ON L_p AFFINE SURFACE AREA AND CURVATURE MEASURES

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Abstract. The relationship between L_p affine surface area and curvature measures is investigated. As a result, a new representation of the existing notion of L_p affine surface area depending only on curvature measures is derived. Direct proofs of the equivalence between this new representation and those previously known are provided. The proofs show that the new representation is, in a sense, "polar" to that of Lutwak's and "dual" to that of Schütt & Werner's.

1. Introduction

Since its introduction by Lutwak in [29], L_p affine surface area (defined below) has become a fundamental concept in the L_p Brunn–Minkowski theory and has appeared in a growing number of works (e.g., Ludwig [20, 21], Paouris & Werner [34], Werner & Ye [47, 48], and Ye [49]). Different approaches to L_p affine surface area have been discussed, e.g., Meyer & Werner [33], Schütt & Werner [41], and Werner [45]. Characterization theorems for L_p affine surface area have been given, e.g., Haberl & Parapatits [12, 13], and Ludwig & Reitzner [23]. Its relation to PDE was explored, e.g., Lutwak & Oliker [30]. Connections between L_p affine surface area and information theory were discovered in Werner [46]. For the $p = 1$ case, L_p affine surface area is an older notion usually referred to simply as affine surface area. Results here are even more numerous. Different approaches to this notion include Leichtweiß [17], Lutwak [25, 26], Meyer & Werner [32], Schütt & Werner [40], and Werner [43, 44]. A characterization of affine surface area was given in Ludwig & Reitzner [22]. Connections between affine surface area and the affine Plateau problem were discussed, e.g., Trudinger & Wang [42]. It is also not surprising to see the appearance of affine surface area in polytopal approximation, e.g., Bárány [3], Böröczky [4], and Gruber [9, 10, 11]. Perhaps, most importantly, it is the crucial ingredient in fundamental affine isoperimetric inequalities and gives rise to some affine analytic inequalities, e.g., Artstein-Avidan, Klartag, Schütt & Werner [2], Caglar & Werner [6], and Lutwak [24, 27].

Let $K \subset \mathbb{R}^n$ be a convex body (compact convex set with non-empty interior). The curvature measures of K are a list of n Borel measures defined on the boundary of K that can be defined via local parallel sets (see e.g., Chapter 4 in [38]). In the current paper, the relationship between L_p affine surface area and curvature measures will be explored. As a result, a new representation of the existing notion of L_p affine surface area using only curvature measures will be derived. New proofs of the important properties of L_p affine surface area, such as the upper semi-continuity and the L_p affine isoperimetric inequality, will be given using the new representation. The proofs given will be ones requiring no prior knowledge of the properties already established using other definitions of L_p affine surface area. It is also the aim of this paper to investigate the relationship between the new representation of L_p affine surface area and the three existing ones. This will be done by providing direct proofs of equivalence between the new representation and those previously given. It will become apparent that the new form of L_p affine surface area is, in a sense, "polar" to that of Lutwak's and "dual" to Schütt & Werner's. In order to establish the equivalence, the Lipschitz property of a sequence of restrictions of the inverse Gauss map will be investigated, which may be of independent interest. This will be discussed in Section 5.

It is important to note that it is the attempt of the current paper to give a new representation of the usual L_p affine surface area (not to define a new (different) L_p affine surface area).

The notion of affine surface area traces back to affine differential geometry. In affine differential geometry, the affine surface area of a convex body K with sufficiently smooth boundary (at least C^2) and everywhere positive Gauss curvature is given by

$$\Omega(K) = \int_{\partial K} H_K^{\frac{1}{n+1}}(x) d\mathcal{H}^{n-1}(x), \tag{1.1}$$

where $H_K(x)$ is the Gauss curvature of K at $x \in \partial K$ (the boundary of K) and $\mathcal{H}^{n-1}$ is $(n-1)$ dimensional Hausdorff measure.

When K has C^2 boundary with positive Gauss curvature, the Gauss map $\nu_K : \partial K \to S^{n-1}$ is nice enough to allow the change of variable $u = \nu_K(x)$ and we get:

$$\Omega(K) = \int_{S^{n-1}} F_K^{\frac{n}{n+1}}(u) d\mathcal{H}^{n-1}(u). \tag{1.2}$$

Here $F_K : S^{n-1} \to \mathbb{R}$ is the curvature function of K.

A very important result in affine differential geometry is the affine isoperimetric inequality which characterizes ellipsoids. For a convex body $K \subset \mathbb{R}^n$ with C^2 boundary and positive Gauss curvature,

$$\Omega(K)^{n+1} \leq n^{n+1}\omega_n^2 V(K)^{n-1}, \tag{1.3}$$

with equality if and only if K is an ellipsoid. Here $V(K)$ is the volume of K and ω_n is the volume of the n-dimensional unit ball.

Extending the definition of affine surface area to one that works for general convex bodies (without smoothness assumptions) and still respects the basic properties of the classical definition was of huge interest during the late 80s and 90s (in the previous century). In particular, is there a way to do the extension so that the affine isoperimetric inequality, with the same equality conditions, still holds? The first attempt was made by Petty [35]. He observed that (1.2) makes sense for convex bodies that possess curvature functions. The affine isoperimetric inequality was also shown to hold under this extension.

Although the Gauss curvature and the curvature function do not necessarily exist for general convex bodies, the generalized Gauss curvature and the generalized curvature function (see [38] or Section 2) exist almost everywhere on ∂K and S^{n-1} with respect to $(n-1)$ dimensional Hausdorff measure. Thus (1.1) and (1.2) already suggest two possible extensions. But, the two extensions are not trivial at all, since the two integrals might not make sense.

With the notion of floating body, Leichtweiß was able to give a geometric meaning to (1.2) when F_K is the generalized curvature function.

Definition 1.1 (Affine surface area by Leichtweiß [17])**.** *Let $K \subset \mathbb{R}^n$ be a convex body. The affine surface area of K is given by*

$$\Omega(K) = \int_{S^{n-1}} F_K^{\frac{n}{n+1}}(u) d\mathcal{H}^{n-1}(u), \tag{1.4}$$

where F_K is the generalized curvature function.

One is tempted to use a strategy similar to how we arrived at (1.2) to get (1.1) to work for general convex bodies. This turns out to be invalid since neither the Gauss map nor the inverse Gauss map, in this case, is smooth enough (in fact, not even Lipschitz) to permit such a change of variable. In spite of this unfortunate fact, there is a natural extension to (1.1). Schütt & Werner [40], via the notion of convex floating body, were able to give a geometric meaning to the integral representation (1.1) with H_K being the generalized Gauss curvature.

Definition 1.2 (Affine surface area by Schütt & Werner [40]). *Let $K \subset \mathbb{R}^n$ be a convex body. The affine surface area of K is given by*

$$\Omega(K) = \int_{\partial K} H_K^{\frac{1}{n+1}}(x) d\mathcal{H}^{n-1}(x), \tag{1.5}$$

where H_K is the generalized Gauss curvature.

One of the major characteristics that distinguishes affine surface area (look at either (1.4) or (1.5)) and other geometric invariants is that affine surface area is not continuous with respect to the Hausdorff metric. For example, any convex body can be approximated by polytopes, but polytopes are always of zero affine surface area. Given this fact, only upper semi-continuity can be expected. But, even establishing the upper semi-continuity of classical affine surface area (in the smooth case) was unsolved in the 80s. One of the difficulties of establishing this lies in the lack of knowledge of the limit behaviors of the Gauss curvature and the curvature function. Since (1.4) and (1.5) take similar formulations, difficulty persists. Note that the lack of continuity also adds to the difficulty of establishing the affine isoperimetric inequality, since we cannot establish the inequality for a dense class of convex bodies and then take a limit.

The long conjectured upper semi-continuity of classical affine surface area was settled by Lutwak. In [26], he found the following characterization of affine surface area for general convex bodies.

Definition 1.3 (Affine surface area by Lutwak [26]). *Let $K \subset \mathbb{R}^n$ be a convex body. The affine surface area of K is given by*

$$\Omega(K) = \inf_h \left\{ \left(\int_{S^{n-1}} h^n(u) d\mathcal{H}^{n-1}(u) \right)^{\frac{1}{n+1}} \left(\int_{S^{n-1}} h^{-1}(u) dS_K(u) \right)^{\frac{n}{n+1}} \right\}, \tag{1.6}$$

where the infimum is taken over all positive continuous functions $h : S^{n-1} \to \mathbb{R}$, and S_K is the surface area measure of K.

Note that Lutwak's definition applies to all convex bodies (even the ones without smoothness assumptions). More importantly, the upper semi-continuity of (1.6) follows directly from the weak continuity of the measures in the integrals and the fact that the infimum of a class of continuous functionals is upper semi-continuous. Since (1.6) agrees with classical affine surface area for smooth convex bodies with everywhere positive curvature function (also shown in [26]), this, in turn, proves the upper semi-continuity of classical affine surface area. As shown in [26], the affine isoperimetric inequality can also be established in this case by using the Blaschke–Santaló inequality.

A natural question to ask is: are Definitions (1.4), (1.5), and (1.6) equivalent? As was explained earlier, the equivalence of (1.4) and (1.5) is by no means a trivial problem for general convex bodies. It was not until 1993 that Schütt, in [39], proved the equivalence—using a somewhat indirect method. A direct proof was given in [15] by Hug. That (1.4) and (1.6) are equivalent was shown by Leitchweiß in [18]. Note that in Lutwak's original definition, the infimum is only taken over all positive functions h such that $\int_{S^{n-1}} uh(u)^{n+1} d\mathcal{H}^{n-1}(u) = o$. The removal of this restriction was proposed by Leichtweiß in [18] and the equivalence between this formulation and Lutwak's original definition was shown by Dolzmann & Hug in [7] using a topological argument.

Note that it is trivial to see that affine surface area is translation invariant. Schütt, in [39], proved that affine surface area is a valuation. Hence, affine surface area is an upper semi-continuous valuation that is invariant under volume preserving affine transformations. In a landmark work, Ludwig & Reitzner [22] established the "converse": if a real-valued upper semi-continuous valuation on the set of convex bodies is invariant under volume preserving affine transformations, then it must be of the form $c_0V_0 + c_1V + c_2\Omega$ with $c_0, c_1, c_2 \in \mathbb{R}$ and $c_2 > 0$. Here V_0 is the Euler characteristic, V is volume, and Ω is affine surface area.

Observe that (1.4) and (1.5) are "polar" to each other in the sense that one is defined as an integral over the boundary of the convex body (domain of the Gauss map), while the other is defined as an integral over the unit sphere (image of the Gauss map). In fact, as the proof provided by Hug in [15] indicates, Definitions (1.4) and (1.5) are linked by the Gauss map. Also note that (1.6) is "dual" to (1.4) as one can see in the proof in [18].

Recall that for a convex body $K \subset \mathbb{R}^n$, curvature measures are a list of n Borel measures defined on ∂K that can be defined via local parallel sets. For details, the reader should consult Chapter 4 in [38]. Among these curvature measures, the 0-th and $(n-1)$-th curvature measures have stronger geometric meanings. More specifically, for a Borel set $\beta \subset \partial K$, the 0-th curvature measure $C_0(K, \beta)$ and the $(n-1)$-th curvature measure $C_{n-1}(K, \beta)$ of β are given by

$$C_0(K, \beta) = \mathcal{H}^{n-1}(\nu(K, \beta)) \quad \text{and} \quad C_{n-1}(K, \beta) = \mathcal{H}^{n-1}(\beta),$$

where $\nu(K, \beta) \subset S^{n-1}$ is the set of outer unit normals of K at points in β.

Note that previous formulations of affine surface area involve $(n-1)$ dimensional Hausdorff measure on ∂K (Definition (1.5)), $(n-1)$ dimensional Hausdorff measure on S^{n-1} (Definitions (1.4) and (1.6)), and the surface area measure of K (Definition (1.6)). Since curvature measures are also a crucial type of measures associated to a convex body K, it is natural to study the relationship between affine surface area and the curvature measures. It is the purpose of this paper to investigate this missing element. To be precise, the following theorem will be proved:

Theorem 1.4. *For a convex body $K \subset \mathbb{R}^n$,*

$$\int_{\partial K} H_K^{\frac{1}{n+1}}(x) d\mathcal{H}^{n-1}(x) = \inf_g \left\{ \left(\int_{\partial K} g^{-n}(x) dC_0(K, x) \right)^{\frac{1}{n+1}} \left(\int_{\partial K} g(x) dC_{n-1}(K, x) \right)^{\frac{n}{n+1}} \right\}, \quad (1.7)$$

where the infimum is taken over all positive continuous functions $g : \partial K \to \mathbb{R}$.

In light of Theorem 1.4, we may view the right side of (1.7) as a new representation of the existing notion of affine surface area.

