Gaussian inequalities for Wulff shapes

HUANG Qingzhong, HE Binwu
(Department of Mathematics, Shanghai University, Shanghai, 200444, P.R.China)
(E-mail: hqz376560571@shu.edu.cn, hebinwu@shu.edu.cn)

Abstract. It is shown that there are two Gaussian inequalities for Wulff shapes corresponding to Schuster and Weberndorfer's results for the Lebesgue measure. Barthe's mean width inequality for continuous isotropic measures and its dual inequality are special cases of these new inequalities. Moreover, two new Gaussian inequalities related to the LYZ ellipsoid are obtained.

1. Introduction

A non-negative Borel measure ν on the unit sphere S^{n-1} is said to be isotropic if, when viewed as a mass distribution on S^{n-1}, it has the same moment of inertia about every 1-dimensional subspace of $\mathbb{R}^n$. The notion is closely related to John's theorem and was exploited by Ball [2] in the proof of his celebrated normalized Brascamp-Lieb inequality. A few years later, Barthe [6] discovered the important reverse Brascamp-Lieb inequality, giving at the same time a simpler approach to the proof of the Brascamp-Lieb inequality and its equality conditions. These inequalities have had a profound influence (see, e.g., [1, 2, 3, 4, 5, 6, 7, 10, 11, 13, 23, 25, 26]). Using his transportation of mass technique, Lutwak, Yang, and Zhang [18, 20, 21] developed a new approach based on the work of Ball and Barthe, which helped them to extend the results of Ball-Barthe for discrete measures to general measures, along with characterizations of all extremals.

In 2010, Lutwak, Yang, and Zhang [21] introduced the notion of isotropic embedding, which can be dated back to Ball. Very recently, using this technique, Li and Leng [14] obtained Barthe's mean width inequality for continuous isotropic measures together with its dual inequality, which are closely related to Gaussian inequalities. Since Schuster and

Keywords. Gauss measure, Wulff shape, Isotropic embedding.
2010 Mathematics Subject Classification. 52A40.
Supported in part by the National Natural Science Foundation of China (Grant NO.11071156) and Shanghai Leading Academic Discipline Project (Project Number: J50101).

Weberndorfer [27] have established inequalities for the Lebesgue measure of Wulff shapes, the natural question arises: What corresponding inequalities for the Gaussian measure of Wulff shapes hold? In this paper, we will answer this question. In order to state our main results, several notations are needed.

Throughout this paper, a convex body K in Euclidean space $\mathbb{R}^n$ is a compact convex set that contains the origin in its interior. The polar body of a convex body K is defined by

$$K^* = \{x \in \mathbb{R}^n : x \cdot y \leq 1 \text{ for all } y \in K\},$$

where $x \cdot y$ denotes the standard inner product of x and y in $\mathbb{R}^n$. We denote the Euclidean norm of x by $\|x\| = \sqrt{x \cdot x}$, and the Euclidean unit sphere by S^{n-1}. Let $\operatorname{supp} \nu$ denote the support of a measure ν, and $\operatorname{conv} L$ denotes the convex hull of a set $L \subset \mathbb{R}^n$. Moreover, γ_n is the standard Gaussian measure with density $\frac{1}{(\sqrt{2\pi})^n} e^{-\|x\|^2/2}$.

The notion of Wulff shape has its roots in the theory of crystal growth, and it is a powerful tool to study geometric objects (see, e.g, [8, 16, 22]).

Definition *Suppose ν is a Borel measure on S^{n-1} and f is a positive continuous function on S^{n-1}. The Wulff shape $W_{\nu,f}$ determined by ν and f is defined by*

$$W_{\nu,f} := \{x \in \mathbb{R}^n : x \cdot u \leq f(u) \ \textit{for all} \ u \in \operatorname{supp} \nu\}.$$

Corresponding to the case of the Lebesgue measure that Schuster and Weberndorfer [27] considered, we obtain the following inequalities for the Gaussian measure of Wulff shapes.

Theorem 1 *Suppose f is a positive continuous function on S^{n-1} such that $\|f\|_{L^2(\nu)} = 1$, and ν is an isotropic f-centered measure. Then,*

$$\int_0^\infty e^{-\frac{r^2}{2}} \gamma_n(rW_{\nu,f})dr \leq \sqrt{\frac{\pi}{2^{2n+1}}}.$$

If f is constant on $\operatorname{supp} \nu$*, then equality holds if and only if* $\operatorname{conv} \operatorname{supp} \nu$ *is a regular simplex inscribed in S^{n-1}.*

Theorem 2 *Suppose f is a positive continuous function on S^{n-1} such that $\|f\|_{L^2(\nu)} = 1$, and ν is an isotropic f-centered measure. Then,*

$$\int_0^\infty e^{-\frac{r^2}{2}} \gamma_n(rW^*_{\nu,f})dr \geq \sqrt{\frac{\pi}{2^{2n+1}}}.$$

If f is constant on $\operatorname{supp} \nu$*, then equality holds if and only if* $\operatorname{conv} \operatorname{supp} \nu$ *is a regular simplex inscribed in S^{n-1}.*

In fact, we establish two more general inequalities with an extra variable λ (see Theorem 4.2 and Theorem 4.3). As has been shown in [5, 14], this variable λ is crucial in the proofs of Barthe's mean width inequality and its dual inequality. Following the same steps, the results of Li and Leng [14] on Barthe's mean width inequality for continuous isotropic measures and its dual inequality are direct consequences of Theorem 4.2 and Theorem 4.3 (see Corollary 5.2 and Corollary 5.3). Moreover, we obtain two inequalities related to the LYZ ellipsoid [17] and Gaussian measure via Theorem 1 and Theorem 2. The ideas and techniques of Ball, Barthe, Lutwak-Yang-Zhang, and Schuster-Weberndorfer play a critical role throughout this paper. It would be impossible to overstate our reliance on their work.

