

Exact Lagrangians in A_n -surface singularities

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Abstract In this paper we classify Lagrangian spheres in A_n -surface singularities up to Hamiltonian isotopy. Combining with a result of Ritter (Geom Funct Anal 20(3):779–816, 2010), this yields a complete classification of exact Lagrangians in A_n -surface singularities. Our main new tool is the application of a technique which we call *ball-swappings* and its relative version.

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1 Introduction

One of the classical problems in symplectic topology is to understand the classification of exact Lagrangians in a given exact symplectic manifold. As appealing as it is, the problem is in general very difficult, even in its simplest form. In particular, Arnold's nearby Lagrangian conjecture asserts that any exact Lagrangian in T^*M is Hamiltonian isotopic to the zero section, which is still quite open. This line of questions have attracted many efforts involving a long list of authors, among which we only mention a few very recent advances [1, 2, 28].

Note that the proofs of above-mentioned works involved advanced techniques from Floer theory, and usually covers symplectic manifolds of arbitrary dimensions. While these deep methods are amazingly powerful in determining the shape of exact Lagrangians coupling with homotopy methods, a general construction of Hamiltonian isotopies is still missing in dimension higher than 4. In contrast, one has more geometric tools available in dimension 4, thus equivalence up to Hamiltonian classes is more approachable. See [8] for an example of a very nice application of foliation techniques

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to this problem, and [9, 16] for another approach which is closer in idea to what we will use here.

In the present paper, we investigate the classification of Lagrangian spheres in an A_n -surface singularity. By definition, an A_n -surface singularity is symplectically identified with the subvariety

$$\{(x, y, z) : x^2 + y^2 + z^{n+1} = 1\} \subset (\mathbb{C}^3, \omega_{std}),$$

endowed with the restricted Kähler form. Throughout the paper we will denote W as the A_{n-1} -surface singularity, which is the main object that we will investigate. It is by now well-known that W is identified symplectically with the plumbing of $n - 1$ copies T^*S^2 . We will call the zero sections of these plumbed copies *standard spheres*. There is an alternative point of view to standard spheres from the Lefschetz fibration structure of W as follows, see [12]. Let $\pi : W \rightarrow \mathbb{C}$ be defined by $\pi(x, y, z) = z$, the fibers over $\zeta_i = e^{\frac{2i\pi\sqrt{-1}}{n}}$ are the n singular fibers. The segments connecting ζ_i to ζ_{i+1} defines a so-called *matching cycles* for $i = 0, \dots, n - 2$. These matching cycles are precisely the standard spheres described earlier. We will adopt the matching cycle point of view in the rest of the paper. The following is our main result:

Theorem 1 *Lagrangian spheres in W are unique up to Hamiltonian isotopy and Lagrangian Dehn twists along the standard spheres.*

A fantastic result showed by Ritter in [22] says that, embedded exact Lagrangians in W are all Lagrangian spheres. We therefore obtain the following corollary, which completely classifies exact Lagrangians in A_n surface singularities up to Hamiltonian isotopy:

Corollary 1.1 *Exact Lagrangians in A_n -surface singularities are isotopic to the zero section of a plumbed copy of T^*S^2 , up to a composition of Lagrangian Dehn twists along the standard spheres.*

Such kind of classification seems desirable but rare in the literature, especially when there exist smoothly isotopic but not Hamiltonian isotopic Lagrangians [25]. Notice that various forms of partial results have been obtained previously. In particular, Hind in [8] proves Theorem 1 for the case of A_1 and A_2 . It was also known that the result is true up to equivalence of objects in the Fukaya category, using the deep computations in algebraic geometry as well as the mirror symmetry of A_n -surface singularities [10, 11]. This Floer-theoretic version already found interesting applications [14, 28]. In principle, Theorem 1 along with computation of [12] on the symplectic side should recover corresponding results in [10, 11] on the mirror side.

En route, we also prove the following result on the compactly supported symplectomorphism group of W :

Theorem 1.2 *Any compactly supported symplectomorphism is Hamiltonian isotopic to a composition of Dehn twists along the standard spheres. In particular, $\pi_0(\text{Symp}_c(W)) = Br_n$.*

This is a refinement of results due to Evans [6, Theorem 4], which asserts that $\pi_0(\text{Symp}_c(W))$ injects into Br_n , and Khovanov–Seidel [12, Corollary 1.4], which proves in any dimension of A_n -singularities, there is an injection from the other direction. Our result shows that these two injections are in fact both isomorphisms in dimension 4.

The paper is structured as follows. In Sect. 2 we set up the notation and describe two different but closely related models for W and its compactification. We then reduce the main theorems to problems in its compactification. Section 3 contains the proof of Proposition 2.3, which implies Theorem 1.2, and Sect. 4 contains the proof of Proposition 2.4, which implies Theorem 1. We conclude the article with some discussions on the ball-swapping symplectomorphisms, which is the main technique involved in this article.

2 Two models of W

In this section we recall two models of W and its compactifications due to J. Evans and I. Smith. Along the way we introduce various notations that we will use throughout the paper.

We first go through the compactification of the A_n -singularities as a complex affine variety following [6].

Let M be the blow-up of $\mathbb{C}\mathbb{P}^2$ at $\{p_i = [\xi_i, 0, 1]\}_{i=1}^n$, which is identified as

$$\{([x, y, z], [a_i, b_i]) : [x, y, z] \in \mathbb{C}\mathbb{P}^2, [a_i, b_i] \in \mathbb{C}\mathbb{P}^1, a_k y = b_k(x - \xi_k z), i = 1, \dots, n\}.$$

Here ξ_i denotes the i -th n -unit root. We therefore obtain a pencil structure of M , which is the blow-up of the pencil of lines passing through $[0, 1, 0]$ in $\mathbb{C}\mathbb{P}^2$. We denote:

- the pencil as $\{P_i\}_{i=[x,z] \in \mathbb{C}\mathbb{P}^1}$,
- C_i is the exceptional curves of the blow-up at $[\xi_i, 0, 1], i = 1, \dots, n$,
- $C_{n+1} = \{[x, y, 0], [x, y], \dots, [x, y]\}$,
- $C_{n+2} = \{[x, 0, z], [0, 1], \dots, [0, 1]\}$.

