OPTIMAL SOBOLEV NORMS IN THE AFFINE CLASS

QINGZHONG HUANG AND AI-JUN LI

Abstract. Optimal Sobolev norms under volume preserving affine transformations are considered. It turns out that this minimal transform is equivalent to the $(p, 2)$ Fisher information matrix defined by Lutwak, Lv, Yang, and Zhang. Furthermore, some analytic inequalities regarding to the L^p affine and L^p sine energies for the optimal function are investigated.

1. Introduction

For $p \geq 1$ and $n \geq 2$, the L^p Sobolev space $W^{1,p}(\mathbb{R}^n)$ is the space of real-valued L^p functions on $\mathbb{R}^n$ with weak L^p partial derivatives. The sharp L^p Sobolev inequality states that if $f \in W^{1,p}(\mathbb{R}^n)$ for $1 \leq p < n$, then there exists a best constant $A_{n,p}$ such that

$$\|\nabla f\|_p \geq A_{n,p} \|f\|_{\frac{np}{n-p}}, \tag{1.1}$$

where $\|\cdot\|_q$ denotes the L^q norm of the Euclidean norm of functions and ∇f is the gradient of f. Inequality (1.1) goes back to Federer and Fleming [12] and Maz'ya [31] for $p = 1$ and to Aubin [3] and Talenti [35] for $1 < p < n$. For strengthened versions of (1.1), see, e.g., [6, 7, 9, 10, 32, 39].

A stronger version of the sharp L^p Sobolev inequality, known as sharp affine L^p Sobolev inequality, were introduced and established by Zhang [42] for $p = 1$ and by Lutwak, Yang and Zhang [24] for $1 < p < n$. It

2000 *Mathematics Subject Classification.* 46E35, 52A40.

Key words and phrases. L^p surface isotropic position, functional L^p surface isotropic position, L^p affine energy, L^p sine energy, $(p, 2)$ Fisher information matrix.

The second-named author was supported by NSFC-Henan Joint Fund (U1204102) and Doctoral Fund of Henan Polytechnic University (B2011-024).

states that if $f \in W^{1,p}(\mathbb{R}^n)$ for $1 \le p < n$, then

$$\mathcal{E}_p(f) \ge A_{n,p} \|f\|_{\frac{np}{n-p}}, \tag{1.2}$$

where $\mathcal{E}_p(f)$ is the L^p affine energy, defined, for $f \in W^{1,p}(\mathbb{R}^n)$, by

$$\mathcal{E}_p(f) = c_{n,p} \Big(\int_{S^{n-1}} \|D_u f\|_p^{-n} du \Big)^{-1/n}, \tag{1.3}$$

with

$$c_{n,p} = \frac{n^{\frac{1}{n}+\frac{1}{p}} \pi^{\frac{1}{2p}+\frac{1}{2}} \Gamma(\frac{n+p}{2})^{\frac{1}{p}}}{2^{\frac{1}{p}} \Gamma(1+\frac{n}{2})^{\frac{1}{n}+\frac{1}{p}} \Gamma(\frac{p+1}{2})^{\frac{1}{p}}}$$

and $D_u f = u \cdot \nabla f(x)$ is the directional derivative of f in the direction u. Note that the L^p affine energy is a fundamental concept in variants of affine Sobolev inequalities (see e.g., [2, 8, 17–19, 24, 28, 36–38, 42]).

We emphasize the remarkable and important fact that $\mathcal{E}_p(f)$ is invariant under volume preserving affine transformations on $\mathbb{R}^n$, while $\|\nabla f\|_p$ is invariant only under rigid motions. Hence the affine L^p Sobolev inequality (1.2) is invariant under affine transformations of $\mathbb{R}^n$, while the classical L^p Sobolev inequality (1.1) is invariant only under rigid motions.

Moreover, it was shown by Lutwak, Yang, and Zhang [24] that

$$\|\nabla f\|_p \ge \mathcal{E}_p(f). \tag{1.4}$$

Hence the affine L^p Sobolev inequality (1.2) is stronger than the L^p Sobolev inequaliy (1.1). Denote $f_T(x) = f(Tx)$ for every $T \in \mathrm{SL}(n)$ and $x \in \mathbb{R}^n$. From the above facts, we may ask the following question: Does there exist a $T_p \in \mathrm{SL}(n)$ and a constant $B_{n,p}$ such that

$$B_{n,p} \|\nabla f_{T_p}\|_p \le \mathcal{E}_p(f_{T_p}) = \mathcal{E}_p(f)? \tag{1.5}$$

In other words, it asks whether there exists a transformations $T_p \in \mathrm{SL}(n)$ minimizes $\|\nabla f_T\|_p$ for all $T \in \mathrm{SL}(n)$.

We answer this question in the following theorem.

Theorem 1.1. *Given $f \in W^{1,p}(\mathbb{R}^n)$. There exists a unique (up to orthogonal transformations) $T_p \in \mathrm{SL}(n)$ minimizes $\{\|\nabla f_T\|_p : T \in$*

SL$(n)\}$. *In particular, when* $p = 2$, *for the minimizer* T_2, *we have*

$$\|\nabla f_{T_2}\|_2^2 I_n = n \int_{\mathbb{R}^n} \nabla f_{T_2}(x) \otimes \nabla f_{T_2}(x) dx, \tag{1.6}$$

where $\nabla f_{T_2} \otimes \nabla f_{T_2}$ *is the rank-one orthogonal projection onto the space spanned by the vector* ∇f_{T_2} *and* I_n *denotes the identity operator on* $\mathbb{R}^n$.

It is shown in Section 6 that T_p in Theorem 1.1 is equivalent to the $(p, 2)$ Fisher information matrix $J_{p,2}(X)$ defined by Lutwak, Lv, Yang, and Zhang [22] in the context of the information theory. For the related problem, please see [28] by Lutwak, Yang, and Zhang.

Thus, the problem (1.5) can be answered as follows.

Theorem 1.2. *Given* $f \in W^{1,p}(\mathbb{R}^n)$, *we have*

$$B_{n,p}\|\nabla f_{T_p}\|_p \leq \mathcal{E}_p(f) \leq \|\nabla f_{T_p}\|_p, \tag{1.7}$$

where

$$B_{n,p} = \frac{\pi^{\frac{1}{2p}+\frac{1}{2}}\Gamma(\frac{n+p}{2})^{\frac{1}{p}}\Gamma(1+\frac{n}{p})^{\frac{1}{n}}}{2^{\frac{1}{p}+1}\Gamma(1+\frac{n}{2})^{\frac{1}{n}+\frac{1}{p}}\Gamma(\frac{p+1}{2})^{\frac{1}{p}}\Gamma(1+\frac{1}{p})}.$$

Moreover, let K *be an origin-symmetric convex body in* $\mathbb{R}^n$ *and let* $f(x) = g(\|x\|_K)$ *with* $g \in C^1(0,\infty)$. *For* $p \neq 2$, *there is equality in the left inequality if and only if* K *is a parallelotope. If* p *is not an even integer, then there is equality in the right inequality if and only if* K *is an ellipsoid.*

Unlike the L_p Sobolev norm $\|\nabla f\|_p$, the quantity

$$\min_{T\in \mathrm{SL}(n)} \|\nabla f_T\|_p$$

is clearly invariant under volume preserving affine transformations on $\mathbb{R}^n$. Thus, the inequality (1.7) is invariant under affine transformations of $\mathbb{R}^n$, while the inequality (1.4) is invariant only under rigid motions.

