^n$. If ν is the unique invariant probability measure on $SO(n)$, and η is the Lebesgue measure on $\mathbb{R}^n$, then μ is chosen as the pull back measure of $\eta \otimes \nu$ under φ^{-1}.

The *inclusion measure* of a convex figure L contained in a convex body K is defined by

$$m_K(L) = m(L \subseteq K) = \int\limits_{\{g \in G(n):\ gL \subseteq K\}} d\mu(g).$$

It gives the measure of the set of copies congruent to L which are contained in a fixed convex body K. Ref. [16] is an excellent survey paper on inclusion measures for which one can consult. The first important work on inclusion measures is due to Hadwiger, who gave bounds for inclusion measures and used the bounds to derive Bonnesen-type isoperimetric inequalities. Hadwiger's work was generalized to higher dimensions in [15]. In [7], D. Ren introduced the notion of generalized support function of a convex body in the plane and used it to establish integral formulas for the inclusion measure of a line segment inside a convex body. He then applied his formulas to solving generalized Buffon needle problems of lattices in [8]. In [12], Xiong showed that if K_i, $i = 1, \ldots, s$, $s > 1$, $s \in \mathbb{N}$, are convex bodies and L is a convex figure in $\mathbb{R}^n$, then

$$m_{K_1 + \cdots + K_s}(L) > m_{K_1}(L) + \cdots + m_{K_s}(L).$$

Specifically, when L is a line segment, then

$$m_{\alpha_1 K_1 + \cdots + \alpha_s K_s}(L) > \alpha_1^n m_{K_1}\left(\frac{L}{\alpha_1}\right) + \cdots + \alpha_s^n m_{K_s}\left(\frac{L}{\alpha_s}\right)$$

for $\alpha_i \geqslant 1$. In particular, the inclusion measure $m_K(L)$ is not linear with respect to the convex body K.

In this paper, we continue to investigate the inclusion measures of convex bodies and obtain upper and lower bounds for inclusion measures by using the theory of mixed volumes. All the associated inequalities proved are new isoperimetric-type inequalities. For the new progress of isoperimetric inequalities, see Refs. [3,5,6,9,13,14]. In some sense, this paper illustrates the powerful and effective applications of the theory of mixed volume to the theory of inclusion measures.

2. Notations and preliminaries

Let K be a convex figure in $\mathbb{R}^n$. Associated with K is its *support function* h_K defined on $\mathbb{R}^n$ by

$$h_K(x) = \max\{\langle x, y\rangle: \ y \in K\},$$

where $\langle x, y\rangle$ is the usual inner product of x and y in $\mathbb{R}^n$. The function h_K is positively homogeneous of degree 1. We will usually be concerned with the restriction of the support function to the unit sphere S^{n-1}.

The *Minkowski addition* of two convex figures K and L is defined as

$$K + L = \{x + y: \ x \in K, \ y \in L\}.$$

The *Minkowski difference* of two convex figures K and L is defined as

$$K \sim L = \{x \in \mathbb{R}^n: \ x + L \subseteq K\}.$$

If L is empty, $K \sim L$ is, by convention, equal to $\mathbb{R}^n$.

The *scalar multiplication* λK of K, where $\lambda \geqslant 0$, is defined as

$$\lambda K = \{\lambda x: \ x \in K\}.$$

For convex figure $\lambda K + \mu L$, the support function satisfies

$$h_{\lambda K+\mu L} = \lambda h_K + \mu h_L.$$

If the convex body K in $\mathbb{R}^n$ is a *Minkowski linear combination* of m convex bodies, i.e., $K = \lambda_1 K_1 + \lambda_2 K_2 + \cdots + \lambda_m K_m$, $\lambda_1, \ldots, \lambda_m \geqslant 0$, then the volume of K can be expressed as an nth degree homogeneous polynomial in the λ_i as follows:

$$V(K) = \sum_{1 \leqslant p_1, \ldots, p_n \leqslant m} V(K_{p_1}, K_{p_2}, \ldots, K_{p_n}) \lambda_{p_1} \lambda_{p_2} \cdots \lambda_{p_n}.$$

Here the summation is extended over all p_i independently as i varies from 1 to n. The coefficients $V(K_{p_1}, K_{p_2}, \ldots, K_{p_n})$ are called *mixed volumes*. Specifically, for convex figure $\lambda K + \mu L$, its volume is a homogeneous polynomial in λ and μ given by

$$V(\lambda K + \mu L) = \sum_{i=0}^{n} \binom{n}{i} V_i(K, L) \lambda^{n-i} \mu^i.$$

The coefficients $V_i(K, L)$ are called *mixed volumes* of K and L. In particular, if B is the unit ball, then the mixed volumes $V(\underbrace{K, \ldots, K}_{n-i}, \underbrace{B, \ldots, B}_{i}) = V_i(K, B)$ are called the *quermassintegrals* of K and denoted by $W_i(K)$.