This paper is organized as follows: In Section 2 we list for quick reference some basic notations and preliminaries. In Section 3 we introduce the notion of isotropic embedding and recall several needed lemmas. Section 4 contains the proofs of our main results. Some applications of these theorems are presented in Section 5.

2. Background and notation

For a general reference, the reader may wish to consult the books of Gardner [9], Gruber [12], and Schneider [24].

If K is a convex body (i.e., a compact, convex subset of $\mathbb{R}^n$ that contains the origin in its interior), then its support function $h_K = h(K, \cdot) : \mathbb{R}^n \to \mathbb{R}$ is defined for $x \in \mathbb{R}^n$ by

$$h_K(x) = h(K, x) := \max\{x \cdot y : y \in K\}.$$

Let $\| \cdot \|_K : \mathbb{R}^n \to [0, \infty)$ denote the Minkowski functional of a convex body K; i.e., $\|x\|_K = \min\{\lambda \geq 0 : x \in \lambda K\}$.

We now list some results from the quadratic Brunn-Minkowski theory, which is the special case $p = 2$ of the evolving L_p-Brunn-Minkowski theory.

For convex bodies K, L, and $\varepsilon > 0$, the quadratic Firey-combination $K +_2 \sqrt{\varepsilon} L$ is defined as the convex body whose support function is given by

$$h(K +_2 \sqrt{\varepsilon} L, \cdot)^2 = h(K, \cdot)^2 + \varepsilon h(L, \cdot)^2.$$

The quadratic mixed volume $V_2(K, L)$ of convex bodies K, L was defined in [16] by

$$\frac{n}{2} V_2(K, L) = \lim_{\varepsilon \to 0^+} \frac{V(K +_2 \sqrt{\varepsilon} L) - V(K)}{\varepsilon}.$$

In particular, for $K = L$,

$$V_2(K, K) = V(K).$$

It was shown in [16] that corresponding to each convex body K, there is a positive Borel measure $S_2(K,\cdot)$ on S^{n-1}, called the quadratic surface area measure of K, such that

$$V_2(K,L)=\frac{1}{n}\int_{S^{n-1}}h_L^2(u)dS_2(K,u),$$

for each convex body L. It was also shown in [16] that the quadratic surface area measures is h_K-centered; i.e.,

$$\int_{S^{n-1}}uh_K(u)dS_2(K,u)=o. \tag{1}$$

Note that the Wulff shape that is determined by the quadratic surface area measure $S_2(K,\cdot)$ and the support function $h(K,\cdot)$ of K is the convex body K:

$$W_{S_2(K,\cdot),h(K,\cdot)}=K. \tag{2}$$

In 2000, Lutwak, Yang, and Zhang [17] introduced a new ellipsoid $\Gamma_{-2}K$ associated with a convex body K, which is defined by

$$\|u\|^2_{\Gamma_{-2}K}=\frac{1}{V(K)}\int_{S^{n-1}}(u\cdot v)^2dS_2(K,v) \tag{3}$$

for $u\in S^{n-1}$ (see also [15, 19]).

Next, we introduce several other notions related to positive continuous functions f on S^{n-1}. A Borel measure ν on S^{n-1} is called f-centered provided that

$$\int_{S^{n-1}}f(u)ud\nu(u)=o.$$

The measure ν is called isotropic if

$$\int_{S^{n-1}}u\otimes ud\nu(u)=I_n, \tag{4}$$

where $u\otimes u$ is the rank 1 linear operator on $\mathbb{R}^n$ that takes x to $(x\cdot u)u$, and I_n denotes the identity operator on $\mathbb{R}^n$. Moreover, taking traces in (4), we get

$$\nu(S^{n-1})=n. \tag{5}$$

The displacement of $W_{\nu,f}$ [27] is defined by

$$\text{disp}(W_{\nu,f})=\frac{1}{V(W_{\nu,f})}\int_{W_{\nu,f}}xdx\cdot\int_{S^{n-1}}\frac{u}{f(u)}d\nu(u).$$

Similarly, we define the Gaussian displacement of $W_{\nu,f}$ by

$$\text{Gdisp}(W_{\nu,f})=\frac{1}{\gamma_n(W_{\nu,f})}\int_{W_{\nu,f}}xd\gamma_n(x)\cdot\int_{S^{n-1}}\frac{u}{f(u)}d\nu(u), \tag{6}$$

where γ_n is the standard Gaussian measure with density $\frac{1}{(\sqrt{2\pi})^n} e^{-\|x\|^2/2}$.

The ℓ-norm of a convex body K in $\mathbb{R}^n$ is an important quantity in local theory of Banach spaces. It is defined by

$$\ell(K) = \int_{\mathbb{R}^n} \|x\|_K d\gamma_n(x).$$

The mean width of a convex body K is

$$W(K) = \int_{S^{n-1}} (h_K(u) + h_K(-u)) d\sigma(u) = 2 \int_{S^{n-1}} h_K(u) d\sigma(u),$$

where σ is the Haar probability measure on the unit sphere S^{n-1}.

Note that

$$2\ell(K^*) = c_n \sqrt{n} W(K).$$

Here c_n is a numerical constant satisfying

$$c_n \to 1, \quad n \to \infty.$$

The following observation is crucial in the proof of Barthe's mean width inequality and its dual inequality. Let K be a convex body, then

$$\begin{aligned} \ell(K) &= \int_{\mathbb{R}^n} \|x\|_K d\gamma_n(x) = \int_{\mathbb{R}^n} \Big(\int_0^{\|x\|_K} dt \Big) d\gamma_n(x) \\ &= \int_0^\infty \int_{\mathbb{R}^n} \mathbf{1}_{\{\|x\|_K > t\}} d\gamma_n(x) dt = \int_0^\infty (1 - \gamma_n(tK)) dt. \end{aligned} \tag{7}$$

3. Isotropic embeddings

The concept of isotropic embeddings was first proposed by Lutwak, Yang, and Zhang [21]. Here, we adopt the definition of this concept given in [27].

