OPTIMAL SOBOLEV NORMS AND THE L^p MINKOWSKI PROBLEM

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Dedicated to Professor Rolf Schneider on the occasion of his sixty-fifth birthday

Abstract. The existence and uniqueness of an optimal L^p Sobolev norm for a function on $\mathbf{R}^n$ is shown to be essentially equivalent to the existence and uniqueness of the solution to the L^p Minkowski problem for even measures. The former is established using the latter. This leads to new affine analytic inequalities, as well as a new proof of the affine L^p Sobolev inequality previously established by the authors.

1. Introduction

Throughout this paper $\|\cdot\|$ denotes a norm on $\mathbf{R}^n$ ($n > 1$ is always assumed) that is normalized so that its unit ball has the same volume as the Euclidean unit ball.

A norm $\|\cdot\|$ on $\mathbf{R}^n$ induces a Sobolev norm for compactly supported functions $f : \mathbf{R}^n \to \mathbf{R}$ with L^1 weak derivative, given by

$$f \mapsto \|\nabla f\|_1 = \int_{\mathbf{R}^n} \|\nabla f(x)\|_* \, dx,$$

where $\|\cdot\|_*$ denotes the norm dual to $\|\cdot\|$ (see §2 for precise definitions).

Cordero, Nazaret and Villani [7] recently used a beautiful mass transportation argument to establish the following family of sharp Gagliardo-Nirenberg inequalities for this Sobolev norm. If $\|\cdot\|$ is a norm on $\mathbf{R}^n$ and $f : \mathbf{R}^n \to \mathbf{R}$ is compactly supported and smooth, then

$$\|\nabla f\|_1 \geq c_{1,r,n} |f|_1^{1-\alpha} |f|_r^{\alpha}, \tag{1}$$

where $0 < r \leq n/(n-1)$, $\alpha \in \mathbf{R}$ is determined by scale invariance, and $|f|_r$ denotes the standard L^r-norm of f. Their work extends earlier results of Maz′ja [26], Gromov [11], Alvino, Ferone, Trombetti, Lions [1], and Del Pino and Dolbeault [8].

The CNV inequality immediately raises the obvious question:

The optimal L^1 Sobolev norm. *Given a function $f : \mathbf{R}^n \to \mathbf{R}$ with L^1 weak derivative, what is the unique norm $\|\cdot\|$ on $\mathbf{R}^n$ that minimizes $\|\nabla f\|_1$?*

An apparently unrelated question is the following:

The even Minkowski problem. *Given a positive even function g on the unit sphere S^{n-1}, what is the unique convex body K such that for each unit vector u, $g(u)$ is the Gauss curvature at the point on the boundary ∂K that has outer unit normal u?*

One aim of this note is to show that the two questions stated above are essentially equivalent. We will in fact consider L^p-generalizations of these questions. This can be stated as follows (see §2 for precise definitions).

If $1 \le p < n$, a norm $\|\cdot\|$ on $\mathbf{R}^n$ induces a Sobolev norm for compactly supported functions $f : \mathbf{R}^n \to \mathbf{R}$ with L^p weak derivative, given by

$$f \mapsto \|\nabla f\|_p = \left(\int_{\mathbf{R}^n} \|\nabla f(x)\|_*^p \, dx \right)^{\frac{1}{p}}, \tag{2}$$

where $\|\cdot\|_*$ denotes the dual norm.

Cordero, Nazaret and Villani [7] extended earlier results of Aubin [2], Talenti [31], Gromov [11], Alvino, Ferone, Trombetti, Lions [1], and Del Pino and Dolbeault [8] and established the following family of sharp L^p Gagliardo-Nirenberg inequalities (throughout this paper they will be called the *CNV inequalities*): If $1 \le p < n$, $\|\cdot\|$ is a norm on $\mathbf{R}^n$, and $f : \mathbf{R}^n \to \mathbf{R}$ is compactly supported and smooth, then

$$\|\nabla f\|_p \ge c_{p,r,n} |f|_q^{1-\alpha} |f|_r^{\alpha}, \tag{3}$$

where $0 < r \le np/(n-p)$, $q = 1 + r(p-1)/p$, $|f|_p$ denotes the standard L^p-norm of f, and $\alpha \in \mathbf{R}$ is determined by scale invariance.

This leads us to ask the following for every $p \ge 1$ (and not just $1 \le p < n$):

The optimal L^p Sobolev norm. *Given a function $f : \mathbf{R}^n \to \mathbf{R}$ with L^p weak derivative, what is the unique norm $\|\cdot\|$ on $\mathbf{R}^n$ that minimizes $\|\nabla f\|_p$?*

In this paper we show that this problem is essentially the same as the apparently unrelated even L^p Minkowski problem. The L^p Minkowski problem, which can be written as a Monge-Ampère equation

$$h^{1-p} \det(h_{ij} + h\delta_{ij}) = g$$

on the unit sphere, is a central question in the L^p Brunn-Minkowski theory of convex bodies. See §3 for more.

A consequence of Theorem 6 in this paper is that all possible L^p Sobolev norms of a function $f : \mathbf{R}^n \to \mathbf{R}$ can be encoded naturally within a single origin-symmetric convex body K. In particular, for each norm on $\mathbf{R}^n$, the corresponding L^p Sobolev norm of f is given by the (normalized) L^p mixed volume of K and the unit ball of the norm. See Theorem 6 and the remark immediately following it for details.