The quermassintegrals are generalizations of the surface area and the volume. Indeed, it can be shown that

$$
\begin{aligned}
W_0(K) &= V(K),\\
nW_1(K) &= S(K),\\
\frac{2}{\omega_n} W_{n-1}(K) &= M(K),\\
W_n(K) &= \omega_n,
\end{aligned}
$$

where ω_n is the volume of the unit ball in $\mathbb{R}^n$ and $M(K)$ is the mean width of K, i.e.,

$$
M(K) = \frac{2}{n\omega_n} \int\limits_{S^{n-1}} h_K(u)\, dS(u). \tag{2.0}
$$

The quermassintegrals arise in many areas of mathematics and have different definitions. If K has a C^2 boundary, they are the integrals of elementary symmetric functions of the principal curvatures over the boundary. In the theory of mixed volumes, the quermassintegrals are called simple mixed volumes. They are also called projection measures, intrinsic volumes, etc. The reader should consult [10] and [11] for details.

The following elementary properties of mixed volumes will be used later. For any convex figures K, L, K_i $(1 \leqslant i \leqslant n)$ in $\mathbb{R}^n$, $K \supseteq L$, and for any $a, b \geqslant 0$,

$$
V(K_1, \ldots, K_{n-1}, aK_n + bL) = aV(K_1, \ldots, K_{n-1}, K_n) + bV(K_1, \ldots, K_{n-1}, L), \tag{2.1}
$$

$$
V(K_1, \ldots, K_{n-1}, K) \geqslant V(K_1, \ldots, K_{n-1}, L). \tag{2.2}
$$

The mixed volume $V_1(K, L)$ has an integral representation given by

$$
V_1(K, L) = \frac{1}{n} \int\limits_{S^{n-1}} h_L(u)\, dS_K(u), \tag{2.3}
$$

where S_K is the *surface area measure* of K (see [4, p. 166]).

The Minkowski inequality states that if K and L are convex bodies in $\mathbb{R}^n$, then

$$
V_1(K, L)^n \geqslant V(K)^{n-1} V(L), \tag{2.4}
$$

with equality if and only if K and L are homothetic. The Minkowski inequality has an equivalent form

$$
V(K + L)^{\frac{1}{n}} \geqslant V(K)^{\frac{1}{n}} + V(L)^{\frac{1}{n}}, \tag{2.5}
$$

which is called the Brunn–Minkowski inequality.

If $K_1, \dots, K_n$ are convex bodies in $\mathbb{R}^n$ and $m < n$ is a natural number, then

$$V(K_1, \dots, K_n)^m \geqslant \prod_{i=1}^{m} V(\underbrace{K_i, \dots, K_i}_{m}, K_{m+1}, \dots, K_n). \tag{2.6}$$

Inequality (2.6) is due to Alexandrov [1].

The *inradius* $r(K, L)$ of K with respect to L is defined by

$$r(K, L) = \sup\{\lambda\colon\ x \in \mathbb{R}^n \text{ and } x + \lambda L \subseteq K\}.$$

If L is the unit ball B in $\mathbb{R}^n$, then $r(K, L)$ will denote the radius of the maximal inscribable ball of K.

Let h_K and h_L be the support functions of K and L, respectively. First, we assume that L is a convex body. For a fixed $\lambda \in [0, r]$, consider the function $h_\lambda = h_K - \lambda h_L$ on the unit sphere, where $r = r(K, L)$ is the inradius of K with respect to L. In general, h_λ is not the support function of a convex body. Denote by $C(K, L, \lambda)$ the intersection of halfspaces $\{x \in \mathbb{R}^n\colon \langle x, u\rangle \leqslant h_\lambda(u)\}$, $u \in S^{n-1}$. The boundaries $\partial C(K, L, \lambda)$ are pairwise disjoint and

$$\bigcup_{0 \leqslant \lambda \leqslant r(K,L)} \partial C(K, L, \lambda) = K.$$

The following formula is known (see [1,4]),

$$\frac{d}{d\lambda} V\big(C(K, L, \lambda)\big) = -n V_1\big(C(K, L, \lambda), L\big). \tag{2.7}$$