Here C_{n+1} plays the role of a generic fiber of the pencil which is a line, and C_{n+2} is the proper transform of the line passing through $\{p_i\}_{i=1}^n$, which is also a section of the pencil. Endow a Kähler form ω to M so that $\int_{C_{n+1}} \omega = 1, \int_{C_i} \omega = r := 1/N$ for all $1 \leq i \leq n$. Here we require $N \gg n$ to be a large integer. This is always possible by the construction of a symplectic blow-up [20].

Let $U = M \setminus (C_{n+1} \cup C_{n+2})$. Then U has a Lefschetz fibration structure induced from M . Lemma 7.1 of [6] showed that U is biholomorphic to the A_n -singularity W . As a result of [4, Lemma 2.1.6], U has a symplectic completion symplectomorphic to W . Therefore, one may obtain a symplectic embedding $\iota : U \hookrightarrow W$. Note that the Lefschetz fibration of U defined above coincides with that of W described in the introduction. In particular they have the same number of Lefschetz singularities and monodromies. Thus one may assume ι preserves the Lefschetz fibration structure. In particular, their Lefschetz thimbles, thus matching cycles coincide. If we identify $C_{n+2} \setminus C_{n+1}$ with the base of the Lefschetz fibration of U , then the standard spheres are

Fig. 1 Evans’s construction

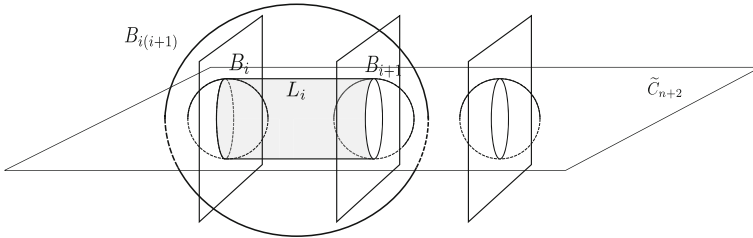
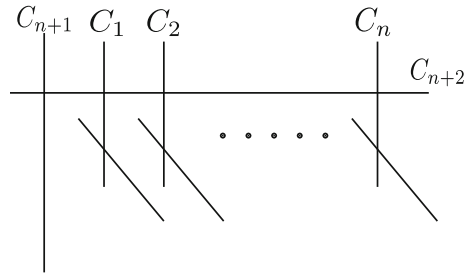


Fig. 2 Smith’s construction

the matching cycles lying above the straight arcs connecting p_i to p_{i+1} in the base. We will use this interpretation of standard spheres throughout, but a more explicit symplectic description is in order (Figs. 1, 2).

The above compactification model is explicit in complex coordinates, but for our purpose of constructing symplectomorphisms we need to recall the following alternative model slightly generalized from [29, Example 4.25].

For $z^0 = (z_1^0, z_2^0) \in \mathbb{C}^2$ and real number R , let

$$B(z^0; R) = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1 - z_1^0|^2 + |z_2 - z_2^0|^2 < R^2\}.$$

Consider $B = B(0, \frac{1}{\sqrt{\pi}})$, $B_i = B(\frac{2i-n-1}{(n+1)\sqrt{\pi}}, \frac{1}{\sqrt{N\pi}})$ for $1 \leq i \leq n$. The circles $\{|z_2| = \frac{1}{\sqrt{N\pi}}\}$ over the arcs

$$\gamma_i = \left\{ \operatorname{Im}(z_1) = 0, \operatorname{Re}(z_1) \in \left[\frac{2i-n-1}{(1+n)\sqrt{\pi}}, \frac{2i-n+1}{(1+n)\sqrt{\pi}} \right] \right\} \subset \{z_2 = 0\}, \quad 1 \leq i \leq n-1,$$

forms $(n - 1)$ Lagrangian tubes. Upon blowing up all B_i , the boundary circles of the tubes are collapsed to points on the exceptional spheres, and the Lagrangian tubes become matching cycles between two singular fibers of the Lefschetz fibration described previously, thus giving the standard spheres up to Hamiltonian isotopy.

The relation between Evans’s and Smith’s construction is clear from the symplectic interpretation of blow-ups. In particular, since the complement of a line in $\mathbb{C}\mathbb{P}^2$ is identified as a 4-ball, U is exactly the complement of the proper transform of $\{z_2 = 0\}$ in the blow up of $B(0, \frac{1}{\sqrt{\pi}})$ along B_i . Notice the following facts:

Lemma 2.1 [6, Proposition 2.1] *Symp_c(U) is weakly homotopic to Symp_c(W).*

Lemma 2.2 *Suppose Lagrangian spheres are unique up to compactly supported symplectomorphisms in U, then the same holds in W.*

To see Lemma 2.2, notice that since U has its symplectization identified with W, any Lagrangian $L \subset W$ is isotopic to one in U through the negative Liouville flow. The two lemmata reduce our main theorems from W to U. Concretely, they show that the following two propositions imply the main theorems:

Proposition 2.3 $\pi_0(\text{Symp}_c(U))$ is generated by the Dehn twists along L_1, \dots, L_n , where L_i are matching cycles of the Lefschetz fibration of U, for $1 \leq i \leq n$.

Proposition 2.4 Any pair of Lagrangian spheres $L, L' \subset U$ are symplectomorphic, i.e., there is a $\phi \in \text{Symp}_c(U)$ such that $\phi(L) = L'$.

The two propositions will be proved in Sects. 3 and 4, respectively. To free up the notation later we also define:

Definition 2.5 Let $f : B(r) \subset \mathbb{C}^2 \rightarrow (M, \omega)$ be a symplectic embedding, and $\Sigma \subset M$ is a symplectic divisor. Then f and Σ is said to intersect normally if $f^{-1}(\Sigma) = B(r) \cap \{z_2 = 0\}$.

3 The mapping class group of W

In this section we introduce a technique of producing symplectomorphism alluded to in [3, 16], which we call the *ball-swapping*. We will show that certain ball-swaps are Hamiltonian isotopic to Lagrangian Dehn twists, and that they generate $\pi_0(\text{Symp}_c(U))$. This will prove Proposition 2.3.