Definition *If ν is a Borel measure on S^{n-1}, then a continuous function $g : S^{n-1} \to \mathbb{R}^{n+1} \backslash \{o\}$ is called an isotropic embedding of ν if the Borel measure $\bar{\nu}$ on S^n, defined by*

$$\int_{S^n} t(w) d\bar{\nu}(w) = \int_{S^{n-1}} t\Big(\frac{g(u)}{\|g(u)\|} \Big) \|g(u)\|^2 d\nu(u) \tag{8}$$

for every continuous $t : S^n \to \mathbb{R}$, is isotropic.

In order to obtain our results, several lemmas from [27] are needed.

Lemma 3.1 *Suppose f is a positive continuous function on S^{n-1} and ν is an isotropic measure on S^{n-1}. Then $g_\pm : S^{n-1} \to \mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$, defined by*

$$g_\pm(u) = (\pm u, f(u)), \tag{9}$$

are isotropic embeddings of ν *if and only if* ν *is* f*-centered and* $\|f\|_{L_2(\nu)} = 1$.

The following two special cases of isotropic embeddings of the form (8) are well known (see [4, 5, 14, 20, 21, 27]).

(i) If ν is a 1-centered isotropic measure, then, by Lemma 3.1, the functions $g_\pm : S^{n-1} \to \mathbb{R}^{n+1}$, defined by

$$g_\pm(u) = (\pm u, \frac{1}{\sqrt{n}}) \tag{10}$$

are isotropic embeddings of ν.

(ii) If $K \subset \mathbb{R}^n$ is a convex body such that $\Gamma_{-2}K = B_2^n$. By (1) and

$$\frac{1}{V(K)} \int_{S^{n-1}} \Big(\frac{h(K,u)}{\sqrt{n}}\Big)^2 dS_2(K,u) = 1,$$

it follows from (3) and Lemma 3.1 that $g_\pm : S^{n-1} \to \mathbb{R}^{n+1}$, defined by

$$g_\pm(u) = \Big(\pm u, \frac{h(K,u)}{\sqrt{n}}\Big) \tag{11}$$

are isotropic embeddings of $S_2(K,\cdot)/V(K)$.

The following critical lemma is due to Ball, Barthe [6]. In [18], Lutwak, Yang, and Zhang give a different proof along with the new equality conditions.

The Ball-Barthe Lemma *If* $\bar{\nu}$ *is an isotropic measure on* S^n *and* t *is a positive continuous function on* supp $\bar{\nu}$*, then*

$$det \int_{S^n} t(w) w \otimes w d\bar{\nu}(w) \geq \exp\Big(\int_{S^n} \log t(w) d\bar{\nu}(w)\Big) \tag{12}$$

with equality if and only if $t(v_1)\cdots t(v_{n+1})$ *is constant for linearly independent* $v_1, \cdots, v_{n+1} \in$ supp $\bar{\nu}$.

The Ball-Barthe Lemma plays a critical role in the proof of statements about isotropic measures including our results. Applying the equality conditions, Schuster and Weberndorfer [27] obtained the following characterization of the support of 1-centered isotropic measures which are embedded by the functions given in (10).

Lemma 3.2 *Let* ν *be a* 1*-centered isotropic measure on* S^{n-1}*, let* $\bar{\nu}_\pm$ *denote the isotropic measures on* S^n *defined by* (8)*, isotropically embedded by* $g_\pm$ *defined in* (10)*, and let* $D \subset \mathbb{R}^{n+1}$ *be an open cone with apex at the origin containing* e_{n+1} *such that* $w \cdot z > 0$ *for every* $w \in$ supp $\bar{\nu}_\pm$ *and* $z \in D$.

For every $z \in D$*, defined* t_z : supp $\bar{\nu}_\pm \to (0, \infty)$ *by*

$$t_z(w) = \phi_w(w \cdot z),$$

where $\phi_w : (0,\infty) \to (0,\infty)$ *is smooth nonconstant such that* $t_z(w)$*, for every fixed* $z \in D$*, depends continuously on* $w \in \operatorname{supp} \bar{\nu}_{\pm}$*. If there is equality in* (12) *for* $\bar{\nu}_+$*, or* $\bar{\nu}_-$ *respectively, and every* t_z*,* $z \in D$*, then* conv supp ν *is a regular simplex inscribed in* S^{n-1}*.*

4. Proof of the main results

A simple use of the Cauchy-Schwarz inequality and the definition of isotropy yields

Lemma 4.1 ([20]) *If* μ *is an isotropic measure on* S^n *and* $J \in L^2(\mu)$*, then*

$$\Big\| \int_{S^n} uJ(u)d\mu(u) \Big\| \le \|J\|_{L^2(\mu)} \tag{13}$$

with equality if and only if $J(u) = u \cdot \int_{S^n} vJ(v)d\mu(v)$ *for almost all* $u \in S^n$ *with respect to the measure* μ*.*

We now prove the more general Gaussian inequalities. Note that Theorem 1 and Theorem 2 are simply deduced from Theorem 4.2 and Theorem 4.3 by setting $\lambda = 0$.

Theorem 4.2 *Suppose* f *is a positive continuous function on* S^{n-1} *such that* $\|f\|_{L^2(\nu)} = 1$*, and* ν *is an isotropic* f*-centered measure. Then, for any real* λ*,*

$$\int_0^\infty e^{-\frac{r^2}{2}+\lambda(n+1)r-\lambda \mathrm{Gdisp}(rW_{\nu,f})}\gamma_n(rW_{\nu,f})dr \le \frac{1}{(2\pi)^{n/2}} \exp\Big(\int_{S^{n-1}} (\log G_{\lambda,u})(1+f^2(u))d\nu(u)\Big),$$

where $G_{\lambda,u} = \int_0^\infty \exp(-\frac{s^2}{2} + \frac{\lambda s\sqrt{1+f^2(u)}}{f(u)})ds$*. If* f *is constant on* supp ν*, then equality holds if and only if* conv supp ν *is a regular simplex inscribed in* S^{n-1}*.*

Proof. Since $\|f\|_{L^2(\nu)} = 1$ and ν is isotropic, by application of Lemma 3.1, let $\bar{\nu}$ denote the measure on S^n defined by (8), isotropically embedded by $g_- = (-u, f(u)),\ u \in S^{n-1}$.