By integrating both sides of (2.7), we get

$$V(K) - V\big(C(K, L, \lambda)\big) = n \int_0^\lambda V_1\big(C(K, L, \sigma), L\big)\, d\sigma \tag{2.8}$$

and

$$V\big(C(K, L, \lambda)\big) = n \int_\lambda^{r(K,L)} V_1\big(C(K, L, \sigma), L\big)\, d\sigma. \tag{2.9}$$

By a limit process, (2.8) and (2.9) are seen to hold for any convex figure L. The following lemma asserts that $C(K, L, \lambda) = \phi$ if $\lambda > r(K, L)$, and thus (2.8) and (2.9) hold for any $\lambda \geqslant 0$.

Lemma 1. *(See [15].) The intersection $C(K, L, \lambda)$ of halfspaces $\{x \in \mathbb{R}^n\colon \langle x, u\rangle \leqslant h_\lambda(u)\}$ is equal to the set $\{x \in \mathbb{R}^n\colon x + \lambda L \subseteq K\}$, i.e.,*

$$C(K, L, \lambda) = \{x \in \mathbb{R}^n\colon\ x + \lambda L \subseteq K\}, \quad \lambda \geqslant 0.$$

According to the definition of *Minkowski difference*, the set $C(K,L,\lambda)$ is actually equal to the set $K \sim \lambda L$, that is, $C(K,L,\lambda) = K \sim \lambda L$.

The following lemma will also be useful in the sequel.

Lemma 2. *(See [15].) If K is a convex body and L is a convex figure in $\mathbb{R}^n$, then the inclusion measure of L contained in K is*

$$m_K(L) = \int\limits_{SO(n)} V\big(C(K,eL,1)\big)\,d\nu(e),$$

where ν is the unique invariant probability measure on $SO(n)$.

3. Main results

For simplicity, if no confusion may arise, we shall abbreviate $C(K,L,\lambda)$ as $C(K)$. In the following, we obtain the lower and upper bounds of the volume for the set $C(K)$.

Lemma 3. *If L is a fixed convex figure and K is any convex body in $\mathbb{R}^n$, then for any $0 \leqslant \lambda \leqslant r(K,L)$, we have*

$$\begin{aligned} V_i\big(C(K),L\big) \leqslant{}& V_i(K,L) - \lambda V_{i+1}(K,L) \\ &- \lambda \sum_{k=1}^{n-i-1} V\big(\underbrace{C(K),\dots,C(K)}_{k},\underbrace{K,\dots,K}_{n-k-i-1},\underbrace{L,\dots,L}_{i+1}\big). \end{aligned} \tag{3.0}$$

Moreover, if $C(K) + \lambda L = K$, then equality (3.0) *holds.*

Proof. From Lemma 1, we have $K \supseteq C(K) + \lambda L$, for $0 \leqslant \lambda \leqslant r(K,L)$. Using properties (2.1) and (2.2) of mixed volumes, we have

$$\begin{aligned} V_i(K,L) &= V(\underbrace{K,\dots,K}_{n-i},\underbrace{L,\dots,L}_{i}) \geqslant V\big(C(K)+\lambda L,\underbrace{K,\dots,K}_{n-i-1},\underbrace{L,\dots,L}_{i}\big) \\ &= V\big(C(K),\underbrace{K,\dots,K}_{n-i-1},\underbrace{L,\dots,L}_{i}\big) + \lambda V(\underbrace{K,\dots,K}_{n-i-1},\underbrace{L,\dots,L}_{i+1}) \\ &= V\big(C(K),\underbrace{K,\dots,K}_{n-i-1},\underbrace{L,\dots,L}_{i}\big) + \lambda V_{i+1}(K,L) \\ &\geqslant V\big(C(K),C(K)+\lambda L,\underbrace{K,\dots,K}_{n-i-2},\underbrace{L,\dots,L}_{i}\big) + \lambda V_{i+1}(K,L) \\ &= V\big(C(K),C(K),\underbrace{K,\dots,K}_{n-i-2},\underbrace{L,\dots,L}_{i}\big) + \lambda V\big(C(K),\underbrace{K,\dots,K}_{n-i-2},\underbrace{L,\dots,L}_{i+1}\big) \\ &\quad + \lambda V_{i+1}(K,L) \\ &\geqslant V\big(C(K),C(K),C(K),\underbrace{K,\dots,K}_{n-i-3},\underbrace{L,\dots,L}_{i}\big) \end{aligned}$$

