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The minimal Orlicz surface area ☆

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Petty proved that a convex body in $\mathbb{R}^n$ has the minimal surface area amongst its $\mathrm{SL}(n)$ images, if, and only if, its surface area measure is isotropic. By introducing a new notion of minimal Orlicz surface area, we generalize this result to the Orlicz setting. The analog of Ball's reverse isoperimetric inequality is established.

1. Introduction

A classical and useful result proved by Petty [41] is the *minimal surface area theorem*, which states: A convex body (i.e., a compact convex set with non-empty interior) in Euclidean n-space $\mathbb{R}^n$ has the minimal surface area amongst its $\mathrm{SL}(n)$ images, if, and only if, its surface area measure is isotropic on the unit sphere S^{n-1}. Its importance was rediscovered in the 1990s. In [9], Clack generalized it to Minkowski space. Later,

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Giannopoulos and Papadimitrakis [14] used isotropic surface area measure to study the hyperplane projections of convex bodies.

As the Brunn–Minkowski theory [44] was extended to the L_p Brunn–Minkowski theory (see, e.g., [7,8,10,20,25–34,40,43,45–47,50,49,51,53]), the notions of surface area and surface area measure were extended to those of L_p surface area and L_p surface area measure, respectively. See the initial works of Lutwak [28,29]. In [34], Lutwak, Yang and Zhang showed that Petty's theorem has a natural L_p generalization: The L_p surface area of a convex body is minimal amongst its $\mathrm{SL}(n)$ images, if, and only if, its L_p surface area measure is isotropic on S^{n-1}.

Beginning with a series of ground-breaking articles [26,35,36,19] and the very recent work [12], a more wide extension of the L_p Brunn–Minkowski theory emerged, which is now called the Orlicz Brunn–Minkowski theory. In this context, the main goal of this paper is to seek an Orlicz extension of the minimal L_p surface area.

Throughout this paper, let $\varphi : [0, \infty) \to [0, \infty)$ be convex and strictly increasing with $\varphi(0) = 0$, and denote by Φ the class of those φ.

Suppose K is a convex body in $\mathbb{R}^n$ with the origin in its interior. Its *Orlicz surface area* $S_\varphi(K)$ with respect to φ is defined by

$$S_\varphi(K) = \int_{\partial K} \varphi\left(\frac{1}{x \cdot \nu(x)}\right) x \cdot \nu(x) d\mathcal{H}^{n-1}(x).$$

Here, $\nu(x)$, $x \cdot \nu(x)$ and $\mathcal{H}^{n-1}$ denote the outer unit normal of ∂K at $x \in \partial K$, the standard inner product of x and $\nu(x)$, and the $(n-1)$-dimensional Hausdorff measure, respectively. Note that $\nu(x)$ exists for $\mathcal{H}^{n-1}$-almost all $x \in \partial K$.

If $\varphi(t) = t^p$, $1 \le p < \infty$, then $S_\varphi(K)$ is precisely the L_p surface area [28,29] of K.

In Section 3, we demonstrate that modulo orthogonal transformations, the body K has a unique $\mathrm{SL}(n)$ image with minimal Orlicz surface area. In view of this fact, we define the *minimal Orlicz surface area* of K with respect to φ by

$$A_\varphi(K) = \min\{S_\varphi(TK) : T \in \mathrm{SL}(n)\}.$$

For $\varphi \in \Phi \cap C^1(0, \infty)$, i.e., for smooth functions φ in Φ, we introduce the transformation, $\varphi \mapsto \varphi^\diamond$, defined by $\varphi^\diamond(t) = t\varphi'(t)$. Then, for each Borel set $\omega \subseteq S^{n-1}$, we write

$$S_{\varphi^\diamond}(K, \omega) = \int_{\nu^{-1}(\omega)} \varphi'\left(\frac{1}{x \cdot \nu(x)}\right) d\mathcal{H}^{n-1}(x).$$

If $\varphi(t) = t^p$, $1 \le p < \infty$, then $S_{\varphi^\diamond}(K, \cdot)/p$ is just the L_p surface area measure of K.

In Section 4, we extend Petty's result to the Orlicz setting.

Theorem 1.1. *Suppose K is a convex body in $\mathbb{R}^n$ with the origin in its interior, and $\varphi \in \Phi \cap C^1(0, \infty)$. Then $A_\varphi(K) = S_\varphi(K)$ if and only if $S_{\varphi^\diamond}(K, \cdot)$ is isotropic on S^{n-1}.*

In the last section, we provide bounds for the minimal Orlicz surface area $A_\varphi(K)$. When the volume of K is fixed, origin-symmetric ellipsoids attain the minimum; the volume of L_∞ John ellipsoid introduced in [34] dominates it from above. Especially, if the *John point* (i.e., center of the John ellipsoid) of body K is at the origin, we prove the following

Theorem 1.2. *Amongst all convex bodies in $\mathbb{R}^n$ with the same volume and John points at the origin, modulo $\mathrm{SL}(n)$ transformations, the n-dimensional simplex uniquely maximizes the minimal Orlicz surface area. If the involved bodies are origin-symmetric, then the n-dimensional parallelotope is the unique maximizer.*

2. Preliminaries

In order to keep the paper self-contained, we collect here some basic facts from Convex Geometry. Good references on the theory of convex bodies are the books by Gardner [11], Gruber [16], Pisier [42], Schneider [44], and Thompson [48], etc.

As usual, $x \cdot y$ denotes the standard inner product of x and y in $\mathbb{R}^n$; $B = \{x \in \mathbb{R}^n : x \cdot x \le 1\}$ and $S^{n-1} = \partial B$ denote the unit ball and unit sphere in $\mathbb{R}^n$, respectively. The volume of B is $\pi^{n/2}/\Gamma(1 + n/2)$.

According to the context, one can catch clearly that the notation $|\cdot|$ has several different meanings: the absolute value, the standard Euclidean norm on $\mathbb{R}^n$, the n-dimensional volume, the absolute value of determinant of an $n \times n$ matrix, and the total mass of a finite measure.

For brevity, we write $\langle x\rangle = |x|^{-1}x$, for $x \in \mathbb{R}^n \setminus \{0\}$.

Given a convex body K in $\mathbb{R}^n$, its *support function* $h_K : \mathbb{R}^n \to \mathbb{R}$ is defined by

$$h_K(x) = \max\{x \cdot y : y \in K\}.$$

The definition immediately gives that for $T \in \mathrm{GL}(n)$ and $x \in \mathbb{R}^n$,

$$h_{TK}(x) = h_K\big(T^t x\big). \tag{2.1}$$

Throughout this paper, $\mathcal{K}_o^n$ denotes the class of convex bodies in $\mathbb{R}^n$ that contain the origin in their interiors. $\mathcal{K}_o^n$ is often equipped with the Hausdorff metric δ_H, which is defined for $K_1, K_2 \in \mathcal{K}_o^n$, by $\delta_H(K_1, K_2) = \max_{S^{n-1}} |h_{K_1} - h_{K_2}|$.