We start with a more general context. Suppose X is a symplectic manifold. Given two symplectic ball embeddings:

$$\iota_{0,1} : \prod_{i=1}^n B(r_i) \rightarrow X,$$

where ι_0 is isotopic to ι_1 through a Hamiltonian path $\{\iota_t\}$. From the interpretation of blow-ups in the symplectic category [19], the blow-ups can be represented as

$$X^{\# \iota_j} = \left(X \setminus \iota_j \left(\prod_{i=1}^n B_i \right) \right) / \sim, \quad \text{for } j = 0, 1.$$

Here the equivalence relation \sim collapses the natural S^1 -action on $\partial B_i = S^3$. Now assume that $K = \iota_0(\prod B_i) = \iota_1(\prod B_i)$ as sets, then ι_t defines a symplectic automorphism $\tilde{\tau}_t$ of $X \setminus K$, which descends to an automorphism τ_t of $X^{\# \iota_t} := X^{\# \iota_0} = X^{\# \iota_1}$. We call τ_t a *ball-swapping symplectomorphism* or *ball-swapping* on $X^{\# \iota_t}$. Notice it is not known to be true (nor false) that any two ball packings are Hamiltonian isotopic.

McDuff gave an affirmative answer to this question for the symplectic 4-manifolds with $b^+ = 1$ [17] which allows more freedom to create ball-swappings in the blow-ups of these manifolds, but the general case of the question is still widely open. This construction is closely related to one in algebraic geometry, see discussions in Sect. 5.

Getting back to the proof of Proposition 2.3, as usual, we denote τ_L the Lagrangian Dehn twist along L for a Lagrangian sphere L [26]. To compare the ball-swapping, the Dehn twists in dimension 4 and the full mapping class group of W , we first need a local refinement of the following result due to Evans:

Theorem 3.1 [6, Theorem 1.4] *The compactly supported symplectomorphism group $Symp_c(W)$ has weakly contractible connected components. Moreover, $\pi_0(Symp_c(W))$ has an injective homomorphism into the full braid group Br_n with n -strands.*

The proof of Theorem 3.1 is a vital part of our arguments. Indeed, we will incorporate ball-swapping refinements into Evans’ proof to show Proposition 2.3. Define a standard configuration $\{S_i\}_{i=1}^n$ in $M = \mathbb{C}\mathbb{P}^2 \# n\overline{\mathbb{C}\mathbb{P}^2}$ as:

- (i) each S_i is an embedded symplectic sphere disjoint from C_{n+1} ,
- (ii) $[S_i] = [C_i]$,
- (iii) there exist $J \in \mathcal{J}_\omega$, the set of almost complex structures compatible with ω , for which all S_i, C_{n+1}, C_{n+2} are J -holomorphic.
- (iv) There is a neighborhood ν of C_{n+2} such that $S_i \cap \nu = P_{t_i} \cap \nu$, for $t_i = S_i \cap C_{n+2}$.

Recall here P_{t_i} is a curve in the pencil defined at the beginning of Sect. 2. Proposition 7.1 of [6] showed that:

Proposition 3.2 *The space of standard configurations \mathcal{C}_0 is weakly contractible.*

Note that this proposition explains part of the motivation of our choice of areas of $\omega(C_i)$, since its proof needs to ensure no bubble occurs from S_i to apply Pinsonnault’s result [21, Lemma 1.2].

Let $Conf(n)$ be the configuration space of n points on a disk D^2 . Proposition 7.2 of [6] shows one has the following homotopy fibration:

$$\begin{aligned} \Psi : \mathcal{C}_0 &\rightarrow Conf(n), \\ S = \bigcup_{i=1}^n S_i &\mapsto \{S_1 \cap C_{n+2}, \dots, S_n \cap C_{n+2}\}. \end{aligned}$$

Here D^2 is identified with $C_{n+2} \setminus C_{n+1}$. The fiber of this fibration is denoted as \mathcal{F} . Explicitly, fix an unordered n -tuple of points $[x_1, \dots, x_n], x_i \in D^2, \mathcal{F}$ is the space of standard configurations $\{S_i\}$ such that $[S_i \cap C_{n+2}]_{i=1}^n = [x_i]_{i=1}^n$ as unordered n -tuples. The associated long exact sequence thus gives the following isomorphism:

$$Br_n = \pi_1(Conf(n)) \xrightarrow{\sim} \pi_0(\mathcal{F}) \tag{3.1}$$

Moreover, [6, Theorem 7.6] shows that:

$$\pi_i(\mathcal{F}) = 0, \quad \text{for all } i > 1, \tag{3.2}$$

that is, the connected components of \mathcal{F} are weakly contractible. The construction of the isomorphism (3.1) amounts to the following lemma:

Lemma 3.3 [6, Proposition 7.2] *Let α be a loop in $Conf(n)$. For a compactly supported Hamiltonian \tilde{f}_α on $C_{n+2} \setminus C_{n+1}$ inducing α , there is an extension of \tilde{f}_α to a Hamiltonian f_α of M supported in a neighborhood of C_{n+2} , such that:*

- f_α preserves $C_{n+1} \cup C_{n+2}$ and fixes C_{n+1} pointwisely,
- f_α preserves the set C_0 .

Take $[S_1, \dots, S_n] \in \mathcal{F}$ and α a loop in $Conf(n)$. Then (3.1) is given by $[f_\alpha(S_1), \dots, f_\alpha(S_n)]$. Note that there is by no means a canonical group structure on $\pi_0(\mathcal{F})$, so one should understand (3.1) as the Br_n -action on $\pi_0(\mathcal{F})$ is free and transitive. This explicit construction of the isomorphism (3.1) will be used later.

We now look closer to the special case when $n = 2$ using Smith’s model. Denote $B = B(0; 1)$, $B_+ = B((\frac{1}{2}, 0); \frac{1}{3})$, $B_- = B((-\frac{1}{2}, 0); \frac{1}{3})$, and $\tilde{C} = \{z_2 = 0\} \cap B$. As in Sect. 2, we may blow up B_\pm and remove C , which is defined to be the proper transform of \tilde{C} . This gives exactly U constructed previously for $n = 2$, $N = \frac{1}{3}$. We denote this open symplectic manifold by U_2 in case of confusions. Therefore, by Lemma 2.1 and [24]

$$Symp_c(U) = Symp_c(T^*S^2) = \mathbb{Z}, \tag{3.3}$$

and it is generated by the Lagrangian Dehn twist along the matching cycle described in Sect. 2. Now we consider a Hamiltonian isotopy ρ swapping B_- and B_+ . Explicitly, we choose a small number $\epsilon > 0$, and a smooth function $f : \mathbb{R} \rightarrow \mathbb{R}$ such that $f(x) = 0$ when $x \geq 1 - \epsilon$ and $f(x) = 1$ when $x \leq 1 - 2\epsilon$. Then ρ is defined by the Hamiltonian function $\pi f(|z_1|^2 + |z_2|^2) \cdot |z_1|^2$. ρ thus defines a ball-swapping symplectomorphism $\tau_\rho \in Symp_c(B\#2\mathbb{C}P^2)$ with corresponding symplectic form. Denote C_+ (C_- resp.) for the exceptional sphere by blowing up $\rho(B_+)$ ($\rho(B_-)$ resp.). Then τ_ρ preserves C and exchanges C_+ , C_- as sets.