Definition 1.5. *Let $K \subset \mathbb{R}^n$ be a convex body. The affine surface area of K can be defined by*

$$\Omega(K) = \inf_g \left\{ \left(\int_{\partial K} g^{-n}(x) dC_0(K, x) \right)^{\frac{1}{n+1}} \left(\int_{\partial K} g(x) dC_{n-1}(K, x) \right)^{\frac{n}{n+1}} \right\}, \quad (1.8)$$

where the infimum is taken over all positive continuous functions $g : \partial K \to \mathbb{R}$.

Note again that this form of affine surface area uses only curvature measures.

A list of properties of (1.8) will be given in Section 4. In particular, among other things, the upper semi-continuity (Theorem 4.3) and the affine isoperimetric inequality with equality condition (Theorem 6.4) will be demonstrated using the new representation (1.8). The author would like to point out that although previously established, none of the results proved in Section 4 require acknowledgement of properties proved under the existing forms.

With the recent development of the L_p Brunn-Minkowski theory, efforts were also made to generalize affine surface area to its L_p analogue. One of the key findings in the L_p Brunn-Minkowski theory is the L_p curvature function discovered by Lutwak [28, 29]. Given a convex body K that possesses a curvature function F_K, the L_p curvature function may be defined by $h_K^{1-p} F_K$ with h_K being the support function of K. Since the generalized curvature function exists almost everywhere for an arbitrary convex body, the generalized L_p curvature function also exists almost everywhere. Given this notion, L_p affine surface area can be defined.

The notion of L_p affine surface area was introduced by Lutwak in [29]. He defined the L_p affine surface area of a convex body $K \subset \mathbb{R}^n$ that contains the origin in its interior to be,

$$\Omega_p(K) = \inf_h \left\{ \left(\int_{S^{n-1}} h^n(u) d\mathcal{H}^{n-1}(u) \right)^{\frac{p}{n+p}} \left(\int_{S^{n-1}} h^{-p}(u) h_K^{1-p}(u) dS_K(u) \right)^{\frac{n}{n+p}} \right\}, \tag{1.9}$$

where the infimum is taken over all positive continuous functions $h : S^{n-1} \to \mathbb{R}$. Although Lutwak originally presented this definition for the case $p \geq 1$, it works perfectly fine for any $0 < p < 1$ as observed by Hug in [15].

As with the classical $p = 1$ case, different forms of L_p affine surface area exist. For each $p > 0$, the following form of L_p affine surface area was given by Lutwak [29] for convex bodies that possess a continuous curvature function and by Hug [15] for general convex bodies,

$$\Omega_p(K) = \int_{S^{n-1}} \left(\frac{F_K(u)}{h_K^{p-1}(u)} \right)^{\frac{n}{n+p}} d\mathcal{H}^{n-1}(u). \tag{1.10}$$

Analogously, for each $p > 0$, another form of L_p affine surface area, was given by Hug in [15],

$$\Omega_p(K) = \int_{\partial K} \left(\frac{H_K(x)}{(h_K(\nu_K(x)))^{(p-1)n/p}} \right)^{\frac{p}{n+p}} d\mathcal{H}^{n-1}(x). \tag{1.11}$$

Note that (1.10) is the L_p extension of (1.4) while (1.11) is the L_p extension of (1.5).

The equivalence of (1.10) and (1.11) was proved by Hug in [15]. That (1.9) and (1.10) are equivalent for convex bodies that possess a positive continuous curvature function was due to Lutwak [29], and can in general be proved in a similar way as Leitchweiß did in [18] for $p = 1$ as pointed out in [15].

It is also possible to prove the analogue of Theorem 1.4 in the L_p setting. Namely,

Theorem 1.6. *Let $p > 0$ be a real number. For each convex body $K \subset \mathbb{R}^n$ that contains the origin in its interior,*

$$\begin{aligned} &\int_{\partial K} \left(\frac{H_K(x)}{(h_K(\nu_K(x)))^{(p-1)n/p}} \right)^{\frac{p}{n+p}} d\mathcal{H}^{n-1}(x) \\ &= \inf_g \left\{ \left(\int_{\partial K} g^{-n}(x) dC_0(K,x) \right)^{\frac{p}{n+p}} \left(\int_{\partial K} g^p(x) (h_K(\nu_K(x)))^{1-p} dC_{n-1}(K,x) \right)^{\frac{n}{n+p}} \right\}, \end{aligned}$$

where the infimum is taken over all positive continuous functions $g : \partial K \to \mathbb{R}$.

2. Preliminaries

We will be working mainly in $\mathbb{R}^n$ with the canonical inner product $\langle \cdot, \cdot \rangle$. The usual Euclidean 2-norm will be denoted by $||\cdot||$ and the open (resp. closed) ball of radius r, which is centered at x, will be denoted by $B(x,r)$ (resp. $B[x,r]$). We write ω_n for the volume of the n-dimensional unit ball. For a subset $A \subset \mathbb{R}^n$, we will write $\bar{A}$ and A^c for the closure of A and the complement of A, respectively. The characteristic function of E, for any set E, is written as $\mathbf{1}_E$.

A subset K of $\mathbb{R}^n$ is called a *convex body* if it is a compact convex set with non-empty interior. The set of all convex bodies that contain the origin in the interior is denoted by $\mathcal{K}_0^n$. The boundary of K will be written as ∂K. For an integer $m \leq n$, we will write $\mathcal{H}^m$ for m dimensional Hausdorff measure. If η is a measure on a topological space X and $A \subset X$ is η measurable, the restriction of η to A will be denoted by $\eta \llcorner A$.

Associated to each convex body K is the support function $h_K : \mathbb{R}^n \to \mathbb{R}$ given by

$$h_K(x) = \max\{\langle x, y \rangle : y \in K\},$$

for each $x \in \mathbb{R}^n$.

The supporting hyperplane $P(K, u)$ of K for each $u \in S^{n-1}$ is given by

$$P(K, u) = \{x \in \mathbb{R}^n : \langle x, u\rangle = h_K(u)\}.$$

At each boundary point $x \in \partial K$, a unit vector u is said to be an *outer unit normal* of K at x if $P(K, u)$ passes through x. For a subset $\beta \subset \partial K$, the spherical image, $\nu(K, \beta)$, of K at β, is the set of all outer unit normal vectors of K at points in β. A boundary point x is *regular* if $\nu(K, \{x\})$ contains exactly one point in S^{n-1}. Denote by $\operatorname{reg} K$ the set of all regular boundary points of K. The *Gauss map*, $\nu_K : \operatorname{reg} K \to S^{n-1}$ is the map that takes each regular boundary point to the unique outer unit normal of that point. Similarly, for each subset $\omega \subset S^{n-1}$, the inverse spherical image, $\tau(K, \omega)$, of K at ω, is the set of all boundary points of K that have outer normal vectors in ω. A unit vector u is *regular* if $\tau(K, \{u\})$ contains exactly one point in ∂K. Denote by $\operatorname{regn} K$ the set of all regular normal vectors. The *inverse Gauss map*, $\tau_K : \operatorname{regn} K \to \partial K$ is the map that takes each regular normal vector to the unique point in $\tau(K, \{u\})$. Both the Gauss map and the inverse Gauss map are continuous (see Lemma 2.2.12 in [38]).

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function. The set

$$\partial f(x) = \{v \in \mathbb{R}^n : f(y) \geq f(x) + \langle v, y - x\rangle, \ \forall y \in \mathbb{R}^n\}$$

is called the *subdifferential* of f at x. If $\vartheta : \mathbb{R}^n \to \mathbb{R}^n$ satisfies $\vartheta(x) \in \partial f(x)$ for each $x \in \mathbb{R}^n$, then it is called a *subgradient choice* of f. Moreover, f is differentiable at x if and only if $\partial f(x)$ contains only $\nabla f(x)$, the gradient of f at x.

The following notion of second order differentiability is useful. We say f is *second order differentiable* at x_0 in the generalized sense if f is differentiable at x_0 in the classical sense and there exists a symmetric linear map $Af(x_0) : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$f(y) = f(x_0) + \langle \nabla f(x_0), y - x_0\rangle + \frac{1}{2}\langle Af(x_0)(y - x_0), y - x_0\rangle + o(||y - x_0||^2),$$

for every $y \in \mathbb{R}^n$. It follows from [1] that a convex function f is *second order differentiable* at x_0 in the generalized sense if and only if there exists a neighborhood V of x_0 and a symmetric linear map $Af(x_0) : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$||\vartheta(y) - \vartheta(x_0) - Af(x_0)(y - x_0)|| = o(||y - x_0||)$$

for all $y \in V$ and all subgradient choices ϑ. Note that the generalized second order differentiability is a local property. Hence, the above notion extends naturally to the case where f is only defined on an open subset of $\mathbb{R}^n$.

For a regular boundary point $x_0 \in \partial K$, suppose $u_0 = \nu_K(x_0)$. The *tangent space*, $T_{x_0}K$, of K at x_0 is the linear subspace $P(K, u_0) - x_0$. Write $\tilde{y}$ for the orthogonal projection of y to $T_{x_0}K$ for each $y \in \mathbb{R}^n$. We can choose a number $\varepsilon > 0$ and a neighborhood $U(x_0)$ of x_0 such that for each $x \in U(x_0) \cap \partial K$,

$$x = x_0 + \tilde{x} - \widetilde{x_0} - f(\tilde{x} - \widetilde{x_0})u_0, \tag{2.1}$$

where $||\tilde{x} - \widetilde{x_0}|| < \varepsilon$ and $f : T_{x_0}K \cap B(o, \varepsilon) \to \mathbb{R}$ is a convex function satisfying $f \geq 0$ and $f(o) = 0$. We say a regular boundary point x_0 is *normal* if f in (2.1) is second order differentiable at o in the generalized sense. Denote by $\operatorname{nor} K$ the set of all normal boundary points of K. With a proper choice of orthonormal basis $\mathfrak{B} = \{e_1, e_2, \ldots, e_n\}$ satisfying $e_1, \ldots, e_{n-1} \in T_{x_0}K$ and $e_n = -u_0$, it is possible to write f as:

$$f(\tilde{x} - \widetilde{x_0}) = \frac{1}{2}\kappa_1(x_0)(x^1 - x_0^1)^2 + \ldots + \frac{1}{2}\kappa_{n-1}(x_0)(x^{n-1} - x_0^{n-1})^2 + o(||\widetilde{x} - \widetilde{x_0}||^2), \tag{2.2}$$

where $(x^1, \ldots, x^n)$ are the coordinates of x under $\mathfrak{B}$. Here, $\kappa_i(x_0)$ is called a generalized principal curvature while $e_i(x_0)$ is the associated generalized principal direction, for $1 \leq i \leq n - 1$. In this case, the generalized Gauss curvature $H_K(x_0)$ of K at x_0 is given by

$$H_K(x_0) = \kappa_1(x_0)\kappa_2(x_0)\cdots\kappa_{n-1}(x_0).$$

It follows from the Alexandrov Theorem [1, 5] that

$$\mathcal{H}^{n-1}(\partial K \setminus \operatorname{nor} K) = 0. \tag{2.3}$$

Hence, $H_K(x)$ is defined for $\mathcal{H}^{n-1}$ almost all $x \in \partial K$. The set H^+ is given by

$$H^+ = \{x \in \partial K : x \text{ is a normal boundary point and } H_K(x) > 0\}. \tag{2.4}$$

The support function h_K is differentiable at $u_0 \in S^{n-1}$ if and only if u_0 is a regular normal vector. In this case, $\nabla h_K(u_0) = \tau_K(u_0)$. (See Corollary 1.7.3 in [38].) It was the result of the Alexandrov Theorem [1, 5] that every convex function $f : \mathbb{R}^n \to \mathbb{R}$ is $\mathcal{H}^n$ almost everywhere second order differentiable in the generalized sense. In particular, the support function h_K is $\mathcal{H}^n$ almost everywhere second order differentiable in the generalized sense. Denote by $D^2(h_K)$ the set of all points at which h_K is second order differentiable in the generalized sense. The following properties can be easily seen from the homogeneity of h_K:

(1) If $u_0 \in D^2(h_K)$, then $tu_0 \in D^2(h_K)$, for all $t \in \mathbb{R} \setminus \{0\}$. Hence

$$\mathcal{H}^{n-1}(S^{n-1} \setminus D^2(h_K)) = 0. \tag{2.5}$$

(2) Given $u_0 \in D^2(h_K)$, we have that u_0 is an eigenvector of $Ah_K(u_0)$ with 0 being the associated eigenvalue. The fact that $Ah_K(u_0)$ is symmetric tells us that $u_0^\perp$ is an invariant subspace of $Ah_K(u_0)$.