Next, motivated by the work of Maresch, Schuster [29] and Guo, Leng [16], we define the L^p *sine energy*, for $f \in W^{1,p}(\mathbb{R}^n)$, by

$$\mathcal{G}_p(f) = d_{n,p}\Big(\int_{S^{n-1}} \|D_{u^\perp} f\|_p^{-n} du\Big)^{-1/n}, \tag{1.8}$$

where

$$d_{n,p} = \frac{n^{\frac{1}{n}+\frac{1}{p}}\pi^{\frac{1}{2}}\Gamma(\frac{n-1}{2})\Gamma(\frac{n+p}{2})}{2\Gamma(1+\frac{n}{2})^{1+\frac{1}{n}}\Gamma(\frac{n+p-1}{2})}$$

and $D_{u^\perp} f = \mathrm{P}_{u^\perp}\nabla f(x)$. Here $u^\perp$ is the central hyperplane perpendicular to u and $\mathrm{P}_{u^\perp}\nabla f(x)$ is the orthogonal projection of the vector $\nabla f(x)$ onto $u^\perp$. Unfortunately, it is easy to see that $\mathcal{G}_p(f)$ is not invariant under volume preserving affine transformations on $\mathbb{R}^n$ but is still invariant under rigid motions. Similar to Theorem 1.2, the following theorem is established.

Theorem 1.3. *Given $f \in W^{1,p}(\mathbb{R}^n)$, we have*

$$C_{n,p}\|\nabla f_{T_p}\|_p \le \mathcal{G}_p(f_{T_p}) \le \|\nabla f_{T_p}\|_p, \tag{1.9}$$

where

$$C_{n,p} = \frac{p^{\frac{1}{n-1}} n^{\frac{1}{p}-1}(n-1)^{\frac{1}{p}-\frac{1}{n-1}+1}\Gamma(1+\frac{n}{p})^{\frac{1}{n}}\Gamma(\frac{n-1}{2})^2\Gamma(\frac{n+p}{2})^2}{4\Gamma(\frac{n-1}{p})^{\frac{1}{n-1}}\Gamma(\frac{n+2}{2})^2\Gamma(\frac{n+p-1}{2})^2}.$$

Moreover, let K be an origin-symmetric convex body in $\mathbb{R}^n$ and let $f(x) = g(\|x\|_K)$ with $g \in C^1(0,\infty)$. If p is not an even integer, then there is equality in the right inequality if and only if K is an ellipsoid.

Furthermore, we establish the following inequalities.

Theorem 1.4. *Given $f \in W^{1,p}(\mathbb{R}^n)$, for every $u \in S^{n-1}$,*

$$n^{-\frac{1}{p}}\|\nabla f_{T_p}\|_p \le \|D_u f_{T_p}\|_p \le n^{-\frac{1}{2}}\|\nabla f_{T_p}\|_p, \quad 1 \le p \le 2,$$

and

$$n^{-\frac{1}{p}}\|\nabla f_{T_p}\|_p \ge \|D_u f_{T_p}\|_p \ge n^{-\frac{1}{2}}\|\nabla f_{T_p}\|_p, \quad p \ge 2.$$

Theorem 1.5. *Given $f \in W^{1,p}(\mathbb{R}^n)$, for every $u \in S^{n-1}$,*

$$\left(\frac{n-1}{n}\right)^{\frac{1}{p}}\|\nabla f_{T_p}\|_p \le \|D_{u^\perp} f_{T_p}\|_p \le \left(\frac{n-1}{n}\right)^{\frac{1}{2}}\|\nabla f_{T_p}\|_p, \quad 1 \le p \le 2,$$

and

$$\left(\frac{n-1}{n}\right)^{\frac{1}{p}}\|\nabla f_{T_p}\|_p \ge \|D_{u^\perp} f_{T_p}\|_p \ge \left(\frac{n-1}{n}\right)^{\frac{1}{2}}\|\nabla f_{T_p}\|_p, \quad p \ge 2.$$

The rest of this paper is organized as follows: In Section 2 the basic notations and preliminaries are provided. To prove our main theorems, some facts about the L^p surface isotropic positions are given in Section 3. The interplay of the geometric inequalities and corresponding analytic inequalities is presented in Section 4 and Section 5. In Section 6, the equivalence of T_p in Theorem 1.1 and the $(p,2)$ Fisher information matrix $J_{p,2}(X)$ is explored.

2. Notations and Preliminaries

We refer to the book [34] for the basic facts about convex bodies and the L^p Brunn-Minkowski theory.

Denote by $p^* \in [1,\infty)$ the Hölder conjugate of $p \in [1,\infty)$; i.e.,

$$\frac{1}{p}+\frac{1}{p^*}=1.$$

Denote by $\kappa_n(p)$ the volume of the unit ball of the ℓ_p^n space, that is,

$$\kappa_n(p)=\frac{2^n(\Gamma(1+\frac{1}{p}))^n}{\Gamma(1+\frac{n}{p})}.$$

Abbreviate $\kappa_n(2)$ by κ_n, the volume of n-dimensional Euclidean unit ball B_2^n. Define

$$\alpha_{n,p}:=\Big(\frac{\Gamma(1+\frac{n}{2})\Gamma(\frac{1+p}{2})}{\Gamma(1+\frac{1}{2})\Gamma(\frac{n+p}{2})}\Big)^{\frac{n}{p}},$$

$$\beta_{n,p}:=\frac{p^{\frac{n}{n-1}}n^{\frac{n}{p}-n}(n-1)^{\frac{n}{p}-\frac{n}{n-1}+n}\Gamma(1+\frac{n}{p})}{\Gamma(\frac{n-1}{p})^{\frac{n}{n-1}}},$$

and

$$\gamma_{n,p}:=\frac{2\Gamma(\frac{n+2}{2})\Gamma(\frac{n+p-1}{2})}{\Gamma(\frac{n-1}{2})\Gamma(\frac{n+p}{2})}.$$

A convex body is a compact convex set in $\mathbb{R}^n$ which is throughout assumed to contain the origin in its interior. We denote by $\mathcal{K}_o^n$ the class of convex bodies. Let $\mathcal{K}_e^n$ denote the class of origin-symmetric members of $\mathcal{K}_o^n$. Each convex body K is uniquely determined by its support function $h_K = h(K,\cdot) : \mathbb{R}^n \to \mathbb{R}$ defined by

$$h_K(x)=h(K,x):=\max\{x\cdot y : y\in K\},$$

where $x \cdot y$ denotes the standard inner product of x and y in $\mathbb{R}^n$. Let $\|\cdot\|_K : \mathbb{R}^n \to [0, \infty)$ denote the Minkowski functional of $K \in \mathcal{K}_o^n$; i.e., $\|x\|_K = \min\{\lambda \geq 0 : x \in \lambda K\}$.

The polar set K^* of $K \in \mathcal{K}_o^n$ is the convex body defined by

$$K^* = \{x \in \mathbb{R}^n : x \cdot y \leq 1 \text{ for all } y \in K\}.$$

Clearly, for $K \in \mathcal{K}_o^n$,

$$h_{K^*}(\cdot) = \|\cdot\|_K. \tag{2.1}$$

For $K, L \in \mathcal{K}_o^n$, and $\varepsilon > 0$, the L^p Minkowski-Firey combination $K +_p \varepsilon \cdot L$ is the convex body whose support function is given by

$$h(K +_p \varepsilon \cdot L, \cdot)^p = h(K, \cdot)^p + \varepsilon h(L, \cdot)^p.$$