Define the cone $C \subset \mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$ by

$$C = \bigcup_{r>0} rW_{\nu,f} \times \{r\} \subset \mathbb{R}^{n+1}.$$

Obviously, $e_{n+1} \in C$. Note that $w \in \operatorname{supp} \bar{\nu} \subset \mathbb{R}^n \times \mathbb{R}$ if and only if

$$w = \frac{(-u, f(u))}{\sqrt{1+f^2(u)}} \tag{14}$$

for some $u \in \operatorname{supp} \nu$. Thus, by the definition of $W_{\nu,f}$, we have that, for every $w \in \operatorname{supp} \bar{\nu}$ and $z = (x, r) \in C$,

$$w \cdot z = \frac{-u \cdot x + rf(u)}{\sqrt{1+f^2(u)}} \ge 0.$$

Now, for $w \in \text{supp}\ \bar{\nu}$, let

$$G_{\lambda,w} = \int_0^\infty e^{-\frac{s^2}{2} + \frac{\lambda s}{e_{n+1}\cdot w}} ds.$$

Define the smooth and strictly increasing function $T_w : (0, \infty) \to \mathbb{R}$ by

$$\frac{1}{G_{\lambda,w}} \int_0^t e^{-\frac{s^2}{2} + \frac{\lambda s}{e_{n+1}\cdot w}} ds = \int_{-\infty}^{T_w(t)} e^{-\pi s^2} ds.$$

Differentiating both sides with respect to t gives

$$\frac{1}{G_{\lambda,w}} (e^{-\frac{t^2}{2} + \frac{\lambda t}{e_{n+1}\cdot w}}) = e^{-\pi T_w^2(t)} T_w'(t).$$

Taking the log of both sides and putting $t = w \cdot z$ for $w \in \text{supp}\ \bar{\nu}$ and $z \in \text{int}\ C$, we get

$$-\frac{(w\cdot z)^2}{2} + \frac{\lambda(w \cdot z)}{e_{n+1}\cdot w} - \log G_{\lambda,w} = -\pi T_w^2(w\cdot z) + \log T_w'(w \cdot z). \tag{15}$$

Define the transformation $T : \text{int}\ C \to \mathbb{R}^{n+1}$ by

$$T(z) = \int_{S^n} T_w(w\cdot z) w d\bar{\nu}(w).$$

Hence, for every $z \in \text{int}\ C$,

$$dT(z) = \int_{S^n} T_w'(w\cdot z) w \otimes w d\bar{\nu}(w). \tag{16}$$

Since $T_w' > 0$ and $\bar{\nu}$ is not concentrated on a proper subspace of $\mathbb{R}^{n+1}$, we conclude that the matrix $dT(z)$ is positive definite for int C. Therefore, the mean value theorem shows that $T :\ \text{int}\ C \to \mathbb{R}^{n+1}$ is globally 1-1 onto its image.

Moreover, by Lemma 4.1 with $J(w) = T_w(w \cdot z)$, we obtain

$$\|T(z)\|^2 \le \int_{S^n} T_w^2(w\cdot z) d\bar{\nu}(w). \tag{17}$$

By (15), the Ball-Barthe Lemma with $t(w) = T_w'(w \cdot z)$, (16), (17), and the change of variables $y = T(z)$, we have

$$\begin{aligned} &\int_{\text{int}\ C} \exp\Big(\int_{S^n} \Big(-\frac{(w\cdot z)^2}{2} + \frac{\lambda(w\cdot z)}{e_{n+1}\cdot w} - \log G_{\lambda,w}\Big) d\bar{\nu}(w)\Big) dz \\ &= \int_{\text{int}\ C} \exp\Big(\int_{S^n} -\pi T_w^2(w\cdot z) d\bar{\nu}(w)\Big) \exp\Big(\int_{S^n} \log T_w'(w\cdot z) d\bar{\nu}(w)\Big) dz \\ &\le \int_{\text{int}\ C} \exp(-\pi \|T(z)\|^2) \det dT(z) dz \le \int_{\mathbb{R}^{n+1}} \exp(-\pi\|y\|^2) dy = 1. \end{aligned} \tag{18}$$

By definition (8) of $\bar{\nu}$, (5), and since ν is f-centered and $\|f\|_{L^2(\nu)} = 1$, we obtain for $z = (x, r) \in C$

$$\begin{aligned}
\int_{S^n} \frac{(x,r)\cdot w}{e_{n+1}\cdot w} d\bar{\nu}(w) &= \int_{S^{n-1}} \frac{(x,r)\cdot(-u, f(u))/\sqrt{1+f^2(u)}}{e_{n+1}\cdot(-u, f(u))/\sqrt{1+f^2(u)}}(1+f^2(u))d\nu(u) \\
&= \int_{S^{n-1}} \Big(-\frac{x\cdot u}{f(u)} + r - x\cdot u f(u) + r f^2(u)\Big) d\nu(u) \\
&= -\int_{S^{n-1}} \frac{x\cdot u}{f(u)} d\nu(u) + (n+1)r.
\end{aligned} \tag{19}$$