The generalized curvature function of K at $u \in D^2(h_K) \cap S^{n-1}$, denoted by $F_K(u)$, is defined to be the determinant of $Ah_K(u)|_{u^\perp}$. Note that $F_K(u)$ is defined for $\mathcal{H}^{n-1}$ almost all $u \in S^{n-1}$. The set F^+ is given by

$$F^+ = \{u \in D^2(h_K) \cap S^{n-1} : F_K(u) > 0\}. \tag{2.6}$$

The surface area measure S_K of a convex body K is a Borel measure on S^{n-1} and is given by

$$S_K(\omega) = \mathcal{H}^{n-1}(\tau(K, \omega)),$$

for each Borel set $\omega \subset S^{n-1}$.

Recall that the 0-th curvature measure $C_0(K, \cdot)$ and the $(n-1)$-th curvature measure $C_{n-1}(K, \cdot)$ are Borel measures on the boundary of K and are given by

$$C_0(K, \beta) = \mathcal{H}^{n-1}(\nu(K, \beta)) \qquad \text{and} \qquad C_{n-1}(K, \beta) = \mathcal{H}^{n-1}(\beta), \tag{2.7}$$

for each Borel set $\beta \subset \partial K$. It is obvious that $C_0(K, \cdot)$ and $C_{n-1}(K, \cdot)$ are finite measures. The 0-th curvature measure $C_0(K, \cdot)$ has the following decomposition (see e.g., Hilfssatz 3.6 in [37] or (2.7) in [16]): for each Borel set $\beta \subset \partial K$,

$$C_0(K, \beta) = \int_\beta H_K(x) d\mathcal{H}^{n-1}(x) + \int_{\beta \cap \hat{\partial} K} dC_0(K, x), \tag{2.8}$$

where $\hat{\partial} K \subset \partial K$ is a Borel set and $\mathcal{H}^{n-1}(\hat{\partial} K) = 0$. In particular, one has

$$\int_{\partial K} H_K(x) d\mathcal{H}^{n-1}(x) < \infty. \tag{2.9}$$

Curvature measures are weakly continuous with respect to the Hausdorff metric (see [38]).

The following definitions are needed for Federer's coarea formula. See [8] for details.

A subset ω of $\mathbb{R}^n$ is said to be $(\mathcal{H}^{n-1}, n-1)$ *rectifiable* if $\mathcal{H}^{n-1}(\omega) < \infty$ and there exists $\{(f_i, E_i)\}_{i \in \mathbb{N}^+}$ such that $E_i \subset \mathbb{R}^{n-1}$ is bounded, $f_i : E_i \to \mathbb{R}^n$ is Lipschitz and $\mathcal{H}^{n-1}(\omega \setminus \cup_{i \in \mathbb{N}^+} f_i(E_i)) = 0$.

Let S be a non-empty subset of $\mathbb{R}^n$. The *tangent cone* of S at a given point $a \in \mathbb{R}^n$, denoted by $\mathrm{Tan}(S,a)$, can be defined as the set of $v \in \mathbb{R}^n$ such that for every $\varepsilon > 0$ there exists $x \in S$ and $r > 0$ with $||x-a|| < \varepsilon$ and $||r(x-a)-v|| < \varepsilon$.

Suppose η is a measure on $\mathbb{R}^n$. The $(n-1)$ *dimensional density* $\Theta^{n-1}(\eta,a)$ at $a \in \mathbb{R}^n$ is given by

$$\Theta^{n-1}(\eta,a) = \lim_{r\to 0+} \omega_{n-1}^{-1} r^{-(n-1)} \eta(B(a,r)),$$

if the limit exists. The $(\eta, n-1)$ *approximate tangent cone* $\mathrm{Tan}^{n-1}(\eta,a)$ at a is given by

$$\mathrm{Tan}^{n-1}(\eta,a) = \bigcap \{\mathrm{Tan}(S,a) : S \subset \mathbb{R}^n, \Theta^{n-1}(\eta \llcorner (\mathbb{R}^n \setminus S), a) = 0\}.$$

Suppose f maps a subset of $\mathbb{R}^n$ into $\mathbb{R}^n$. We say that f is $(\eta, n-1)$ *approximately differentiable* at a if and only if there exists $\xi \in \mathbb{R}^n$ and a continuous linear map $\zeta : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$\Theta^{n-1}\left(\eta \llcorner (\mathbb{R}^n \setminus \{x : ||f(x) - \xi - \zeta(x-a)|| \le \varepsilon\, ||x-a||\}), a\right) = 0,$$

for every $\varepsilon > 0$. In this case, the $(\eta, n-1)$ *approximate differential* of f at a, denoted by $(\eta, n-1)\,\mathrm{ap}\ Df(a)$, is given by

$$(\eta, n-1)\,\mathrm{ap}\ Df(a) = \zeta|_{\mathrm{Tan}^{n-1}(\eta,a)}.$$

Suppose V, W are two $(n-1)$-dimensional Hilbert spaces. Let $\bigwedge^{n-1} V$ and $\bigwedge^{n-1} W$ be the $(n-1)$th exterior power of V and W equipped with the induced inner products from V and W respectively. Every linear map $f : V \to W$ induces a map $\bigwedge^{n-1} f : \bigwedge^{n-1} V \to \bigwedge^{n-1} W$. By $\left|\left|\bigwedge^{n-1} f\right|\right|$, we mean the operator norm of $\bigwedge^{n-1} f$. Note that $\left|\left|\bigwedge^{n-1} f\right|\right|$ is just the absolute value of the determinant of f when $V = W$. See Chapter 1 in [8] for details.

When η is the restriction of $\mathcal{H}^{n-1}$ to some $\mathcal{H}^{n-1}$ measurable and $(\mathcal{H}^{n-1}, n-1)$ rectifiable subset of $\mathbb{R}^n$, by Theorem 3.2.19 in [8], the approximate tangent cone $\mathrm{Tan}^{n-1}(\eta,a)$ is an $(n-1)$ dimensional subspace of $\mathbb{R}^n$ for $\mathcal{H}^{n-1}$ almost all a in that subset. In this case (when $\mathrm{Tan}^{n-1}(\eta,a)$ is an $(n-1)$ dimensional subspace of $\mathbb{R}^n$), we call $\left|\left|\bigwedge^{n-1}(\eta, n-1)\,\mathrm{ap}\ Df(a)\right|\right|$ the $(\eta, n-1)$ *approximate Jacobian* of f at a and denote it by $(\eta, n-1)\,\mathrm{ap}\ Jf(a)$.

The following is a special case of Federer's coarea formula [8, Theorem 3.2.22]. Note that the original theorem works for any non-negative measurable function by the obvious application of the monotone convergence theorem.

Theorem 2.1 (Federer's coarea formula)**.** *Suppose $W, Z \subset \mathbb{R}^n$ are $\mathcal{H}^{n-1}$ measurable and $(\mathcal{H}^{n-1}, n-1)$ rectifiable. If $f : W \to Z$ is Lipschitz, then for each $\mathcal{H}^{n-1} \llcorner W$ measurable non-negative function g on W,*

$$\int_W g(x) \cdot (\mathcal{H}^{n-1} \llcorner W, n-1)\,\mathrm{ap}\ Jf(x) d\mathcal{H}^{n-1}(x) = \int_Z \int_{f^{-1}(z)} g(y) d\mathcal{H}^0(y) d\mathcal{H}^{n-1}(z).$$

It is implied in Theorem 2.1 that $\int_{f^{-1}(z)} g(y) d\mathcal{H}^0(y)$ is $\mathcal{H}^{n-1}$ measurable as a function in z.

3. Curvature Measures and L_p Affine Surface Area

In this section, we will prove the promised Theorem 1.6, which will reveal the relationship between L_p affine surface area and curvature measures. Notice that Theorem 1.4 follows by setting $p = 1$ in Theorem 1.6 and the obvious fact that both sides of (1.7) are translation invariant.

The following notations will be needed.

Let

$$T_1 = \left\{ g : \partial K \to \mathbb{R}\ \mathcal{H}^{n-1} \text{ measurable: } 0 < g < \infty \text{ and } \int_{\partial K} g^p(x) (h_K(\nu_K(x)))^{1-p} d\mathcal{H}^{n-1}(x) < \infty \right\},$$

$$T_2 = \{ g : \partial K \to \mathbb{R} \text{ continuous} : g > 0 \}.$$

Note that when $K \in \mathcal{K}_0^n$, the sets T_1 and T_2 have the following relationship:

$$T_1 \supset T_2. \tag{3.1}$$

Recall that H^+ is the set of normal boundary points with positive Gauss curvature (see (2.4)).

Proof of Theorem 1.6. For the sake of simplicity, let us introduce the following notations:

$$L_1(g) = \left(\int_{\partial K} g^{-n}(x) H_K(x) d\mathcal{H}^{n-1}(x)\right)^{\frac{p}{n+p}} \left(\int_{\partial K} g^p(x)(h_K(\nu_K(x)))^{1-p} d\mathcal{H}^{n-1}(x)\right)^{\frac{n}{n+p}}, \tag{3.2}$$

$$L_2(g) = \left(\int_{\partial K} g^{-n}(x) dC_0(K,x)\right)^{\frac{p}{n+p}} \left(\int_{\partial K} g^p(x)(h_K(\nu_K(x)))^{1-p} dC_{n-1}(K,x)\right)^{\frac{n}{n+p}}. \tag{3.3}$$

Note that $L_1(g)$ is defined for $g \in T_1$, while $L_2(g)$ is defined for any positive Borel measurable function $g : \partial K \to \mathbb{R}^n$ satisfying $\int_{\partial K} g^p(x)(h_K(\nu_K(x)))^{1-p} dC_{n-1}(K,x) < \infty$.

For each $g \in T_1$, by Hölder's inequality,

$$\begin{aligned}\int_{\partial K} \left(\frac{H_K(x)}{(h_K(\nu_K(x)))^{(p-1)\frac{n}{p}}}\right)^{\frac{p}{n+p}} d\mathcal{H}^{n-1}(x) &= \int_{\partial K} \left(\frac{H_K(x)}{(h_K(\nu_K(x)))^{(p-1)\frac{n}{p}}}\right)^{\frac{p}{n+p}} g^{-\frac{np}{n+p}}(x) g^{\frac{np}{n+p}}(x) d\mathcal{H}^{n-1}(x) \\ &\le L_1(g). \end{aligned} \tag{3.4}$$

For each $g \in T_2$, by (2.8),

$$\begin{aligned}\int_{\partial K} g^{-n}(x) dC_0(K,x) &= \int_{\hat{\partial} K} g^{-n}(x) dC_0(K,x) + \int_{\partial K} g^{-n}(x) H_K(x) d\mathcal{H}^{n-1}(x) \\ &\ge \int_{\partial K} g^{-n}(x) H_K(x) d\mathcal{H}^{n-1}(x).\end{aligned}$$

This, the fact that $\mathcal{H}^{n-1}(\beta) = C_{n-1}(K,\beta)$ for each Borel set $\beta \subset \partial K$, (3.2), and (3.3) imply that,

$$L_1(g) \le L_2(g), \tag{3.5}$$

for each $g \in T_2$.

Equations (3.4), (3.5), and the fact that $T_1 \supset T_2$ show

$$\int_{\partial K} \left(\frac{H_K(x)}{(h_K(\nu_K(x)))^{(p-1)\frac{n}{p}}}\right)^{\frac{p}{n+p}} d\mathcal{H}^{n-1}(x) \le \inf_{g \in T_1} L_1(g) \le \inf_{g \in T_2} L_1(g) \le \inf_{g \in T_2} L_2(g). \tag{3.6}$$

To complete the proof, let us now show

$$\inf_{g \in T_2} L_2(g) \le \int_{\partial K} \left(\frac{H_K(x)}{(h_K(\nu_K(x)))^{(p-1)\frac{n}{p}}}\right)^{\frac{p}{n+p}} d\mathcal{H}^{n-1}(x). \tag{3.7}$$

It suffices, for every $\varepsilon > 0$, to find a $g \in T_2$ such that

$$L_2(g) \le \int_{\partial K} \left(\frac{H_K(x)}{(h_K(\nu_K(x)))^{(p-1)\frac{n}{p}}}\right)^{\frac{p}{n+p}} d\mathcal{H}^{n-1}(x) + \varepsilon. \tag{3.8}$$

Let $\tilde{f}_i : \partial K \to (0,\infty)$ be defined by

$$\tilde{f}_i(x) = \begin{cases} i, & \text{if } x \in \hat{\partial} K, \\ h_K(\nu_K(x))^{\frac{p-1}{n+p}} H_K^{\frac{1}{n+p}}(x), & \text{if } x \in H^+ \setminus \hat{\partial} K, \\ \frac{1}{i}, & \text{if } x \notin H^+ \cup \hat{\partial} K. \end{cases} \tag{3.9}$$

Note that for each $\mathcal{H}^{n-1}$ measurable subset $A \subset \partial K$, there exists a Borel measurable set $\bar{A} \subset \partial K$ such that $\bar{A} \supset A$ and $\mathcal{H}^{n-1}(\bar{A}\setminus A) = 0$. This and the fact that $\hat{\partial} K$ is a Borel set ensure that we can modify the value of $\tilde{f}_i$ on a subset $Z \subset \partial K \setminus \hat{\partial} K$ with $\mathcal{H}^{n-1}(Z) = 0$, such that the resulting function is a Borel measurable function. Denote the resulting function by f_i. Clearly $f_i(x) = \tilde{f}_i(x) = i$ for any $x \in \hat{\partial} K$ and $f_i(x) = \tilde{f}_i(x)$ for $\mathcal{H}^{n-1}$ almost all $x \in \partial K$. Define

$$h_i(x) = \begin{cases} \frac{1}{i}, & \text{if } f_i(x) < \frac{1}{i}, \\ f_i, & \text{if } \frac{1}{i} \le f_i(x) \le i, \\ i, & \text{if } i < f_i(x). \end{cases} \tag{3.10}$$

Note that both f_i and h_i are Borel measurable, and $\frac{1}{i} \le h_i \le i$.