$$
\begin{aligned}
&+\lambda V\big(C(K), C(K), \underbrace{K, \dots, K}_{n-i-3}, \underbrace{L, \dots, L}_{i+1}\big)\\
&+\lambda V\big(C(K), \underbrace{K, \dots, K}_{n-i-2}, \underbrace{L, \dots, L}_{i+1}\big) + \lambda V_{i+1}(K, L)\\
&\ \vdots\\
&\geqslant V_i\big(C(K), L\big) + \lambda V_{i+1}(K, L) + \lambda \sum_{k=1}^{n-i-1} V\big(\underbrace{C(K), \dots, C(K)}_{k}, \underbrace{K, \dots, K}_{n-k-i-1}, \underbrace{L, \dots, L}_{i+1}\big).
\end{aligned}
$$

Hence the inequality (3.0).

If $C(K) + \lambda L = K$, from the property (2.1) of mixed volume, each equality in the above derivation holds, so equality (3.0) holds too.

This completes the proof. □

Theorem 1. *Let L be a convex figure and K a convex body in $\mathbb{R}^n$. If $r(K, L) \geqslant 1$, then*

$$
\begin{aligned}
m_K(L) \leqslant{} & V(K) - \frac{1}{2n} S(K) M(L)\\
& - \int_{SO(n)} \sum_{k=1}^{n-1} V\big(\underbrace{C(K, eL, 1), \dots, C(K, eL, 1)}_{k}, \underbrace{K, \dots, K}_{n-k-1}, eL\big)\, d\nu(e). \qquad (3.1)
\end{aligned}
$$

Moreover, if $C(K, eL, 1) + eL = K$ for any $e \in SO(n)$, then equality (3.1) *holds.*

Proof. From Lemma 3, when $i = 0$ we have

$$
V\big(C(K)\big) \leqslant V(K) - \lambda V_1(K, L) - \lambda \sum_{k=1}^{n-1} V\big(\underbrace{C(K), \dots, C(K)}_{k}, \underbrace{K, \dots, K}_{n-k-1}, L\big).
$$

With the condition $r(K, L) \geqslant 1$, from Lemma 2 we have

$$
\begin{aligned}
m_K(L) &= \int_{SO(n)} V\big(C(K, eL, 1)\big)\, d\nu(e)\\
&\leqslant \int_{SO(n)} \bigg[V(K) - V_1(K, eL)\\
&\quad - \sum_{k=1}^{n-1} V\big(\underbrace{C(K, eL, 1), \dots, C(K, eL, 1)}_{k}, \underbrace{K, \dots, K}_{n-k-1}, eL\big) \bigg]\, d\nu(e)\\
&= V(K) - \int_{SO(n)} V_1(K, eL)\, d\nu(e)
\end{aligned}
$$

$$- \int_{SO(n)} \sum_{k=1}^{n-1} V\big(\underbrace{C(K,eL,1),\dots,C(K,eL,1)}_{k}, \underbrace{K,\dots,K}_{n-k-1}, eL\big)\,d\nu(e). \qquad (*)$$

From (2.3), (2.0), and Fubini's theorem, we obtain

$$\int_{SO(n)} V_1(K,eL)\,d\nu(e) = \frac{1}{n}\int_{S^{n-1}} dS_K(u) \int_{SO(n)} h_{eL}(u)\,d\nu(e)$$

$$= \frac{1}{n}\int_{S^{n-1}} dS_K(u)\cdot\frac{1}{n\omega_n}\int_{S^{n-1}} h_L(u)\,dS(u) = \frac{1}{2n}S(K)M(L).$$

This gives (3.1).

If $C(K,eL,1)+eL=K$ for any $e\in SO(n)$, by Lemma 3, equality $(*)$ holds. So equality (3.1) holds too.

This completes the proof. □

Theorem 2. *Let L be a convex figure and K a convex body in $\mathbb{R}^n$. If $r(K,L)\geqslant 1$, then*

$$m_K(L) \geqslant V(K) - \frac{1}{2}S(K)M(L) + \frac{n}{2\omega_n}W_2(K)W_{n-2}(L)$$
$$+ n\int_{SO(n)} \left[\sum_{k=1}^{n-2}\int_0^1 tV\big(\underbrace{C(K,eL,t),\dots,C(K,eL,t)}_{k}, \underbrace{K,\dots,K}_{n-k-2}, eL, eL\big)\,dt\right]d\nu(e). \qquad (3.2)$$