Moreover, the (suitably normalized) volume of this convex body is precisely equal to the optimal L^p Sobolev norm of f. We show in §6 that minimizing the left side of the CNV inequality (3) over all norms on $\mathbf{R}^n$ establishes an affine version of the Cordero-Nazaret-Villani inequalities.

Zhang [34] and the authors [23] recently established a sharp L^p affine Sobolev inequality (a version of the L^1 affine Sobolev involving capacity was recently established by Xiao [33]). The proof in [23] is rather involved, using the L^p Petty projection inequality established by the authors [21] and a rearrangement argument, where the solution to the even L^p Minkowski problem is applied to each level set of a function. A less circuitous proof also using the L^p Petty projection inequality, as well as the optimal L^p Sobolev norm and the CNV inequality (3), is presented in §7.

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2. Preliminaries

Throughout this paper, $u \cdot x$ denotes the standard inner product of $u, x \in \mathbf{R}^n$, and $|\cdot|$ denotes the standard Euclidean norm on $\mathbf{R}^n$. For $1 \le p < \infty$ and a measurable function $f : \mathbf{R}^n \to \mathbf{R}$, let $|f|_p$ denote the L^p norm of f and $L^p(\mathbf{R}^n)$ the corresponding space of L^p-bounded functions on $\mathbf{R}^n$.

An L^p_{loc} function $f : \mathbf{R}^n \to \mathbf{R}$ *has L^p weak derivative,* if there exists a measurable function $\nabla f : \mathbf{R}^n \to \mathbf{R}^n$ such that $|\nabla f| \in L^p(\mathbf{R}^n)$ and

$$\int_{\mathbf{R}^n} v(x) \cdot \nabla f(x)\, dx = -\int_{\mathbf{R}^n} f(x) \nabla \cdot v(x)\, dx,$$

for every compactly supported smooth vector field $v : \mathbf{R}^n \to \mathbf{R}^n$. The function ∇f is called the *weak gradient* of f, and the L^p norm of $|\nabla f|$ is denoted by $|\nabla f|_p$.

The norm dual to $\|\cdot\|$ is denoted by $\|\cdot\|_*$, where

$$\|u\|_* = \sup\{(u \cdot x)/\|x\| \ : \ x \in \mathbf{R}^n \backslash \{0\}\},$$

for each $u \in \mathbf{R}^n$. Given a function $f : \mathbf{R}^n \to \mathbf{R}$ with L^p weak derivative, we denote

$$\|\nabla f\|_p = \left(\int_{\mathbf{R}^n} \|\nabla f(x)\|_*^p \, dx \right)^{1/p}. \tag{4}$$

We will call this an L^p *Sobolev norm of* f, even though it is only a semi-norm.

Throughout this paper, a *convex body* is always assumed to be an origin-symmetric compact convex set in $\mathbf{R}^n$ with nonempty interior. A measure is always assumed to be a positive finite Borel measure.

The volume (i.e., Lebesgue measure) of a convex body K will be denoted by $V(K)$. A convex body K defines a norm $|\cdot|_K$ on $\mathbf{R}^n$ given by

$$|x|_K = \inf\{t > 0 \ : \ x/t \in K\}$$

for each $x \in \mathbf{R}^n$. The polar body K^* of K is defined by

$$K^* = \{u \in \mathbf{R}^n \ : \ u \cdot x \le 1 \text{ for each } x \in K\}.$$

Note that $|\cdot|_{K^*}$ is the norm dual to $|\cdot|_K$ and also the support function of K. The boundary of K will be denoted ∂K.

The standard unit ball in $\mathbf{R}^n$ will be denoted B and its volume ω_n.

3. The L^p Minkowski problem

We begin by recalling basics that we need from the Brunn-Minkowski theory of convex bodies and its L^p extension. (See Schneider [29] for details regarding the classical Brunn-Minkowski theory.)

If K, L are convex bodies and $0 < t < \infty$, then the Minkowski combination $K + tL$ is defined by

$$|\cdot|_{(K+tL)^*} = |\cdot|_{K^*} + t|\cdot|_{L^*}.$$

As an aside, note that $K + tL = \{x + ty \ : \ x \in K, \ y \in L\}$.

The *mixed volume* $V_1(K, L)$ *of* K *and* L is defined by

$$V_1(K, L) = \frac{1}{n} \lim_{t \to 0+} \frac{V(K + tL) - V(K)}{t}.$$

A fundamental fact is that corresponding to each convex body K is a unique Borel measure $S(K, \cdot)$ on the unit sphere S^{n-1} such that

$$V_1(K, L) = \frac{1}{n} \int_{S^{n-1}} |u|_{L^*} \, dS(K, u),$$

for each convex body L. The measure $S(K, \cdot)$ is called the *surface area measure* of K.

Let $h = |\cdot|_{K^*}$ denote the support function of K, and $h^* = |\cdot|_K$ the support function of K^*. Note that

$$h^*(x) = 1, \tag{5}$$

for each $x \in \partial K$. Recall that the Gauss map assigns to each point of the boundary of a sufficiently smooth convex body in $\mathbf{R}^n$ its outer unit normal. Since h^* is a convex function (and therefore differentiable almost everywhere) and constant along the boundary of K, the Gauss map $\gamma : \partial K \to S^{n-1}$ can be defined almost everywhere on ∂K by

$$\gamma = \nabla h^* / |\nabla h^*|. \tag{6}$$

It follows from the definition of the dual norm that $h(\nabla h^*(x)) = 1$, for almost every $x \in \mathbf{R}^n$. This and the homogeneity (of degree 1) of h, give

$$h(\gamma(x)) = 1/|\nabla h^*(x)|. \tag{7}$$

Let $\sigma(\partial K, \cdot)$ be the $(n-1)$-dimensional volume measure induced on ∂K by the standard Euclidean structure on $\mathbf{R}^n$. It turns out that the surface area measure is given by

$$S(K, \cdot) = \gamma_* \sigma(\partial K, \cdot), \tag{8}$$

where γ_* denotes the pushforward induced by the Gauss map γ. If the boundary ∂K is strictly convex and smooth, then

$$S(K, \cdot) = du/\kappa(\gamma^{-1}(u)),$$

where du is the standard Lebesgue measure on S^{n-1}, and $\kappa : \partial K \to \mathbf{R}$ is the Gauss curvature of the hypersurface ∂K.