By the fact that both $C_0(K,\cdot)$ and $C_{n-1}(K,\cdot)$ are finite measures, the assumption that K contains the origin in its interior, (2.8), the fact that $\mathcal{H}^{n-1}(\beta) = C_{n-1}(K,\beta)$ for each Borel set $\beta \subset \partial K$, the choice of f_i, and $\mathcal{H}^{n-1}(\hat{\partial} K) = 0$, we can compute the following limit,

$$\begin{aligned}
&\lim_{i\to\infty} \left(\int_{\partial K} (f_i^{-n}(x) + \frac{1}{i^n}) dC_0(K,x)\right)^{\frac{p}{n+p}} \left(\int_{\partial K} (f_i^p(x) + \frac{1}{i^p})(h_K(\nu_K(x)))^{1-p} dC_{n-1}(K,x)\right)^{\frac{n}{n+p}} \\
&= \lim_{i\to\infty} \left(\int_{\partial K} f_i^{-n}(x) dC_0(K,x)\right)^{\frac{p}{n+p}} \left(\int_{\partial K} f_i^p(x)(h_K(\nu_K(x)))^{1-p} dC_{n-1}(K,x)\right)^{\frac{n}{n+p}} \\
&= \lim_{i\to\infty} \left(\int_{\partial K} f_i^{-n}(x) H_K(x) d\mathcal{H}^{n-1}(x) + \frac{1}{i^n}\int_{\hat{\partial} K} dC_0(K,x)\right)^{\frac{p}{n+p}} \\
&\qquad\qquad \cdot \left(\int_{\partial K} f_i^p(x)(h_K(\nu_K(x)))^{1-p} dC_{n-1}(K,x)\right)^{\frac{n}{n+p}} \\
&= \lim_{i\to\infty} \left(\int_{\partial K} f_i^{-n}(x) H_K(x) d\mathcal{H}^{n-1}(x)\right)^{\frac{p}{n+p}} \left(\int_{\partial K} f_i^p(x)(h_K(\nu_K(x)))^{1-p} d\mathcal{H}^{n-1}(x)\right)^{\frac{n}{n+p}} \\
&= \lim_{i\to\infty} \left(\int_{H^+} f_i^{-n}(x) H_K(x) d\mathcal{H}^{n-1}(x)\right)^{\frac{p}{n+p}} \\
&\qquad \cdot \left(\int_{H^+} f_i^p(x)(h_K(\nu_K(x)))^{1-p} d\mathcal{H}^{n-1}(x) + \frac{1}{i^p}\int_{\partial K\setminus H^+} h_K(\nu_K(x))^{1-p} d\mathcal{H}^{n-1}(x)\right)^{\frac{n}{n+p}} \\
&= \lim_{i\to\infty} \left(\int_{H^+} f_i^{-n}(x) H_K(x) d\mathcal{H}^{n-1}(x)\right)^{\frac{p}{n+p}} \left(\int_{H^+} f_i^p(x)(h_K(\nu_K(x)))^{1-p} d\mathcal{H}^{n-1}(x)\right)^{\frac{n}{n+p}} \\
&= \int_{\partial K} \left(\frac{H_K(x)}{(h_K(\nu_K(x)))^{(p-1)\frac{n}{p}}}\right)^{\frac{p}{n+p}} d\mathcal{H}^{n-1}(x).
\end{aligned} \tag{3.11}$$

By (3.3), and the fact that $h_i^{-n} \le f_i^{-n} + \frac{1}{i^n}$, $h_i^p \le f_i^p + \frac{1}{i^p}$,

$$\begin{aligned}
&\limsup_{i\to\infty} L_2(h_i) \\
&\le \lim_{i\to\infty} \left(\int_{\partial K} (f_i^{-n}(x) + \frac{1}{i^n}) dC_0(K,x)\right)^{\frac{p}{n+p}} \left(\int_{\partial K} (f_i^p(x) + \frac{1}{i^p})(h_K(\nu_K(x)))^{1-p} dC_{n-1}(K,x)\right)^{\frac{n}{n+p}}.
\end{aligned} \tag{3.12}$$

Equations (3.11) and (3.12) imply that there exists i_0 such that

$$L_2(h_{i_0}) \leq \int_{\partial K} \left(\frac{H_K(x)}{(h_K(\nu_K(x)))^{(p-1)\frac{n}{p}}} \right)^{\frac{p}{n+p}} d\mathcal{H}^{n-1}(x) + \varepsilon/2. \tag{3.13}$$

Note that $\frac{1}{i_0} \leq h_{i_0} \leq i_0$. Let $d\eta(x) = (h_K(\nu_K(x)))^{1-p} dC_{n-1}(K,x) + dC_0(K,x)$. Note that η is a finite positive Borel measure on a compact metric space. Hence η is regular. Since $\eta(\partial K) < \infty$, by Lusin's Theorem (see Theorem 2.24 in [36]), there exists $\tilde{g}_j \in C(\partial K)$ such that

$$\eta(\{x \in \partial K : \tilde{g}_j(x) \neq h_{i_0}(x)\}) < \min\{\frac{1}{2i_0^n}, \frac{1}{2i_0^p}\}\frac{1}{j}. \tag{3.14}$$

Let

$$g_j(x) = \begin{cases} \frac{1}{i_0}, & \text{if } \tilde{g}_j(x) < \frac{1}{i_0}, \\ \tilde{g}_j(x), & \text{if } \frac{1}{i_0} \leq \tilde{g}_j(x) \leq i_0, \\ i_0, & \text{if } i_0 < \tilde{g}_j. \end{cases}$$

It is easy to see that g_j is still continuous and $\frac{1}{i_0} \leq g_j \leq i_0$. Moreover, since whenever $g_j(x) \neq h_{i_0}(x)$, it must be the case that $\tilde{g}_j(x) \neq h_{i_0}(x)$, we have by (3.14),

$$\eta(\{x \in \partial K : g_j \neq h_{i_0}\}) < \min\{\frac{1}{2i_0^n}, \frac{1}{2i_0^p}\}\frac{1}{j}.$$

Hence,

$$\left| \int_{\partial K} h_{i_0}^{-n}(x) dC_0(K,x) - \int_{\partial K} g_j^{-n}(x) dC_0(K,x) \right| \leq \frac{1}{2i_0^n}\frac{1}{j} 2i_0^n = \frac{1}{j},$$
$$\left| \int_{\partial K} h_{i_0}^p(x)(h_K(\nu_K(x)))^{1-p} dC_{n-1}(K,x) - \int_{\partial K} g_j^p(x)(h_K(\nu_K(x)))^{1-p} dC_{n-1}(K,x) \right| \leq \frac{1}{2i_0^p}\frac{1}{j}2i_0^p = \frac{1}{j}.$$

This implies that $\lim_{j\to\infty} L_2(g_j) = L_2(h_{i_0})$. As a result, there exists j_0 such that

$$L_2(g_{j_0}) \leq L_2(h_{i_0}) + \varepsilon/2. \tag{3.15}$$

Choose $g = g_{j_0}$. By (3.13) and (3.15), such a g will satisfy (3.8). □

It is immediate from (3.6) and (3.7) that

$$\begin{aligned} &\inf_{g\in T_1} \left\{ \left(\int_{\partial K} g^{-n}(x) H_K(x) d\mathcal{H}^{n-1}(x) \right)^{\frac{p}{n+p}} \left(\int_{\partial K} g^p(x)(h_K(\nu_K(x)))^{1-p} d\mathcal{H}^{n-1}(x) \right)^{\frac{n}{n+p}} \right\} \\ &= \inf_{g\in T_2} \left\{ \left(\int_{\partial K} g^{-n}(x) dC_0(K,x) \right)^{\frac{p}{n+p}} \left(\int_{\partial K} g^p(x)(h_K(\nu_K(x)))^{1-p} dC_{n-1}(K,x) \right)^{\frac{n}{n+p}} \right\}. \end{aligned} \tag{3.16}$$

Theorem 1.6 suggests, in addition to (1.9), (1.10), and (1.11), there is a new representation of the existing notion of L_p affine surface area:

Definition 3.1. *Let $p > 0$ be a real number and $K \subset \mathbb{R}^n$ be a convex body that contains the origin in its interior. The L_p affine surface area of K can be defined by*

$$\Omega_p(K) = \inf_g \left\{ \left(\int_{\partial K} g^{-n}(x) dC_0(K,x) \right)^{\frac{p}{n+p}} \left(\int_{\partial K} g^p(x)(h_K(\nu_K(x)))^{1-p} dC_{n-1}(K,x) \right)^{\frac{n}{n+p}} \right\}, \tag{3.17}$$

where the infimum is taken over all positive continuous functions $g : \partial K \to \mathbb{R}$.

Note that (3.17) is the L_p analogue of (1.8).

It is worthwhile to point out that, as can be seen from the proofs in this section, Definition (3.17) is "dual" to Definition (1.11). Since it has already been established that Definitions (1.9), (1.10), (1.11) are the same, it follows from Theorem 1.6 that Definition (3.17) is equivalent to Definition (1.9). However, we wish to give a direct proof of the equivalence between Definitions (3.17) and (1.9) as this will further reveal the relationship between the two formulations of L_p affine surface area. This will be carried out in Section 6.

4. Properties of L_p Affine Surface Area

In this section, some basic properties of L_p affine surface area will be shown using the new representation (3.17) ((1.8) if $p = 1$). Although the properties given in this section are not new, different proofs are given using Definition (3.17) that do not depend on any results about L_p affine surface area established using the previously known forms.

The following proposition shows that Ω_p is a homogeneous functional on the set of convex bodies containing the origin in their interiors.

Proposition 4.1. *Let $p > 0$ be a real number. Suppose $K \in \mathcal{K}_0^n$. For $\lambda > 0$, $\Omega_p(\lambda K) = \lambda^{\frac{n(n-p)}{n+p}}\Omega_p(K)$.*

Proof. For each positive continuous g on ∂K, define $\tilde{g}$ on $\partial(\lambda K)$ by

$$\tilde{g}(y) = g(x), \qquad \text{if } y = \lambda x. \tag{4.1}$$

Notice that for each Borel set $\beta \subset \partial K$,

$$C_0(\lambda K, \lambda\beta) = \mathcal{H}^{n-1}(\nu(\lambda K, \lambda\beta)) = \mathcal{H}^{n-1}(\nu(K, \beta)) = C_0(K, \beta). \tag{4.2}$$

By (4.2) and (4.1),

$$\begin{aligned} \int_{\partial(\lambda K)} \tilde{g}^{-n}(y)dC_0(\lambda K, y) &= \int_{\partial K} \tilde{g}^{-n}(\lambda x)dC_0(K, x) \\ &= \int_{\partial K} g^{-n}(x)dC_0(K, x). \end{aligned} \tag{4.3}$$

By the homogeneity of the support function,

$$\begin{aligned} \int_{\partial(\lambda K)} \tilde{g}^p(y)(h_{\lambda K}(\nu_{\lambda K}(y)))^{1-p}dC_{n-1}(\lambda K, y) &= \int_{\partial K} \tilde{g}^p(\lambda x)\lambda^{1-p}(h_K(\nu_K(x)))^{1-p}\lambda^{n-1}dC_{n-1}(K, x) \\ &= \lambda^{n-p}\int_{\partial K} g^p(x)(h_K(\nu_K(x)))^{1-p}dC_{n-1}(K, x). \end{aligned} \tag{4.4}$$

By (3.17), (4.3), and (4.4), we get the desired result. ☐

It is trivial to see that when $p = 1$, translation invariance is satisfied by Ω_1.

Proposition 4.2. *Let $p > 0$ be a real number. Suppose K is polytope that contains the origin in its interior. Then,*

$$\Omega_p(K) = 0.$$

Proof. Note that the measure $C_0(K, \cdot)$ in this case is concentrated on the set of vertices of K, which has only finitely many points. This implies that we can let the second integral in (3.17) be arbitrarily small while holding the value of the first integral constant. Hence $\Omega_p(K) = 0$. ☐

Another important property of L_p affine surface area that historically took a long time to prove (settled in [26, 29]) is its upper semi-continuity.