Consider $l := \tau_\rho|_C : (C, [x_+, x_-]) \rightarrow (C, [x_+, x_-])$, where $x_\pm = C_\pm \cap C$ and $[x_+, x_-]$ denotes the unordered pair of points they form on \tilde{C} . l can be symplectically isotoped to id on C by a compactly supported Hamiltonian path l_t^{-1} which induces the generator of $\pi_1(Conf(2))$. We may then lift this isotopy to $l_t^{-1} \times id \in Ham(D_2(1) \times D_2(\epsilon))$ for some small $\epsilon > 0$. the corresponding Hamiltonian function vanishes near $(\partial D_2(1)) \times D_2(\epsilon)$, and one can further cut off the function near $D_2(1) \times (\partial D_2(\epsilon))$, then extend the new function trivially to the rest of B . We denote the resulting Hamiltonian diffeomorphism as \bar{l}^{-1} . From our way of constructing the symplectomorphisms, it is clear that $\bar{l}^{-1} \circ \tau_\rho = id$ on a neighborhood of C . Therefore, $\bar{l}^{-1} \circ \tau_\rho \in Symp_c(U_2)$. Figure 3 provides a graphic summary of the construction described above, and the commutativity follows from the definition of τ_ρ .

Lemma 3.4 $\bar{l}^{-1} \circ \tau_\rho$ is isotopic to a Lagrangian Dehn twist which generates $\pi_0(Symp_c(U_2))$.

Proof Consider the action of $\bar{l}^{-1} \circ \tau_\rho$ on $([C_+, C_-]) \in \mathcal{F}$. Since τ_ρ exchanges C_+ and C_- as sets, the action of τ_ρ on the unordered pair $[C_+, C_-]$ is actually trivial.

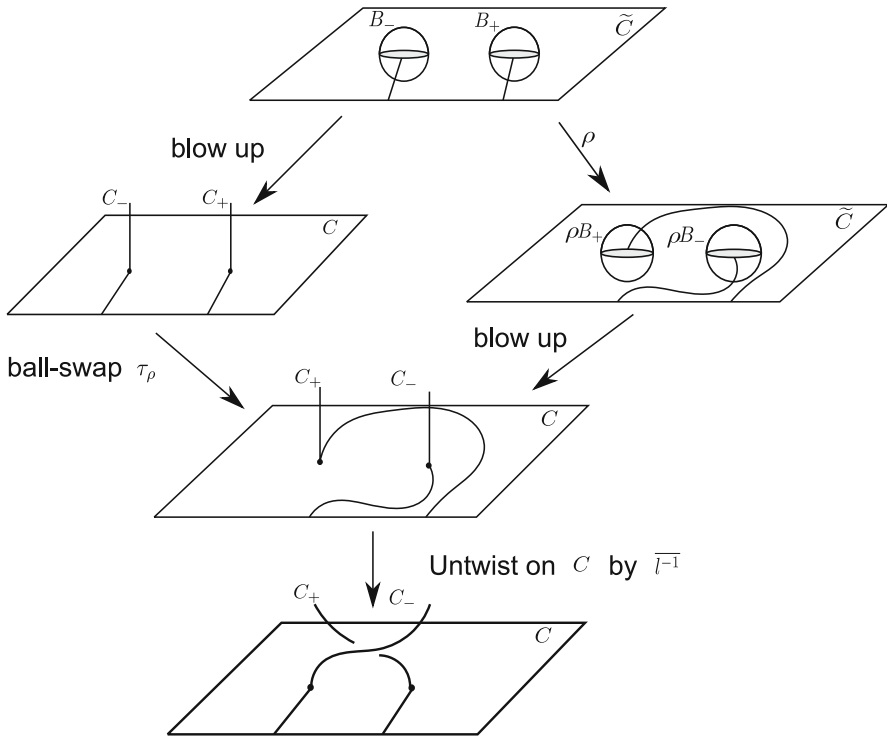


Fig. 3 Swapping two balls

Also, notice that $l^{-1}([x_+, x_-])$ is the generator of $\pi_1(\text{Conf}(2))$, from (3.1) we see that $\{(\overline{l^{-1}} \circ \tau_\rho)^k([C_+, C_-]) : k \in \mathbb{Z}\}$ contains exactly one point in each component of \mathcal{F} . On the other hand, since τ_L is the generator of $\pi_0(\text{Symp}_c(U))$, $\overline{l^{-1}} \circ \tau_\rho$ is compactly isotopic to τ_L^m for some integer m . Therefore, τ_L also acts transitively on $\pi_0(\mathcal{F})$. Now [6, Lemma 7.6] showed that the action $\mathbb{Z} = \pi_0(\text{Symp}_c(U))$ is free on its orbit in $\pi_0(\mathcal{F})$. Therefore, the action of τ_L on $\pi_0(\mathcal{F})$ is also free and transitive, so actions of the generators τ_L and $\overline{l^{-1}} \circ \tau_\rho$ have to match, that is, $[\tau_L(C_+), \tau_L(C_-)] = [\overline{l^{-1}} \circ \tau_\rho(C_+), \overline{l^{-1}} \circ \tau_\rho(C_-)] \in \pi_0(\mathcal{F})$ up to a change of the orientation of L . This shows that τ_L is Hamiltonian isotopic to $\overline{l^{-1}} \circ \tau_\rho$ in $\text{Symp}_c(U_2)$, since the stabilizer of the action of $\text{Symp}_c(U_2)$ on \mathcal{F} is weakly contractible by (3.2).

Remark 1 From the proof it is clear that the particular sizes of balls involved in Lemma 3.4 is not relevant, as long as the Kähler packing is possible. By abuse of notation, we will denote U_2 by open symplectic manifolds obtained this way.

Remark 2 One may also construct a proof of Lemma 3.4 from a more classical algebro-geometric point of view. See discussions in Sect. 5. We adopt the current more indirect approach for that it consists part of the proof for Lemma 3.5.