Recall that a measure μ on the unit sphere S^{n-1} is said to be *even,* if it assumes the same values on antipodal Borel sets. The even Minkowski problem can be stated as follows: Given an even Borel measure μ on the unit sphere S^{n-1}, does there exist a convex body K whose surface area measure is μ? Or, equivalently, does there exist a convex body K such that

$$V_1(K, L) = \frac{1}{n}\int_{S^{n-1}} |u|_{L^*}\, d\mu(u),$$

for each convex body L?

Lutwak [18] showed how elements of the classical Brunn-Minkowski theory can be extended to a more general L^p Brunn-Minkowski theory (see, e.g., [5,6,12–14,16–25,27,28,30]) by using L^p Minkowski sums first introduced by Firey. The essential details are reviewed below.

Suppose $1 \le p < \infty$. If K, L are convex bodies, and $0 < t < \infty$, then the L^p *Minkowski-Firey combination* $K +_p tL$ is defined by

$$|\cdot|^p_{(K+_p tL)^*} = |\cdot|^p_{K^*} + t^p |\cdot|^p_{L^*}.$$

The L^p *mixed volume of* K *and* L, is defined by

$$V_p(K, L) = \frac{p}{n}\lim_{t\to 0}\frac{V(K +_p t^{\frac{1}{p}} L) - V(K)}{t},$$

and can be viewed as an L^p surface area of ∂K with respect to the geometric structure induced by the norm $|\cdot|_L$. It generalizes the Euclidean surface area of K, which is given by $nV_1(K, B)$, where B is the standard unit ball in $\mathbf{R}^n$. Note that

$$V_p(K, K) = V(K). \tag{9}$$

A fundamental inequality that we need is the following special case of the L^p *Minkowski inequality* [18].

Lemma 1. *If $1 \le p < \infty$ and K, L are origin-symmetric convex bodies in $\mathbf{R}^n$, then*

$$V_p(K, L) \ge V(K)^{1-\frac{p}{n}} V(L)^{\frac{p}{n}}. \tag{10}$$

Equality holds if and only if $L = tK$ for some $t > 0$.

This inequality generalizes the classical isoperimetric inequality, where $p = 1$ and $L = B$. The following is a well-known and useful consequence.

Lemma 2. *If K and L are convex bodies such that*

$$\frac{V_p(K, Q)}{V(K)} = \frac{V_p(L, Q)}{V(L)}$$

for each convex body Q, then $K = L$.

Proof. Setting $Q = K$ gives, by (9) and Lemma 1,

$$1 = \frac{V_p(K, K)}{V(K)} = \frac{V_p(L, K)}{V(L)} \ge \left(\frac{V(K)}{V(L)}\right)^{\frac{p}{n}}.$$

This gives $V(K) \le V(L)$; setting $Q = L$ gives the reverse inequality. From the equality conditions of Lemma 1, L is a dilate of K. Since $V(K) = V(L)$, the bodies must be the same. □

It was shown in [18] that corresponding to each convex body K, is a unique Borel measure $S_p(K,\cdot)$ on the unit sphere S^{n-1} such that

$$V_p(K,L) = \frac{1}{n}\int_{S^{n-1}} |u|_{L^*}^p \, dS_p(K,u), \tag{11}$$

for each convex body L. The measure $S_p(K,\cdot)$ is called the L^p *surface area measure* of K. One easily observes that for every $t > 0$,

$$S_p(tK,\cdot) = t^{n-p} S_p(K,\cdot). \tag{12}$$

It was also shown in [18] that the L^p surface area measure $S_p(K,\cdot)$ is absolutely continuous with respect to $S(K,\cdot) = S_1(K,\cdot)$, and that for the Radon-Nikodym derivative we have

$$\frac{dS_p(K,\cdot)}{dS(K,\cdot)} = h^{1-p}, \tag{13}$$

where $h = |\cdot|_{K^*}$ is the support function of K.

The even L^p Minkowski problem. *Given an even Borel measure μ on S^{n-1}, does there exist a convex body K such that $\mu = S_p(K,\cdot)$? Or, equivalently, does there exist a convex body K such that*

$$V_p(K,L) = \frac{1}{n}\int_{S^{n-1}} |u|_{L^*}^p \, d\mu(u),$$

for each convex body L?

Lutwak [18] gave an affirmative answer to this problem when $p \neq n$. The authors [24] introduced the *volume-normalized L^p Minkowski problem*, for which the case $p = n$ can be handled as well (The volume-normalized L^1 Minkowski problem was used earlier by Ball [3] to construct convex bodies with large shadow areas in all directions). See [5, 6, 12, 14] for recent progress on the L^p Minkowski problem when the given measure is not assumed to be even.

In particular, the authors solved the even case of the volume-normalized L^p Minkowski problem and proved the following in [24].