Theorem 4.3. *Let $p > 0$ be a real number. Then, Ω_p is upper semi-continuous with respect to the Hausdorff metric.*

Proof. This is a direct result from the weak continuity of $C_0(K,\cdot)$, $C_{n-1}(K,\cdot)$ and the fact that the infimum of continuous functionals is upper semi-continuous. $\square$

By (3.16), the following variant of (3.17) will give us more flexibility in choosing the function g:

$$\Omega_p(K) = \inf_{g\in T_1}\left\{\left(\int_{\partial K} g^{-n}(x)H_K(x)d\mathcal{H}^{n-1}(x)\right)^{\frac{p}{n+p}}\left(\int_{\partial K} g^{p}(x)(h_K(\nu_K(x)))^{1-p}d\mathcal{H}^{n-1}(x)\right)^{\frac{n}{n+p}}\right\}. \tag{4.5}$$

We will need the following two lemmas, which were established by Schütt & Werner in [41].

Lemma 4.4. *Let $K \in \mathcal{K}_0^n$ and $\phi : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map with determinant either 1 or -1. For each integrable function $f : \partial K \to \mathbb{R}$,*

$$\int_{\partial K} f(x)d\mathcal{H}^{n-1}(x) = \int_{\partial(\phi K)} f(\phi^{-1}(y))\left|\left|\phi^{-t}(\nu_K(\phi^{-1}(y)))\right|\right|^{-1}d\mathcal{H}^{n-1}(y). \tag{4.6}$$

Lemma 4.5. *Let $K \in \mathcal{K}_0^n$ and $\phi : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map with determinant either 1 or -1. Suppose x is a normal boundary point of K. Then $\phi(x)$ is a normal boundary point of ϕK and moreover,*

$$H_K(x) = \left|\left|\phi^{-t}(\nu_K(x))\right|\right|^{n+1} H_{\phi K}(\phi(x)). \tag{4.7}$$

The next proposition shows that Ω_p is invariant under volume preserving linear transformations.

Proposition 4.6. *Let $p > 0$ be a real number. Suppose $K \in \mathcal{K}_0^n$ and $\phi : \mathbb{R}^n \to \mathbb{R}^n$ is a linear map with determinant either 1 or -1. Then*

$$\Omega_p(\phi K) = \Omega_p(K).$$

Proof. Note that $|\det(\phi^{-1})| = 1$ and $\phi^{-1}(\phi K) = K$. Thus, we only need to show $\Omega_p(K) \geq \Omega_p(\phi K)$.

If x is a regular boundary point of K, then $\phi(x)$ is a regular boundary point of ϕK and

$$\nu_{\phi K}(\phi(x)) = \frac{\phi^{-t}(\nu_K(x))}{||\phi^{-t}(\nu_K(x))||}. \tag{4.8}$$

Hence,

$$\left|\left|\phi^{t}(\nu_{\phi K}(\phi(x)))\right|\right| = \left|\left|\phi^{-t}(\nu_K(x))\right|\right|^{-1}. \tag{4.9}$$

The definitions of support function and outer unit normal, together with (4.8) and (4.9), imply

$$h_K(\nu_K(x)) = \left|\left|\phi^{t}(\nu_{\phi K}(\phi(x)))\right|\right|^{-1} h_{\phi K}(\nu_{\phi K}(\phi(x))). \tag{4.10}$$

Let $g \in T_1$ be a function defined on ∂K such that $\int_{\partial K} g^{-n}(x)H_K(x)d\mathcal{H}^{n-1}(x) < \infty$. By (4.6), (4.7), and (4.9),

$$\int_{\partial K} g^{-n}(x)H_K(x)d\mathcal{H}^{n-1}(x) = \int_{\partial(\phi K)}\left|\left|\phi^{t}(\nu_{\phi K}(y))\right|\right|^{-n} g^{-n}(\phi^{-1}(y))H_{\phi K}(y)d\mathcal{H}^{n-1}(y). \tag{4.11}$$

By (4.6), (4.9), and (4.10),

$$\int_{\partial K} g^{p}(x)(h_K\circ\nu_K(x))^{1-p}d\mathcal{H}^{n-1}(x) = \int_{\partial(\phi K)}\left|\left|\phi^{t}(\nu_{\phi K}(y))\right|\right|^{p} g^{p}(\phi^{-1}(y))(h_{\phi K}(\nu_{\phi K}(y)))^{1-p}d\mathcal{H}^{n-1}(y). \tag{4.12}$$

Let $\tilde{g} : \partial(\phi K) \to \mathbb{R}$ be defined as

$$\tilde{g}(x) = \begin{cases} g(\phi^{-1}(y))\left|\left|\phi^{t}(\nu_{\phi K}(y))\right|\right|, & \text{if } y \text{ is a regular boundary point of } K,\\ 1, & \text{otherwise.}\end{cases}$$

The fact that $\mathcal{H}^{n-1}$ almost all points on the boundary of a convex body are regular, the choice of g, and (4.12) show that $\tilde{g}$ is a positive, $\mathcal{H}^{n-1}$ measurable function on $\partial(\phi K)$ and

$$\int_{\partial(\phi K)} \tilde{g}^p(y)(h_{\phi K}(\nu_{\phi K}(y)))^{1-p} d\mathcal{H}^{n-1}(y) < \infty.$$

By (4.11), (4.12), and (4.5), we immediately have $\Omega_p(K) \geq \Omega_p(\phi K)$. □

An immediate corollary of Proposition 4.1 and Proposition 4.6 is:

Corollary 4.7. *Let $p > 0$ be a real number. Suppose $K \in \mathcal{K}_0^n$ and $\phi : \mathbb{R}^n \to \mathbb{R}^n$ is an invertible linear map. Then*

$$\Omega_p(\phi K) = |\det(\phi)|^{\frac{n-p}{n+p}} \Omega_p(K).$$

The L_p affine surface area functional is also a valuation. That is, if $K, L \in \mathcal{K}_0^n$ are such that $K \cup L \in \mathcal{K}_0^n$, then

$$\Omega_p(K \cap L) + \Omega_p(K \cup L) = \Omega_p(K) + \Omega_p(L).$$

This property, however, is not immediate under the new form (3.17). The reader is recommended to see e.g., Schütt [39], Ludwig & Reitzner [22, 23] (and the references therein) for a proof of the valuation property of L_p affine surface area and the role of L_p affine surface area in the theory of valuation.

For each $g \in T_1$, denote

$$V_p(K, g) = \frac{1}{n} \int_{\partial K} g^p(x)(h_K(\nu_K(x)))^{1-p} d\mathcal{H}^{n-1}(x),$$
$$W(K, g) = \frac{1}{n} \int_{\partial K} g^{-n}(x) H_K(x) d\mathcal{H}^{n-1}(x).$$

Notice that for a convex body L that contains the origin in its interior,

$$V_p(K, h_L \circ \nu_K) = V_p(K, L).$$

Here $V_p(K, L)$ is the L_p mixed volume of K and L and can be defined by

$$V_p(K, L) = \frac{1}{n} \int_{S^{n-1}} h_L^p(u) h_K^{1-p}(u) dS_K(u).$$

Built in (4.5) is the following L_p mixed volume type inequality,

$$\frac{1}{n}\Omega_p(K) \leq W(K, h_L \circ \nu_K)^{\frac{p}{n+p}} V_p(K, L)^{\frac{n}{n+p}}. \tag{4.13}$$

We will postpone the proof of the L_p affine isoperimetric inequality (the L_p analogue of the celebrated affine isoperimetric inequality (1.3), which was given by Lutwak in [29] for $p \geq 1$, and by Werner & Ye in [47] for all other p) to Section 6. The proof will utilize (4.13).

5. Lipschitz Property of Restrictions of τ_K

Recall that ν_K is the Gauss map defined on $\operatorname{reg} K$, the set of regular boundary points of K, and τ_K is the inverse Gauss map defined on $\operatorname{regn} K$, the set of regular normal vectors of K. See Section 2.

One of the essential difficulties people encounter when trying to prove the equivalence between Definitions (1.10) and (1.11), as well as Definitions (1.9) and (3.17), is linking an integral over the unit sphere (image of the Gauss map or domain of the inverse Gauss map) with an integral over the boundary of a convex body (domain of the Gauss map or image of the inverse Gauss map). One of the direct bridges was built in [15] by exploring the Lipschitz property of restrictions of ν_K.

In particular, for each $r > 0$ and each convex body $K \subset \mathbb{R}^n$, denote

$$(\partial K)_r = \left\{x \in \partial K : \exists a \in S^{n-1} \text{ such that } B(x - ra, r) \subset K\right\}.$$

The following lemma was shown in [31]:

Lemma 5.1. *For each convex body $K \subset \mathbb{R}^n$,*

$$\mathcal{H}^{n-1}(\partial K \setminus (\cup_{r>0}(\partial K)_r) = 0.$$

Denote $\nu_K|_{(\partial K)_r}$ by ν_r. Hug in [15, Lemmas 2.1, 2.3] showed that ν_r is a Lipschitz map and calculated the approximate Jacobian of ν_r:

Lemma 5.2. *Let $K \subset \mathbb{R}^n$ be a convex body and $r > 0$. The following results are true:*

(1) *$(\partial K)_r$ is a closed subset of ∂K, and ν_r is Lipschitz;*
(2) *For $\mathcal{H}^{n-1}$ almost all $x \in (\partial K)_r$, we have*

$$(\mathcal{H}^{n-1} \llcorner (\partial K)_r, n-1) \text{ ap } J\nu_r(x) = H_K(x).$$

The following characterization of points at which the generalized curvature function is positive was established in [15, Lemma 2.7]:

Lemma 5.3. *Let $K \subset \mathbb{R}^n$ be a convex body. Suppose $u_0 \in D^2(h_K) \cap S^{n-1}$. Then the following two conditions are equivalent:*

(1) *There is some $r > 0$ such that $B(\tau_K(u_0) - ru_0, r) \subset K$.*
(2) *$F_K(u_0) > 0$.*

In this section, the Lipschitz property of restrictions of τ_K will be discussed, which will be useful in proving the equivalence between Definitions (1.9) and (3.17). In particular, we will divide the unit sphere into a countable collection of subsets (up to a set of measure 0), such that τ_K restricted to each subset is Lipschitz. Thus a change of variable will be made possible by using Federer's coarea formula.

Let $K \subset \mathbb{R}^n$ be a convex body. For each $i \in \mathbb{N}^+$, we define $A_i \subset S^{n-1}$ by

$$A_i = \{u \in S^{n-1} : \exists x \in \tau(K, \{u\}) \text{ such that } K \subset B[x - iu, i]\}.$$

Here $\tau(K, \{u\})$ is the inverse spherical image of $\{u\}$ (see Section 2).

Remark 5.4. *Note that it is easily seen that if $u \in A_i$, then u must be a regular normal vector of K. Hence,*

$$A_i = \{u \in \operatorname{regn} K : K \subset B[x - iu, i], \textit{where } x = \tau_K(u)\}. \tag{5.1}$$

We will denote the restriction of τ_K to A_i by τ_i, for each $i \in \mathbb{N}^+$.

The following lemma was observed by Hug in [15], which can be proved similarly to Lemma 2.7 in [15].

Lemma 5.5. *Let $K \subset \mathbb{R}^n$ be a convex body. For each $u_0 \in D^2(h_K) \cap S^{n-1}$, there exists $i \in \mathbb{N}^+$ such that*

$$K \subset B[x_0 - iu_0, i], \tag{5.2}$$

where $x_0 = \tau_K(u_0)$.

The following corollary follows immediately from Lemma 5.5 and (2.5).

Corollary 5.6. *Let $K \subset \mathbb{R}^n$ be a convex body. With respect to $\mathcal{H}^{n-1}$, almost every point in S^{n-1} is contained in $\cup_{i=1}^{\infty} A_i$, i.e.,*

$$\mathcal{H}^{n-1}(S^{n-1} \setminus \cup_{i=1}^{\infty} A_i) = 0.$$

The following lemma will be needed.