The L^p mixed volume $V_p(K, L)$ of $K, L \in \mathcal{K}_o^n$ was defined in [21] by

$$V_p(K, L) = \frac{p}{n} \lim_{\varepsilon \to 0^+} \frac{V(K +_p \varepsilon \cdot L) - V(K)}{\varepsilon}.$$

where V is the n-dimensional volume (i.e. Lebesgue measure in $\mathbb{R}^n$). In particular, $V_p(K, K) = V(K)$. It was shown in [21] that corresponding to each $K \in \mathcal{K}_o^n$, there is a positive Borel measure $S_p(K, \cdot)$ on S^{n-1}, called the L^p surface area measure of K, such that for every $L \in \mathcal{K}_o^n$,

$$V_p(K, L) = \frac{1}{n} \int_{S^{n-1}} h_L^p(u) dS_p(K, u).$$

Moreover, the L^p surface area measure is absolutely continuous with respect to the surface area measure $S(K, \cdot)$ of K:

$$dS_p(K, \cdot) = h_K^{1-p}(\cdot) dS(K, \cdot).$$

Recall that for a Borel set $\omega \subset S^{n-1}$, $S(K, \omega)$ is the $(n-1)$-dimensional Hausdorff measure $\mathcal{H}^{n-1}$ of the set of all boundary points of K for which there exists a normal vector of K belonging to ω. When $L = B_2^n$, the L^p surface area $S_p(K)$ of K is given by

$$S_p(K) = nV(K, B_2^n).$$

Clearly,

$$S_p(tK, \cdot)/V(tK) = t^{-p} S_p(K, \cdot)/V(K) \tag{2.2}$$

for all $t > 0$ and $K \in \mathcal{K}_o^n$.

In [25], Lutwak, Yang, and Zhang established the following remarkable result.

The solution of normalized even L^p Minkowski problem. Let $p \geq 1$. If μ is an even Borel measure on S^{n-1} whose support is not contained in a great subsphere of S^{n-1}, then there exists a unique origin-symmetric convex body K such that $S_p(K, \cdot)/V(K) = \mu$.

A finite nonnegative Borel measure μ on S^{n-1} is called isotropic if

$$\int_{S^{n-1}} v \otimes v d\mu(v) = I_n. \tag{2.3}$$

The measure μ is said to be even if it assumes the same value on antipodal sets. The two most important examples of even isotropic measures on S^{n-1} are suitably normalized spherical Lebesgue measure and the cross measure, i.e., measures concentrated uniformly on $\{\pm b_1, \ldots, \pm b_n\}$, where $b_1, \ldots, b_n$ denote orthonormal basis vectors of $\mathbb{R}^n$.

3. The L^p surface isotropic position

A convex body K in $\mathbb{R}^n$ is said in L^p *surface isotropic position* if its normalized L^p surface area measure $nS_p(K, \cdot)/S_p(K)$ is isotropic on S^{n-1}. This concept was first introduced by Lutwak, Yang, Zhang [27], while the L^1 case was first defined in [33]. Moreover, Lutwak, Yang, Zhang [27] obtained the following result (see also [40] by Yu). For $p = 1$, it was due to Petty [33] and Giannopoulos, Papadimitrakis [13].

Proposition 3.1. *Suppose $p \in [1, \infty)$ and $K \in \mathcal{K}_o^n$. Then $S_p(K) = \min\{S_p(TK) : T \in \mathrm{SL}(n)\}$ if and only if K is in L^p surface isotropic position. The minimal L^p surface area position is unique up to orthogonal transformations.*

Other extensions of this notion can be found in [1, 13–15, 30, 41].

Define the convex body $\mathcal{C}_p(\mu)$ in $\mathbb{R}^n$ whose support function, for $u \in S^{n-1}$, is given by

$$h_{\mathcal{C}_p(\mu)}(u) = \Big(\int_{S^{n-1}} |u \cdot v|^p d\mu(v) \Big)^{1/p}. \tag{3.1}$$

The following lemma, extending the results of Ball [4] and Barthe[5], was due to Lutwak, Yang and Zhang [26].

Lemma 3.2. *Suppose $p \in [1, \infty)$. If μ is an even isotropic measure on S^{n-1}, then*

$$\frac{\kappa_n}{\alpha_{n,p}} \le V(\mathcal{C}_p^* \mu) \le \kappa_n(p). \tag{3.2}$$

If p is not an even integer, then there is equality in the left inequality if and only if μ is suitably normalized Lebesgue measure. For $p \neq 2$, there is equality in the right inequality if and only if μ is a cross measure.

Define the convex body $\mathcal{S}_p(\mu)$ in $\mathbb{R}^n$ whose support function, for $u \in S^{n-1}$, is given by

$$h_{\mathcal{S}_p(\mu)}(u) = \Big(\int_{S^{n-1}} (1 - |u \cdot v|^2)^{p/2} d\mu(v) \Big)^{1/p} = \Big(\int_{S^{n-1}} |\mathrm{P}_{v^\perp} u|^p d\mu(v) \Big)^{1/p}.$$

The following lemma, extending the case $p = 1$ due to Maresch and Schuster [29], was proved by Guo and Leng [16].

Lemma 3.3. *Suppose $p \in [1, \infty)$. If μ is an even isotropic measure on S^{n-1}, then*

$$\frac{\kappa_n}{\gamma_{n,p}^n} \le V(\mathcal{S}_p^* \mu) \le \frac{\kappa_n \gamma_{n,p}^n}{\beta_{n,p}}. \tag{3.3}$$

If p is not an even integer, then there is equality in the left inequality if and only if μ is suitably normalized Lebesgue measure.

As shown in [29] and [16], the right inequality in (3.3) is asymptotically optimal.

For each convex body $K \in \mathcal{K}_o^n$, the L^p projection body, $\Pi_p K$, of K was introduced by Lutwak, Yang, and Zhang [23], which is the origin-symmetric convex body whose support function, for $u \in S^{n-1}$, is given by

$$h_{\Pi_p K}(u) = \Big(\frac{1}{V(K)} \int_{S^{n-1}} |u \cdot v|^p dS_p(K, v) \Big)^{1/p}. \tag{3.4}$$

From (2.2), we have

$$\Pi_p(cK) = c^{-1}\Pi_p K. \tag{3.5}$$

The following lemma without the equality conditions was due to Yu [40], which extended the case $p = 1$ by Giannopoulos and Papadimitrakis [13]. Here we characterize the equality conditions.

Lemma 3.4. *Suppose $p \in [1, \infty)$. If $K \in \mathcal{K}_e^n$ is in L^p surface isotropic position, then*

$$\frac{n^{n/p}\kappa_n}{\alpha_{n,p}} \le V(\Pi_p^* K)\Big(\frac{S_p(K)}{V(K)}\Big)^{n/p} \le n^{n/p}\kappa_n(p). \tag{3.6}$$

If p is not an even integer, then there is equality in the left inequality if and only if K is a ball. For $p \neq 2$, there is equality in the right inequality if and only if K is a cube.

Proof. Define the even measure μ on S^{n-1} by

$$\mu = \frac{n}{S_p(K)} S_p(K, \cdot).$$

Since K is in L^p surface isotropic position, it follows that μ is isotropic. By the definitions of $\mathcal{C}_p\mu$ (3.1) and $\Pi_p K$ (3.4), we have

$$\mathcal{C}_p^*\mu = \Big(\frac{S_p(K)}{nV(K)}\Big)^{1/p}\Pi_p^* K.$$

Then, inequalities (3.6) immediately follows from (3.2).