Applying Jensen's inequality and (6) shows that

$$\begin{aligned}
&\frac{1}{(2\pi)^{\frac{n}{2}}\gamma_n(rW_{\nu,f})} \int_{rW_{\nu,f}} \exp\Big(-\lambda \int_{S^{n-1}} \frac{x\cdot u}{f(u)} d\nu(u)\Big) e^{-\frac{\|x\|^2}{2}} dx \\
&\qquad \geq \exp\Big(\frac{-\lambda}{(2\pi)^{\frac{n}{2}}\gamma_n(rW_{\nu,f})} \int_{rW_{\nu,f}} \int_{S^{n-1}} \frac{x\cdot u}{f(u)} d\nu(u) e^{-\frac{\|x\|^2}{2}} dx\Big) \\
&\qquad = \exp(-\lambda \mathrm{Gdisp}(rW_{\nu,f})).
\end{aligned} \tag{20}$$

Now, we consider the left-hand side of (18). By the isotropy of $\bar{\nu}$, (19), and (20), we have

$$\begin{aligned}
&\int_{\mathrm{int}\ C} \exp\Big(-\frac{\|z\|^2}{2} + \int_{S^n} \frac{\lambda(w\cdot z)}{e_{n+1}\cdot w} d\bar{\nu}(w)\Big) dz \\
&= \int_0^\infty \int_{rW_{\nu,f}} \exp\Big(-\frac{r^2 + \|x\|^2}{2} - \lambda \int_{S^{n-1}} \frac{x\cdot u}{f(u)} d\nu(u) + \lambda(n+1)r\Big) dx dr \\
&\geq (2\pi)^{\frac{n}{2}} \int_0^\infty \exp\Big(-\frac{r^2}{2} + \lambda(n+1)r - \lambda \mathrm{Gdisp}(rW_{\nu,f})\Big) \gamma_n(rW_{\nu,f}) dr.
\end{aligned}$$

Hence

$$\int_0^\infty \exp\Big(-\frac{r^2}{2} + \lambda(n+1)r - \lambda \mathrm{Gdisp}(rW_{\nu,f})\Big)\gamma_n(rW_{\nu,f}) dr \leq \frac{1}{(2\pi)^{\frac{n}{2}}} \exp\Big(\int_{S^n} \log G_{\lambda,w} d\bar{\nu}(w)\Big).$$

Now if f is constant on the support of ν, we have $f \equiv \frac{1}{\sqrt{n}}$ because $\|f\|_{L^2(\nu)} = 1$. Thus, applying Lemma 3.2, where $D = \mathrm{int}\ C$ and $\phi_w = T'_w$, concludes the proof. $\square$

Theorem 4.3 *Suppose f is a positive continuous function on S^{n-1} such that $\|f\|_{L^2(\nu)} = 1$, and ν is an isotropic f-centered measure. Then, for any real λ,*

$$\int_0^\infty \exp\Big(-\frac{r^2}{2} + \lambda r\Big)\gamma_n(rW^*_{\nu,f}) dr \geq \frac{1}{(2\pi)^{n/2}} \exp\Big(\int_{S^{n-1}} (\log \widehat{G}_{\lambda,u})(1+f^2(u)) d\nu(u)\Big),$$

where $\widehat{G}_{\lambda,u} = \int_0^\infty \exp(-\frac{s^2}{2} + \frac{\lambda f(u) s}{\sqrt{1+f^2(u)}}) ds$. If f is constant on supp ν*, then equality holds if and only if* conv supp ν *is a regular simplex inscribed in S^{n-1}.*

Proof. Since $\|f\|_{L^2(\nu)} = 1$ and ν is isotropic, let $\bar{\nu}$ denote the measure defined on S^n by (8), isotropically embedded by $g_+(u) = (u, f(u)),\ u \in S^{n-1}$ (see Lemma 3.1).

For $w \in \operatorname{supp} \bar{\nu}$, let

$$\widehat{G}_{\lambda,w} = \int_0^\infty e^{-\frac{s^2}{2}+(e_{n+1}\cdot w)\lambda s} ds.$$

Define the smooth and strictly increasing function $\widehat{T}_w : \mathbb{R} \to (0, \infty)$ by

$$\frac{1}{\widehat{G}_{\lambda,w}} \int_0^{\widehat{T}_w(t)} e^{-\frac{s^2}{2}+(e_{n+1}\cdot w)\lambda s} ds = \int_{-\infty}^t e^{-\pi s^2} ds.$$

Differentiating both sides with respect to t gives

$$\frac{1}{\widehat{G}_{\lambda,w}} (e^{-\frac{\widehat{T}_w^2(t)}{2}+(e_{n+1}\cdot w)\lambda \widehat{T}_w(t)}) \widehat{T}_w'(t) = e^{-\pi t^2}.$$

Taking the log of both sides and putting $t = w \cdot z$ for $w \in \operatorname{supp} \bar{\nu}$ and $z \in \mathbb{R}^{n+1}$, we get

$$-\frac{\widehat{T}_w^2(w\cdot z)}{2} + (e_{n+1}\cdot w)\lambda \widehat{T}_w(w\cdot z) + \log \widehat{T}_w'(w\cdot z) - \log \widehat{G}_{\lambda,w} = -\pi (w\cdot z)^2. \tag{21}$$

Define the transformation $\widehat{T} : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ by

$$\widehat{T}(z) := \int_{S^n} \widehat{T}_w(w \cdot z) w d\bar{\nu}(w). \tag{22}$$