Moreover, if $C(K,eL,1)+eL=K$ for any $e\in SO(n)$, then equality (3.2) *holds.*

Proof. From Lemma 3, when $i=1$ we have

$$V_1\big(C(K),L\big) \leqslant V_1(K,L) - \lambda V_2(K,L) - \lambda\sum_{k=1}^{n-2} V\big(\underbrace{C(K),\dots,C(K)}_{k}, \underbrace{K,\dots,K}_{n-k-2}, L, L\big)$$

for all $0\leqslant\lambda\leqslant r(K,L)$. By integrating both sides of this inequality from 0 to t with respect to λ, and applying (2.8) we have

$$\frac{1}{n}\big[V(K) - V\big(C(K,L,t)\big)\big]$$
$$\leqslant tV_1(K,L) - \frac{t^2}{2}V_2(K,L) - \sum_{k=1}^{n-2}\int_0^t \lambda V\big(\underbrace{C(K),\dots,C(K)}_{k}, \underbrace{K,\dots,K}_{n-k-2}, L, L\big)\,d\lambda.$$

Write $t=\lambda$. Then

$$
\begin{aligned}
V\big(C(K)\big) \geqslant{} & V(K) - n\lambda V_1(K,L) \\
& + n\frac{\lambda^2}{2} V_2(K,L) + n\sum_{k=1}^{n-2}\int_0^{\lambda} tV\big(\underbrace{C(K,L,t),\dots,C(K,L,t)}_{k},\underbrace{K,\dots,K}_{n-k-2},L,L\big)\,dt.
\end{aligned}
$$

With the condition $r(K,L) \geqslant 1$, from Lemma 2 we have

$$
\begin{aligned}
m_K(L) &= \int_{SO(n)} V\big(C(K,eL,1)\big)\,d\nu(e) \\
&\geqslant \int_{SO(n)} \Bigg[V(K) - nV_1(K,eL) + \frac{n}{2}V_2(K,eL) \\
&\quad + n\sum_{k=1}^{n-2}\int_0^1 tV\big(\underbrace{C(K,eL,t),\dots,C(K,eL,t)}_{k},\underbrace{K,\dots,K}_{n-k-2},eL,eL\big)\,dt\Bigg]\,d\nu(e) \\
&= V(K) - \frac{1}{2}S(K)M(L) + \frac{n}{2}\int_{SO(n)} V_2(K,eL)\,d\nu(e) \\
&\quad + n\int_{SO(n)} \Bigg[\sum_{k=1}^{n-2}\int_0^1 tV\big(\underbrace{C(K,eL,t),\dots,C(K,eL,t)}_{k},\underbrace{K,\dots,K}_{n-k-2},eL,eL\big)\,dt\Bigg]\,d\nu(e).
\end{aligned} \tag{$*$}
$$

Using the formula

$$\int_{SO(n)} V_i(K,eL)\,d\nu(e) = \frac{1}{\omega_n} W_i(K)W_{n-i}(L),$$

we have

$$\int_{SO(n)} V_2(K,eL)\,d\nu(e) = \frac{1}{\omega_n} W_2(K)W_{n-2}(L).$$

From this (3.2) follows.

If $C(K,eL,1) + eL = K$ for any $e \in SO(n)$, then by Lemma 3, equality $(*)$ holds, so equality (3.2) holds too.

This completes the proof. □

For the case $n = 3$, we have a comparatively simpler estimate than (3.1) and (3.2). The following lemma is useful to our proof.

Lemma 4. *(See [2,15].) If the convex body $C(\lambda) = C(K,L,\lambda)$ is defined as in Lemma 1, then the functions $V_{n-i}(C(\lambda),L)^{\frac{1}{i}}$ and $V(C(\lambda))^{\frac{1}{n}}$ are concave functions of λ.*

Corollary 1. *Let L be a convex figure and K a convex body in $\mathbb{R}^3$. If $r(K,L)\geqslant 1$, then*

$$m_K(L)\leqslant V(K)-\frac{1}{6}S(K)M(L)-2V^{\frac{1}{3}}(L)\int\limits_{SO(3)}V^{\frac{2}{3}}\big(C(K,eL,1)\big)\,d\nu(e)$$
$$-V^{\frac{2}{3}}(L)\int\limits_{SO(3)}V^{\frac{1}{3}}\big(C(K,eL,1)\big)\,d\nu(e). \tag{3.3}$$

Moreover, if $C(K,eL,1)+eL=K$ and $C(K,eL,1)$ is homothetic to eL for any $e\in SO(3)$, then equality (3.3) *holds. Specifically, if K and L are balls, or L is a point, then equality* (3.3) *holds.*