The classical *surface area measure* S_K, of a convex body K, is the unique Borel measure on S^{n-1} such that

$$\int_{S^{n-1}} f(u)\,dS_K(u) = \int_{\partial K} f\big(\nu_K(y)\big)\,d\mathcal{H}^{n-1}(y),$$

for each continuous $f : S^{n-1} \to \mathbb{R}$.

The *cone-volume measure* V_K, of a convex body K, is a Borel measure on S^{n-1} defined for a Borel set $\omega \subseteq S^{n-1}$ by

$$V_K(\omega) = \frac{1}{n} \int_{\omega} h_K \, dS_K.$$

It is convenient to use the *normalized* cone-volume measure $\overline{V}_K = \frac{V_K}{|K|}$, of K. Observe that $\overline{V}_K$ is a probability measure on S^{n-1}. Also, $\overline{V}_K$ is $\mathrm{GL}(n)$-invariant, i.e., for $T \in \mathrm{GL}(n)$ and a Borel subset $\omega \subseteq S^{n-1}$, it yields

$$\overline{V}_{T^t K}(\omega) = \overline{V}_K\big(\langle T\omega\rangle\big), \tag{2.2}$$

where $\langle T\omega\rangle = \{\langle Tu\rangle : u \in \omega\}$.

The cone-volume measure has been appeared and investigated widely in various contexts recently, see e.g., [5,7,8,15,21,20,26,27,38–40,46,47,51].

What follows recalls some fundamental facts established in [12].

For $K \in \mathcal{K}_o^n$, $\varphi \in \Phi$, and $\varepsilon \geq 0$, define the function $h_{\varphi,\varepsilon} : \mathbb{R}^n \to [0, \infty)$ by

$$h_{\varphi,\varepsilon}(x) = \inf\left\{\lambda > 0 : \varphi\left(\frac{h_K(x)}{\lambda}\right) + \varepsilon\varphi\left(\frac{|x|}{\lambda}\right) \leq \varphi(1)\right\}.$$

Observe that $h_{\varphi,\varepsilon}$ is both sublinear and positive definite. Thus, there exists a unique convex body $K_{\varphi,\varepsilon} \in \mathcal{K}_o^n$ such that its support function is precisely $h_{\varphi,\varepsilon}$.

According to Lemmas 8.2 and 8.4 in [12], we have $K_{\varphi,\varepsilon} \to K$, as $\varepsilon \to 0^+$, and

$$\left.\frac{\partial}{\partial\varepsilon}\right|_{\varepsilon=0^+} h_{\varphi,\varepsilon} = \frac{h_K(u)}{\varphi'_-(1)} \varphi\left(\frac{1}{h_K(u)}\right), \quad \text{uniformly for } u \in S^{n-1}.$$

Here, $\varphi'_-(1)$ denotes the left derivative of φ at 1.

Note that $h(\varepsilon, u) = h_{\varphi,\varepsilon}(u) : (0, \infty) \times S^{n-1} \to (0, \infty)$ is continuous ([54]). Thus, by Aleksandrov's variational principle (see Lemma 3.1 in [8], Lemma 8.3 in [12] or Lemma 1 in [19]), it yields that

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0^+} |K_{\varphi,\varepsilon}| = \frac{n}{\varphi'_-(1)} \int_{S^{n-1}} \varphi\left(\frac{1}{h_K}\right) dV_K.$$

If $\varphi(t) = t$, then $K_{\varphi,\varepsilon} = K_\varepsilon := \{x + y : x \in K, y \in \varepsilon B\}$ is precisely the outer parallel body [44] of K. Correspondingly, the above formula reduces to the classical version

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0^+} |K_\varepsilon| = |S_K| = \mathcal{H}^{n-1}(\partial K).$$

In this sense, we introduce the following

Definition 2.1. The *Orlicz surface area* $S_\varphi(K)$, of a convex body $K \in \mathcal{K}_o^n$ with respect to a function $\varphi \in \Phi$, is defined by

$$S_\varphi(K) = n \int_{S^{n-1}} \varphi\left(\frac{1}{h_K}\right) dV_K.$$

If $\varphi(t) = t^p$, $1 \le p < \infty$, then $S_\varphi(K)$ turns to $S_p(K)$, the L_p surface area of K.

Recall that the *Orlicz mixed volume* $V_\varphi(K, L)$, of $K, L \in \mathcal{K}_o^n$, is defined by

$$V_\varphi(K, L) = \frac{1}{n} \int_{S^{n-1}} \varphi\left(\frac{h_L}{h_K}\right) dV_K.$$

(See, e.g., [12,52,54]). Thus, $S_\varphi(K) = nV_\varphi(K, B)$.

3. The minimal Orlicz surface area

In order to demonstrate the existence and uniqueness of minimal Orlicz surface area, we begin by proving two lemmas.

Lemma 3.1. *Suppose $K \in \mathcal{K}_o^n$, $\varphi \in \Phi$ and $T \in \mathrm{GL}(n)$. Then*

$$S_\varphi(TK) = n|T| \int_{S^{n-1}} \varphi\left(\frac{h_{T^{-1}B}}{h_K}\right) dV_K.$$

Proof. From Definition 2.1, (2.1) and (2.2), we have

$$\begin{aligned}
S_\varphi(TK) &= n|TK| \int_{S^{n-1}} \varphi\left(\frac{1}{h_{TK}(u)}\right) d\overline{V}_{TK}(u) \\
&= n|T||K| \int_{S^{n-1}} \varphi\left(\frac{|T^{-t}\langle T^t u\rangle|}{h_K(\langle T^t u\rangle)}\right) d\overline{V}_K(\langle T^t u\rangle) \\
&= n|T||K| \int_{S^{n-1}} \varphi\left(\frac{h_{T^{-1}B}(\langle T^t u\rangle)}{h_K(\langle T^t u\rangle)}\right) d\overline{V}_K(\langle T^t u\rangle) \\
&= n|T| \int_{S^{n-1}} \varphi\left(\frac{h_{T^{-1}B}}{h_K}\right) dV_K,
\end{aligned}$$

as desired. □

Given an origin-symmetric ellipsoid E in $\mathbb{R}^n$, let $d_E = \max\{h_E(u) : u \in S^{n-1}\}$. Then there exists a $v_E \in S^{n-1}$ such that for all $u \in S^{n-1}$,

$$d_E|u \cdot v_E| \le h_E(u). \tag{3.1}$$

Lemma 3.2. *Suppose $K \in \mathcal{K}_o^n$, $\varphi \in \Phi$ and $T \in \mathrm{SL}(n)$. Then*

$$d_{T^{-1}B} \le \frac{n|K|\varphi^{-1}(\frac{S_\varphi(TK)}{n|K|})}{\min_{v \in S^{n-1}} \int_{S^{n-1}} |u \cdot v| dS_K(u)},$$

where φ^{-1} is the inverse function of φ.