Theorem 3. *If $1 \le p < \infty$ and μ is an even Borel measure on the unit sphere S^{n-1}, then there exists a unique origin-symmetric convex body $\bar{K}$ such that*

$$\frac{S_p(\bar{K},\cdot)}{V(\bar{K})} = \mu$$

if and only if the support of μ is not contained in any $(n-1)$-dimensional linear subspace.

If $p \neq n$, Theorem 3 is equivalent to the solution to the even L^p Minkowski problem. Given a measure μ satisfying the assumptions of Theorem 3, then it follows from (12) that the unique solution to the even L^p Minkowski problem is obtained by letting

$$K = V(\bar{K})^{\frac{1}{p-n}} \bar{K},$$

where $\bar{K}$ is the origin-symmetric convex body given by Theorem 3.

4. The functional L^p Minkowski problem

We begin by defining the L^p surface area measure of a Sobolev function.

Lemma 4. *Given $1 \leq p < \infty$ and a function $f : \mathbf{R}^n \to \mathbf{R}$ with L^p weak derivative, there exists a unique finite Borel measure $S_p(f, \cdot)$ on S^{n-1} such that*

$$\int_{\mathbf{R}^n} \varphi(-\nabla f(x))^p \, dx = \int_{S^{n-1}} \varphi(u)^p \, dS_p(f, u), \tag{14}$$

for every nonnegative continuous function $\varphi : \mathbf{R}^n \to \mathbf{R}$ homogeneous of degree 1. If f is not equal to a constant function almost everywhere, then the support of $S_p(f, \cdot)$ cannot be contained in any $(n-1)$-dimensional linear subspace.

We call the measure $S_p(f, \cdot)$ given by the lemma above the L^p *surface area measure of the function* f.

Proof. Let $\Sigma = \{x \;:\; \nabla f(x) = 0\}$. Since

$$\psi \mapsto \int_{\mathbf{R}^n \setminus \Sigma} \psi(-\nabla f(x)/|\nabla f(x)|)|\nabla f(x)|^p \, dx, \tag{15}$$

defines a nonnnegative bounded linear functional on the space of continuous functions on S^{n-1}, it follows by the Riesz representation theorem that there exists a unique Borel measure $S_p(f, \cdot)$ on S^{n-1} such that

$$\int_{\mathbf{R}^n \setminus \Sigma} \psi(-\nabla f(x)/|\nabla f(x)|)|\nabla f(x)|^p \, dx = \int_{S^{n-1}} \psi(u) \, dS_p(f, u), \tag{16}$$

for each continuous function $\psi : S^{n-1} \to \mathbf{R}$.

If $\varphi : \mathbf{R}^n \to [0, \infty)$ is continuous and homogeneous of degree 1, then $\varphi(-\nabla f(x)) = 0$, for each $x \in \Sigma$. This, the homogeneity of φ, and (16) with $\psi = \varphi^p$ (restricted to the unit sphere) give

$$\begin{aligned} \int_{\mathbf{R}^n} \varphi(-\nabla f(x))^p \, dx &= \int_{\mathbf{R}^n \setminus \Sigma} \varphi(-\nabla f(x))^p \, dx \\ &= \int_{\mathbf{R}^n \setminus \Sigma} \varphi(-\nabla f(x)/|\nabla f(x)|)^p |\nabla f(x)|^p \, dx \\ &= \int_{S^{n-1}} \varphi(u)^p \, dS_p(f, u). \end{aligned} \tag{17}$$

Thus, the measure $S_p(f, \cdot)$ satisfies (14) for each nonnegative continuous function $\varphi : \mathbf{R}^n \to \mathbf{R}$ homogeneous of degree 1. The uniqueness of $S_p(f, \cdot)$ follows by observing that any measure $S_p(f, \cdot)$ on S^{n-1} that satisfies (14) defines the same linear functional as given by (15).

If the support of $S_p(f, \cdot)$ is contained in $H \cap S^{n-1}$, where, say, $H = \{x \in \mathbf{R}^n \;:\; x_n = 0\}$, then by (14),

$$\begin{aligned} 0 &= \int_{S^{n-1}} |u_n|^p \, dS_p(f, u) \\ &= \int_{\mathbf{R}^n} |\partial_n f(x)|^p \, dx. \end{aligned}$$

It follows that $\partial_n f = 0$ almost everywhere, and therefore, for each $i = 1, \dots, n$ and compactly supported smooth function χ,

$$\partial_n \partial_i (f \star \chi) = 0, \tag{18}$$

where $f \star \chi$ denotes the convolution of f and χ. Since $\partial_i f \in L^p(\mathbf{R}^n)$,

$$\partial_i (f \star \chi) \in L^p(\mathbf{R}^n), \tag{19}$$

and since $f \star \chi$ is smooth, it follows by (18), (19), and the mean value theorem that $\partial_i(f \star \chi)$ is identically zero and $f \star \chi$ is constant. This holds for every compactly supported smooth function χ, and therefore the function f must be constant almost everywhere. □

This leads naturally to the following.

The functional L^p Minkowski problem. *Given a Borel measure μ on S^{n-1}, does there exist a function $f : \mathbf{R}^n \to \mathbf{R}$ with L^p weak derivative such that $S_p(f, \cdot) = \mu$?*

We answer this question for even measures by using the solution to the normalized even L^p Minkowski problem.