Proof. From Theorem 1, with the properties (2.1) and (2.2) of mixed volumes and the fact that $C(K,eL,1)+eL\subseteq K$, we have

$$m_K(L)\leqslant V(K)-\frac{1}{6}S(K)M(L)-\int\limits_{SO(3)}V_1\big(C(K,eL,1),eL\big)\,d\nu(e)$$
$$-\int\limits_{SO(3)}V\big(C(K,eL,1),K,eL\big)\,d\nu(e)$$
$$\leqslant V(K)-\frac{1}{6}S(K)M(L)-2\int\limits_{SO(3)}V_1\big(C(K,eL,1),eL\big)\,d\nu(e)$$
$$-\int\limits_{SO(3)}V_2\big(C(K,eL,1),eL\big)\,d\nu(e).$$

Using the Minkowski inequality, we have

$$V_1\big(C(K,eL,1),eL\big)\geqslant V^{\frac{2}{3}}\big(C(K,eL,1)\big)V^{\frac{1}{3}}(eL)=V^{\frac{2}{3}}\big(C(K,eL,1)\big)V^{\frac{1}{3}}(L)$$

and

$$V_2\big(C(K,eL,1),eL\big)\geqslant V^{\frac{1}{3}}\big(C(K,eL,1)\big)V^{\frac{2}{3}}(eL)=V^{\frac{1}{3}}\big(C(K,eL,1)\big)V^{\frac{2}{3}}(L).$$

Therefore,

$$m_K(L)\leqslant V(K)-\frac{1}{6}S(K)M(L)-2V^{\frac{1}{3}}(L)\int\limits_{SO(3)}V^{\frac{2}{3}}\big(C(K,eL,1)\big)\,d\nu(e)$$
$$-V^{\frac{2}{3}}(L)\int\limits_{SO(3)}V^{\frac{1}{3}}\big(C(K,eL,1)\big)\,d\nu(e).$$

Assume that $C(K, eL, 1) + eL = K$ and $C(K, eL, 1)$ is homothetic to eL for any $e \in SO(3)$. According to Theorem 1 and the equality condition of the Minkowski inequality, each equality in the above arguments has to hold, so equality (3.3) holds too.

Specifically, when L is a point, then $m_K(L) = V(K)$. It is easy to compute the right-hand side of (3.3) directly and it is precisely equal to $V(K)$.

If both K and L are balls with the same radius r, then $m_K(L) = 0$. Computing the right-hand side of (3.3), we get

$$\frac{4\pi r^3}{3} - \frac{8\pi r^3}{6} - 2r \times \left(\frac{4\pi}{3}\right)^{\frac{1}{3}} \times 0 - r^2 \times \left(\frac{4\pi}{3}\right)^{\frac{2}{3}} \times 0 = 0.$$

Hence, the equality holds in this case.

Assume that both K and L are balls with radii r_K and r_L, $r_K > r_L$, respectively. From Lemma 2, we have $m_K(L) = \frac{4\pi(r_K - r_L)^3}{3}$. Computing the right-hand side of (3.3), we arrive at

$$\frac{4\pi r_K^3}{3} - \frac{8\pi r_K^2 r_L}{6} - \frac{8\pi r_L (r_K - r_L)^2}{3} - \frac{4\pi r_L^2 (r_K - r_L)}{3},$$

which is precisely $\frac{4\pi(r_K - r_L)^3}{3}$. Hence, the equality holds in this case.

This completes the proof. □

Corollary 2. *Let L be a convex figure and K be a convex body in $\mathbb{R}^3$. If $r(K, L) \geqslant 1$, then*

$$m_K(L) \geqslant V(K) - \frac{1}{2}S(K)M(L) + \frac{1}{3}M(K)S(L) + V^{\frac{2}{3}}(L) \int_{SO(3)} V^{\frac{1}{3}}\big(C(K, eL, 1)\big)\, d\nu(e). \tag{3.4}$$

Specifically, if K and L are balls, or L is a point, then equality (3.4) *holds.*

Proof. From Theorem 2, we have

$$m_K(L) \geqslant V(K) - \frac{1}{2}S(K)M(L) + \frac{3}{2} \int_{SO(3)} V_2(K, eL)\, d\nu(e)$$
$$+ 3 \int_{SO(3)} \int_0^1 t V_2\big(C(K, eL, t), eL\big)\, dt\, d\nu(e).$$