Now, we deal with the characterization of the equalities in (3.6). The equality conditions of (3.2) yield that if p is not an even integer, then there is equality in the left inequality if and only if $nS_p(K,\cdot)/S_p(K)$ is suitably normalized Lebesgue measure; For $p \neq 2$, there is equality in the right inequality if and only if $nS_p(K,\cdot)/S_p(K)$ is a cross measure. On the other hand, it is easy to verify that $nS_p(B_2^n,\cdot)/S_p(B_2^n)$ is a suitably normalized Lebesgue measure and $nS_p(OC_0,\cdot)/S_p(OC_0)$ is a cross measure on S^{n-1} for some $O \in \mathrm{O}(n)$, where $C_0 = [-1,1]^n$. It follows from (2.2) that

$$\frac{nS_p(B_2^n,\cdot)}{S_p(B_2^n)} = \frac{S_p(\lambda_1 B_2^n,\cdot)}{V(\lambda_1 B_2^n)} \quad \text{with } \lambda_1 = \Big(\frac{nV(B_2^n)}{S_p(B_2^n)}\Big)^{-\frac{1}{p}}$$

and

$$\frac{nS_p(OC_0,\cdot)}{S_p(OC_0)}=\frac{S_p(\lambda_2 OC_0,\cdot)}{V(\lambda_2 OC_0)}\quad\text{with } \lambda_2=\Big(\frac{nV(C_0)}{S_p(C_0)}\Big)^{-\frac{1}{p}}.$$

Thus, the equality conditions follow from the uniqueness of the solution of the normalized even L_p Minkowski problem and the fact that

$$\Big(\frac{nV(tK)}{S_p(tK)}\Big)^{-\frac{1}{p}}tK=\Big(\frac{nV(K)}{S_p(K)}\Big)^{-\frac{1}{p}}K$$

for all $t>0$. □

Corresponding to the L^p projection body Π_pK, Maresch, Schuster [29] and Guo, Leng [16] introduced a new convex body Ψ_pK whose support function, for $u\in S^{n-1}$, is given by

$$h_{\Psi_pK}(u)=\Big(\frac{1}{V(K)}\int_{S^{n-1}}|\mathrm{P}_{v^\perp}u|^p dS_p(K,v)\Big)^{1/p}. \tag{3.7}$$

From (2.2), we have

$$\Psi_p(cK)=c^{-1}\Psi_pK. \tag{3.8}$$

Using the similar argument of Lemma 3.4, Guo and Leng [16] established the following result.

Lemma 3.5. *Suppose $p\in[1,\infty)$. If $K\in\mathcal{K}_e^n$ is in L^p surface isotropic position, then*

$$\frac{n^{n/p}\kappa_n}{\gamma_{n,p}^n}\le V(\Psi_p^*K)\Big(\frac{S_p(K)}{V(K)}\Big)^{n/p}\le\frac{n^{n/p}\kappa_n\gamma_{n,p}^n}{\beta_{n,p}}. \tag{3.9}$$

If p is not an even integer, then there is equality in the left inequality if and only if K is a ball.

The following observation, extending the case $p=1$ by Giannopoulos and Papadimitrakis [13], was due to Yu [40]. For the completeness, we present the proof here.

Lemma 3.6. *Suppose $p\in[1,\infty)$. If $K\in\mathcal{K}_o^n$ is in L^p surface isotropic position, then for every $u\in S^{n-1}$,*

$$n^{-\frac{1}{p}}\Big(\frac{S_p(K)}{V(K)}\Big)^{\frac{1}{p}}\le h(\Pi_pK,u)\le n^{-\frac{1}{2}}\Big(\frac{S_p(K)}{V(K)}\Big)^{\frac{1}{p}},\quad 1\le p\le 2,$$

and

$$n^{-\frac{1}{p}}\Big(\frac{S_p(K)}{V(K)}\Big)^{\frac{1}{p}} \geq h(\Pi_p K, u) \geq n^{-\frac{1}{2}}\Big(\frac{S_p(K)}{V(K)}\Big)^{\frac{1}{p}}, \quad p \geq 2.$$

Proof. For $p = 2$, by the definition of $\Pi_p K$ (3.4), we have

$$h(\Pi_2 K, u) = \Big(\frac{1}{V(K)}\int_{S^{n-1}} |u\cdot v|^2 dS_2(K,v)\Big)^{\frac{1}{2}} = \Big(\frac{S_2(K)}{nV(K)}\Big)^{\frac{1}{2}},$$

for every $u \in S^{n-1}$.

For $1 \leq p < 2$, by (3.4) and the Hölder inequality, we obtain

$$\begin{aligned} h(\Pi_p K, u) &= \Big(\frac{1}{V(K)}\int_{S^{n-1}} |u\cdot v|^p dS_p(K,v)\Big)^{\frac{1}{p}} \\ &\leq \Big[\Big(\frac{1}{V(K)}\int_{S^{n-1}} |u\cdot v|^2 dS_p(K,v)\Big)^{\frac{p}{2}} \Big(\frac{S_p(K)}{V(K)}\Big)^{1-\frac{p}{2}}\Big]^{\frac{1}{p}} \\ &= n^{-\frac{1}{2}}\Big(\frac{S_p(K)}{V(K)}\Big)^{\frac{1}{p}}, \end{aligned}$$

for every $u \in S^{n-1}$. On the other hand, we have

$$h(\Pi_p K, u) \geq \Big(\frac{1}{V(K)}\int_{S^{n-1}} |u\cdot v|^2 dS_p(K,v)\Big)^{\frac{1}{p}} = \Big(\frac{S_p(K)}{nV(K)}\Big)^{\frac{1}{p}}.$$

For $p > 2$, the argument is similar as the above arguments. □

To obtain the similar result of Lemma 3.6 about $\Psi_p K$, we need the following lemma.

Lemma 3.7. *If μ is an isotropic measure on S^{n-1}, then*

$$\frac{1}{n-1}\int_{S^{n-1}} |\mathrm{P}_{v^\perp} u|^2 d\mu(v) = 1, \quad \textit{for every } u \in S^{n-1}.$$

Proof. It follows from the isotropicity assumption (2.3) that

$$\mathrm{P}_{u^\perp} = \int_{S^{n-1}} \mathrm{P}_{u^\perp} v \otimes \mathrm{P}_{u^\perp} v \, d\mu(v). \tag{3.10}$$

Notice that $\mathrm{P}_{u^\perp}$ has rank $n-1$. Taking traces in (3.10), we get

$$\int_{S^{n-1}} |\mathrm{P}_{u^\perp} v|^2 d\mu(v) = n-1, \quad \text{for every } u \in S^{n-1}.$$

The result now follows from the fact that $|\mathrm{P}_{v^\perp} u| = |\mathrm{P}_{u^\perp} v|$ for every $u, v \in S^{n-1}$. □

Then we have

Lemma 3.8. *Suppose* $p \in [1, \infty)$. *If* $K \in \mathcal{K}_o^n$ *is in* L^p *surface isotropic position, then for every* $u \in S^{n-1}$,

$$\Big(\frac{n-1}{n}\Big)^{\frac{1}{p}}\Big(\frac{S_p(K)}{V(K)}\Big)^{\frac{1}{p}} \le h(\Psi_p K, u) \le \Big(\frac{n-1}{n}\Big)^{\frac{1}{2}}\Big(\frac{S_p(K)}{V(K)}\Big)^{\frac{1}{p}}, \quad 1 \le p \le 2,$$

and

$$\Big(\frac{n-1}{n}\Big)^{\frac{1}{p}}\Big(\frac{S_p(K)}{V(K)}\Big)^{\frac{1}{p}} \ge h(\Psi_p K, u) \ge \Big(\frac{n-1}{n}\Big)^{\frac{1}{2}}\Big(\frac{S_p(K)}{V(K)}\Big)^{\frac{1}{p}}, \quad p \ge 2.$$

Proof. Using Lemma 3.7 with the isotropic measure $\mu = n dS_p(K, \cdot)/S_p(K)$, we have

$$\frac{1}{n-1}\int_{S^{n-1}} |\mathrm{P}_{v^\perp} u|^2 \frac{n}{S_p(K)} dS_p(K, v) = 1, \quad \text{for every } u \in S^{n-1}.$$

For $p = 2$, by (3.7), we have

$$h(\Psi_2 K, u) = \Big(\frac{1}{V(K)}\int_{S^{n-1}} |\mathrm{P}_{v^\perp} u|^2 dS_2(K, v)\Big)^{\frac{1}{2}} = \Big(\frac{(n-1)S_2(K)}{nV(K)}\Big)^{\frac{1}{2}},$$

for every $u \in S^{n-1}$.