Theorem 5. *If $1 \le p < \infty$ and μ is an even Borel measure on S^{n-1} whose support is not contained in any $(n-1)$-dimensional linear subspace, then there exists a function $f : \mathbf{R}^n \to \mathbf{R}$ with L^p weak derivative such that $S_p(f, \cdot) = \mu$.*

Proof. By Theorem 3, there exists an origin-symmetric convex body K such that

$$\frac{S_p(K, \cdot)}{V(K)} = \mu. \tag{20}$$

Let $\chi : [0, \infty) \to [0, \infty)$ be a smooth decreasing compactly ssuch that

$$\int_0^\infty (-\chi'(t))^p t^{n-1}\, dt = \frac{1}{V(K)}, \tag{21}$$

and define $f : \mathbf{R}^n \to \mathbf{R}$ by

$$f(x) = \chi(h^*(x)),$$

where, as before, $h^* = |\cdot|_K$ and $h = |\cdot|_{K^*}$. Since f is compactly supported, χ is smooth, and h^* is Lipschitz, it follows that the function f has weak L^p derivative.

Observe that the $(n-1)$-dimensional volume measure on $(h^*)^{-1}(t) = t\partial K$ induced by the standard Euclidean structure on $\mathbf{R}^n$ is given by

$$\sigma(t\partial K, t\omega) = t^{n-1} \sigma(\partial K, \omega),$$

for each $t > 0$ and Borel set $\omega \subset \partial K$. Therefore, by the coarea formula (see, for example, Federer [9]) and the observation that ∇h^* is homogeneous of degree 0,

$$\begin{aligned} \int_{\mathbf{R}^n} F(x) |\nabla h^*(x)|^p \, dx &= \int_0^\infty \int_{(h^*)^{-1}(t)} F(x) |\nabla h^*(x)|^{p-1} \, d\sigma(t\partial K, x)\, dt \\ &= \int_0^\infty \int_{\partial K} F(tv) |\nabla h^*(v)|^{p-1} t^{n-1} \, d\sigma(\partial K, v)\, dt, \end{aligned} \tag{22}$$

for every $F \in L^1(\mathbf{R}^n)$.

Let $\varphi : \mathbf{R}^n \to [0,\infty)$ be continuous and homogeneous of degree 1. By the chain rule, (22) and the homogeneity of φ, (5), (6) and (7), (21) and (8), and (13) and (20),

$$\begin{aligned}\int_{\mathbf{R}^n} \varphi(-\nabla f(x))^p\,dx &= \int_{\mathbf{R}^n} \varphi(-\chi'(h^*(x))\nabla h^*(x))^p\,dx\\ &= \int_0^\infty \int_{\partial K} t^{n-1}(-\chi'(t))^p \varphi(\nabla h^*(v)/|\nabla h^*(v)|)^p |\nabla h^*(v)|^{p-1}\,d\sigma(\partial K, v)\,dt\\ &= \int_0^\infty (-\chi'(t))^p t^{n-1}\,dt \int_{\partial K} \varphi(\gamma(v))^p h(\gamma(v))^{1-p}\,d\sigma(\partial K, v)\\ &= \frac{1}{V(K)} \int_{S^{n-1}} \varphi(u)^p h(u)^{1-p}\,dS(K,u)\\ &= \int_{S^{n-1}} \varphi(u)^p \frac{dS_p(K,u)}{V(K)}\\ &= \int_{S^{n-1}} \varphi(u)^p\,d\mu(u).\end{aligned}$$

□

5. Existence and uniqueness of an optimal Sobolev norm

We use the even L^p Minkowski problem to show that for each function $f : \mathbf{R}^n \to \mathbf{R}$ with L^p weak derivative, there is a unique origin-symmetric convex body K whose L^p surface area measure is equal to the L^p surface area measure of f and that a dilate of this body is the unit ball for the optimal L^p Sobolev norm of f. This construction establishes a fundamental connection between functions on $\mathbf{R}^n$ and convex bodies in $\mathbf{R}^n$.

Theorem 6. *If $1 \le p < \infty$ and $f : \mathbf{R}^n \to \mathbf{R}$ has L^p weak derivative, then there exists a unique origin-symmetric convex body $K = \mathrm{K}_p f$ such that*

$$\int_{\mathbf{R}^n} \varphi(-\nabla f(x))^p\,dx = \frac{1}{V(K)} \int_{S^{n-1}} \varphi(u)^p\,dS_p(K,u), \tag{23}$$

for every even continuous function $\varphi : \mathbf{R}^n \to [0,\infty)$ that is homogeneous of degree 1.

A consequence of this theorem is that the L^p Sobolev norm of f is equal to a suitably normalized L^p mixed volume of the convex body $\mathrm{K}_p f$ and the unit ball of the norm used to define the Sobolev norm. Specifically, if we set $\varphi = |\cdot|_{Q^*}$, then it follows by (23) and (11) that

$$\frac{1}{n}\int_{\mathbf{R}^n} |\nabla f(x)|_{Q^*}^p\,dx = \frac{V_p(K,Q)}{V(K)}, \tag{24}$$

for each origin-symmetric convex body Q.

A still elusive complete solution to the L^p Minkowski problem should provide necessary and sufficient conditions on a function f to guarantee the existence of a not necessarily origin-symmetric convex body K for which (23) holds for all continuous nonnegative functions φ that are homogeneous of degree 1 (but not necessarily even).