Proof. From Lemma 3.1, the strict monotonicity of φ together with (3.1), the convexity of φ together with Jensen's inequality, we have

$$\begin{aligned}
\frac{S_\varphi(TK)}{n|K|} &= \int_{S^{n-1}} \varphi\left(\frac{h_{T^{-1}B}}{h_K}\right) d\overline{V}_K \\
&\ge \int_{S^{n-1}} \varphi\left(\frac{d_{T^{-1}B}|u \cdot v_{T^{-1}B}|}{h_K(u)}\right) d\overline{V}_K(u) \\
&\ge \varphi\left(\int_{S^{n-1}} \frac{d_{T^{-1}B}|u \cdot v_{T^{-1}B}|}{h_K(u)} d\overline{V}_K(u)\right) \\
&\ge \varphi\left(\frac{d_{T^{-1}B}}{n|K|} \min_{v \in S^{n-1}} \int_{S^{n-1}} |u \cdot v| dS_K(u)\right).
\end{aligned}$$

Since φ^{-1} is strictly increasing in $[0, \infty)$, and $\min_{v \in S^{n-1}} \int_{S^{n-1}} |u \cdot v| dS_K(u) > 0$ by the fact that S_K is not concentrated on any great subsphere, the desired inequality is derived. □

With the previous lemmas in hand, we can show that the minimal Orlicz surface area is well-defined.

Theorem 3.3. *Suppose $K \in \mathcal{K}_o^n$ and $\varphi \in \Phi$. Then modulo orthogonal transformations, there exists a unique solution to the minimization problem*

$$\min_{T \in \mathrm{SL}(n)} S_\varphi(TK).$$

Alternatively, by using Orlicz mixed volume, we can reformulate this theorem as follows: The unit ball B has a unique $\mathrm{SL}(n)$ image E_0 such that

$$V_\varphi(K, E_0) = \min\{V_\varphi(K, TB) : T \in \mathrm{SL}(n)\}.$$

Proof. Let $\{T_k\}_k \subset \mathrm{SL}(n)$ be a minimizing sequence for the problem, that is,

$$\lim_{k\to\infty} S_\varphi(T_k K) = \inf\{S_\varphi(TK) : T \in \mathrm{SL}(n)\}.$$

Note that

$$\inf\{S_\varphi(TK) : T \in \mathrm{SL}(n)\} \le S_\varphi(K) < \infty,$$

this implies $\{S_\varphi(T_k K)\}_k$, therefore $\{\varphi^{-1}(\frac{S_\varphi(T_k K)}{n|K|})\}_k$, is bounded from above. So, by Lemma 3.2, $\{T_k^{-1}B\}_k$ is bounded with respect to the Hausdorff metric. From the Blaschke selection theorem, $\{T_k^{-1}B\}_k$ has a convergent subsequence $\{T_{k_j}^{-1}B\}_j$ that converges to a body E. Since volume functional is continuous with respect to the Hausdorff metric, and $|T_{k_j}^{-1}B| = \omega_n$ for each j, it yields that $|E| = \omega_n$; Since the convergence of $\{T_{k_j}^{-1}B\}_j$ is equivalent to the uniform convergence of $\{h_{T_{k_j}^{-1}B}\}_j$ on S^{n-1}, and $h_{T_{k_j}^{-1}B}(u) = h_{T_{k_j}^{-1}B}(-u)$ for all $u \in S^{n-1}$, it yields that $h_E(u) = h_E(-u)$ for all $u \in S^{n-1}$. Thus, E is a non-degenerated origin-symmetric ellipsoid.

Consequently, there exists a transformation $T_0 \in \mathrm{SL}(n)$ such that $E = T_0^{-1}B$. This demonstrates the existence of solutions to the considered problem.

Now, we prove the uniqueness by contradiction. Assume there are two solutions $T_1, T_2 \in \mathrm{SL}(n)$ to the considered problem, and they don't differ only by an orthogonal transformation. It is known that T_i^{-1} can be represented in the form $T_i^{-1} = P_i Q_i$, $i = 1, 2$, where P_i is a symmetric positive definite matrix and Q_i is an orthogonal matrix.

The above assumption implies that $P_1 \ne \lambda P_2$, for all $\lambda > 0$. Hence, the Minkowski inequality for symmetric positive definite matrices shows that

$$\det\left(\frac{P_1 + P_2}{2}\right)^{\frac{1}{n}} > \frac{1}{2}\det(P_1)^{\frac{1}{n}} + \frac{1}{2}\det(P_2)^{\frac{1}{n}} = 1.$$

Let

$$T_3 = \left(\det\left(\frac{P_1 + P_2}{2}\right)^{-\frac{1}{n}} \frac{(P_1 + P_2)}{2}\right)^{-1}.$$

Then, $T_3 \in \mathrm{SL}(n)$ and $h_{T_3^{-1}B}(u) < h_{\frac{1}{2}(P_1+P_2)B}(u)$, for all $u \in S^{n-1}$. Since φ is strictly increasing and convex in $[0, \infty)$, we have

$$\varphi\left(\frac{h_{T_3^{-1}B}}{h_K}\right) < \varphi\left(\frac{h_{\frac{1}{2}(P_1+P_2)B}}{h_K}\right) \le \frac{1}{2}\varphi\left(\frac{h_{P_1B}}{h_K}\right) + \frac{1}{2}\varphi\left(\frac{h_{P_2B}}{h_K}\right).$$

Thus, by Lemma 3.1, we have

$$S_\varphi(T_3 K) = n \int_{S^{n-1}} \varphi\left(\frac{h_{T_3^{-1}B}}{h_K}\right) dV_K$$

$$\begin{aligned}
&< \frac{n}{2}\int_{S^{n-1}} \varphi\left(\frac{h_{P_1B}}{h_K}\right)dV_K + \frac{n}{2}\int_{S^{n-1}} \varphi\left(\frac{h_{P_2B}}{h_K}\right)dV_K \\
&= \frac{1}{2}S_\varphi(T_1K) + \frac{1}{2}S_\varphi(T_2K) \\
&= S_\varphi(T_1K) \\
&= S_\varphi(T_2K).
\end{aligned}$$

That is,

$$S_\varphi(T_3K) < S_\varphi(T_1K) = S_\varphi(T_2K).$$

However, by the previous assumption on T_1 and T_2, we have

$$S_\varphi(T_3K) \geq S_\varphi(T_1K) = S_\varphi(T_2K),$$

which is a contradiction. The proof is complete. □

In view of Theorem 3.3, naturally, we introduce the following

Definition 3.4. Suppose $K \in \mathcal{K}_o^n$ and $\varphi \in \Phi$. The quantity

$$A_\varphi(K) = \min\{S_\varphi(TK) : T \in \mathrm{SL}(n)\}$$

is called the *minimal Orlicz surface area* of the convex body K with respect to φ.