We are now ready to return to the general case of U . From (3.1) we have a Br_n -action on $\pi_0(\mathcal{F})$, which is free and transitive. By comparing with the free

action of $\pi_0(\text{Sym}_p_c(U_2))$ on the same space, Evans obtains the monomorphism in Theorem 3.1:

$$e : \pi_0(\text{Sym}_p_c(U)) \hookrightarrow Br_n. \tag{3.4}$$

This is precisely the identification taken in Lemma 3.4 in the case of U_2 . Therefore, one may interpret Lemma 3.4 as showing that e is an isomorphism for $n = 2$. The following result shows that e is an isomorphism for any n , and that $\pi_0(\text{Sym}_p_c(U))$ is generated by Dehn twists along matching cycles, hence implying Proposition 2.3.

Lemma 3.5 *Let $T \subset \pi_0(\text{Sym}_p_c(U))$ be the subgroup generated by Lagrangian Dehn twists of matching cycles. Then there exist an isomorphism $\kappa : T \rightarrow Br_n$, such that the following diagram commutes:*

$$\begin{array}{ccc}
 T \subset \pi_0(\text{Sym}_p_c(U)) & \xrightarrow{c} & \pi_0(\text{Sym}_p_c(U)) \\
 & \searrow \kappa & \downarrow e \\
 & & Br_n
 \end{array} \tag{3.5}$$

Proof We adopt Smith’s model and notation from Sect. 2. Consider the blow-down of $M \setminus C_{n+1}$ along C_i , $i \leq n$. The blow-down is identified with a symplectic ball B coming with the embedded balls B_i resulted from C_i for $i \leq n$. Denote $\tilde{C}_{n+2} \subset B$ as the proper transformation of C_{n+2} under the blow-down. As long as $r = \omega(C_i)$ is sufficiently small, it is clear that one has an embedded symplectic ball $B_{i(i+1)} \hookrightarrow B$ for $1 \leq i \leq n - 1$, such that (see Fig. 2):

- $B_{i(i+1)}$ intersects \tilde{C}_{n+2} normally,
- $B_i \cup B_{i+1} \subset B_{i(i+1)}$,
- $B_{i(i+1)} \cap B_k = \emptyset$, for any $k \neq i, i + 1$.

For example, one may take $B_{i(i+1)} = B(\frac{2i-n}{(n+1)\sqrt{\pi}}, \frac{1}{(1+n)\sqrt{\pi}} + \frac{2}{\sqrt{N\pi}})$. From the local construction Lemma 3.4, one sees that the action of each generator of the braid group σ_i on $\pi_0(\mathcal{F})$ is explicitly realized by a symplectomorphism coming from ball-swapping which is compactly supported in $B_{i(i+1)}$. Moreover, these symplectomorphisms are isotopic to Lagrangian Dehn twists of matching cycles supported in $B_{i(i+1)}$, and such an identification is given precisely by $e \circ c$ in (3.5) by the discussion preceding Lemma 3.5. Since both e and c are injective, all arrows in (3.5) are indeed isomorphisms.

4 A symplectomorphism by ball-swappings

4.1 Constructing ϕ for Proposition 2.4

In this section, we will prove Proposition 2.4. The basic idea of constructing ϕ is again to use the ball-swapping in the compactification.

Notice first that one may assume L and L' are homologous in $M = \mathbb{C}P^2 \# n \overline{\mathbb{C}P^2}$. This follows from the classification of homology classes of Lagrangian spheres in rational manifolds [16, Theorem 1.4]. Since L, L' are disjoint from C_{n+1} which is a line in M , their classes have to have the form of $[C_i] - [C_j]$ for $i, j \leq n$. By certain Dehn twists along the standard Lagrangian spheres, one may assume $[L] = [L'] = [C_1] - [C_2]$ in M .

Recall also the following result from [16].

Theorem 4.1 [16, Theorem 1.1, 1.2] *Let $b^+(M) = 1$ and the Gromov–Taubes invariant $GT(A) \neq 0$, L be a Lagrangian sphere. Then A has an embedded representative with minimal intersection with L . If A is represented by an embedded sphere, then any given representatives of A can be symplectically isotoped so that they achieve minimal intersections.*

Let $L, L' \subset U = M \setminus (C_{n+1} \cup C_{n+2})$, Theorem 4.1 implies that one may isotope $\{C_3, \dots, C_n\}$ to another standard configuration $\{S_3, \dots, S_n\}$, which are disjoint from L . Extend this isotopy of spheres to a Hamiltonian isotopy Ψ_t , then $\Psi_t^{-1}(L)$ isotopes L away from $\{C_3, \dots, C_n\}$. By performing the same type of isotopy to L' , we may assume $\{C_3, \dots, C_n\}$ are disjoint from both L and L' .

We now blow down along the set $\mathcal{E} = \{C_3, \dots, C_n\}$ to obtain a set of balls $\mathcal{B} = \{B_3, \dots, B_n\}$, and the resulting manifold is denoted as $M_2 = \mathbb{C}P^2 \# 2 \overline{\mathbb{C}P^2}$, where we have the proper transformations of L and L' denoted as \tilde{L} and \tilde{L}' . Note that by removing the proper transform of C_{n+1} and C_{n+2} we have $M_2 \setminus (\tilde{C}_{n+1} \cup \tilde{C}_{n+2}) = U_2$. One also have from Remark 1:

Lemma 4.2 ([3], Lemma A.1 and the discussions following it) *Lagrangian spheres in $U_2 = M_2 \setminus (\tilde{C}_{n+2} \cup \tilde{C}_{n+1})$ are unique up to compactly supported Hamiltonian isotopy.*

Combining discussions above, there is a compactly supported Hamiltonian isotopy $\tilde{\Phi}_t : U_2 \rightarrow U_2$, such that $\tilde{\Phi}_t(\tilde{L}) = \tilde{L}'$ where $\tilde{\Phi}_t = id$ near $\tilde{C}_{n+1} \cup \tilde{C}_{n+2}$ (Fig. 4).