Proof. Let μ be the even part of the measure $S_p(f,\cdot)$ on S^{n-1}. By Theorem 3, there exists an origin-symmetric convex body K such that

$$\frac{S_p(K,\cdot)}{V(K)} = \mu.$$

If $\varphi : \mathbf{R}^n \to [0,\infty)$ is an even continuous function that is homogeneous of degree 1, then it follows by Lemma 4 and (20) that

$$\begin{aligned}\int_{\mathbf{R}^n} \varphi(-\nabla f(x))^p\,dx &= \int_{S^{n-1}} \varphi(u)^p\,dS_p(f,u)\\ &= \int_{S^{n-1}} \varphi(u)^p\,d\mu(u)\\ &= \frac{1}{V(K)}\int_{S^{n-1}} \varphi(u)^p\,dS_p(K,u).\end{aligned}$$

If K_1 and K_2 are both origin-symmetric convex bodies satisfying (23), then by (24) and Lemma 2, $K_1 = K_2$. □

Corollary 7. *Suppose $1 \le p < \infty$. If $f : \mathbf{R}^n \to \mathbf{R}$ has L^p weak derivative, then there is, among all norms on $\mathbf{R}^n$ whose unit ball has the same volume as the Euclidean unit ball, a unique norm $\|\cdot\|$ that minimizes $\|\nabla f\|_p$. That norm is given by*

$$\|\cdot\| = \left(\frac{V(K)}{\omega_n}\right)^{\frac{1}{n}} |\cdot|_K, \tag{25}$$

where $K = \mathrm{K}_p f$. Moreover,

$$\|\nabla f\|_p = n^{1/p}\omega_n^{1/n} V(\mathrm{K}_p f)^{-1/n}. \tag{26}$$

Proof. Note that (25) is equivalent to

$$\|\cdot\|_* = \left(\frac{\omega_n}{V(K)}\right)^{\frac{1}{n}} |\cdot|_{K^*}. \tag{27}$$

For each norm $|\cdot|_L$ such that $V(L) = \omega_n$, it follows by (24), the L^p Minkowski inequality (10), (9), (24) again, and (27) that

$$\begin{aligned}\int_{\mathbf{R}^n} |\nabla f(x)|_{L^*}^p\,dx &= n\frac{V_p(K,L)}{V(K)}\\ &\ge nV(K)^{-\frac{p}{n}}\omega_n^{\frac{p}{n}}\\ &= n\left(\frac{\omega_n}{V(K)}\right)^{\frac{p}{n}} \frac{V_p(K,K)}{V(K)}\\ &= \left(\frac{\omega_n}{V(K)}\right)^{\frac{p}{n}} \int_{\mathbf{R}^n} |\nabla f(x)|_{K^*}^p\,dx\\ &= \int_{\mathbf{R}^n} \|\nabla f(x)\|_*^p\,dx.\end{aligned} \tag{28}$$

Uniqueness of the norm $\|\cdot\|$ follows from the equality condition of the L^p Minkowski inequality (10).

Note that equation (26) is contained in the last four lines of (28). □

Theorems 5 and 3 imply the following converse to Theorem 6.

Proposition 8. *Suppose* $1 \le p < \infty$. *If* K *is an origin-symmetric convex body, then there exists a function* $f : \mathbf{R}^n \to \mathbf{R}$ *with weak* L^p *derivative such that* $\mathrm{K}_p f = K$.

The convex body $\mathrm{K}_p f$ encodes the geometry of the level sets of f. In particular, if all of the level sets are dilates of an origin-symmetric convex body K, then $\mathrm{K}_p f$ is a dilate of K.

The following proposition describes how $\mathrm{K}_p f$ behaves if f is composed with an invertible linear transformation.

Proposition 9. *Suppose* $1 \le p < \infty$. *If* $f : \mathbf{R}^n \to \mathbf{R}$ *has* L^p *weak derivative, and* $\phi \in \mathrm{SL}(n)$, *then*

$$\mathrm{K}_p(f \circ \phi^{-1}) = \phi(\mathrm{K}_p f).$$

Proof. Using the definitions of the L^p Minkowski-Firey combination and L^p mixed volume, it is straightforward to verify the well-known fact that for every pair of convex bodies K and L,

$$\frac{V_p(\phi K, L)}{V(\phi K)} = \frac{V_p(K, \phi^{-1} L)}{V(K)}. \tag{29}$$

Using the identity $|\phi^{-t} \cdot|_{L^*} = |\cdot|_{(\phi^{-1}L)^*}$, where ϕ^{-t} denotes the inverse transpose of ϕ, and making the change of variables $y = \phi(x)$ gives

$$\int_{\mathbf{R}^n} |\nabla(f \circ \phi^{-1})(y)|_{L^*}^p \, dy = \int_{\mathbf{R}^n} |\nabla f(x)|_{(\phi^{-1}L)^*}^p \, dx. \tag{30}$$

Let $K = \mathrm{K}_p f$ and $K_\phi = \mathrm{K}_p(f \circ \phi^{-1})$. By (11), (24), (30), (24) again, and (29),

$$\frac{V_p(K_\phi, L)}{V(K_\phi)} = \frac{V_p(\phi K, L)}{V(\phi K)},$$

for each convex body L. The proposition now follows by Lemma 2. □

It is easily verified that if $f : \mathbf{R}^n \to \mathbf{R}$ has weak L^p derivative and $g : \mathbf{R}^n \to \mathbf{R}$ is given by

$$g(x) = t f(cx + y),$$

for each $x \in \mathbf{R}^n$, where $t, c > 0$ and $y \in \mathbf{R}^n$, then $\mathrm{K}_p g = t^{-1} c^{n/p-1} \mathrm{K}_p f$. Combining this with Proposition 9 gives

$$\mathrm{K}_p(tf \circ \Phi^{-1}) = t^{-1} |\phi|^{-1/p} \phi(\mathrm{K}_p f), \tag{31}$$

for each $t > 0$ and invertible affine transformation $\Phi : \mathbf{R}^n \to \mathbf{R}^n$ given by

$$\Phi(x) = \phi(x) + y,$$

where $y \in \mathbf{R}^n$, $\phi \in \mathrm{GL}(n)$, and $|\phi|$ denotes the absolute value of the determinant of ϕ.