Obviously, $A_\varphi(K)$ is $\mathrm{SL}(n)$ invariant and a generalization of Petty's minimal surface area. If $\varphi(t) = t^p$, $1 \leq p < \infty$, then the notion of minimal Orlicz surface area reduces to that of minimal L_p surface area and particularly that of minimal total L_p curvature, whenever the boundary ∂K is of class C_+^2. See [34].

4. A characterization of the minimal Orlicz surface area

Throughout this section, we impose a condition on $\varphi \in \Phi$, that φ is smooth in $(0, \infty)$. Suppose $K \in \mathcal{K}_o^n$ and $\varphi \in \Phi \cap C^1(0, \infty)$. Recall that

$$S_{\varphi^\diamond}(K, \omega) = \int_\omega \varphi'\left(\frac{1}{h_K}\right)dS_K,$$

for each Borel subset $\omega \subseteq S^{n-1}$.

For further discussion, we introduce the important notion of isotropy of measures. A nonnegative Borel measure μ on S^{n-1} is said to be *isotropic* if

$$\int_{S^{n-1}} (u\cdot v)^2 d\mu(u) = \frac{|\mu|}{n}, \quad \text{for all } v \in S^{n-1}.$$

The definition immediately yields

$$\int_{S^{n-1}} u_i^2 d\mu(u) = \frac{|\mu|}{n},$$

where u_i denotes the ith component of the coordinate of u. For more information about the isotropy, we refer to [1,2,6,13,14,37].

For $x \in \mathbb{R}^n \setminus \{0\}$, the map $x \otimes x : \mathbb{R}^n \to \mathbb{R}^n$ is the rank 1 linear operator $y \mapsto (x \cdot y)x$. So, it follows that $\operatorname{tr}(x \otimes x) = |x|^2$.

The next theorem characterizes the convex body with minimal Orlicz surface area.

Theorem 4.1. *Suppose $K \in \mathcal{K}_o^n$, $\varphi \in \Phi \cap C^1(0,\infty)$ and $T_0 \in \mathrm{SL}(n)$. Then the following assertions are equivalent:*

(1) $A_\varphi(K) = S_\varphi(T_0 K)$.
(2) *The measure $S_{\varphi^\diamond}(T_0 K, \cdot)$ is isotropic on S^{n-1}.*
(3) *For all $x \in \mathbb{R}^n$, the transformation T_0 satisfies*

$$|x|^2 \int_{\omega} |T_0^{-t} u| \varphi'\left(\frac{|T_0^{-t} u|}{h_K(u)}\right) dS_K(u) = n \int_{S^{n-1}} \frac{|x \cdot T_0^{-t} u|^2}{|T_0^{-t} u|} \varphi'\left(\frac{|T_0^{-t} u|}{h_K(u)}\right) dS_K(u).$$

Proof. First, we prove the equivalence of (1) and (2).

Suppose that (1) holds. Since $A_\varphi(K)$ is $\mathrm{SL}(n)$ invariant, we may assume that T_0 is the $n \times n$ identity matrix I_n.

Let T be a linear transformation. Then there exists $\varepsilon_0 > 0$ such that for each $\varepsilon \in (0, \varepsilon_0)$, the matrices $I_n + \varepsilon T$ and $I_n - \varepsilon T$ are still positive definite. For $\varepsilon \in (0, \varepsilon_0)$, define

$$T_\varepsilon = \frac{I_n + \varepsilon T}{|I_n + \varepsilon T|^{\frac{1}{n}}}.$$

If ε is sufficiently small, from Lemma 3.1, the smoothness of φ, together with the two equalities

$$|I_n + \varepsilon T|^{\frac{1}{n}} = 1 + \frac{\varepsilon}{n} \operatorname{tr} T + o(\varepsilon^2),$$
$$\left|(I_n + \varepsilon T)^{-t} u\right| = 1 - \varepsilon u \cdot T^t u + o(\varepsilon^2),$$

and the definition of $S_{\varphi^\diamond}$, we have

$$S_\varphi(T_\varepsilon K) = \int_{S^{n-1}} \varphi\left(\frac{|T_\varepsilon^{-t} u|}{h_K(u)}\right) h_K(u) dS_K(u)$$

$$= \int\limits_{S^{n-1}} \varphi\biggl(\frac{|I_n + \varepsilon T|^{\frac{1}{n}} |(I_n + \varepsilon T)^{-t} u|}{h_K(u)}\biggr) h_K(u) dS_K(u)$$

$$= S_\varphi(K) + \varepsilon \int\limits_{S^{n-1}} \left(\frac{\operatorname{tr} T}{n} - u \cdot T^t u\right) \varphi'\left(\frac{1}{h_K(u)}\right) dS_K(u) + o\bigl(\varepsilon^2\bigr)$$

$$= S_\varphi(K) + \varepsilon \int\limits_{S^{n-1}} \left(\frac{\operatorname{tr} T}{n} - u \cdot T^t u\right) dS_{\varphi^\diamond}(u) + o\bigl(\varepsilon^2\bigr).$$

By the assumption that $S_\varphi(T_\varepsilon K) \geq S_\varphi(K)$, it immediately yields

$$\int\limits_{S^{n-1}} \left(\frac{\operatorname{tr} T}{n} - u \cdot T^t u\right) dS_{\varphi^\diamond}(u) \geq 0.$$

Replacing T by $-T$ in the above inequality, we get

$$\int\limits_{S^{n-1}} \left(\frac{\operatorname{tr} T}{n} - u \cdot T^t u\right) dS_{\varphi^\diamond}(u) \leq 0.$$

Hence,

$$\int\limits_{S^{n-1}} u \cdot T^t u dS_{\varphi^\diamond}(K, u) = \frac{\operatorname{tr} T}{n} \bigl|S_{\varphi^\diamond}(K, \cdot)\bigr|.$$

Taking $T = x \otimes x$, with $x \in \mathbb{R}^n \setminus \{0\}$, and using the facts $u \cdot (x \otimes x)^t u = |x \cdot u|^2$ and $\operatorname{tr}(x \otimes x) = |x|^2$, we have

$$\int\limits_{S^{n-1}} (x \cdot u)^2 dS_{\varphi^\diamond}(K, u) = \frac{|S_{\varphi^\diamond}(K, \cdot)|}{n} |x|^2,$$

which shows the isotropy of $S_{\varphi^\diamond}(K, \cdot)$.

Next, we show the implication "(2) $\Longrightarrow$ (1)". The proof will be completed by two steps.

Firstly, for a point $a = (a_1, \cdots, a_n) \in [0, \infty)^n$, let

$$F(a) = \int\limits_{S^{n-1}} \varphi\biggl(\frac{|\operatorname{diag}(a_1, \cdots, a_n) u|}{h_K(u)}\biggr) dV_K(u),$$

where $\operatorname{diag}(a_1, \cdots, a_n)$ denotes the diagonal matrix with diagonal elements $a_1, \cdots, a_n$.