For $1 \le p < 2$, by (3.7) and the Hölder inequality, we obtain

$$\begin{aligned} h(\Psi_p K, u) &= \Big(\frac{1}{V(K)}\int_{S^{n-1}} |\mathrm{P}_{v^\perp} u|^p dS_p(K, v)\Big)^{\frac{1}{p}} \\ &\le \Big[\Big(\frac{1}{V(K)}\int_{S^{n-1}} |\mathrm{P}_{v^\perp} u|^2 dS_p(K, v)\Big)^{\frac{p}{2}}\Big(\frac{S_p(K)}{V(K)}\Big)^{1-\frac{p}{2}}\Big]^{\frac{1}{p}} \\ &= \Big(\frac{n-1}{n}\Big)^{\frac{1}{2}}\Big(\frac{S_p(K)}{V(K)}\Big)^{\frac{1}{p}}, \end{aligned}$$

for every $u \in S^{n-1}$. On the other hand, we have

$$h(\Psi_p K, u) \ge \Big(\frac{1}{V(K)}\int_{S^{n-1}} |\mathrm{P}_{v^\perp} u|^2 dS_p(K, v)\Big)^{\frac{1}{p}} = \Big(\frac{(n-1)S_p(K)}{nV(K)}\Big)^{\frac{1}{p}}.$$

For $p > 2$, the argument is similar as the previous. ∎

4. The functional L^p surface isotropic position

The even functional Minkowski problem on $W^{1,p}(\mathbb{R}^n)$ was solved by Lutwak, Yang, and Zhang [28].

Theorem 4.1. *Given a function $f \in W^{1,p}(\mathbb{R}^n)$, there exists a unique origin-symmetric convex body $\langle f \rangle_p$ such that*

$$\int_{\mathbb{R}^n} \Phi^p(\nabla f(x))dx = \frac{1}{V(\langle f \rangle_p)} \int_{S^{n-1}} \Phi(v)^p dS_p(\langle f \rangle_p, v), \tag{4.1}$$

for every even continuous function $\Phi : \mathbb{R}^n \to [0, \infty)$ that is homogeneous of degree 1.

The lemma below (see [28, Proposition 5.4.]) describes how $\langle f \rangle_p$ behaves if f is composed with a volume preserving linear transformation.

Lemma 4.2. *Suppose $f \in W^{1,p}(\mathbb{R}^n)$. Then for $T \in \mathrm{SL}(n)$,*

$$\langle f \circ T^{-1} \rangle_p = T \langle f \rangle_p.$$

We say that a function $f \in W^{1,p}(\mathbb{R}^n)$ is in the *functional L^p surface isotropic position* if $\langle f \rangle_p$ is in the L^p surface isotropic position.

Theorem 4.3. *Suppose $f \in W^{1,p}(\mathbb{R}^n)$ is in the functional L^p surface isotropic position. Then*

$$\|\nabla f\|_p = \min\{\|\nabla(f \circ T)\|_p : T \in \mathrm{SL}(n)\}.$$

The minimal position is unique up to orthogonal transformations. Moreover, if f is in the functional L^2 surface isotropic position, then

$$\|\nabla f\|_2^2 I_n = n \int_{\mathbb{R}^n} \nabla f(x) \otimes \nabla f(x) dx.$$

Proof. Taking $\Phi(v) = |v|$ in (4.1), we get

$$\|\nabla f\|_p^p = \int_{\mathbb{R}^n} |\nabla f(x)|^p dx = \frac{S_p(\langle f \rangle_p)}{V(\langle f \rangle_p)}. \tag{4.2}$$

From (4.2), Proposition 3.1 with $K = \langle f \rangle_p$, Lemma 4.2, and (4.2) again, we have

$$\|\nabla f\|_p^p = S_p(\langle f \rangle_p)/V(\langle f \rangle_p)$$

$$
\begin{aligned}
&= \min\{S_p(T\langle f\rangle_p)/V(T\langle f\rangle_p) : T \in \mathrm{SL}(n)\} \\
&= \min\{S_p(\langle f \circ T^{-1}\rangle_p)/V(\langle f \circ T^{-1}\rangle_p) : T \in \mathrm{SL}(n)\} \\
&= \min\{\|\nabla(f \circ T^{-1})\|_p : T \in \mathrm{SL}(n)\}.
\end{aligned}
$$

Now, we consider the case $p = 2$. Taking $\Phi(v) = |u \cdot v|$ in (4.1), we get

$$
\int_{\mathbb{R}^n} |u \cdot \nabla f(x)|^2 dx = \frac{1}{V(\langle f\rangle_2)} \int_{S^{n-1}} |u \cdot v|^2 dS_2(\langle f\rangle_2, v). \tag{4.3}
$$

Since f is in the functional L^2 surface isotropic position, it follows that $nS_2(\langle f\rangle_2, \cdot)/S_2(\langle f\rangle_2)$ is isotropic. Thus, from (4.3) and (4.2), we have

$$
\int_{\mathbb{R}^n} |u \cdot \nabla f(x)|^2 dx = \frac{S_2(\langle f\rangle_2)}{nV(\langle f\rangle_2)} = \frac{\|\nabla f\|_2^2}{n}.
$$

□

Theorem 4.4. *Suppose $f \in W^{1,p}(\mathbb{R}^n)$ is in the functional L^p surface isotropic position. Then*

$$
\frac{c_{n,p}}{n^{\frac{1}{n}+\frac{1}{p}} \kappa_n(p)^{\frac{1}{n}}} \|\nabla f\|_p \le \mathcal{E}_p(f) \le \|\nabla f\|_p. \tag{4.4}
$$

For $p \neq 2$, there is equality in the left inequality if and only if $\langle f\rangle_p$ is a cube. If p is not an even integer, then there is equality in the right inequality if and only if $\langle f\rangle_p$ is a ball.

Proof. Taking $\Phi(v) = |u \cdot v|$ in (4.1) and (3.4) , we have

$$
\begin{aligned}
\|D_u f\|_p^p &= \int_{\mathbb{R}^n} |u \cdot \nabla f(x)|^p dx \\
&= \frac{1}{V(\langle f\rangle_p)} \int_{S^{n-1}} |u \cdot v|^p dS_p(\langle f\rangle_p, v) = h(\Pi_p\langle f\rangle_p, u)^p.
\end{aligned} \tag{4.5}
$$

From (1.3) and the polar coordinate formula for volume, we obtain

$$
\begin{aligned}
\mathcal{E}_p(f) &= c_{n,p}\Big(\int_{S^{n-1}} \|D_u f\|_p^{-n} du\Big)^{-1/n} \\
&= c_{n,p}\Big(\int_{S^{n-1}} h(\Pi_p\langle f\rangle_p, u)^{-n} du\Big)^{-1/n} \\
&= c_{n,p} n^{-\frac{1}{n}} V(\Pi_p^*\langle f\rangle_p)^{-\frac{1}{n}}.
\end{aligned} \tag{4.6}
$$

The inequalities (4.4) and their equality conditions follow by combining (4.2), (4.6) and Lemma 3.4. □

Theorem 4.5. *Suppose $f \in W^{1,p}(\mathbb{R}^n)$ is in the functional L^p surface isotropic position. Then*

$$\frac{d_{n,p}\beta_{n,p}^{\frac{1}{n}}}{n^{\frac{1}{n}+\frac{1}{p}}\kappa_n^{\frac{1}{n}}\gamma_{n,p}}\|\nabla f\|_p \le \mathcal{G}_p(f) \le \|\nabla f\|_p. \tag{4.7}$$

If p is not an even integer, then there is equality in the right inequality if and only if $\langle f\rangle_p$ is a ball.