6. Sharp affine inequalities

Let $\mathrm{Aff}(n)$ denote the group of invertible affine transformations of $\mathbf{R}^n$. There is a natural left action of $\mathbf{R}\backslash\{0\} \times \mathrm{Aff}(n)$ on functions $f : \mathbf{R}^n \to \mathbf{R}$, given by $f \mapsto tf \circ \Phi^{-1}$, for each $(t, \Phi) \in \mathbf{R}\backslash\{0\} \times \mathrm{Aff}(n)$. An *affine inequality* for a class of functions $f : \mathbf{R}^n \to \mathbf{R}$ is an inequality $L[f] \leq R[f]$, where L and R are functionals such that

$$\frac{L[tf \circ \Phi]}{R[tf \circ \Phi]} = \frac{L[f]}{R[f]},$$

for each $(t, \Phi) \in \mathbf{R}\backslash\{0\} \times \mathrm{Aff}(n)$. The inequality is *sharp* if there exists a function f for which equality holds, and such a function is called an *extremal* function for the inequality. If f is extremal, then so is $tf \circ \Phi$. In other words, the set of extremal functions is invariant under the left action of $\mathbf{R}\backslash\{0\} \times \mathrm{Aff}(n)$.

Corollary 7 can be used to establish sharp affine Sobolev inequalities. For example, it leads to a family of sharp affine inequalities, stated below in Theorem 10, that extend the Cordero-Nazaret-Villani inequalities.

For $x \in \mathbf{R}$, denote $x_+ = \max\{x, 0\}$. If $1 < p < n$, and $r \in (0, np/(n-p)]$, define $w : [0,\infty) \to [0,\infty)$ by

$$w(t) = \begin{cases} (1 + (r-p)t^{\frac{p}{p-1}})_+^{\frac{p}{p-r}} & \text{if } r \neq p \\ \exp(-pt^{p/(p-1)}) & \text{if } r = p. \end{cases} \tag{32}$$

For $p = 1$, let

$$w(t) = \begin{cases} 1 & \text{if } t \leq 1 \\ 0 & \text{if } t > 1. \end{cases} \tag{33}$$

Let

$$W(x) = w(|x|).$$

Let $q, \alpha, c_{p,r,n} \in \mathbf{R}$ satisfy

$$\begin{aligned} q &= \left(1 - \frac{1}{p}\right) r + 1 \\ \frac{1-\alpha}{q} + \frac{\alpha}{r} &= \frac{1}{p} - \frac{1}{n} \\ c_{p,r,n} &= \frac{|\nabla W|_p}{|W|_q^{1-\alpha} |W|_r^{\alpha}}. \end{aligned} \tag{34}$$

By Corollary 7, there is a norm $\|\cdot\|$ such that the volume of $\mathrm{K}_p f$ suitably normalized is equal to $\|\nabla f\|_p$. By this and the CNV inequalities (3), we get the following:

Theorem 10. *Suppose* $1 \leq p < n$ *and* $0 < r \leq np/(n-p)$. *If* $f \in L^q(\mathbf{R}^n) \cap L^r(\mathbf{R}^n)$ *has* L^p *weak derivative, then* f *satisfies the sharp affine inequality*

$$V(\mathrm{K}_p f)^{-1/n} \geq \tilde{c}_{p,r,n} |f|_q^{1-\alpha} |f|_r^{\alpha}, \tag{35}$$

where

$$\tilde{c}_{p,r,n} = n^{-1/p}\omega_n^{-1/n} c_{p,r,n},$$

and q*,* α*, and* $c_{p,r,n}$ *are given by* (34)*. Equality holds if there exists a norm* $\|\cdot\|$ *on* $\mathbf{R}^n$*,* $a \in \mathbf{R}$*,* $\sigma \in (0,\infty)$*, and* $x_0 \in \mathbf{R}^n$*, such that*

$$f(x) = aw(\|x - x_0\|/\sigma),$$

for all $x \in \mathbf{R}^n$*, where* w *is given by* (32) *if* $p > 1$ *and* (33) *if* $p = 1$*.*

That the sharp inequality (35) is affine follows from (31). Note that the set of extremal functions in Theorem 10 is infinite-dimensional, and therefore the group $\mathbf{R} \times \mathrm{Aff}(n)$ does not act transitively on this set. This is in contrast to Theorem 12, where the set of extremal functions is finite-dimensional, and the group $\mathbf{R} \times \mathrm{Aff}(n)$ acts transitively on that set.

7. The sharp affine L^p Gagliardo-Nirenberg inequalities

In this section we show how the optimal Sobolev norm can be used to give a new straightforward proof of the sharp affine L^p Sobolev inequality proved by Zhang [34] for the case $p = 1$ and the authors [23] for the case $1 < p < n$. We begin by recalling the crucial geometric inequality underlying the analytic inequality, as well as some definitions needed to state the theorem.