We aim to show that

$$F(a) \geq F(e), \quad \text{whenever } a_1 a_2 \cdots a_n = 1. \tag{4.1}$$

Here, e denotes the point $(1, \cdots, 1)$.

It can be checked that $F : [0,\infty)^n \to [0,\infty)$ is continuous and convex, and $F(\lambda a)$ is strictly increasing in $\lambda \in [0,\infty)$, for any $a \in (0,\infty)^n$. Thus, $F^{-1}([0,F(e)])$ is compact, convex and of non-empty interior. Precisely, it is a convex body.

By the smoothness of φ and that $|\mathrm{diag}(a_1,\cdots,a_n)u|$ is smooth in $(a_1,\cdots,a_n)$ uniformly for $u \in S^{n-1}$, we have

$$\begin{aligned}\frac{\partial}{\partial a_j}\bigg|_{a=e} F(a) &= \int\limits_{S^{n-1}} \frac{\partial}{\partial a_j}\bigg|_{a=e} \varphi\left(\frac{|\mathrm{diag}(a_1,\cdots,a_n)u|}{h_K(u)}\right) dV_K(u) \\ &= \frac{1}{n}\int\limits_{S^{n-1}} u_j^2 dS_{\varphi^\diamond}(K,u).\end{aligned}$$

Meanwhile, since the boundary of $F^{-1}([0,F(e)])$ is given by the equation $F(a) = F(e)$ with $a \in \mathbb{R}_+^n$, so the vector $\int_{S^{n-1}}(u_1^2,\cdots,u_n^2)dS_{\varphi^\diamond}(K,u)$ is an outer normal of $F^{-1}([0,F(e)])$ at the boundary point e. Note that $S_{\varphi^\diamond}(K,\cdot)$ is isotropic, it yields

$$\int\limits_{S^{n-1}} (u_1^2,\cdots,u_n^2)dS_{\varphi^\diamond}(K,u) = \frac{|S_{\varphi^\diamond}(K,\cdot)|}{n}(1,\cdots,1) = \frac{|S_{\varphi^\diamond}(K,\cdot)|}{n}e.$$

Thus, e is an outer normal of $F^{-1}([0,F(e)])$ at the boundary point e. Consequently,

$$F^{-1}\big(\big[0,F(e)\big]\big) \subset \big\{a \in \mathbb{R}^n : a \cdot e \le n\big\}.$$

That is to say, for all $a \in [0,\infty)^n$, if $F(a) \le F(e)$, then $a \cdot e \le n$. In contrast, for all $b = (b_1,\cdots,b_n) \in (0,\infty)^n$ with $b_1 \cdots b_n = 1$, the AM-GM inequality yields that $b \cdot e \ge n$, with equality if and only if $b = e$. Thus, (4.1) is derived.

Secondly, with (4.1) in hand, we aim to show that for all $T \in \mathrm{SL}(n)$, $S_\varphi(TK) \ge S_\varphi(K)$, with equality if and only if T is orthogonal.

It is known that T^{-t} can be represented in the form $T^{-t} = PAQ$, where P and Q are $n \times n$ orthogonal matrices, and $A = \mathrm{diag}(a_1,\cdots,a_n)$ is diagonal and positive definite. So, by Lemma 3.1, (2.2), (4.1), and Lemma 3.1 again, we have

$$\begin{aligned}S_\varphi(TK) &= n\int\limits_{S^{n-1}} \varphi\left(\frac{|Au|}{h_{QK}(u)}\right) dV_{QK}(u) \\ &= n\int\limits_{S^{n-1}} \varphi\left(\frac{|\mathrm{diag}(a_1,\cdots,a_n)u|}{h_{QK}(u)}\right) dV_{QK}(u) \\ &\ge n\int\limits_{S^{n-1}} \varphi\left(\frac{|\mathrm{diag}(1,\cdots,1)u|}{h_{QK}(u)}\right) dV_{QK}(u)\end{aligned}$$

$$
\begin{aligned}
&= n \int_{S^{n-1}} \varphi\left(\frac{1}{h_{QK}(u)}\right) dV_{QK}(u) \\
&= S_\varphi(QK) \\
&= S_\varphi(K),
\end{aligned}
$$

equality holds if and only if $(a_1, \cdots, a_n) = (1, \cdots, 1)$, equivalently, if and only if T is orthogonal. Thus, the implication "(2) $\Longrightarrow$ (1)" is shown.

In the rest, we prove the equivalence of (2) and (3).

From the definitions of $S_{\varphi^\diamond}(T_0K, \cdot)$ and cone-volume measure, (2.1), and (2.2), we have

$$
\begin{aligned}
dS_{\varphi^\diamond}(T_0K, u) &= n\varphi'\left(\frac{1}{h_{T_0K}(u)}\right) \frac{1}{h_{T_0K}(u)} dV_{T_0K}(u) \\
&= n\varphi'\left(\frac{|T_0^{-t}\langle T_0^t u\rangle|}{h_K(\langle T_0^t u\rangle)}\right) \frac{|T_0^{-t}\langle T_0^t u\rangle|}{h_K(\langle T_0^t u\rangle)} dV_K\big(\langle T_0^t u\rangle\big) \\
&= \varphi'\left(\frac{|T_0^{-t}\langle T_0^t u\rangle|}{h_K(\langle T_0^t u\rangle)}\right) \big|T_0^{-t}\langle T_0^t u\rangle\big| dS_K\big(\langle T_0^t u\rangle\big),
\end{aligned}
$$

which immediately yields that

$$
\int_{S^{n-1}} \big|T_0^{-t}v\big| \varphi'\left(\frac{|T_0^{-t}v|}{h_K(v)}\right) dS_K(v) = \big|S_{\varphi^\diamond}(T_0K, \cdot)\big|.
$$

Meanwhile, for $x \in \mathbb{R}^n$ we have

$$
\begin{aligned}
&\int_{S^{n-1}} |x \cdot u|^2 dS_{\varphi^\diamond}(T_0K, u) \\
&\quad = \int_{S^{n-1}} \big|x \cdot T_0^{-t}\langle T_0^t u\rangle\big|^2 \big|T_0^t u\big|^2 \varphi'\left(\frac{|T_0^{-t}\langle T_0^t u\rangle|}{h_K(\langle T_0^t u\rangle)}\right) \big|T_0^{-t}\langle T_0^t u\rangle\big| dS_K\big(\langle T_0^t u\rangle\big) \\
&\quad = \int_{S^{n-1}} \big|x \cdot T_0^{-t}v\big|^2 \big|T_0^{-t}v\big|^{-2} \varphi'\left(\frac{|T_0^{-t}v|}{h_K(v)}\right) \big|T_0^{-t}v\big| dS_K(v) \\
&\quad = \int_{S^{n-1}} \big|x \cdot T_0^{-t}v\big|^2 \big|T_0^{-t}v\big|^{-1} \varphi'\left(\frac{|T_0^{-t}v|}{h_K(v)}\right) dS_K(v).
\end{aligned}
$$

With these, the equivalence of (2) and (3) is shown.