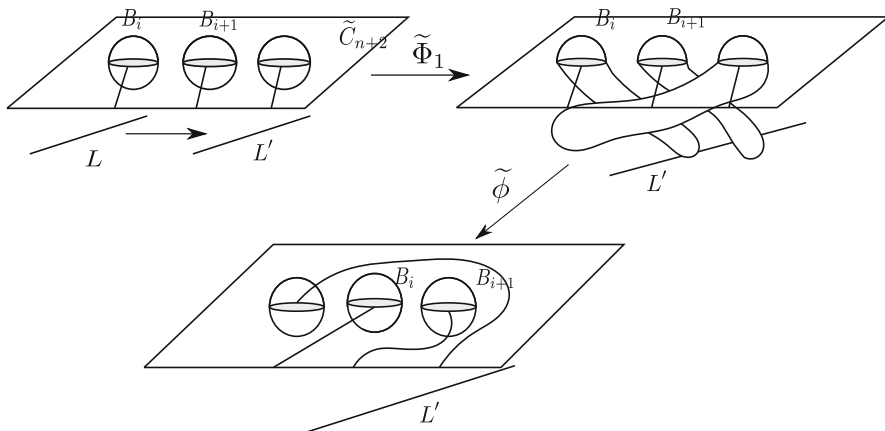


Fig. 4 A pictorial proof of Proposition 2.4

To obtain a compactly supported symplectomorphism on $M \setminus (C_{n+1} \cup C_{n+2})$, it amounts to showing the following lemma, whose proof will be given in the next section.

Lemma 4.3 *There is $\tilde{\phi} \in \text{Ham}(M_2)$, such that:*

- (i) $\tilde{\phi}(\tilde{\Phi}_1(B_i)) = B_i$ for $i \geq 3$, $\tilde{\phi}(\tilde{C}_{n+2}) = \tilde{C}_{n+2}$,
- (ii) for some neighborhood \mathcal{N} of $\tilde{C}_{n+1} \cup L'$, $\tilde{\phi}|_{\mathcal{N}} = \text{id}$.

Lemma 4.3 concludes the proof of Proposition 2.4 because $\tilde{\phi} \circ \tilde{\Phi}$ sends L to L' and fixes B_i for $i \geq 3$, and it clearly is compactly supported in $M_2 \setminus \tilde{C}_{n+1}$. Therefore, it can be lifted to a compactly supported symplectomorphism ϕ of $M \setminus C_{n+1}$ by blowing up the balls B_i for $i \geq 3$, which preserves C_{n+2} . One then apply Lemma 3.3 again to obtain a symplectomorphism ϕ which fixes C_{n+2} pointwisely. This is always possible because $\text{Ham}_c(C_{n+2} \setminus C_{n+1}) = \text{Ham}_c(D_2) \sim pt$ by the Smale’s theorem. Since the gauge group of the normal bundle of $C_{n+2} \setminus (C_{n+1} \cup C_{n+2})$ is homotopic equivalent to $\text{Map}(S, *)$, $SL_2(\mathbb{R}) \sim pt$, by composing another symplectomorphism fixing C_{n+2} pointwisely one may assume ϕ fixes also the normal bundle of C_{n+2} . All these adjustments can be made supported in a small neighborhood of C_{n+2} thus not affecting L' . Therefore, ϕ indeed descends to a compactly supported symplectomorphism of $U = M \setminus (C_{n+1} \cup C_{n+2})$, which concludes our proof.

Remark 3 For our purpose it suffices to prove $\tilde{\phi} \in \text{Symp}(M_2)$ in Lemma 4.3, which will be slightly easier. From the proof it is clear that the lemma indeed holds for more general cases of packing relative to a symplectic divisors, coupling with results in [18] and [3], but we restrict ourselves for ease of expositions.

Remark 4 One recognizes that the braiding in the symplectomorphism group required for sending L to L' comes exactly from the restriction of $\tilde{\phi}$ on $C_{n+1} \setminus C_{n+2}$, which swaps the shadows $B_i \cap \tilde{C}_{n+2}$ by $\text{Diff}(D_2)$. This nicely matches the pictures of Sect. 3.

4.2 Connectedness of ball packing relative to a divisor

We prove Lemma 4.3 in this section. The overall idea is not new. In the absence of \tilde{C}_{n+2} , this is just the ball-packing connectedness problem in the complement of a Lagrangian spheres in rational manifolds. This was settled in [3] using McDuff–Opshstein’s non-generic inflation lemma [18, Lemma 4.3.3] and her ideas in the original proof of connectedness of ball-packings [17, Section 3]. We will adapt these ideas in our relative case and point out necessary modifications. We will continue to use notations defined in previous sections.

Let $\iota_0 : B_i(r) \hookrightarrow M$ be the inclusion and $\iota_1 = \tilde{\Phi}_1 : B_i \hookrightarrow M$. Fix some small $\delta > 0$ and choose an extension of the embedding $\iota_j : B_i(r) \rightarrow M$ to $B_i(r + \delta)$ for $j = 0, 1$, so that the extensions still intersect \tilde{C}_{n+2} normally (Definition 2.5). Take two families of diffeomorphism ϕ_j^s , $j = 0, 1$, so that the following holds:

- (1) $\phi_j^0 = id$,
- (2) $\phi_j^s|_{B(sr_i)}$ is a radial contraction from $B(sr_i)$ to $B(\delta)$, and identity near $\partial B(sr_i + \delta)$,
- (3) $\phi_j^s(\tilde{C}_{n+2}) = \tilde{C}_{n+2}$.

This is not hard to achieve if we have B_i intersecting \tilde{C}_{n+2} normally in the first place.

The push-forward of ω by ϕ_j^s endows a family of symplectic forms $\{\omega_j^s\}_{0 \leq s \leq 1}$ on M for $j = 0, 1$. Notice in our situation, $\tilde{\Phi}_1$ in Lemma 4.3 is compactly supported in U_2 , which implies $\iota_0(B_i(\delta)) = \iota_1(B_i(\delta))$ for sufficiently small $\delta > 0$. Therefore, performing a blow-up on $\iota_j(B_i(\delta))$, $j = 0, 1$ gives a family of symplectic forms $\{\{\tilde{\omega}_j^s\}\}$ on the same smooth manifold M_2 without further identification. The resulting exceptional curves formed by blowing up B_i gets back to C_i in $\mathbb{C}\mathbb{P}^2 \# n\overline{\mathbb{C}\mathbb{P}^2}$, but the form near it is deformed. Notice $\int_{C_i} \tilde{\omega}_j^s = \pi(sr_i)^2$. We claim:

Lemma 4.4 *For any given $s \in [0, 1]$, $\tilde{\omega}_0^s$ are isotopic to $\tilde{\omega}_1^s$ via a family of diffeomorphism $\psi^s(t) : M \rightarrow M$ such that $\psi^s(t)(C_i) = C_i$ and $\psi^s(t)(L') = L'$ for any given $s \in [0, 1]$ and $3 \leq i \leq n + 2$.*