Lemma 5.7. *Let $K \subset \mathbb{R}^n$ be a convex body. For each $i \in \mathbb{N}^+$, A_i is closed.*

Proof. Suppose $\{u_j\}_{j=1}^\infty$ is a convergent sequence in A_i. Denote by u_0 its limit. Let $x_j = \tau_K(u_j)$. Since ∂K is compact, we can take a convergent subsequence of $\{x_j\}_{j=1}^\infty$. We denote the subsequence and the corresponding subsequence of $\{u_j\}_{j=1}^\infty$ again by $\{x_j\}_{j=1}^\infty$ and $\{u_j\}_{j=1}^\infty$. Denote by x_0 the limit of $\{x_j\}_{j=1}^\infty$. Let P_0 be the hyperplane that passes x_0 and has u_0 as its normal. Since $x_j = \tau_K(u_j)$,

$$\langle x, u_j\rangle \leq \langle x_j, u_j\rangle,$$

for each $x \in K$. Let $j \to \infty$, we have

$$\langle x, u_0\rangle \leq \langle x_0, u_0\rangle,$$

for each $x \in K$. Hence P_0 is a supporting hyperplane of K at x_0 and $x_0 \in \tau(K, \{u_0\})$. Since $\{u_j\}_{j=1}^\infty \subset A_i$, we have $K \subset B[x_j - iu_j, i]$. Let $j \to \infty$. We have $K \subset B[x_0 - iu_0, i]$, where $x_0 \in \tau(K, \{u_0\})$. This implies that $u_0 \in A_i$. Hence A_i is closed. □

Lemma 5.8. *For each $i \in \mathbb{N}^+$ and each convex body $K \subset \mathbb{R}^n$, τ_i is Lipschitz.*

Proof. Let $u_1, u_2 \in A_i$. Denote by $\theta \in [0, \pi]$ the angle formed by u_1 and u_2. Let $x_1 = \tau_i(u_1)$, and $x_2 = \tau_i(u_2)$.

We first assume $0 < \theta < \pi/2$. Suppose $x_1 \neq x_2$. Otherwise, there is nothing to prove. Since u_1, u_2 are not parallel to each other, the points $x_1 - iu_1$ and $x_2 - iu_2$ cannot both lie on the line passing x_1, x_2. Suppose $x_1 - iu_1$ does not lie on the line passing x_1, x_2. Denote $x_1 - iu_1$ by C. Without loss of generality, since this lemma is invariant under translation, we may assume that C is the origin. Let P be the two dimensional subspace spanned by x_1 and x_2. Note that u_1 is parallel to x_1. Write

$$u_2 = v_2 + w_2, \tag{5.3}$$

where $v_2 \in P$ and $w_2 \in P^\perp$. Since u_1 and u_2 are not perpendicular, we have $v_2 \neq 0$. Let $\tilde{u_2} = \frac{v_2}{||v_2||} \in P$. Denote by $\tilde{\theta} \in [0, \pi]$ the angle formed by u_1 and $\tilde{u_2}$. Notice that by the definition of $\tilde{u_2}$, (5.3), and the fact that $u_1 \in P$,

$$\cos\tilde{\theta} = \langle \tilde{u_2}, u_1\rangle = \frac{1}{||v_2||}\langle v_2, u_1\rangle = \frac{1}{||v_2||}\langle u_2, u_1\rangle = \frac{1}{||v_2||}\cos\theta. \tag{5.4}$$

This implies that if $0 < \theta < \pi/2$, we have $0 \leq \tilde{\theta} < \pi/2$. In this case, by (5.4) and that $||v_2|| \leq 1$, we have $\cos\tilde{\theta} \geq \cos\theta$. By the monotonicity of the cosine function on $[0, \pi/2)$, we have

$$\tilde{\theta} \leq \theta, \tag{5.5}$$

when $0 < \theta < \pi/2$.

By the definition of A_i, the fact that $u_1 \in A_i$ implies that $P \cap K$ is a subset of the disc $P \cap B[x_1 - iu_1, i]$. Note that $P \cap K$ is non-empty and is either the line segment joining x_1 and x_2 or a convex body in P. In either case, for any $x \in P \cap K$, by the definition of $\tilde{u_2}$, (5.3), the fact that $x_2 = \tau_i(u_2)$, (5.3), and the definition of $\tilde{u_2}$ once again,

$$\langle \tilde{u_2}, x\rangle = \frac{1}{||v_2||}\langle v_2, x\rangle = \frac{1}{||v_2||}\langle u_2, x\rangle \leq \frac{1}{||v_2||}\langle u_2, x_2\rangle = \frac{1}{||v_2||}\langle v_2, x_2\rangle = \langle \tilde{u_2}, x_2\rangle. \tag{5.6}$$

Now, we show that $\tilde{\theta} \neq 0$ as a result of $x_1 \neq x_2$. If otherwise, $\tilde{u_2} = u_1$. Since $P \cap K \subset P \cap B[x_1 - iu_1, i]$, the line $l_{x_1} \subset P$ passing x_1 and perpendicular to u_1 can only intersect $P \cap K$ at x_1. Note also that since $x_1 = \tau_i(u_1)$, we have $\langle u_1, x\rangle \leq \langle u_1, x_1\rangle$ for each $x \in P \cap K$. But (5.6) implies that $\langle u_1, x_1\rangle = \langle \tilde{u_2}, x_1\rangle \leq \langle \tilde{u_2}, x_2\rangle = \langle u_1, x_2\rangle$. Hence $\langle u_1, x_2\rangle = \langle u_1, x_1\rangle$ and as a result, we have $x_2 \in l_{x_1}$. This immediately implies $x_1 = x_2$, which contradicts with our assumption. Thus, it suffices to look at the case when $0 < \tilde{\theta} \leq \theta < \pi/2$, with $x_1 \neq x_2$. In this case, let $l \subset P$ be the line passing through x_2 and is perpendicular to $\tilde{u_2}$. Extend Cx_1 so that it intersects l at D. Starting

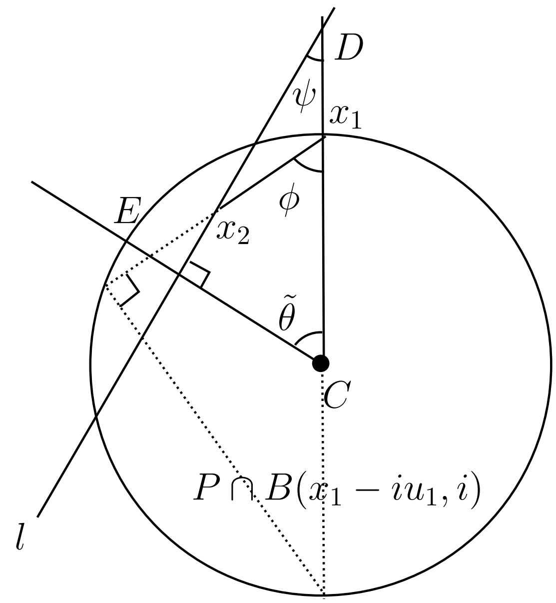

FIGURE 1.

from C, make an ray in the direction of $\tilde{u_2}$ (perpendicular to l as a result), so that it crosses the boundary of $B[x_1 - iu_1, i]$ at E. Let ϕ, ψ be the angles indicated in Figure 1. By (5.6) and that $0 < \tilde{\theta} < \pi/2$, the point D does not belong to the interior of $B(x_1 - iu_1, i)$. Hence $\phi \geq \psi = \pi/2 - \tilde{\theta}$. The fact that $x_2 \in B[x_1 - iu_1, i]$, together with (5.5), implies

$$||x_1 - x_2|| \leq 2i\cos\phi \leq 2i\cos(\pi/2 - \tilde{\theta}) = 2i\sin\tilde{\theta} \leq 2i\sin\theta. \tag{5.7}$$

Observe that $||u_1 - u_2|| = 2\sin\frac{\theta}{2}$. Since

$$\lim_{\theta\to 0} \frac{2i\sin\theta}{2\sin\frac{\theta}{2}} = 2i,$$

we conclude that there exists $0 < \delta_0 < \pi/2$, such that

$$||x_1 - x_2|| \leq 3i\,||u_1 - u_2||\,, \tag{5.8}$$

for any $u_1, u_2 \in A_i$, satisfying $0 < \theta < \delta_0$. (Note that (5.8) trivially holds if $x_1 = x_2$.)

For the case $\delta_0 \leq \theta \leq \pi$, since $||u_1 - u_2|| = 2\sin\frac{\theta}{2}$ and K is bounded, we have

$$\frac{||x_1 - x_2||}{||u_1 - u_2||}$$

is bounded from above. This and (5.8) prove the existence of $M > 0$ such that

$$||x_1 - x_2|| \leq M\,||u_1 - u_2||\,,$$

for any $u_1, u_2 \in A_i$. Hence τ_i is Lipschitz. $\square$

The following characterization of normal boundary points with positive curvature was shown in [15] as Corollary 3.2 (see also [19]).

Lemma 5.9. *Let $K \subset \mathbb{R}^n$ be a convex body and x_0 be a normal boundary point. The following two conditions are equivalent:*

(1) $H_K(x_0) > 0$;
(2) *there exists $i \in \mathbb{N}^+$ such that $K \subset B[x_0 - iu_0, i]$ where $u_0 = \nu_K(x_0)$.*

The following lemma was proved in [15, Lemma 2.5].

Lemma 5.10. *Let $K \subset \mathbb{R}^n$ be a convex body. Suppose x_0 is a normal boundary point and $u_0 = \nu_K(x_0) \in D^2(h_K)$. Then $H_K(x_0)F_K(u_0) = 1$.*

There is still one piece missing that hinders us from applying Federer's coarea formula to τ_i, namely, the Jacobian of τ_i. Taking the Jacobian of τ_i in the classical sense is impossible since the classical Jacobian requires τ_i to be defined in an open set. Therefore we have to consider the approximate Jacobian of τ_i instead.

Lemma 5.11. *Let $K \subset \mathbb{R}^n$ be a convex body. For each given $i \in \mathbb{N}^+$,*

$$(\mathcal{H}^{n-1} \llcorner A_i, n-1)\, ap\, J\tau_i(u) = F_K(u), \tag{5.9}$$

for $\mathcal{H}^{n-1}$ almost all $u \in A_i$.

Proof. Observe that A_i is an $(\mathcal{H}^{n-1}, n-1)$ rectifiable and $\mathcal{H}^{n-1}$ measurable subset (since A_i is closed by Lemma 5.7) of $\mathbb{R}^n$. By Theorem 3.2.19 in [8], for $\mathcal{H}^{n-1}$ almost all $u_0 \in A_i$, $\mathrm{Tan}^{n-1}(\mathcal{H}^{n-1} \llcorner A_i, u_0)$ is an $(n-1)$ dimensional subspace of $\mathbb{R}^n$. By definition,

$$\mathrm{Tan}^{n-1}(\mathcal{H}^{n-1} \llcorner A_i, u_0) \subset \mathrm{Tan}(S^{n-1}, u_0) = u_0^{\perp}.$$

Hence,

$$\mathrm{Tan}^{n-1}(\mathcal{H}^{n-1} \llcorner A_i, u_0) = u_0^{\perp}, \tag{5.10}$$

for $\mathcal{H}^{n-1}$ almost all $u_0 \in A_i$. This and (2.5) imply that $\mathcal{H}^{n-1}$ almost all vectors in A_i are in $D^2(h_K)$ and satisfy (5.10). Thus, to prove this lemma, we may assume that $u_0 \in D^2(h_K)$ and u_0 satisfies (5.10).

Let $\varepsilon > 0$ be a real number and $\vartheta : \mathbb{R}^n \to \mathbb{R}^n$ be an arbitrary subgradient choice of h_K. The fact that $u_0 \in D^2(h_K)$ implies that there exists $\delta_0 > 0$ and a symmetric linear map $Ah_K(u_0) : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$\{u \in A_i : ||\vartheta(u) - \vartheta(u_0) - Ah_K(u_0)(u-u_0)|| > \varepsilon\, ||u-u_0||\} \cap B(u_0, \delta) = \emptyset,$$

for $0 < \delta \le \delta_0$.

Since each $u \in A_i$ is a regular normal vector, we have $\vartheta(u) = \tau_i(u)$ for each $u \in A_i$. Hence by the definition of Θ^{n-1}, we have

$$\begin{aligned}\Theta^{n-1}\big(\{(\mathcal{H}^{n-1} \llcorner A_i) \llcorner (\mathbb{R}^n \backslash \{u \in A_i : ||\tau_i(u) - \tau_i(u_0) - Ah_K(u_0)(u-u_0)|| \\ \le\ \varepsilon\, ||u-u_0||\}), u_0\}\big) = 0.\end{aligned}$$

By the definition of $(\mathcal{H}^{n-1} \llcorner A_i, n-1)$ ap $D\tau_i(u_0)$ and (5.10), we have

$$(\mathcal{H}^{n-1} \llcorner A_i, n-1)\ \mathrm{ap}\ D\tau_i(u_0) = Ah_K(u_0)|_{\mathrm{Tan}^{n-1}(\mathcal{H}^{n-1} \llcorner A_i, u_0)} = Ah_K(u_0)|_{u_0^{\perp}}.$$

By this, the definition of $(\mathcal{H}^{n-1} \llcorner A_i, n-1)$ ap $J\tau_i(u_0)$, that $u_0^{\perp}$ is an invariant subspace of $Ah_K(u_0)$, and the definition of the generalized curvature function, we have

$$(\mathcal{H}^{n-1} \llcorner A_i, n-1)\ \mathrm{ap}\ J\tau_i(u_0) = F_K(u_0).$$

□

6. Equivalence between Definitions (1.9) and (3.17)

In this section, the relationship between Definitions (1.9) and (3.17) will be unveiled by providing a direct proof of the equivalence between the two. The proof utilizes the tool we established in Section 5. At the end of this section, the L_p affine isoperimetric inequality (with equality condition) is established using the new representation (3.17) of L_p affine surface area.