Associated with an origin-symmetric convex body K is the convex body $\Gamma_{-p}K$, which is the unit ball of the norm on $\mathbf{R}^n$ given by

$$|x|^p_{\Gamma_{-p}K} = \frac{1}{V(K)} \int_{S^{n-1}} |u \cdot x|^p \, dS_p(K, u). \tag{36}$$

The body $\Gamma_{-p}K$ is called the *normalized* L^p *polar projection body of* K*.*

The range of the operator Γ_{-p} is the class of n-dimensional central slices of the L_p-ball. The integral transform used to define Γ_{-p} is the L^p*-cosine transform*, which is also the Fourier transform for even homogeneous functions of degree $-n-p$ on $\mathbf{R}^n$. See Koldobsky [15] for details.

Petty established the case $p = 1$ (see, for example, [10, 29, 32]) and the authors [21] established the case $p > 1$ of the following geometric inequality (See Campi and Gronchi [4] for a different approach).

Theorem 11 (L^p Petty projection inequality)**.** *If* $1 \le p < \infty$ *and* K *is a convex body, then*

$$V(\Gamma_{-p}K) \le a_{p,n} V(K),$$

where

$$a_{p,n} = \left[\frac{\sqrt{\pi}\,\Gamma(\frac{p+n}{2})}{2\Gamma(\frac{n}{2}+1)\Gamma(\frac{p+1}{2})}\right]^{n/p}. \tag{37}$$

Equality holds if and only if K *is an ellipsoid.*

These concepts were extended from bodies to functions by Zhang [34] for $p = 1$ and the authors [23] for $p > 1$.

If $1 \le p < \infty$ and $f : \mathbf{R}^n \to \mathbf{R}$ has L^p weak derivative, then the L^p *polar projection body* of f is defined to be the unit ball $\mathrm{B}_p f$ of the norm on $\mathbf{R}^n$ given by

$$|x|_{\mathrm{B}_p f} = \left(\int_{\mathbf{R}^n} |x \cdot \nabla f(y)|^p \, dy \right)^{1/p}. \tag{38}$$

We observe the volume of $\mathrm{B}_p f$ can be given directly in terms of the function f by

$$V(\mathrm{B}_p f) = \frac{1}{\Gamma(\frac{n}{p}+1)} \int_{\mathbf{R}^n} \exp\left(- \int_{\mathbf{R}^n} |x \cdot \nabla f(y)|^p \, dy \right) dx.$$

By (38), Theorem 6, and (36),

$$\mathrm{B}_p f = \Gamma_{-p} \mathrm{K}_p f, \tag{39}$$

In [21] it is shown that for each $\phi \in \mathrm{GL}(n)$ and each convex body K, we have $\Gamma_{-p}\phi K = \phi \Gamma_{-p} K$. This and (31) gives

$$\mathrm{B}_p(tf \circ \Phi^{-1}) = t^{-1} |\phi|^{-1/p} \phi(\mathrm{B}_p f), \tag{40}$$

for each $t > 0$ and invertible affine transformation $\Phi : \mathbf{R}^n \to \mathbf{R}^n$ given by

$$\Phi(x) = \phi(x) + y,$$

where $y \in \mathbf{R}^n$ and $\phi \in \mathrm{GL}(n)$.

The identity (39), the L^p Petty projection inequality (Theorem 11), and Corollary 7 lead to the following sharp affine L^p Gagliardo-Nirenberg inequalities. In contrast to Theorem 10, the extremal functions for this theorem are defined in terms of an inner product norm and have ellipsoids as level sets.

Theorem 12. *Suppose* $1 \le p < n$ *and* $0 < r \le np/(n-p)$. *If* $f \in L^q(\mathbf{R}^n) \cap L^r(\mathbf{R}^n)$ *has* L^p *weak derivative, then* f *satisfies the sharp affine inequality*

$$V(\mathrm{B}_p f)^{-1/n} \ge C_{p,r,n} |f|_q^{1-\alpha} |f|_r^{\alpha}, \tag{41}$$

where

$$C_{p,r,n} = n^{-1/p} (\omega_n a_{p,n})^{-1/n} c_{p,r,n},$$

q, α, *and* $c_{p,r,n}$ *are given by* (34), *and* $a_{p,n}$ *is given by* (37). *Equality holds if there exists an inner product norm* $\|\cdot\|$ *on* $\mathbf{R}^n$, $a \in \mathbf{R}$, $\sigma \in (0,\infty)$, *and* $x_0 \in \mathbf{R}^n$, *such that*

$$f(x) = a w(\|x - x_0\|/\sigma),$$

for all $x \in \mathbf{R}^n$, *where* w *is given by* (32) *if* $p > 1$ *and* (33) *if* $p = 1$.

Proof. That the sharp inequality (41) is affine follows from (40).

By (39) and the L^p Petty projection inequality (Theorem 11), (26), and the CNV inequality (3),

$$\begin{aligned} V(\mathrm{B}_p f)^{-1/n} &\ge a_{p,n}^{-1/n} V(\mathrm{K}_p f)^{-1/n} \\ &= n^{-1/p} (\omega_n a_{p,n})^{-1/n} \|\nabla f\|_p \\ &\ge C_{p,r,n} |f|_q^{1-\alpha} |f|_r^{\alpha}, \end{aligned} \tag{42}$$

where $\|\nabla f\|_p$ is the optimal L^p Sobolev norm of f. This proves (41).

The equality conditions for (41) follow from the equality conditions for the L^p Petty projection inequality (Theorem 11) and the affine CNV inequality (Theorem 10). □

The case $r = pn/(n-p)$ of Theorem 12 is the sharp affine L^p Sobolev inequality established for $p = 1$ by Zhang [34] and for $1 < p < n$ by the authors [23].

8. Problem

Is there a direct solution to the functional even L^p Minkowski problem that does not make use of solution to even L^p Minkowski problem?

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