Using the formula

$$\int_{SO(n)} V_i(K, eL)\, d\nu(e) = \frac{1}{\omega_n} W_i(K) W_{n-i}(L),$$

we have

$$\frac{3}{2}\int_{SO(3)} V_2(K,eL)\,d\nu(e)=\frac{3}{2}\cdot\frac{1}{\omega_3}W_2(K)W_1(L)=\frac{1}{4}M(K)S(L).$$

For the set $V_2(C(K,eL,t),eL), 0\leqslant t\leqslant 1,$ since it is a concave function of t, by Lemma 4, we have

$$tV_2\big(C(K,eL,t),eL\big)\geqslant t\big[(1-t)V_2(K,eL)+tV_2\big(C(K,eL,1),eL\big)\big].$$

Therefore,

$$\begin{aligned}
&3\int_{SO(3)}\int_0^1 tV_2\big(C(K,eL,t),eL\big)\,dt\,d\nu(e)\\
&\quad\geqslant 3\int_{SO(3)}\int_0^1\big[\big(t-t^2\big)V_2(K,eL)+t^2V_2\big(C(K,eL,1),eL\big)\big]\,dt\,d\nu(e)\\
&\quad=\frac{1}{2}\int_{SO(3)} V_2(K,eL)\,d\nu(e)+\int_{SO(3)} V_2\big(C(K,eL,1),eL\big)\,d\nu(e)\\
&\quad=\frac{1}{12}M(K)S(L)+\int_{SO(3)} V_2\big(C(K,eL,1),eL\big)\,d\nu(e).
\end{aligned}$$

Furthermore, for the set $V_2(C(K,eL,1),eL)$, using the Minkowski inequality, we have

$$V_2\big(C(K,eL,1),eL\big)\geqslant V^{\frac{1}{3}}\big(C(K,eL,1)\big)V^{\frac{2}{3}}(eL)=V^{\frac{1}{3}}\big(C(K,eL,1)\big)V^{\frac{2}{3}}(L),$$

therefore,

$$\int_{SO(3)} V_2\big(C(K,eL,1),eL\big)\,d\nu(e)\geqslant V^{\frac{2}{3}}(L)\int_{SO(3)} V^{\frac{1}{3}}\big(C(K,eL,1)\big)\,d\nu(e).$$

Hence

$$m_K(L)\geqslant V(K)-\frac{1}{2}S(K)M(L)+\frac{1}{3}M(K)S(L)+V^{\frac{2}{3}}(L)\int_{SO(3)} V^{\frac{1}{3}}\big(C(K,eL,1)\big)\,d\nu(e).$$

When L is a point, or K and L are balls, we can verify directly as in Corollary 1 that equality (3.4) holds.

This completes the proof. □

Problem 1. Assume that $C(K,eL,1)$ is homothetic to eL for any $e\in SO(3)$. Is it affirmative that K and L are balls?

If the answer to the problem is positive, then equalities (3.3) and (3.4) hold, if and only if K and L are balls.

In the special case where L is the unit ball B in $\mathbb{R}^n$, we can obtain beautiful bounds for inclusion measures of convex bodies. For simplicity, in what follows we will denote $r(K, L)$, i.e., the radius of the maximal inscribable ball of K, by r.

Theorem 3. *Let K be a convex body and B the unit ball in $\mathbb{R}^n$. If $r \geqslant 1$, then*

$$m_K(B) \leqslant V(K) - \frac{1}{n}S(K) - \big((r-1)r^{n-1} - (r-1)^n\big)\omega_n. \tag{3.5}$$

Specifically, if K is a ball with radius r, then equality (3.5) *holds.*

Proof. When $L = B$, by (3.1) we have

$$m_K(B) \leqslant V(K) - \frac{1}{n}S(K) - \int_{SO(n)} \sum_{k=1}^{n-1} V\big(\underbrace{C(K, eB, 1), \ldots, C(K, eB, 1)}_{k}, \underbrace{K, \ldots, K}_{n-k-1}, eB\big)\, d\nu(e).$$

Since $rB \subseteq K$ and $(r-1)B \subseteq C(K, B, 1)$, we have

$$\begin{aligned}
&V\big(\underbrace{C(K, eB, 1), \ldots, C(K, eB, 1)}_{k}, \underbrace{K, \ldots, K}_{n-k-1}, eB\big)\\
&\quad = V\big(\underbrace{C(K, B, 1), \ldots, C(K, B, 1)}_{k}, \underbrace{K, \ldots, K}_{n-k-1}, B\big)\\
&\quad \geqslant V\big(\underbrace{(r-1)B, \ldots, (r-1)B}_{k}, \underbrace{rB, \ldots, rB}_{n-k-1}, B\big)\\
&\quad = (r-1)^k r^{n-k-1}\omega_n, \quad k = 1, 2, \ldots, n-1.
\end{aligned}$$