Let us first introduce some notations.

Let

$$\begin{aligned}
T_3 &= \left\{ h : S^{n-1} \to \mathbb{R}\ \mathcal{H}^{n-1}\text{measurable} : 0 < h < \infty \text{ and } \int_{S^{n-1}} h^n(u) d\mathcal{H}^{n-1}(u) < \infty \right\}, \\
T_4 &= \left\{ h : S^{n-1} \to \mathbb{R} \text{ continuous} : h > 0 \right\}.
\end{aligned}$$

Note that $T_3 \supset T_4$.

We claim that for each positive real number p and each $K \in \mathcal{K}_0^n$,

$$\begin{aligned}
&\inf_{g \in T_1} \left\{ \left(\int_{\partial K} g^{-n}(x) H_K(x) d\mathcal{H}^{n-1}(x) \right)^{\frac{p}{n+p}} \left(\int_{\partial K} g^p(x) (h_K(\nu_K(x)))^{1-p} d\mathcal{H}^{n-1}(x) \right)^{\frac{n}{n+p}} \right\} \\
= &\inf_{g \in T_1} \left\{ \left(\int_{H^+} g^{-n}(x) H_K(x) d\mathcal{H}^{n-1}(x) \right)^{\frac{p}{n+p}} \left(\int_{H^+} g^p(x) (h_K(\nu_K(x)))^{1-p} d\mathcal{H}^{n-1}(x) \right)^{\frac{n}{n+p}} \right\}.
\end{aligned} \tag{6.1}$$

Indeed, for each $g \in T_1$, let

$$g_\varepsilon(x) = \begin{cases} \varepsilon, & \text{if } x \notin H^+ \text{ and } g(x) > \varepsilon, \\ g(x), & \text{otherwise.} \end{cases}$$

Notice that $g_\varepsilon \le \varepsilon$ on $\partial K \setminus H^+$. By (2.3), (2.4), the choice of g_ε, and the fact that

$$\int_{\partial K \setminus H^+} h_K(\nu_K(x))^{1-p} d\mathcal{H}^{n-1}(x)$$

is a finite number,

$$\begin{aligned}
&\lim_{\varepsilon \to 0} \left(\int_{\partial K} g_\varepsilon^{-n}(x) H_K(x) d\mathcal{H}^{n-1}(x) \right)^{\frac{p}{n+p}} \left(\int_{\partial K} g_\varepsilon^p(x) (h_K(\nu_K(x)))^{1-p} d\mathcal{H}^{n-1}(x) \right)^{\frac{n}{n+p}} \\
\le &\left(\int_{H^+} g^{-n}(x) H_K(x) d\mathcal{H}^{n-1}(x) \right)^{\frac{p}{n+p}} \left(\int_{H^+} g^p(x) (h_K(\nu_K(x)))^{1-p} d\mathcal{H}^{n-1}(x) \right. \\
&\qquad\qquad \left. + \lim_{\varepsilon \to 0} \varepsilon^p \int_{\partial K \setminus H^+} (h_K(\nu_K(x)))^{1-p} d\mathcal{H}^{n-1}(x) \right)^{\frac{n}{n+p}} \\
= &\left(\int_{H^+} g^{-n}(x) H_K(x) d\mathcal{H}^{n-1}(x) \right)^{\frac{p}{n+p}} \left(\int_{H^+} g^p(x) (h_K(\nu_K(x)))^{1-p} d\mathcal{H}^{n-1}(x) \right)^{\frac{n}{n+p}}.
\end{aligned}$$

This shows that the left side is less than or equal to the right side in (6.1). By the fact that $H^+ \subset \partial K$ and that all integrands in (6.1) are non-negative, the left side is greater than or equal to the right side in (6.1).

By (3.16), we can further show that

Lemma 6.1. *Let $p > 0$ be a real number. Suppose $K \in \mathcal{K}_0^n$. We have*

$$\begin{aligned}
&\inf_{g \in T_2} \left\{ \left(\int_{\partial K} g^{-n}(x) dC_0(K, x) \right)^{\frac{p}{n+p}} \left(\int_{\partial K} g^p(x) (h_K(\nu_K(x)))^{1-p} dC_{n-1}(K, x) \right)^{\frac{n}{n+p}} \right\} \\
= &\inf_{g \in T_1} \left\{ \left(\int_{H^+} g^{-n}(x) H_K(x) d\mathcal{H}^{n-1}(x) \right)^{\frac{p}{n+p}} \left(\int_{H^+} g^p(x) (h_K(\nu_K(x)))^{1-p} d\mathcal{H}^{n-1}(x) \right)^{\frac{n}{n+p}} \right\}.
\end{aligned}$$

Recall that F^+ is the set of unit vectors at which the curvature function F_K is positive (see (2.6)). By a similar argument from above,

$$\begin{aligned}&\inf_{h\in T_3}\left\{\left(\int_{S^{n-1}} h^n(u)d\mathcal{H}^{n-1}(u)\right)^{\frac{p}{n+p}}\left(\int_{S^{n-1}} h^{-p}(u)h_K^{1-p}(u)F_K(u)d\mathcal{H}^{n-1}(u)\right)^{\frac{n}{n+p}}\right\}\\ =&\inf_{h\in T_3}\left\{\left(\int_{F^+} h^n(u)d\mathcal{H}^{n-1}(u)\right)^{\frac{p}{n+p}}\left(\int_{F^+} h^{-p}(u)h_K^{1-p}(u)F_K(u)d\mathcal{H}^{n-1}(u)\right)^{\frac{n}{n+p}}\right\}.\end{aligned}$$

This, together with Hilfssatz 2 and 3 in [18] (see also [19]), immediately implies

Lemma 6.2. *Let $p>0$ be a real number. Suppose $K\in\mathcal{K}_0^n$. We have*

$$\begin{aligned}&\inf_{h\in T_4}\left\{\left(\int_{S^{n-1}} h^n(u)d\mathcal{H}^{n-1}(u)\right)^{\frac{p}{n+p}}\left(\int_{S^{n-1}} h^{-p}(u)h_K^{1-p}(u)dS_K(u)\right)^{\frac{n}{n+p}}\right\}\\ =&\inf_{h\in T_3}\left\{\left(\int_{F^+} h^n(u)d\mathcal{H}^{n-1}(u)\right)^{\frac{p}{n+p}}\left(\int_{F^+} h^{-p}(u)h_K^{1-p}(u)F_K(u)d\mathcal{H}^{n-1}(u)\right)^{\frac{n}{n+p}}\right\}.\end{aligned}$$

Now, we shall prove the equivalence between Definitions (1.9) and (3.17).

Theorem 6.3. *Let $p>0$ be a real number. Suppose $K\in\mathcal{K}_0^n$. We have*

$$\begin{aligned}&\inf_{h}\left\{\left(\int_{S^{n-1}} h^n(u)d\mathcal{H}^{n-1}(u)\right)^{\frac{p}{n+p}}\left(\int_{S^{n-1}} h^{-p}(u)h_K^{1-p}(u)dS_K(u)\right)^{\frac{n}{n+p}}\right\}\\ =&\inf_{g}\left\{\left(\int_{\partial K} g^{-n}(x)dC_0(K,x)\right)^{\frac{p}{n+p}}\left(\int_{\partial K} g^{p}(x)(h_K(\nu_K(x)))^{1-p}dC_{n-1}(K,x)\right)^{\frac{n}{n+p}}\right\},\end{aligned}$$

where the infimums are taken over all positive continuous functions $h: S^{n-1}\to\mathbb{R}$ and $g:\partial K\to\mathbb{R}$ respectively.

Proof. By Lemmas 6.1 and 6.2, it suffices to show

$$\begin{aligned}&\inf_{g\in T_1}\left\{\left(\int_{H^+} g^{-n}(x)H_K(x)d\mathcal{H}^{n-1}(x)\right)^{\frac{p}{n+p}}\left(\int_{H^+} g^{p}(x)(h_K(\nu_K(x)))^{1-p}d\mathcal{H}^{n-1}(x)\right)^{\frac{n}{n+p}}\right\}\\ =&\inf_{h\in T_3}\left\{\left(\int_{F^+} h^n(u)d\mathcal{H}^{n-1}(u)\right)^{\frac{p}{n+p}}\left(\int_{F^+} h^{-p}(u)h_K^{1-p}(u)F_K(u)d\mathcal{H}^{n-1}(u)\right)^{\frac{n}{n+p}}\right\}.\end{aligned}\tag{6.2}$$

Let us first show that the left side is greater than or equal to the right side in (6.2).

Let $g\in T_1$ be such that $\int_{H^+} g^{-n}(x)H_K(x)d\mathcal{H}^{n-1}(x)<\infty$ and $r>0$. We first observe that $(\partial K)_r$ and S^{n-1} are $(\mathcal{H}^{n-1},n-1)$ rectifiable and $\mathcal{H}^{n-1}$ measurable (since $(\partial K)_r$ is closed by Lemma 5.2).

By Lemma 5.2, ν_r, the restriction of ν_K to $(\partial K)_r$, is Lipschitz and

$$(\mathcal{H}^{n-1}\llcorner(\partial K)_r, n-1)\text{ ap } J\nu_r(x)=H_K(x),\tag{6.3}$$

for $\mathcal{H}^{n-1}$ almost all $x\in(\partial K)_r$.

The fact that ν_r is Lipschitz and (2.3) give,

$$\mathcal{H}^{n-1}(\nu_r((\partial K)_r\setminus\text{nor }K))=0.\tag{6.4}$$

By (6.3) and Federer's coarea formula,

$$\begin{aligned}\int_{(\partial K)_r\cap H^+} g^{-n}(x)H_K(x)d\mathcal{H}^{n-1}(x) &= \int_{(\partial K)_r} \mathbf{1}_{H^+}(x)g^{-n}(x)H_K(x)d\mathcal{H}^{n-1}(x)\\ &= \int_{S^{n-1}}\left(\int_{(\nu_r)^{-1}(u)} \mathbf{1}_{H^+}(x)g^{-n}(x)d\mathcal{H}^0(x)\right)d\mathcal{H}^{n-1}(u).\end{aligned} \tag{6.5}$$

It is implied in Federer's coarea formula that

$$\int_{(\nu_r)^{-1}(u)} \mathbf{1}_{H^+}(x)g^{-n}(x)d\mathcal{H}^0(x)$$

is $\mathcal{H}^{n-1}$ measurable on S^{n-1} in u. By (2.5), (6.4), Lemma 5.10, and (2.5), (6.4) once again, the following holds for $\mathcal{H}^{n-1}$ almost all $u\in S^{n-1}$,

$$\begin{aligned}\int_{(\nu_r)^{-1}(u)} \mathbf{1}_{H^+}(x)g^{-n}(x)d\mathcal{H}^0(x) =&\mathbf{1}_{\nu_r((\partial K)_r))\cap D^2(h_K)}(u)\mathbf{1}_{H^+}(\tau_K(u))g(\tau_K(u))^{-n}\\ =&\mathbf{1}_{\nu_r((\partial K)_r)\cap \operatorname{nor} K)\cap D^2(h_K)}(u)\mathbf{1}_{H^+}(\tau_K(u))g(\tau_K(u))^{-n}\\ =&\mathbf{1}_{\nu_r((\partial K)_r)\cap \operatorname{nor} K)\cap D^2(h_K)}(u)\mathbf{1}_{F^+}(u)g(\tau_K(u))^{-n}\\ =&\mathbf{1}_{\nu_r((\partial K)_r)}(u)\mathbf{1}_{F^+}(u)g(\tau_K(u))^{-n}.\end{aligned} \tag{6.6}$$

Hence $\mathbf{1}_{\nu_r((\partial K)_r)}(u)\mathbf{1}_{F^+}(u)(g(\tau_K(u)))^{-n}$, as a function of u, is $\mathcal{H}^{n-1}$ measurable on S^{n-1}. By (6.5) and (6.6),

$$\int_{(\partial K)_r\cap H^+} g^{-n}(x)H_K(x)d\mathcal{H}^{n-1}(x) = \int_{S^{n-1}} \mathbf{1}_{\nu_r((\partial K)_r)}(u)\mathbf{1}_{F^+}(u)(g(\tau_K(u)))^{-n}d\mathcal{H}^{n-1}(u). \tag{6.7}$$

Note that $(\partial K)_r$ is increasing as $r\to 0$. Let $r\to 0$ in (6.7). It follows from the monotone convergence theorem, Lemma 5.1, and Lemma 5.3 that,

$$\int_{H^+} g^{-n}(x)H_K(x)d\mathcal{H}^{n-1}(x) = \int_{F^+}(g(\tau_K(u)))^{-n}d\mathcal{H}^{n-1}(u). \tag{6.8}$$

With the same technique, one can prove that

$$\int_{H^+} g^p(x)(h_K(\nu_K(x)))^{1-p}d\mathcal{H}^{n-1}(x) = \int_{F^+}(g(\tau_K(u)))^p h_K^{1-p}(u)F_K(u)d\mathcal{H}^{n-1}(u). \tag{6.9}$$

Set

$$h(u) = \begin{cases} 1/g(\tau_K(u)), & \text{if } u\in F^+,\\ 1, & \text{otherwise.}\end{cases}$$

Note that $0<h<\infty$ and h is $\mathcal{H}^{n-1}$ measurable on S^{n-1}. By (6.8), h^n is $\mathcal{H}^{n-1}$ integrable on S^{n-1}. Hence $h\in T_3$. By (6.8), (6.9), the left side is greater than or equal to the right side in (6.2).