The proof is complete. □

5. Bounds for the minimal Orlicz surface area

In this section, we estimate the minimal Orlicz surface area $A_\varphi(K)$. Lemma 5.1 and Theorem 5.2 give lower bounds. Theorem 5.3 and Theorem 5.7 give upper bounds.

Write $A(K) = \min\{\mathcal{H}^{n-1}(\partial(TK)) : T \in \mathrm{SL}(n)\}$ for the minimal surface area of K. Note that for our purpose, this quantity is a little different from that of Giannopoulos and Papadimitrakis [14].

The next lemma shows the relationship between $A_\varphi(K)$ and $A(K)$, and is needed in the proofs of Theorem 5.2, Lemma 5.4 and Lemma 5.5.

Lemma 5.1. *Suppose $K \in \mathcal{K}_o^n$ and $\varphi \in \Phi$. Then*

$$A_\varphi(K) \geq n|K|\varphi\left(\frac{A(K)}{n|K|}\right). \tag{5.1}$$

If K has an $\mathrm{SL}(n)$ image K' such that: (1) $S_{K'}$ is isotropic; (2) $h_{K'}|_{\mathrm{supp}\, S_{K'}}$, that is, the restriction of $h_{K'}$ to the support set of $S_{K'}$, is constant, then equality holds in (5.1).

Conversely, if φ is strictly convex, then equality holds in (5.1) only if K has an $\mathrm{SL}(n)$ image K' which satisfies (1) and (2).

Proof. For $T \in \mathrm{SL}(n)$, recall that

$$\frac{S_\varphi(TK)}{n|K|} = \int_{S^{n-1}} \varphi\left(\frac{1}{h_{TK}}\right) d\overline{V}_{TK},$$

and

$$\frac{\mathcal{H}^{n-1}(\partial(TK))}{n|K|} = \frac{\int_{S^{n-1}} dS_{TK}}{n|K|} = \int_{S^{n-1}} \frac{1}{h_{TK}} d\overline{V}_{TK}.$$

Since φ is convex and $\overline{V}_{TK}$ is a probability measure, by Jensen's inequality, we have

$$\frac{S_\varphi(TK)}{n|K|} \geq \varphi\left(\int_{S^{n-1}} \frac{1}{h_{TK}} d\overline{V}_{TK}\right) = \varphi\left(\frac{\mathcal{H}^{n-1}(\partial(TK))}{n|K|}\right),$$

which yields (5.1) by the existence of $A_\varphi(K)$ and $A(K)$.

We proceed to prove the equality condition.

On one hand, by Petty's minimal surface area theorem and (1), we have

$$A(K) = \mathcal{H}^{n-1}(\partial K').$$

By Theorem 4.1, (1) and (2), we have

$$A_\varphi(K) = S_\varphi(K') = n|K|\varphi\left(\frac{\mathcal{H}^{n-1}(\partial K')}{n|K|}\right).$$

Thus, $A_\varphi(K) = n|K|\varphi(\frac{A(K)}{n|K|})$.

Conversely, the equality $A_\varphi(K) = n|K|\varphi(\frac{A(K)}{n|K|})$, as well as the existence of $A(K)$ and $A_\varphi(K)$, implies that K has two $\mathrm{SL}(n)$ images K_1 and K_2 which satisfy the following:

(3) $S_\varphi(K_1) = A_\varphi(K)$.
(4) $S_\varphi(K_1) = n|K|\varphi(\frac{\mathcal{H}^{n-1}(\partial K_2)}{n|K|})$.
(5) For all $T \in \mathrm{SL}(n)$, $\mathcal{H}^{n-1}(\partial(TK_2)) \geq \mathcal{H}^{n-1}(\partial K_2)$, with equality if and only if T is orthogonal.

The proved inequality $S_\varphi(K_1) \geq n|K|\varphi(\frac{\mathcal{H}^{n-1}(\partial K_1)}{n|K|})$ together with (4), yields

$$\varphi\left(\frac{\mathcal{H}^{n-1}(\partial K_2)}{n|K|}\right) \geq \varphi\left(\frac{\mathcal{H}^{n-1}(\partial K_1)}{n|K|}\right).$$

Since φ is strictly increasing, we have

$$\mathcal{H}^{n-1}(\partial K_2) \geq \mathcal{H}^{n-1}(\partial K_1).$$

With this and (5), we conclude that K_1 differs from K_2 only by an orthogonal transformation. Thus, by Petty's minimal surface area theorem, we know that S_{K_1} is isotropic on S^{n-1}. Moreover, by the orthogonal invariance of $\mathcal{H}^{n-1}$ and (4), we have

$$S_\varphi(K_1) = n|K|\varphi\left(\frac{\mathcal{H}^{n-1}(\partial K_1)}{n|K|}\right).$$

That is,

$$\int_{S^{n-1}} \varphi\left(\frac{1}{h_{K_1}}\right) d\overline{V}_{K_1} = \varphi\left(\int_{S^{n-1}} \frac{1}{h_{K_1}} d\overline{V}_{K_1}\right).$$

Since $\overline{V}_{K_1}$ is a probability measure and φ is strictly convex, by the equality condition of Jensen's inequality, it follows that $h_{K_1}|_{\operatorname{supp} \overline{V}_{K_1}}$, i.e., $h_{K_1}|_{\operatorname{supp} S_{K_1}}$, is constant. □

Theorem 5.2. *Suppose $K \in \mathcal{K}_o^n$ and $\varphi \in \Phi$. Then*

$$A_\varphi(K) \geq n|K|\varphi\left(\left(\frac{|B|}{|K|}\right)^{\frac{1}{n}}\right). \tag{5.2}$$

If φ is strictly convex, then equality holds in (5.2) if and only if K is an origin-symmetric ellipsoid.

Proof. Applying the classical isoperimetric inequality to $A(K)$, we have

$$\frac{A(K)}{n|K|} \geq \left(\frac{|B|}{|K|}\right)^{\frac{1}{n}},$$

with equality if and only if K is an ellipsoid. With this and (5.1), (5.2) is established.

Suppose φ is strictly convex. Since $K \in \mathcal{K}_o^n$, by Lemma 5.1 and the equality condition in the above inequality, it follows that equality holds in (5.2) if and only if K is an origin-symmetric ellipsoid. □

In what follows, we use the L_∞ John ellipsoid discovered in [34] to estimate the minimal Orlicz surface area $A_\varphi(K)$.