So,

$$\begin{aligned}
m_K(B) &\leqslant V(K) - \frac{1}{n}S(K) - \omega_n \sum_{k=1}^{n-1} (r-1)^k r^{n-k-1}\\
&= V(K) - \frac{1}{n}S(K) - \big((r-1)r^{n-1} - (r-1)^n\big)\omega_n.
\end{aligned}$$

In case K is a ball with radius r, we can compute the right-hand side of (3.5) directly and get $(r-1)^n\omega_n$, which is precisely equal to $m_K(B)$.

This completes the proof. □

Theorem 4. *Let K be a convex body and B the unit ball in $\mathbb{R}^n$. If $r \geqslant 1$, then*

$$m_K(B) \geqslant V(K) - S(K) + \frac{n}{2} r^{n-2}\omega_n + \left(\frac{n}{2} - 1\right) r^n \omega_n - n\omega_n \sum_{k=1}^{n-2} r^{n-k-2}(r-1)^{k+1} \frac{r+k+1}{(k+1)(k+2)}. \tag{3.6}$$

Specifically, if K is a ball with radius r, then equality (3.6) *holds.*

Proof. When $L = B$, by (3.2) we have

$$m_K(B) \geqslant V(K) - S(K) + \frac{n}{2} W_2(K) + n \int_{SO(n)} \left[\sum_{k=1}^{n-2} \int_0^1 t V\big(\underbrace{C(K, eB, t), \ldots, C(K, eB, t)}_{k}, \underbrace{K, \ldots, K}_{n-k-2}, eB, eB\big)\, dt \right] d\nu(e).$$

Since $rB \subseteq K$ and $(r-t)B \subseteq C(K, B, t)$, $0 \leqslant t \leqslant 1$, we have

$$W_2(K) = V_2(K, B) \geqslant V(\underbrace{rB, \ldots, rB}_{n-2}, B, B) \geqslant r^{n-2}\omega_n,$$

and

$$\begin{aligned} &V\big(\underbrace{C(K, eB, t), \ldots, C(K, eB, t)}_{k}, \underbrace{K, \ldots, K}_{n-k-2}, eB, eB\big) \\ &\quad = V\big(\underbrace{C(K, B, t), \ldots, C(K, B, t)}_{k}, \underbrace{K, \ldots, K}_{n-k-2}, B, B\big) \\ &\quad \geqslant V\big(\underbrace{(r-t)B, \ldots, (r-t)B}_{k}, \underbrace{rB, \ldots, rB}_{n-k-2}, B, B\big) = (r-t)^k r^{n-k-2}\omega_n, \quad k = 1, 2, \ldots, n-2. \end{aligned}$$

So,

$$\begin{aligned} m_K(B) &\geqslant V(K) - S(K) + \frac{n}{2} r^{n-2}\omega_n + n\omega_n \sum_{k=1}^{n-2} \int_0^1 t(r-t)^k r^{n-k-2}\, dt \\ &= V(K) - S(K) + \frac{n}{2} r^{n-2}\omega_n + n\omega_n \sum_{k=1}^{n-2} r^{n-k-2} \int_0^1 t(r-t)^k\, dt \\ &= V(K) - S(K) + \frac{n}{2} r^{n-2}\omega_n + n\omega_n r^n \sum_{k=1}^{n-2} \int_0^{\frac{1}{r}} t(1-t)^k\, dt \\ &= V(K) - S(K) + \frac{n}{2} r^{n-2}\omega_n + \frac{n}{2} r^n \omega_n - r^n \omega_n \end{aligned}$$

$$-n\omega_n\sum_{k=1}^{n-2}r^{n-k-2}(r-1)^{k+1}\frac{r+k+1}{(k+1)(k+2)}$$

$$=V(K)-S(K)+\frac{n}{2}r^{n-2}\omega_n+\left(\frac{n}{2}-1\right)r^n\omega_n$$

$$-n\omega_n\sum_{k=1}^{n-2}r^{n-k-2}(r-1)^{k+1}\frac{r+k+1}{(k+1)(k+2)}.$$

In case K is a ball with radius r, each equality in the arguments has to hold. So equality (3.6) holds too.

This completes the proof. □

Problem 2. What is the necessary condition for equalities (3.5) and (3.6) to hold?

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