Proof. Taking $\Phi(v) = |\mathrm{P}_{u^\perp}v|$ in (4.1) and (3.7) , we have

$$\begin{aligned}\|D_{u^\perp}f\|_p^p &= \int_{\mathbb{R}^n}|\mathrm{P}_{u^\perp}\nabla f(x)|^p dx\\ &= \frac{1}{V(\langle f\rangle_p)}\int_{S^{n-1}}|\mathrm{P}_{u^\perp}v|^p dS_p(\langle f\rangle_p, v)\\ &= \frac{1}{V(\langle f\rangle_p)}\int_{S^{n-1}}|\mathrm{P}_{v^\perp}u|^p dS_p(\langle f\rangle_p, v)\\ &= h(\Psi_p\langle f\rangle_p, u)^p. \end{aligned}\tag{4.8}$$

The third equality follows from the fact that $|\mathrm{P}_{u^\perp}v| = |\mathrm{P}_{v^\perp}u|$ for every $u, v \in S^{n-1}$. From (1.8) and the polar coordinate formula for volume, we obtain

$$\begin{aligned}\mathcal{G}_p(f) &= d_{n,p}\Big(\int_{S^{n-1}}\|D_{u^\perp}f\|_p^{-n}du\Big)^{-1/n}\\ &= d_{n,p}\Big(\int_{S^{n-1}}h(\Psi_p\langle f\rangle_p, u)^{-n}du\Big)^{-1/n}\\ &= d_{n,p}n^{-\frac{1}{n}}V(\Psi_p^*\langle f\rangle_p)^{-\frac{1}{n}}. \end{aligned}\tag{4.9}$$

The inequalities (4.7) and their equality conditions follow by combining (4.2), (4.9) and Lemma 3.5. □

By Lemma 3.6 and 3.8, together with (4.2), (4.5) and (4.8), we have

Theorem 4.6. *Suppose $f \in W^{1,p}(\mathbb{R}^n)$ is in the functional L^p surface isotropic position. Then for every $u \in S^{n-1}$,*

$$n^{-\frac{1}{p}}\|\nabla f\|_p \le \|D_u f\|_p \le n^{-\frac{1}{2}}\|\nabla f\|_p, \qquad 1 \le p \le 2,$$

and

$$n^{-\frac{1}{p}}\|\nabla f\|_p \geq \|D_u f\|_p \geq n^{-\frac{1}{2}}\|\nabla f\|_p, \qquad p \geq 2.$$

Theorem 4.7. *Suppose $f \in W^{1,p}(\mathbb{R}^n)$ is in the functional L^p surface isotropic position. Then for every $u \in S^{n-1}$,*

$$\left(\frac{n-1}{n}\right)^{\frac{1}{p}}\|\nabla f\|_p \leq \|D_{u^\perp} f\|_p \leq \left(\frac{n-1}{n}\right)^{\frac{1}{2}}\|\nabla f\|_p, \qquad 1 \leq p \leq 2,$$

and

$$\left(\frac{n-1}{n}\right)^{\frac{1}{p}}\|\nabla f\|_p \geq \|D_{u^\perp} f\|_p \geq \left(\frac{n-1}{n}\right)^{\frac{1}{2}}\|\nabla f\|_p, \qquad p \geq 2.$$

Therefore, the previous theorems together with Proposition 3.1 immediately yield our main results (Theorem 1.1–1.5).

5. From analytic to geometric inequalities

To obtain geometric inequalities from their analytic inequalities, the following lemma will be needed (see e.g., [20, 37, 38]). Since the argument is slightly different, we give the proof for completeness.

Lemma 5.1. *Suppose $K \in \mathcal{K}_e^n$ and $f = g(\|x\|_K)$ with $g \in C^1(0,\infty)$. Then $\langle f\rangle_p$ is a dilate of K; that is*

$$\langle f\rangle_p = c(f)^{-\frac{1}{p}}K, \tag{5.1}$$

where $c(f) = V(K)\int_0^\infty t^{n-1}|g'(t)|^p dt$.

Proof. Since h_{K^*} is a Lipschitz function and $h_{K^*}(x) = 1$ on ∂K, for almost every $x \in \partial K$, we have

$$\nu_K(x) = \frac{\nabla h_{K^*}(x)}{|\nabla h_{K^*}(x)|},$$

where $\nu_K(x)$ is the outer unit normal vector of K at the point x. Note that $h_K(\nabla h_{K^*}(x)) = 1$ for almost every $x \in \mathbb{R}^n$. Hence

$$h_K(\nu_K(x)) = \frac{1}{|\nabla h_{K^*}(x)|}. \tag{5.2}$$

Then by (4.1) and (2.1), the co-area formula (see, e.g., [11, p.258]), the fact that ∇h_{K^*} is homogeneous of degree 0, and (5.2), we get

$$\begin{aligned}
&\frac{1}{V(\langle f\rangle_p)}\int_{S^{n-1}}\Phi^p(v)dS_p(\langle f\rangle_p,v)\\
=&\int_{\mathbb{R}^n}\Phi^p(\nabla f(x))dx\\
=&\int_{\mathbb{R}^n}\Phi^p(g'(h_{K^*}(x)\nabla h_{K^*}(x)))dx\\
=&\int_{\mathbb{R}^n}|g'(h_{K^*}(x))|^p\Phi^p\Big(\frac{\nabla h_{K^*}(x)}{|\nabla h_{K^*}(x)|}\Big)|\nabla h_{K^*}(x)|^p dx\\
=&\int_0^\infty\int_{\partial K}t^{n-1}|g'(t)|^p\Phi^p\Big(\frac{\nabla h_{K^*}(x)}{|\nabla h_{K^*}(x)|}\Big)|\nabla h_{K^*}(x)|^{p-1}d\mathcal{H}^{n-1}(x)dt\\
=&\int_0^\infty t^{n-1}|g'(t)|^p dt\int_{\partial K}\Phi^p(\nu_K(x))h_K(\nu_K(x))^{1-p}d\mathcal{H}^{n-1}(x)\\
=&\int_0^\infty t^{n-1}|g'(t)|^p dt\int_{S^{n-1}}\Phi^p(v)dS_p(K,v)\\
=&\frac{1}{V(K_0)}\int_{S^{n-1}}\Phi^p(v)dS_p(K_0,v)
\end{aligned}$$

for every even continuous function Φ that is homogeneous of degree 1. From (2.2), we get $K_0 = K/c(f)^{1/p}$ with $c(f) = V(K)\int_0^\infty t^{n-1}|g'(t)|^p dt$. Thus, the uniqueness of the solution of the normalized even L^p Minkowski problem yields

$$\langle f\rangle_p = K_0 = K/c(f)^{1/p}.$$

$\square$

By Lemma 5.1, we see that the function f is in the functional L^p surface isotropic position if and only if the convex body K is in the L^p surface isotropic position when $f = g(\|x\|_K)$ with $g \in C^1(0,\infty)$ and $K \in \mathcal{K}_e^n$.