Proof This is essentially Theorem 1.2.10 and its immediate Corollary 1.2.11 of [18] with minor adaption. To put L' into this framework, perform a symplectic cut near it, which yields a symplectic (-2) -sphere S (cf. [3, Remark 2.2]). This will not affect C_i for $i \geq 3$ following discussions after Theorem 4.1. Set $\mathcal{S} = \bigcup_{i=3}^{n+2} C_i \cup S$. All curves in \mathcal{S} intersect in a symplectically orthogonal way. So \mathcal{S} satisfies conditions of Definition 1.2.1 of [18] which is required by Corollary 1.2.11, except we have included exceptional curves C_i , $3 \leq i \leq n$ and a line C_{n+1} , which does not satisfy $c_1(C_i) \leq -g = 0$. However, \mathcal{S} satisfies Condition 5.1.1 in [18], which says the configuration should be a union of finitely many symplectic submanifolds which intersect in a symplectically orthogonal way. All proofs of 5.1.2–5.1.6 thus 1.2.10 therein were carried out for the class configurations satisfying Condition 5.1.1 hence applies tautologically to our choice of \mathcal{S} . This means one can choose $\overline{\psi}^s(t)(C_i) = C_i$ and $\overline{\psi}^s(t)(S) = S$ from Corollary 1.2.11 of [18].

The last part is to use symplectic fiber sum (cf. [7]) to glue a copy of $S^2 \times S^2$ back along S . Recall that the fiber sum construction in our case is the following procedure. First take away $S = \Sigma^+ \subset \bar{M}$ and $\Delta = \Sigma^- \subset S^2 \times S^2$, where \bar{M} is the symplectic manifold after the symplectic cut, and Δ is the standard diagonal. Then we identify a neighborhood of Σ^- and Σ^+ as $[-\epsilon, 0) \times V$ and $(0, \epsilon] \times V$ respectively, endowed with a symplectic form from symplectization of V , where V is $\mathbb{R}\mathbb{P}^3$ endowed with a standard contact form. Now $\overline{\psi}^s(t)$ defines a Hamiltonian path on $S = V/S^1$, which can be lifted to a path of diffeomorphism f_t of V preserving the contact form. This diffeomorphism is therefore extended to $[-\epsilon, 0) \times V \subset M$ (identified as the glued manifold) using $id \times f_t$ after the gluing. To extend this Hamiltonian path to the rest of M , notice that the part of Hamiltonian path we already constructed restricted to $[-\epsilon, 0) \times V$ has a Hamiltonian function F_t lifted from the projection $[-\epsilon, 0) \times V \rightarrow S = V/S^1$. Therefore one may further extend this diffeomorphism to the whole M by multiplying a cut-off function to F_t so that its support is away from $L' = \bar{\Delta}$, the anti-diagonal of $S^2 \times S^2$, then consider its Hamiltonian flow in $S^2 \times S^2 \setminus \Sigma^-$. Clearly this extension provides a Hamiltonian path in $S^2 \times S^2 \setminus \Sigma^-$ and coincides with $\overline{\psi}^s(t)$ in $\bar{M} \setminus \Sigma^+$.

Such an extended hamiltonian path on M , denoted $\psi^s(t)$, hence satisfies our claim in the lemma.

To finish the proof of Lemma 4.3, we already constructed a symplectomorphism $(\phi_1^s)^{-1}\psi^s\phi_0^s$ interpolating $\tilde{\omega}_0^s$ and $\tilde{\omega}_1^s$ for each s from Lemma 4.4. By blowing down C_i , one obtains a symplectomorphism F^s sending $\iota_0(B_i(sr_i))$ to $\iota_1(B_i(sr_i))$. From the smooth dependence of the family ψ^s on s , one learns that ψ^1 is an isotopy of M . By precomposing another isotopy h preserving C_i for all $3 \leq i \leq n + 1$ as in the proof of [17, Corollary 1.5], one may further assume ψ^1 is identity near the C_i for $i \leq n + 1$. One thus obtains a family of ball-packing by blowing down along C_3, \dots, C_n , which connects ι_0 and ι_1 through a normal intersection family with \tilde{C}_{n+2} as desired.

5 Concluding remarks on ball-swappings

The ball-swapping technique we used in this paper seems to have rich structures and could be of independent interests. This concluding remark summarizes several possibly interesting directions of further study of this class of objects.

- It was pointed out to the author that, it seems instructive to compare ball-swappings with a closely related construction in algebraic geometry which is classical. The author first learned about the following construction from Seidel’s excellent lecture notes [26]. Consider \mathbb{C}^n and its 2-point configuration space $\text{Conf}_2(\mathbb{C}^n)$. One may associate to this space a fibration $\mathcal{E} \rightarrow \text{Conf}_2(\mathbb{C}^n)$, where \mathcal{E}_b is the corresponding complex blow-up of at $b \in \text{Conf}_2(\mathbb{C}^n)$. Seidel demonstrated in [26, Example 1.12] that, when $n = 2$, one may partially compactify this family to $\bar{\mathcal{E}}$ by allowing the two points to collide, where the discriminant Δ is a smooth divisor. A local normal disk D centered at $p \in \Delta$ thus gives a sub-fibration over D , where the fiber over p is a surface with a single ordinary double point. Then [26, Lemma 1.11] shows the monodromy around ∂D is precisely a Dehn twist when an appropriate Kähler form is endowed on the fiber. But this monodromy is equally clearly a ball-swapping, which establishes the relation between ball-swapping and the Dehn twists as monodromies in algebraic geometry. This algebro-geometric point of view also provides another (more elegant) proof for Lemma 3.4, except now one needs to sort out details for the last step of untwisting on the removed divisor. In a Lefschetz fibration point of view, this untwist is equivalent to slowing down the Hamiltonian at infinity in the description of Seidel’s Dehn twist. One may consult [23] for further details. However, the construction of ball-swapping is a priori richer than Dehn twists even in dimension 4. Formally, given symplectic manifold M^4 , consider the action of Hamiltonian group $\text{Ham}(M)$ on the space of ball-embeddings $\text{Emb}_\epsilon(M) = \{\phi : \coprod_{i=1}^n B(\epsilon_i) \rightarrow M\}$ for $\epsilon = (\epsilon_1, \dots, \epsilon_n)$. Take an orbit \mathcal{O} of the action, the stabilizer of the action descends to a subgroup of $\text{Symp}(M\#\overline{n\mathbb{C}\mathbb{P}^2})$. Then isotopy classes of ball-swappings are the images of $\pi_0(\text{Symp}(M\#\overline{n\mathbb{C}\mathbb{P}^2}))$ under the connecting map from $\pi_1(\text{Emb}_\epsilon(M))$. From this formal point of view, the above algebro-geometric construction amounts to the ball-swappings when restricted to images of

$$\pi_1(\text{Conf}_n(M)) \rightarrow \pi_1(\text{Emb}_\epsilon(M)) \rightarrow \pi_0(\text{Symp}(M\#\overline{n\mathbb{C}\mathbb{P}^2})),$$