Define the cone $\widehat{C} \subset \mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$ by

$$\widehat{C} := \bigcup_{r>0} rW^*_{\nu,f} \times \{r\}.$$

Next, we will show that $\widehat{T}(z) \subset \widehat{C}$ for all $z \in \mathbb{R}^{n+1}$. It is sufficient to prove that if $\widehat{T}(z) = (x, r) \in \mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$ and $y \in W_{\nu,f}$, then $x \cdot y \le r$. By definition (8) of $\bar{\nu}$, the definition of $\widehat{T}$, and the fact that $u \cdot y \le f(u)$ for every $u \in \operatorname{supp} \nu$, we obtain

$$\begin{aligned}
x \cdot y &= \int_{S^{n-1}} \widehat{T}_{\frac{(u,f(u))}{\sqrt{1+f^2(u)}}} \Big(\frac{(u, f(u))}{\sqrt{1+f^2(u)}} \cdot z \Big) \Big(\frac{u}{\sqrt{1+f^2(u)}} \cdot y \Big) (1 + f^2(u)) d\nu(u) \\
&= \int_{S^{n-1}} \widehat{T}_{\frac{(u,f(u))}{\sqrt{1+f^2(u)}}} \Big(\frac{(u, f(u))}{\sqrt{1+f^2(u)}} \cdot z \Big) (u \cdot y) \sqrt{1+f^2(u)} d\nu(u) \\
&\le \int_{S^{n-1}} \widehat{T}_{\frac{(u,f(u))}{\sqrt{1+f^2(u)}}} \Big(\frac{(u, f(u))}{\sqrt{1+f^2(u)}} \cdot z \Big) f(u) \sqrt{1+f^2(u)} d\nu(u) \\
&= \int_{S^n} \widehat{T}_w \Big(w \cdot z \Big) (e_{n+1} \cdot w) d\bar{\nu}(w) = r.
\end{aligned}$$

From (22) it follows that

$$d\widehat{T}(z) = \int_{S^n} \widehat{T}'_w(w\cdot z) w \otimes w d\bar{\nu}(w). \tag{23}$$

Since $\widehat{T}'_w > 0$ and $\bar{\nu}$ is not concentrated on a proper subspace of $\mathbb{R}^{n+1}$, we conclude that the matrix $d\widehat{T}(z)$ is positive definite for $\mathbb{R}^{n+1}$. Therefore, the mean value theorem shows that $\widehat{T} : \mathbb{R}^{n+1} \to \widehat{C}$ is globally 1-1 onto its image.

By the isotropy of $\bar{\nu}$, it follows that

$$\|z\|^2 = \int_{S^n} (w\cdot z)^2 d\bar{\nu}(w). \tag{24}$$

Moreover, by Lemma 4.1 with $J(w) = \widehat{T}_w(w\cdot z)$, we obtain

$$\|\widehat{T}(z)\|^2 \le \int_{S^n} \widehat{T}^2_w(w\cdot z) d\bar{\nu}(w).$$

Together with (21), (22), and (24), this implies

$$\begin{aligned}\int_{S^n} \log \widehat{T}'_w(w\cdot z) d\bar{\nu}(w) \;&= \int_{S^n} \Big(-\pi (w\cdot z)^2 + \frac{\widehat{T}^2_w(w\cdot z)}{2} - \\ &\quad (e_{n+1}\cdot w)\lambda \widehat{T}_w(w\cdot z) + \log \widehat{G}_{\lambda,w}\Big) d\bar{\nu}(w) \\ &\ge -\pi\|z\|^2 + \frac{\|\widehat{T}(z)\|^2}{2} - \lambda\widehat{T}(z)\cdot e_{n+1} + \int_{S^n} \log \widehat{G}_{\lambda,w} d\bar{\nu}(w).\end{aligned}$$

Consequently,

$$\begin{aligned} 1 \;&= \int_{\mathbb{R}^{n+1}} e^{-\pi\|z\|^2} dz \\ &\le \int_{\mathbb{R}^{n+1}} \exp\Big(\int_{S^n} \log \widehat{T}'_w(w\cdot z) d\bar{\nu}(w) - \frac{\|\widehat{T}(z)\|^2}{2} + \lambda\widehat{T}(z)\cdot e_{n+1}\Big) \\ &\quad \exp\Big(-\int_{S^n} \log \widehat{G}_{\lambda,w} d\bar{\nu}(w)\Big) dz \\ &= \exp\Big(-\int_{S^n} \log \widehat{G}_{\lambda,w} d\bar{\nu}(w)\Big) \int_{\mathbb{R}^{n+1}} \exp\Big(-\frac{\|\widehat{T}(z)\|^2}{2} + \lambda\widehat{T}(z)\cdot e_{n+1}\Big) \\ &\quad \exp\Big(\int_{S^n} \log \widehat{T}'_w(w\cdot z) d\bar{\nu}(w)\Big) dz. \end{aligned} \tag{25}$$

Now, we consider the right-hand side of (25). The Ball-Barthe Lemma with $t(w) = \widehat{T}'_w(w\cdot z)$ and the change of variables $y = \widehat{T}(z)$ give

$$\begin{aligned}
&\int_{\mathbb{R}^{n+1}} \exp\Big(-\frac{\|\widehat{T}(z)\|^2}{2} + \lambda \widehat{T}(z)\cdot e_{n+1}\Big) \exp\Big(\int_{S^n} \log \widehat{T}'_w(w\cdot z) d\bar{\nu}(w)\Big) dz \\
&\quad \le \int_{\mathbb{R}^{n+1}} \exp\Big(-\frac{\|\widehat{T}(z)\|^2}{2} + \lambda \widehat{T}(z)\cdot e_{n+1}\Big) \det d\widehat{T}(z) dz \\
&\quad \le \int_{\widehat{C}} \exp\Big(-\frac{\|y\|^2}{2} + \lambda y\cdot e_{n+1}\Big) dy \\
&\quad = \int_0^\infty \int_{rW^*_{\nu,f}} \exp\Big(-\frac{r^2+\|x\|^2}{2} + \lambda r\Big) dx dr \\
&\quad =(2\pi)^{\frac{n}{2}} \int_0^\infty \exp\Big(-\frac{r^2}{2} + \lambda r\Big) \gamma_n(rW^*_{\nu,f}) dr.
\end{aligned}$$

Hence

$$\int_0^\infty \exp\Big(-\frac{r^2}{2} + \lambda r\Big)\gamma_n(rW^*_{\nu,f})dr \ge \frac{1}{(2\pi)^{\frac{n}{2}}} \exp\Big(\int_{S^n} \log \widehat{G}_{\lambda,w} d\bar{\nu}(w)\Big).$$

If f is constant on the support of ν, then we have that $f \equiv \frac{1}{\sqrt{n}}$ because $\|f\|_{L^2(\nu)} = 1$. In order to apply Lemma 3.2, define the open cone $D \subset \mathbb{R}^{n+1}$ by

$$D = \{z \in \mathbb{R}^{n+1} : w\cdot z > 0 \text{ for every } w \in \text{supp } \bar{\nu}\},$$

and let $\phi_w = \widehat{T}'_w$. This concludes the proof. ☐

5. Applications

Let $\triangle_n$ denote a regular n-simplex inscribed in S^{n-1}. Then, its polar body $\triangle_n^*$ is a regular n-simplex that contains S^{n-1}.