The assumption that $f(x) = g(\|x\|_K)$, together with (4.2), (5.1) and (2.2), give that

$$\|\nabla f\|_p = \Big(\frac{S_p(\langle f\rangle_p)}{V(\langle f\rangle_p)}\Big)^{\frac{1}{p}} = \Big(\frac{S_p(c(f)^{-\frac{1}{p}}K)}{V(c(f)^{-\frac{1}{p}}K)}\Big)^{\frac{1}{p}} = c(f)^{\frac{1}{p}}\Big(\frac{S_p(K)}{V(K)}\Big)^{\frac{1}{p}}.$$

From (4.5), (4.6), (5.1) and (3.5), we have

$$\begin{aligned}\|D_u f\|_p &= h(\Pi_p\langle f\rangle_p, u) = h(\Pi_p(c(f)^{-\frac{1}{p}}K), u)\\ &= c(f)^{\frac{1}{p}}h(\Pi_p K, u),\end{aligned}$$

for $u \in S^{n-1}$, and

$$\begin{aligned}\mathcal{E}_p(f) &= c_{n,p}n^{-\frac{1}{n}}V(\Pi_p^*\langle f\rangle_p)^{-\frac{1}{n}}\\ &= c_{n,p}n^{-\frac{1}{n}}V(\Pi_p^*(c(f)^{-\frac{1}{p}}K))^{-\frac{1}{n}}\\ &= c(f)^{\frac{1}{p}}c_{n,p}n^{-\frac{1}{n}}V(\Pi_p^*K)^{-\frac{1}{n}}.\end{aligned}$$

Similarly, from (4.8), (4.9), (5.1) and (3.8), we get

$$\|D_{u^\perp} f\|_p = c(f)^{\frac{1}{p}}h(\Psi_p K, u),$$

for $u \in S^{n-1}$, and

$$\mathcal{G}_p(f) = c(f)^{\frac{1}{p}}d_{n,p}n^{-\frac{1}{n}}V(\Psi_p^*K)^{-\frac{1}{n}}.$$

Consequently, we see that our analytic inequalities in Section 4 could imply corresponding geometry inequalities in Section 3. Moreover, the equality conditions of Theorem 1.2 and Theorem 1.3 follow from Lemma 5.1 and the equality conditions of Theorem 4.4 and Theorem 4.5.

6. Relation to the $(p, 2)$ Fisher information Matrix

The last section is dedicate to explain that T_p in Theorem 1.1 and the $(p, 2)$ Fisher information matrix $J_{p,2}(X)$ are actually equivalent.

The following two minimization problem will be considered:

(i) Given $f \in W^{1,p}(\mathbb{R}^n)$, define

$$m_1(f) = \inf\{\|T\nabla f\|_p^p : T \in \mathrm{SL}(n)\}.$$

Does there exist $Q_1 \in \mathrm{SL}(n)$ such that

$$\|Q_1\nabla f\|_p^p = m_1(f)?$$

(ii) Given $f \in W^{1,p}(\mathbb{R}^n)$, define

$$m_2(f) = \inf\{|T|^{-1} : \|T\nabla f\|_p^p = n \text{ and } T \in \mathrm{GL}(n)\}.$$

Does there exist $Q_2 \in \mathrm{GL}(n)$ such that

$$|Q_2|^{-1} = m_2(f)?$$

Here $|T|$ denotes the absolute value of the determinant of the matrix T.

The extremal problems (i) and (ii) are actually equivalent

Proposition 6.1. *If Q_1 is a solution of the problem (i), then $Q_2 = n^{\frac{1}{p}}\|Q_1\nabla f\|_p^{-1}Q_1$ is a solution of the problem (ii). Conversely, if Q_2 is a solution of the problem (ii), then $Q_1 = |Q_2|^{-\frac{1}{n}}Q_2$ is a solution of the problem (i).*

Proof. Suppose Q_1 is a solution of (i). Clearly, the matrix $Q_2 = n^{\frac{1}{p}}\|Q_1\nabla f\|_p^{-1}Q_1$ satisfies $\|Q_2\nabla f\|_p^p = n$. Then for each $T \in \mathrm{GL}(n)$ such that $\|T\nabla f\|_p^p = n$, we have

$$\begin{aligned}
|Q_2|^{-1} &= \frac{\|Q_1\nabla f\|_p^n}{n^{n/p}}|Q_1|^{-1} \\
&= \||Q_1|^{-1/n}Q_1\nabla f\|_p^n n^{-n/p} \\
&\le \||T|^{-1/n}T\nabla f\|_p^n n^{-n/p} \\
&= \frac{\|T\nabla f\|_p^n}{n^{n/p}}|T|^{-1} \\
&= |T|^{-1}.
\end{aligned}$$

Thus, Q_2 is a solution of (ii).

Conversely, suppose Q_2 is a solution of (ii), then the matrix $Q_1 = |Q_2|^{-\frac{1}{n}}Q_2 \in \mathrm{SL}(n)$. Note that for each $T \in \mathrm{SL}(n)$, the matrix $\bar{T} = n^{\frac{1}{p}}\|T\nabla f\|_p^{-1}T$ satisfies $\|\bar{T}\nabla f\|_p^p = n$. Since Q_2 is a solution of (ii), then

$$|Q_2|^{-1} \le |\bar{T}|^{-1} = |n^{\frac{1}{p}}\|T\nabla f\|_p^{-1}T|^{-1} = n^{-\frac{n}{p}}\|T\nabla f\|_p^n.$$

Therefore,

$$\begin{aligned}
\|Q_1\nabla f\|_p^p &= |Q_2|^{-\frac{p}{n}}\|Q_2\nabla f\|_p^p \\
&= n|Q_2|^{-\frac{p}{n}} \\
&\le \|T\nabla f\|_p^p,
\end{aligned}$$

which means Q_1 is a solution of (i).

□

For $T \in \mathrm{GL}(n)$, there exists an orthogonal matrix O and a positive definite symmetric matrix A such that $T = OA$. Let $\mathcal{S}$ denote the class of positive definite symmetric n-by-n matrices. Then the problem (ii) can be rewritten as: given $f \in W^{1,p}(\mathbb{R}^n)$,

$$\begin{aligned} m_2(f) &= \inf\{|A|^{-1} : \|A\nabla f\|_p^p = n \text{ and } A \in \mathcal{S}\} \\ &= \inf\{|A| : \|A^{-1}\nabla f\|_p^p = n \text{ and } A \in \mathcal{S}\}. \end{aligned}$$

It follows from [22, Theorem 4.3] that there is a unique $A_p \in \mathcal{S}$ such that $\|A_p^{-1}\nabla f\|_p^p = n$ and $|A_p| = m_2(f)$. Thus,

$$Q_2 = O_2 A_p^{-1},$$

for some $O_2 \in \mathrm{O}(n)$. In [22, Definition 4.2], the $(p, 2)$ Fisher information matrix $J_{p,2}(X)$ is defined as $J_{p,2}(X) = A_p^p$. Hence

$$Q_2 = O_2 J_{p,2}(X)^{-\frac{1}{p}}. \tag{6.1}$$

However, comparing the problem (i) and Theorem 1.1, we have

$$Q_1 = O_1 T_p^t, \tag{6.2}$$

for some $O_1 \in \mathrm{O}(n)$.

Therefore, it follows from Proposition 6.1, (6.1) and (6.2) that

$$J_{p,2}(X)^{-\frac{1}{p}} = n^{\frac{1}{p}} \|T_p^t \nabla f\|_p^{-1} O_3 T_p^t, \tag{6.3}$$

and

$$T_p^t = |J_{p,2}(X)|^{\frac{1}{np}} O_4 J_{p,2}(X)^{-\frac{1}{p}}, \tag{6.4}$$

for some $O_3, O_4 \in \mathrm{O}(n)$. Moreover, if we rewrite (6.3) and (6.4) with the relation $A_p = J_{p,2}(X)^{1/p}$, then

$$A_p^{-1} = n^{\frac{1}{p}} \|T_p^t \nabla f\|_p^{-1} O_3 T_p^t,$$

and

$$T_p^t = |A_p|^{\frac{1}{n}} O_4 A_p^{-1}.$$

When $p = 2$, it is easy to verify that the identity (1.6):

$$\|T_2^t \nabla f\|_2^2 I_n = n \int_{\mathbb{R}^n} T_2^t \nabla f(x) \otimes T_2^t \nabla f(x) dx,$$

is equivalent to [22, (16)]:

$$I_n = \int_{\mathbb{R}^n} A_2^{-1} \nabla f(x) \otimes A_2^{-1} \nabla f(x) dx.$$

Acknowledgements

The authors are indebted to the referee for many valuable suggestions, especially for the last section.