- while the image of the second arrow forms the full ball-swapping subgroup. In general, the inclusion $\text{Conf}_n(M) \rightarrow \text{Emb}_\epsilon(M)$ is not a homotopy equivalence even when $n = 1$ and M is as simple as $S^2 \times S^2$ with non-monotone forms [15] (but is still 1-connected!). In this particular example one already sees the sizes of packed balls come into play. In higher dimensions, the topology of space of ball-embedding in even \mathbb{C}^n is completely open. Therefore, it seems interesting to clarify the gap between the algebro-geometric construction and the full ball-swapping subgroup.
- The ball-swapping symplectomorphisms seem particularly useful in problems involving $\pi_0(\text{Symp}(M))$, when M is a rational or ruled manifold. From examples known to date and the algebro-geometric constructions above, it seems reasonable to speculate that ball-swappings in dimension 4 is generally related to Lagrangian Dehn twists, at least for those with $b^+ = 1$. A particular tempting question asks that, a blow-up at n balls on $\mathbb{C}\mathbb{P}^2$ with generic sizes has a connected symplectomorphism group (because they do not admit any Lagrangian Dehn twists, see [16]). But to reduce the subgroup generated by ball-swapping into problems of braids, as in what appeared in this note (and also [6, 26]) seems to require independent efforts in finding an appropriate symplectic spheres, as well as a good control on the bubbles. Here we roughly sketch a viewpoint through ball-swapping to $\pi_0(\text{Symp}(M))$ when $M = \mathbb{C}\mathbb{P}^2 \# 5\mathbb{C}\mathbb{P}^2$ where all extra technicalities can be proved irrelevant. In this case, Evans [6] showed that $\pi_0(\text{Symp}(M)) = \text{Diff}(S^2, *)$, where $*$ is a fixed set on S^2 consisting of 5 points, which can be identified with a braid group on S^2 . This coincides with an observation of Seidel [26, Example 1.13]. Jonny Evans also explained to the author that these braids can be seen to be generated by Lagrangian Dehn twists, following from results of [6].¹
- From the ball-swapping point of view, by blowing down the 5 exceptional spheres of classes E_1, \dots, E_5 , we have 5 embedded balls B_1, \dots, B_5 intersecting a $2H$ -sphere C normally. Then the restriction of swapping these 5 balls on the $2H$ -sphere gives precisely a copy of the group $\text{Diff}(S^2, *)$, where the 5 points now is actually the 5 disks coming from intersections $B_i \cap C$. To get the actual statement of Evans, one still needs to switch between blow-ups and downs and go through Evans's proof to show all contributions from other components (configuration space of curves involved, automorphisms of normal bundles, etc.) cancels.
- The ball-swapping also gives a way of seeing these braidings actually come from Lagrangian Dehn twists, but we will not give full details. A naive attempt is to include two embedded balls above to a larger ball normally intersecting C as in the proof of Lemma 3.5. However, this cannot work for packing size restrictions. Instead, Evans shows that there is an embedded Lagrangian $\mathbb{R}\mathbb{P}^2$ in the complement of a configuration of symplectic spheres consisting of classes $\{2H, E_1, \dots, E_5\}$. By removing this $\mathbb{R}\mathbb{P}^2$, one can show that the remainder of M can be identified with a symplectic fibration over C , where a generic fiber is a disk, and there are 5 singular fibers consisting of the union of a E_i -sphere and disk. One then choose a disk in C separating the intersections of the E_1 and E_2 -spheres with C with other E_i 's, then the fibration restricted to this disk is a product of two D_2 's blown-up 2

¹ Private communications.

points, denoted as V_2 . At this point one easily reproduces the proof of Lemma 3.5 with an identification of the the symplectization of $V_2 \setminus C$ with T^*S^2 .

- In higher dimensions ball-swapping seems a new way of constructing monodromies and quite symplectic in flavor. For example, when the ball-packing is sufficiently small, because of Darboux theorem one could essentially move the embedded balls as if they were just points. The question when such symplectomorphisms are actually Hamiltonian seems intriguing.

One particular situation in question is when there is only one embedded ball. Suppose a small embedded ball moves along a loop which is nontrivial in $\pi_1(M)$, is it true that the resulting ball-swapping always lies outside the Hamiltonian group? [29] explained the existence of a Lagrangian torus near a small blow-up. It is not difficult to find a symplectomorphism in the component of $\text{Symp}(M)$ where the ball-swapping lies, so that this torus is invariant under it. Thus, in the sense of [27], one constructs a family of objects in the Fukaya category of the mapping torus of $M \# \overline{\mathbb{C}P^2}$ naturally associated to a ball-swapping.

- In principle, one may extend the “swapping philosophy” to a more general background of surgeries to obtain automorphisms, even in non-symplectic situations. As a simple example, consider a Hamiltonian loop of Lagrangian 2-torus in M^4 , by which we mean a path of Hamiltonians $\phi_t \subset \text{Ham}(M)$ such that $\phi_{0,1}(L) = L$. One may perform a Luttinger surgery on L [13], which gives a new symplectic manifold M^L . Then the “swapping” ϕ_1 is lifted to an automorphism of M^L .

One may also choose any symplectic neighborhood of a symplectic submanifold, as long as one has interesting loops of these objects. Naively thinking, when the submanifold is actually a divisor, the swapping symplectomorphism should always be Hamiltonian, but the author has no proof for that. Otherwise, since blowing up a divisor only gives a deformation of the symplectic form, one may always deform the form first and then swap the divisor. This will yield a symplectomorphism of the original manifold M , which should contain interesting information if not a Hamiltonian.

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