Now we show that the left side is less than or equal to the right side in (6.2).

Let $h\in T_3$ be such that $\int_{F^+} h^{-p}(u)h_K^{1-p}(u)F_K(u)d\mathcal{H}^{n-1}(u)<\infty$ and $i\in\mathbb{N}^+$. We first observe that A_i and ∂K are $(\mathcal{H}^{n-1}, n-1)$ rectifiable and $\mathcal{H}^{n-1}$ measurable (since A_i is closed by Lemma 5.7). By Lemma 5.8, τ_i, the restriction of τ_K to A_i, is Lipschitz. By Lemma 5.11, for $\mathcal{H}^{n-1}$ almost all $u\in A_i$,

$$(\mathcal{H}^{n-1}\llcorner A_i, n-1)\ \mathrm{ap}\ J\tau_i(u) = F_K(u). \tag{6.10}$$

The fact that τ_i is Lipschitz and (2.5) give,

$$\mathcal{H}^{n-1}(\tau_i(A_i\setminus D^2(h_K))) = 0. \tag{6.11}$$

By (6.10) and Federer's coarea formula,

$$\begin{aligned}\int_{A_i\cap F^+} h^{-p}(u)h_K^{1-p}(u)F_K(u)d\mathcal{H}^{n-1}(u) &= \int_{A_i} \mathbf{1}_{F^+}(u)h^{-p}(u)h_K^{1-p}(u)F_K(u)d\mathcal{H}^{n-1}(u)\\ &= \int_{\partial K}\left(\int_{(\tau_i)^{-1}(x)} \mathbf{1}_{F^+}(u)h^{-p}(u)h_K^{1-p}(u)d\mathcal{H}^0(u)\right)d\mathcal{H}^{n-1}(x).\end{aligned}\tag{6.12}$$

It is implied in Federer's coarea formula that

$$\int_{(\tau_i)^{-1}(x)} \mathbf{1}_{F^+}(u)h^{-p}(u)h_K^{1-p}(u)d\mathcal{H}^0(u)$$

is $\mathcal{H}^{n-1}$ measurable on ∂K in x. By (2.3), (6.11), Lemma 5.10 and (2.3), (6.11) once again, the following holds for $\mathcal{H}^{n-1}$ almost all $x \in \partial K$,

$$\begin{aligned}\int_{(\tau_i)^{-1}(x)} \mathbf{1}_{F^+}(u)h^{-p}(u)h_K^{1-p}(u)d\mathcal{H}^0(u) =&\mathbf{1}_{\tau_i(A_i)\cap \operatorname{nor} K}(x)\mathbf{1}_{F^+}(\nu_K(x))h(\nu_K(x))^{-p}h_K(\nu_K(x))^{1-p}\\ =&\mathbf{1}_{\tau_i(A_i\cap D^2(h_K))\cap \operatorname{nor} K}(x)\mathbf{1}_{F^+}(\nu_K(x))h(\nu_K(x))^{-p}h_K(\nu_K(x))^{1-p}\\ =&\mathbf{1}_{\tau_i(A_i\cap D^2(h_K))\cap \operatorname{nor} K}(x)\mathbf{1}_{H^+}(x)h(\nu_K(x))^{-p}h_K(\nu_K(x))^{1-p}\\ =&\mathbf{1}_{\tau_i(A_i)}(x)\mathbf{1}_{H^+}(x)h(\nu_K(x))^{-p}h_K(\nu_K(x))^{1-p}.\end{aligned}\tag{6.13}$$

Hence $\mathbf{1}_{\tau_i(A_i)}(x)\mathbf{1}_{H^+}(x)(h(\nu_K(x)))^{-p}(h_K(\nu_K(x)))^{1-p}$, as a function of x, is $\mathcal{H}^{n-1}$ measurable on ∂K. By (6.12) and (6.13),

$$\int_{A_i\cap F^+} h^{-p}(u)h_K^{1-p}(u)F_K(u)d\mathcal{H}^{n-1}(u) = \int_{\partial K} \mathbf{1}_{\tau_i(A_i)}(x)\mathbf{1}_{H^+}(x)(h(\nu_K(x)))^{-p}(h_K(\nu_K(x)))^{1-p}d\mathcal{H}^{n-1}(x).\tag{6.14}$$

Note that A_i is increasing as $i \to \infty$. Let $i \to \infty$ in (6.14). It follows from the monotone convergence theorem, Corollary 5.6, and Lemma 5.9 that

$$\int_{F^+} h^{-p}(u)h_K^{1-p}(u)F_K(u)d\mathcal{H}^{n-1}(u) = \int_{H^+} (h(\nu_K(x)))^{-p}(h_K(\nu_K(x)))^{1-p}d\mathcal{H}^{n-1}(x).\tag{6.15}$$

With the same technique, one can prove

$$\int_{F^+} h^n(u)d\mathcal{H}^{n-1}(u) = \int_{H^+} (h(\nu_K(x)))^n H_K(x)d\mathcal{H}^{n-1}(x).\tag{6.16}$$

Set

$$g(x) = \begin{cases} 1/h(\nu_K(x)), & \text{if } x \in H^+,\\ 1, & \text{otherwise.}\end{cases}$$

Note that $0 < g < \infty$ and g is $\mathcal{H}^{n-1}$ measurable. By (6.15), $g^p(h_K \circ \nu_K)^{1-p}$ is $\mathcal{H}^{n-1}$ integrable on ∂K. Hence $g \in T_1$. By (6.15), (6.16), the left side is less than or equal to the right side in (6.2). □

The proof given above reveals that Definition (3.17) is "polar" to (1.9) and they are linked by the Gauss map and the inverse Gauss map.

The L_p affine isoperimetric inequality, which is the extension of the affine isoperimetric inequality (1.3) of affine differential geometry, was first established by Lutwak in [29]. Thanks to (4.13), we are ready to prove the L_p affine isoperimetric inequality using the new representation (3.17).

Theorem 6.4. *Let $p > 0$ be a real number. Suppose $K \in \mathcal{K}_0^n$ and has the origin as its centroid. We have,*

$$\Omega_p(K) \le n\omega_n^{\frac{2p}{n+p}} V(K)^{\frac{n-p}{n+p}}, \tag{6.17}$$

with equality if and only if K is an ellipsoid.

Proof. Recall from (4.13) that, for each convex body L that has the origin as its centroid,

$$\frac{1}{n}\Omega_p(K) \le W(K, h_L \circ \nu_K)^{\frac{p}{n+p}} V_p(K, L)^{\frac{n}{n+p}}. \tag{6.18}$$

Based on the change of variable formula established in the proof of Theorem 6.3,

$$W(K, h_L \circ \nu_K) \le V(L^*), \tag{6.19}$$

where L^* is the polar body of L. Combining (6.18) and (6.19), we get,

$$\frac{1}{n}\Omega_p(K) \le V(L^*)^{\frac{p}{n+p}} V_p(K, L)^{\frac{n}{n+p}}. \tag{6.20}$$

It was observed in [26, 29] that (6.20) is stronger than (6.17). Indeed, just by replacing L in (6.20) by K and using the Blaschke-Santaló inequality, we get (6.17).

Equality holds only if the equality holds for the Blaschke-Santaló inequality, i.e., K is an ellipsoid. That the equality does hold for ellipsoids follows from a direct calculation. □

7. A Further Application of Section 5

Hug in [15] proved the equivalence between Definitions (1.10) and (1.11) by applying Federer's coarea formula to the Lipschitz map ν_r, the restriction of ν_K to $(\partial K)_r$. Here we provide a dual proof of the same result by applying Federer's coarea formula to the Lipschitz map τ_i, the restriction of τ_K to A_i, which is made possible by the discussion in Section 5.

Theorem 7.1. *Let $p > 0$ be a real number. Suppose $K \in \mathcal{K}_0^n$. We have*

$$\int_{S^{n-1}} \left(\frac{F_K(u)}{h_K^{p-1}(u)} \right)^{\frac{n}{n+p}} d\mathcal{H}^{n-1}(u) = \int_{\partial K} \left(\frac{H_K(x)}{(h_K(\nu_K(x)))^{(p-1)\frac{n}{p}}} \right)^{\frac{p}{n+p}} d\mathcal{H}^{n-1}(x).$$

Proof. For each $i \in \mathbb{N}^+$, observe that A_i and ∂K are $(\mathcal{H}^{n-1}, n-1)$ rectifiable and $\mathcal{H}^{n-1}$ measurable (since A_i is closed by Lemma 5.7).

By Lemmas 5.8, 5.11, Federer's coarea formula, (2.3), (6.11), Lemma 5.10, and (6.11) once again, we have

$$\begin{aligned}
\int_{A_i}\left(\frac{F_K(u)}{h_K^{p-1}(u)}\right)^{\frac{n}{n+p}} d\mathcal{H}^{n-1}(u) &= \int_{A_i\cap F^+}\left(\frac{F_K(u)}{h_K^{p-1}(u)}\right)^{\frac{n}{n+p}} d\mathcal{H}^{n-1}(u)\\
&= \int_{A_i} \mathbf{1}_{F^+}(u)\left(\frac{F_K^{-\frac{p}{n}}(u)}{h_K^{p-1}(u)}\right)^{\frac{n}{n+p}} F_K(u)d\mathcal{H}^{n-1}(u)\\
&= \int_{\partial K}\left(\int_{\tau_i^{-1}(x)} \mathbf{1}_{F^+}(u)\left(\frac{F_K^{-\frac{p}{n}}(u)}{h_K^{p-1}(u)}\right)^{\frac{n}{n+p}} d\mathcal{H}^0(u)\right) d\mathcal{H}^{n-1}(x)\\
&= \int_{\tau_i(A_i)\cap \operatorname{nor} K} \mathbf{1}_{F^+}(\nu_K(x))\left(\frac{(F_K(\nu_K(x)))^{-\frac{p}{n}}}{(h_K(\nu_K(x)))^{p-1}}\right)^{\frac{n}{n+p}} d\mathcal{H}^{n-1}(x)\\
&= \int_{\tau_i(A_i\cap D^2(h_K))\cap \operatorname{nor} K} \mathbf{1}_{F^+}(\nu_K(x))\left(\frac{(F_K(\nu_K(x)))^{-\frac{p}{n}}}{(h_K(\nu_K(x)))^{p-1}}\right)^{\frac{n}{n+p}} d\mathcal{H}^{n-1}(x)\\
&= \int_{\tau_i(A_i\cap D^2(h_K))\cap \operatorname{nor} K} \mathbf{1}_{H^+}(x)\left(\frac{H_K(x)}{(h_K(\nu_K(x)))^{(p-1)\frac{n}{p}}}\right)^{\frac{p}{n+p}} d\mathcal{H}^{n-1}(x)\\
&= \int_{\tau_i(A_i)\cap \operatorname{nor} K}\left(\frac{H_K(x)}{(h_K(\nu_K(x)))^{(p-1)\frac{n}{p}}}\right)^{\frac{p}{n+p}} d\mathcal{H}^{n-1}(x).
\end{aligned} \tag{7.1}$$

Now, let $i\to\infty$ in (7.1). It follows from the monotone convergence theorem, Corollary 5.6, and Lemma 5.9 that

$$\begin{aligned}
\int_{S^{n-1}}\left(\frac{F_K(u)}{h_K^{p-1}(u)}\right)^{\frac{n}{n+p}} d\mathcal{H}^{n-1}(u) &= \int_{H^+}\left(\frac{H_K(x)}{(h_K(\nu_K(x)))^{(p-1)\frac{n}{p}}}\right)^{\frac{p}{n+p}} d\mathcal{H}^{n-1}(x)\\
&= \int_{\partial K}\left(\frac{H_K(x)}{(h_K(\nu_K(x)))^{(p-1)\frac{n}{p}}}\right)^{\frac{p}{n+p}} d\mathcal{H}^{n-1}(x).
\end{aligned}$$

□

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