Suppose $K \in \mathcal{K}_o^n$. Recall that the L_∞ *John ellipsoid*, $\mathrm{E}_\infty K$, is the unique origin-symmetric ellipsoid contained in K with maximal volume. That is, amongst all origin-symmetric ellipsoids E, $\mathrm{E}_\infty K$ is the unique one that solves the constrained maximization problem

$$\max_E |E| \quad \text{subject to } \overline{V}_\infty(K, E) \leq 1,$$

where $\overline{V}_\infty(K, E) = \max\{\frac{h_E(u)}{h_K(u)} : u \in \operatorname{supp} S_K\}$. Indeed, $\mathrm{E}_\infty K$ necessarily satisfies

$$\overline{V}_\infty(K, \mathrm{E}_\infty K) = 1.$$

Write $\overline{\mathrm{E}}_\infty K$ for $(\frac{|B|}{|\mathrm{E}_\infty K|})^{1/n}\mathrm{E}_\infty K$. As it was shown in [34], $\overline{\mathrm{E}}_\infty K$ is the unique $\mathrm{SL}(n)$ image of B which satisfies

$$\overline{V}_\infty(K, \overline{\mathrm{E}}_\infty K) = \min\bigl\{\overline{V}_\infty(K, TB) : T \in \mathrm{SL}(n)\bigr\}.$$

It is known that the classical John ellipsoid $\mathrm{J}K$ of K, is the unique ellipsoid contained in K with maximal volume. Note that if the John point of K, i.e., the center of $\mathrm{J}K$, is at the origin, then $\mathrm{E}_\infty K$ is precisely $\mathrm{J}K$. For more information about the John ellipsoid, we refer to [3,13,18,17,22–24].

Theorem 5.3. *Suppose $K \in \mathcal{K}_o^n$ and $\varphi \in \Phi$. Then*

$$A_\varphi(K) \leq n|K|\varphi\left(\left(\frac{|B|}{|\mathrm{E}_\infty K|}\right)^{\frac{1}{n}}\right).$$

Proof. Suppose $T \in \mathrm{SL}(n)$ and $1 \leq p < \infty$. From Lemma 3.1, Jensen's inequality, and the definition of $\overline{V}_\infty$, we have

$$\begin{aligned}
\frac{S_\varphi(TK)}{n|K|} &= \int_{S^{n-1}} \varphi\left(\frac{h_{T^{-1}B}}{h_K}\right) d\overline{V}_K \\
&\le \left(\int_{S^{n-1}} \varphi\left(\frac{h_{T^{-1}B}}{h_K}\right)^p d\overline{V}_K\right)^{1/p} \\
&\le \lim_{p\to\infty} \left(\int_{S^{n-1}} \varphi\left(\frac{h_{T^{-1}B}}{h_K}\right)^p d\overline{V}_K\right)^{1/p} \\
&= \max\left\{\varphi\left(\frac{h_{T^{-1}B}(u)}{h_K(u)}\right) : u \in \operatorname{supp} S_K\right\} \\
&= \varphi\left(\max\left\{\frac{h_{T^{-1}B}(u)}{h_K(u)} : u \in \operatorname{supp} S_K\right\}\right) \\
&= \varphi\big(\overline{V}_\infty\big(K, T^{-1}B\big)\big).
\end{aligned}$$

That is,

$$\frac{S_\varphi(TK)}{n|K|} \le \varphi\big(\overline{V}_\infty\big(K, T^{-1}B\big)\big). \tag{5.3}$$

Now, from (5.3) and the definitions of $A_\varphi(K)$, $\overline{\mathrm{E}}_\infty K$, $\overline{V}_\infty$ and $\mathrm{E}_\infty K$, it follows that

$$\begin{aligned}
\frac{A_\varphi(TK)}{n|K|} &\le \min\big\{\varphi\big(\overline{V}_\infty\big(K, T^{-1}B\big)\big) : T \in \mathrm{SL}(n)\big\} \\
&= \varphi\big(\min\big\{\overline{V}_\infty\big(K, T^{-1}B\big) : T \in \mathrm{SL}(n)\big\}\big) \\
&= \varphi\big(\overline{V}_\infty(K, \overline{\mathrm{E}}_\infty K)\big) \\
&= \varphi\left(\left(\frac{|B|}{|\mathrm{E}_\infty K|}\right)^{1/n} \overline{V}_\infty(K, \mathrm{E}_\infty K)\right) \\
&= \varphi\left(\left(\frac{|B|}{|\mathrm{E}_\infty K|}\right)^{1/n}\right),
\end{aligned}$$

as desired. □

If the John point of K is at the origin, two precise upper bounds for $A_\varphi(K)$ can be obtained.

Lemma 5.4. *Suppose T_n is an n-dimensional regular simplex in $\mathbb{R}^n$. Then*

(1) *The surface area measure S_{T_n} is isotropic on S^{n-1}.*
(2) $A(T_n) = \mathcal{H}^{n-1}(\partial T_n)$.

(3) *If the John point of* T_n *is at the origin, then for all* $\varphi \in \Phi$,

$$A_\varphi(T_n) = S_\varphi(T_n) = n|T_n|\varphi\left(\frac{A(T_n)}{n|T_n|}\right).$$

Proof. Let $e_1, \cdots, e_n$ be the standard orthonormal basis of $\mathbb{R}^n$ and take $\mathbb{R}^{n+1}$ as $\mathbb{R}^n \times \mathbb{R}$. Let $u_1^{(n)}, \cdots, u_{n+1}^{(n)}$ be the outer unit normal vectors of T_n. Without loss of generality, we may assume $u_{n+1}^{(n)} = -e_n$. Then S_{T_n} can be represented by

$$S_{T_n} = \frac{|S_{T_n}|}{n+1} \sum_{j=1}^{n+1} \delta_{u_j^{(n)}},$$

where $\delta_{u_j^{(n)}}$ denotes the Dirac measure at $u_j^{(n)}$.

We prove (1) by induction on the dimension n. If $n = 1$, it is obvious.

Assume (1) holds for n. Then, for all $x_1 \in \mathbb{R}^n$, from the definition of isotropy, we have

$$\sum_{j=1}^{n+1} \left(x_1 \cdot u_j^{(n)}\right)^2 = \frac{n+1}{|S_{T_n}|} \cdot \frac{|S_{T_n}|}{n+1} \sum_{j=1}^{n+1} \left(x_1 \cdot u_j^{(n)}\right)^2 = \frac{n+1}{|S_{T_n}|} \cdot \frac{|S_{T_n}|}{n}|x_1|^2 = \frac{n+1}{n}|x_1|^2.$$