References

[1] D. Alonso-Gutierrez, *On a reverse Petty projection inequality for projections of convex bodies*, Adv. Geom. **14** (2014), 215-223.

[2] D. Alonso-Gutierrez, J. Bastero, and J. Bernués, *Factoring Sobolev inequalities through classes of functions*, Proc. Amer. Math. Soc. **140** (2012), 3557-3566.

[3] T. Aubin, *Problèmes isopérimétriques et espaces de Sobolev*, J. Differ. Geom. **11** (1976), 573-598.

[4] K. Ball, *Volume ratios and a reverse isoperimetric inequality*, J. London Math. Soc. **44** (1991), 351-359.

[5] F. Barthe, *On a reverse form of the Brascamp-Lieb inequality*, Invent. Math. **134** (1998), 685-693.

[6] W. Beckner, *Sharp Sobolev inequalities on the sphere and the Moser-Trudinger inequality*, Ann. Math. **138** (1993), 13-242.

[7] S. Bobkov and M. Ledoux, *From Brunn-Minkowski to sharp Sobolev inequalities*, Ann. Mat. Pura Appl. **187** (2008), 369-384.

[8] A. Cianchi, E. Lutwak, D. Yang, and G. Zhang, *Affine Moser-Trudinger and Morrey-Sobolev inequalities*, Calc. Var. Partial Differ. Equ. **36** (2009), 419-436.

[9] D. Cordero-Erausquin, D. Nazaret, and C. Villani, *A mass-transportation approach to sharp Sobolev and Gagliardo-Nirenberg inequalities*, Adv. Math. **182** (2004), 307-332.

[10] M. Del Pino and J. Dolbeault, *Best constants for Gagliardo-Nirenberg inequalities and applications to nonlinear diffusions*. J. Math. Pures Appl. **81** (2002), 847-875.

[11] H. Federer, *Geometric Measure Theory*, Springer, Berlin, 1969.

[12] H. Federer and W. Fleming, *Normal and integral currents*, Ann. Math. **72** (1960), 458-520.

[13] A. Giannopoulos and M. Papadimitrakis, *Isotropic surface area measures*, Mathematika **46** (1999), 1-13.

[14] A. Giannopoulos and V.D. Milman, *Extremal problems and isotropic positions of convex bodies*, Israel J. Math. **117** (2000), 29-60.

[15] A. Giannopoulos, V.D. Milman, and M. Rudelson, *Convex bodies with minimal mean width*, In: Geom. Aspects of Funct. Analysis. Lecture Notes in Mathematics, vol. **1745** (2000), 81-93.

[16] L. Guo and G. Leng, *Volume inequalities for the L_p-sine transform of isotropic measures*, Bull. Korean Math. Soc. **52** (2015), 837-849.

[17] C. Haberl and F. Schuster, *Asymmetric affine L_p Sobolev inequalities*, J. Funct. Anal. **257** (2009), 641-658.

[18] C. Haberl, F. Schuster, and J. Xiao, *An asymmetric affine Pólya-Szegö principle*, Math. Ann. **352** (2012), 517-542.

[19] J. Haddad, C.H. Jiménez, and M. Montenegro, *Sharp affine Sobolev type inequalities*, arXiv: 1505.07763v1.

[20] M. Ludwig, J. Xiao, and G. Zhang, *Sharp convex Lorentz-Sobolev inequalities*, Math. Ann. **350** (2011), 169-197.

[21] E. Lutwak, *The Brunn-Minkowski-Firey theory. I. Mixed volumes and the Minkowski problem*, J. Differential Geom. **38** (1993), 131-150.

[22] E. Lutwak, S. Lv, D. Yang, and G. Zhang. *Extensions of Fisher information and Stam's inequality*, IEEE Trans. Inform. Theory **58** (2012), 1319-1327.

[23] E. Lutwak, D. Yang, and G. Zhang, *L_p affine isoperimetric inequalities*, J. Differential Geom. **56** (2000), 111-132.

[24] E. Lutwak, D. Yang, and G. Zhang, *Sharp affine L_p Sobolev inequalities*, J. Differential Geom. **62** (2002), 17-38.

[25] E. Lutwak, D. Yang, and G. Zhang, *On the L_p-Minkowski problem*, Trans. Amer. Math. Soc. **356** (2004), 4359-4370.

[26] E. Lutwak, D. Yang, and G. Zhang, *Volume inequalities for subspaces of L_p*, J. Differential Geom. **68** (2004), 159-184.

[27] E. Lutwak, D. Yang, and G. Zhang, *L_p John ellipsoids*, Proc. London Math. Soc. **90** (2005), 497-520.

[28] E. Lutwak, D. Yang, and G. Zhang, *Optimal Sobolev norms and the L^p Minkowski problem*, Int. Math. Res. Not. **2006**, Art. ID 62978, 21 pp.

[29] G. Maresch, and F. Schuster, *The sine transform of isotropic measures*, Int. Math. Res. Not. **2012**, No. 4, 717-739.

[30] E. Markessinis, G. Paouris, and C. Saroglou, *Comparing the M-position with some classical positions of convex bodies*, Math. Proc. Cambridge Philos. Soc. **152** (2012), 131-152.

[31] V.G. Maz'ya, *Classes of domains and imbedding theorems for function spaces*, Soviet Math. Dokl. **1** (1960), 882-885.

[32] V.G. Maz'ya, *Conductor and capacitary inequalities for functions on topological spaces and their applications to Sobolev-type imbeddings*, J. Funct. Anal. **224** (2005), 408-430.

[33] C. M. Petty, *Surface area of a convex body under affine transformations*, Proc. Amer. Math. Soc. **12** (1961), 824-828.

[34] R. Schneider, *Convex bodies: the Brunn-Minkowski theory*, second expanded edition, Encyclopedia of Mathematics and its Applications, Cambridge University Press, Cambridge, 2014.

[35] G. Talenti, *Best constant in Sobolev inequality*, Ann. Mat. Pura Appl. **110** (1976), 353-372.

[36] T. Wang, *The affine Sobolev-Zhang inequality on $BV(\mathbb{R}^n)$*, Adv. Math. **230** (2012), 2457-2473.

[37] T. Wang, *The affine Pólya-Szegö principle: Equality cases and stability*, J. Funct. Anal. **265** (2013), 1728-1748.

[38] T. Wang, *On the Discrete Functional L_p Minkowski Problem*, Int. Math. Res. Not. **2015**, Art. ID rnu256.

[39] J. Xiao, *The sharp Sobolev and isoperimetric inequalities split twice*, Adv. Math. **211** (2007), 417-435.

[40] W. Yu, *Isotropic p-surface area measure*, (Chinese) Acta Math. Sci. Ser. A Chin. Ed. **31** (2011), 644-651.

[41] J. Yuan, G. Leng, and W. Cheung, *Convex bodies with minimal p-mean width*, Houst. J. Math. **36** (2010), 499-511.

[42] G. Zhang, *The affine Sobolev inequality*, J. Differential Geom. **53** (1999), 183-202.

(Q. Huang) College of Mathematics, Physics and Information Engineering, Jiaxing University, Jiaxing 314001, China

E-mail address: hqz376560571@163.com

(A.-J. Li) School of Mathematics and Information Science, Henan Polytechnic University, Jiaozuo City 454000, China

E-mail address: liaijun72@163.com