Lemma 5.1 ([5, 14]) *Suppose ν is an isotropic measure. If*

$$\int_0^\infty e^{-\frac{r^2}{2}+\lambda(n+1)r}\gamma_n(\frac{r}{\sqrt{n}}(\text{conv supp }\nu)^*)dr \le \int_0^\infty e^{-\frac{r^2}{2}+\lambda(n+1)r}\gamma_n(\frac{r}{\sqrt{n}}\triangle_n^*)dr$$

holds for every real λ, then

$$\ell((\text{conv supp }\nu)^*) \ge \ell(\triangle_n^*).$$

If

$$\int_0^\infty e^{-\frac{r^2}{2}+\lambda r}\gamma_n(r\sqrt{n}\text{ conv supp }\nu)dr \ge \int_0^\infty e^{-\frac{r^2}{2}+\lambda r}\gamma_n(r\sqrt{n}\triangle_n)dr$$

holds for every real λ, then

$$\ell(\text{conv supp }\nu) \le \ell(\triangle_n).$$

Proof. It is sufficient to prove the first case. The proof of the second case is analog.

Since λ is arbitrary, making the change of variables $t = \frac{r}{\sqrt{n}}$, it follows that for all real a, b

$$\int_0^\infty e^{-\frac{n^2}{2}(t^2+at+b)} \gamma_n(t(\text{conv supp } \nu)^*)dt \le \int_0^\infty e^{-\frac{n^2}{2}(t^2+at+b)} \gamma_n(t\triangle_n^*)dt.$$

In particular, for all $\alpha \in \mathbb{R}$,

$$\int_0^\infty e^{-\frac{n^2}{2}(t-\alpha)^2} \gamma_n(t(\text{conv supp } \nu)^*)dt \le \int_0^\infty e^{-\frac{n^2}{2}(t-\alpha)^2} \gamma_n(t\triangle_n^*)dt.$$

Thus, for all $\alpha \in \mathbb{R}$,

$$\int_0^\infty e^{-\frac{n^2}{2}(t-\alpha)^2} (1 - \gamma_n(t(\text{conv supp } \nu)^*))dt \ge \int_0^\infty e^{-\frac{n^2}{2}(t-\alpha)^2} (1 - \gamma_n(t\triangle_n^*))dt.$$

Integration with respect to α yields

$$\int_0^\infty \Big(\int_{\mathbb{R}} e^{-\frac{n^2}{2}(t-\alpha)^2} d\alpha\Big)(1-\gamma_n(t(\text{conv supp } \nu)^*))dt \ge \int_0^\infty \Big(\int_{\mathbb{R}} e^{-\frac{n^2}{2}(t-\alpha)^2} d\alpha\Big)(1-\gamma_n(t\triangle_n^*))dt.$$

Notice that the inmost integral does not depend on t, hence

$$\int_0^\infty (1 - \gamma_n(t(\text{conv supp } \nu)^*))dt \ge \int_0^\infty (1 - \gamma_n(t\triangle_n^*))dt.$$

Together with (7), we obtain

$$\ell((\text{conv supp } \nu)^*) \ge \ell(\triangle_n^*).$$

□

For discrete measures, the following two results (Corollary 5.2 and Corollary 5.3) were proved by Schmuckenschläger [26] and Barthe [5], respectively. Recently, Li and Leng [14] extended these inequalities to continuous isotropic measures using the isotropic embedding (10). Here, we also apply this isotropic embedding.

Corollary 5.2 *If ν is a 1-centered isotropic measure on S^{n-1}, then*

$$\ell((\text{conv supp } \nu)^*) \ge \ell(\triangle_n^*)$$

with equality if and only if conv supp $\nu = \triangle_n$.

Proof. Let $f \equiv \frac{1}{\sqrt{n}}$. Since ν is a 1-centered isotropic measure on S^{n-1}, we have

$$W_{\nu,\frac{1}{\sqrt{n}}} = \frac{1}{\sqrt{n}}(\text{conv supp } \nu)^* \quad \text{and} \quad \text{Gdisp}(W_{\nu,\frac{1}{\sqrt{n}}}) = 0.$$

Applying Theorem 4.2, we obtain

$$\int_0^\infty e^{-\frac{r^2}{2}+\lambda(n+1)r}\gamma_n\Big(\frac{r}{\sqrt{n}}(\text{conv supp }\nu)^*\Big)dr \le \frac{1}{(2\pi)^{\frac{n}{2}}}\Big(\int_0^\infty e^{-\frac{s^2}{2}+\lambda s\sqrt{n+1}}ds\Big)^{n+1}.$$

Now an application of Lemma 5.1 concludes the proof. □

The same arguments, together with Theorem 4.3, yield the following result.

Corollary 5.3 *If ν is a 1-centered isotropic measure on S^{n-1}, then*

$$\ell(\text{conv supp }\nu) \le \ell(\triangle_n)$$

with equality if and only if conv supp $\nu = \triangle_n$.

By an application of the isotropic embedding (11), we obtain the following two corollaries.

Corollary 5.4 *If $K \subset \mathbb{R}^n$ is a convex body such that $\Gamma_{-2}K = B_2^n$, then*

$$\int_0^\infty e^{-\frac{r^2}{2}}\gamma_n\Big(\frac{r}{\sqrt{n}}K\Big)dr \le \sqrt{\frac{\pi}{2^{2n+1}}}.$$

Moreover, if $K = \triangle_n^*$, then equality holds.