The fact that the centroid of S_{T_n} is at the origin, i.e., $\sum_{j=1}^{n+1} \frac{|S_{T_n}|}{n+1} u_j^{(n)} = 0$, implies that for all $x_1 \in \mathbb{R}^n$,

$$\sum_{j=1}^{n+1} x_1 \cdot u_j^{(n)} = 0.$$

Note that for each $j = 1, \cdots, n+1$,

$$u_j^{(n+1)} = \frac{1}{n+1}\left(\sqrt{n^2+2n}\,u_j^{(n)}, 1\right).$$

Hence, for all $x = (x_1, x_2) \in \mathbb{R}^n \times \mathbb{R}$,

$$\begin{aligned}
\frac{|S_{T_{n+1}}|}{n+2} \sum_{j=1}^{n+2} \left|x \cdot u_j^{(n+1)}\right|^2 &= \frac{|S_{T_{n+1}}|}{n+2}\left(\frac{n^2+2n}{(n+1)^2} \sum_{j=1}^{n+1} \left(x_1 \cdot u_j^{(n)}\right)^2 + \frac{n+2}{n+1}{x_2}^2\right) \\
&= \frac{|S_{T_{n+1}}|}{n+2}\left(\frac{n^2+2n}{(n+1)^2} \cdot \frac{n+1}{n}|x_1|^2 + \frac{n+2}{n+1}x_2^2\right) \\
&= \frac{|S_{T_{n+1}}|}{n+1}|x|^2,
\end{aligned}$$

which shows the isotropy of $S_{T_{n+1}}$, and therefore concludes (1).

Petty's minimal surface area theorem implies that (1) and (2) are equivalent.

If the John point of T_n is at the origin, then T_n satisfies the minimizer conditions given in Lemma 5.1, and therefore (3) follows. □

Lemma 5.5. *Suppose Q_n is an origin-symmetric n-dimensional cube in $\mathbb{R}^n$. Then*

$$A(Q_n) = \mathcal{H}^{n-1}(\partial Q_n),$$

and for all $\varphi \in \Phi$,

$$A_\varphi(Q_n) = S_\varphi(Q_n) = n|Q_n|\varphi\left(\frac{A(Q_n)}{n|Q_n|}\right).$$

Proof. Since S_{Q_n} is isotropic on S^{n-1}, and $h_{Q_n}|_{\operatorname{supp} S_{Q_n}}$ is constant, the desired equalities are obtained by Petty's minimal surface area theorem and Lemma 5.1. □

The following Ball volume-ratio inequalities [2] are needed. The only if part for the equality condition was established by Barthe [4].

Lemma 5.6. *Suppose K is a convex body in $\mathbb{R}^n$, T_n is an n-dimensional simplex, and Q_n is an n-dimensional parallelotope. Then*

$$\frac{|K|}{|\mathrm{J}K|} \le \frac{|T_n|}{|\mathrm{J}T_n|}, \tag{5.4}$$

with equality if and only if K is an affine image of T_n.

If K is centrally symmetric, then

$$\frac{|K|}{|\mathrm{J}K|} \le \frac{|Q_n|}{|\mathrm{J}Q_n|}, \tag{5.5}$$

with equality if and only if K is an affine image of Q_n.

Theorem 5.7. *Suppose $\varphi \in \Phi$, $K \in \mathcal{K}_o^n$, and the John point of K is at the origin. Then*

$$A_\varphi(K) \le n|K|\varphi\left(\left(\frac{n^{n/2}(n+1)^{(n+1)/2}}{n!|K|}\right)^{1/n}\right), \tag{5.6}$$

with equality if and only if K is an n-dimensional simplex.

If K is origin-symmetric, then

$$A_\varphi(K) \le n|K|\varphi\left(\left(\frac{2^n}{|K|}\right)^{\frac{1}{n}}\right), \tag{5.7}$$

with equality if and only if K is an n-dimensional parallelotope.

The theorem is an equivalent statement of Theorem 1.2.

Proof. Since the John point of K is at the origin, it implies that $\mathrm{E}_\infty K = \mathrm{J}K$. Let T_n be an n-dimensional regular simplex with $\mathrm{J}T_n = B$. As an affine invariant inequality, (5.4) can be rewritten as

$$\frac{|B|}{|\mathrm{J}K|} \le \frac{|T_n|}{|K|},$$

with equality if and only if K is an n-dimensional simplex. Since φ is strictly increasing, combining this inequality with Theorem 5.3, we have

$$\frac{A_\varphi(K)}{n|K|} \le \varphi\left(\left(\frac{|B|}{|\mathrm{J}K|}\right)^{\frac{1}{n}}\right) \le \varphi\left(\left(\frac{|T_n|}{|K|}\right)^{\frac{1}{n}}\right). \tag{5.8}$$

Note that $|T_n| = n^{n/2}(n+1)^{(n+1)/2}/n!$. Thus, (5.6) is obtained.

Now, we verify the equality condition in (5.6). Since (5.6), A_φ and volume are all $\mathrm{SL}(n)$ invariant, we may assume $K = \lambda T_n$, $\lambda > 0$. Then, by Lemma 5.4, we have

$$\frac{A_\varphi(K)}{n|K|} = \varphi\left(\frac{n\lambda^{n-1}|T_n|}{n|K|}\right) = \varphi\left(\left(\frac{|T_n|}{|K|}\right)^{\frac{1}{n}}\right),$$

that is, equality holds in (5.6). Conversely, by (5.4), (5.8) and the injectivity of φ, that equality holds in (5.6) necessarily yields the equality

$$\frac{|B|}{|\mathrm{J}K|} = \frac{|T_n|}{|K|},$$

which implies that K is an n-dimensional simplex by Lemma 5.6.

Let $Q_n = [-1,1]^n$. Then $\mathrm{J}Q_n = B$ and $|Q_n| = 2^n$. From Lemma 5.5 and inequality (5.5), using similar arguments to the above, inequality (5.7) and its equality conditions can be derived. □

Closely related to Ball's volume-ratio inequalities are Ball's reverse isoperimetric inequalities [2]. The last corollary shows that there exists an Orlicz version.

Corollary 5.8. *Suppose $\varphi \in \Phi$, T_n is an n-dimensional regular simplex in $\mathbb{R}^n$ with its John point at the origin, and Q_n is an n-dimensional origin-symmetric cube in $\mathbb{R}^n$. Then*

(1) *Each convex body K in $\mathbb{R}^n$ has an affine image $\widetilde{K}$ that satisfies*

$$|\widetilde{K}| = |T_n| \quad \textit{and} \quad S_\varphi(\widetilde{K}) \le S_\varphi(T_n).$$

(2) *Each symmetric convex body K in $\mathbb{R}^n$ has an affine image $\widetilde{K}$ that satisfies*

$$|\widetilde{K}| = |Q_n| \quad \textit{and} \quad S_\varphi(\widetilde{K}) \le S_\varphi(Q_n).$$

Proof. We can always find an affine image K' of K, with its John point at the origin and $|K'| = |T_n|$ (or $|Q_n|$). By Theorem 5.7, it yields that $A_\varphi(K') \le S_\varphi(T_n)$ (or $S_\varphi(Q_n)$). Then, by Theorem 3.3, we know that K' has an $\mathrm{SL}(n)$ image $\widetilde{K}$ satisfying $S_\varphi(\widetilde{K}) = A_\varphi(K')$. Thus, the proof is completed. □

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