Proof. Let $f \equiv \frac{h_K}{\sqrt{n}}$. By the fact that the measure $\nu := \frac{1}{V(K)}S_2(K,\cdot)$ is f-centered and (2), we have

$$W_{\nu,h_K/\sqrt{n}} = \frac{1}{\sqrt{n}}K \quad\text{and}\quad \Big\|\frac{h(K,\cdot)}{\sqrt{n}}\Big\|_{L^2(\nu)} = 1.$$

By Theorem 1, the inequality follows.

A straightforward computation yields $\Gamma_{-2}\triangle_n^* = B_2^n$, and the equality case in Theorem 1 is satisfied. □

The same arguments, together with Theorem 2, yield the following result.

Corollary 5.5 *If $K \subset \mathbb{R}^n$ is a convex body such that $\Gamma_{-2}K = B_2^n$, then*

$$\int_0^\infty e^{-\frac{r^2}{2}}\gamma_n\Big(r\sqrt{n}K^*\Big)dr \ge \sqrt{\frac{\pi}{2^{2n+1}}}.$$

Moreover, if $K = \triangle_n^*$, then equality holds.

Acknowledgement

The authors are grateful for the referee's careful reading and his/her valuable suggestions.

References

[1] D. Alonso-Gutierrez, Volume estimates for L_p zonotopes and best best constant in Brascamp-Lieb inequalities, *Mathematika* **56** (2010), 45-60.

[2] K. Ball, Volumes of sections of cubes and related problems, Geometric aspects of functional analysis, Lecture Notes in Math., **1376**, Springer, Berlin, (1989), 251-260.

[3] K. Ball, Shadows of convex bodies, *Trans. Amer. Math. Soc.* **327** (1991), 891-901.

[4] K. Ball, Volume ratios and a reverse isoperimetric inequality, *J. London Math. Soc.* **44** (1991), 351-359.

[5] F. Barthe, An extremal property of the mean width of the simplex, *Math. Ann.* **310** (1998), 685-693.

[6] F. Barthe, On a reverse form of the Brascamp-Lieb inequality, *Invent. Math.* **134** (1998), 335-361.

[7] F. Barthe, A continuous version of the Brascamp-Lieb inequalities, Geometric aspects of functional analysis, Lecture Notes in Math., **1850**, Springer, Berlin, (2004), 53-63.

[8] R.J. Gardner, The Brunn-Minkowski inequality, *Bull. Amer. Math. Soc.* **39** (2002), 355-405.

[9] R.J. Gardner, Geometric Tomography, 2nd edition, Encyclopedia of Mathematics and its Applications, vol. 58, Cambridge University Press, Cambridge, 2006.

[10] A.A. Giannopoulos and M. Papadimitrakis, Isotropic surface area measures, *Mathematika* **46** (1999), 1-13.

[11] A.A. Giannopoulos, V. D. Milman, and M. Rudelson, Convex bodies with minimal mean width, Geometric aspects of functional analysis, Lecture Notes in Math., **1745**, Springer, Berlin, (2000), 81-93.

[12] P.M. Gruber, Convex and Discrete Geometry, Grundlehren Math. Wiss., vol. 336, Springer, Berlin, 2007.

[13] P.M. Gruber, Application of an idea of Voronoi to John type problems, *Adv. Math.* **218** (2008), 309-351.

[14] A.J. Li and G. Leng, Mean width inequalities for isotropic measures, *Math. Z.* **270** (2012), 1089-1110.

[15] M. Ludwig, Ellipsoids and matrix valued valuations, *Duke Math. J.* **119** (2003), 159-188.

[16] E. Lutwak, The Brunn-Minkowski-Firey theory. I. Mixed volumes and the Minkowski problem, *J. Differential Geom.* **38** (1993), 131-150.

[17] E. Lutwak, D. Yang, and G. Zhang, A new ellipsoid associated with convex bodies, *Duke Math. J.* **104** (2000), 375-390.

[18] E. Lutwak, D. Yang, and G. Zhang, Volume inequalities for subspaces of L_p, *J. Differential Geom.* **68** (2004), 159-184.

[19] E. Lutwak, D. Yang, and G. Zhang, L_p John ellipsoids, *Proc. London Math. Soc.* **90** (2005), 497-520.

[20] E. Lutwak, D. Yang, and G. Zhang, Volume inequalities for isotropic measures, *Amer. J. Math.* **129** (2007), 1711-1723.

[21] E. Lutwak, D. Yang, and G. Zhang, A volume inequality for polar bodies, *J. Differential Geom.* **84** (2010), 163-178.

[22] C. Haberl, E. Lutwak, D. Yang, and G. Zhang, The even Orlicz Minkowski problem, *Adv. Math.* **224** (2010), 2485-2510.

[23] G. Maresch and F. Schuster, The sine transform of isotropic measures, *Int. Math. Res. Notices.* **4** (2012), 717-739.

[24] R. Schneider, Convex Bodies: The Brunn-Minkowski Theory, Cambridge Univ. Press, Cambridge 1993.

[25] G. Schechtman and M. Schmuckenschläger, A concentration inequality for harmonic measures on the sphere, Geometric aspects of functional analysis (Israel, 1992-1994), Oper. Theory Adv. Appl., vol. 77, Birkhäuser, Basel, 1995, 255-273.

[26] M. Schmuckenschläer, An extremal property of the regular simplex, Convex geometric analysis (Berkeley, CA, 1996), Math. Sci. Res. Inst. Publ., vol. 34, Cambridge Univ. Press, Cambridge, 1999, 199-202.

[27] F.E. Schuster and M. Weberndorfer, Volume inequalities for asymmetric Wulff shapes, *J. Differential Geom.* **92** (2012), 263-284.