The Busemann theorem for complex p-convex bodies

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Abstract. The Busemann theorem states that the intersection body of an origin-symmetric convex body is also convex. In this paper, we prove a version of Busemann theorem for complex p-convex bodies. Namely, the complex intersection body of an origin-symmetric complex p-convex body is γ-convex for certain γ. The result is the complex analogue of Kim, Yaskin, Zvavitch's work on (real) p-convex bodies. Furthermore, we show that the generalized radial qth mean body of a p-convex body is γ-convex for certain γ.

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1. Introduction

As usual, S^{n-1} denotes the unit sphere and o the origin in Euclidean n-space $\mathbb{R}^n$. If $\xi \in S^{n-1}$, we denote by $\xi^\perp$ the central hyperplane perpendicular to the vector ξ. We use the notation $|K|$ for the volume of a compact set K.

A body is a compact set with nonempty interior. A set K is star-shaped with respect to the origin if every line through the origin which meets K does so in a closed line segment. Let $K \subset \mathbb{R}^n$ be a star shaped body (with respect to the origin), the radial function $\rho(K, \cdot) : \mathbb{R}^n \backslash \{o\} \to \mathbb{R}$ is defined by

$$\rho(K, x) = \rho_K(x) = \max\{\lambda \geq 0 : \lambda x \in K\}.$$

The Minkowski functional $\|\cdot\|_K : \mathbb{R}^n \to \mathbb{R}$ is defined by $\|x\|_K = \min\{\lambda \geq 0 : x \in \lambda K\}$. Obviously, $\rho_K(x) = \|x\|_K^{-1}$, for $x \in \mathbb{R}^n \backslash \{o\}$.

The concept of an intersection body was introduced by Lutwak [15] in 1988. Let K and L be origin symmetric star bodies in $\mathbb{R}^n$. We say that K is the intersection body of L and write $K = I(L)$ if the radius of K in every direction

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is equal to the volume of the central hyperplane section of L perpendicular to this direction, i.e. for every $\xi \in S^{n-1}$,

$$\|\xi\|_{I(L)}^{-1} = |L \cap \xi^{\perp}|.$$

For additional references regarding intersection bodies and their applications, the reader may wish to consult the books by Gardner [12] and Koldobsky [5].

One celebrated result for intersection bodies is the classical Busemann theorem (see [3, 12, 16]):

Theorem 1.1. *Let K be an origin-symmetric convex body in $\mathbb{R}^n$. Then the intersection body $I(K)$ of K is an origin symmetric convex body.*

Let $p \in (0, 1]$. A body K is p-convex if, for all $x, y \in \mathbb{R}^n$,

$$\|x + y\|_K^p \leq \|x\|_K^p + \|y\|_K^p, \tag{1.1}$$

or, equivalently $t^{1/p}x + (1-t)^{1/p}y \in K$ for any points $x, y \in K$ and $t \in (0, 1)$. Note that p-convex sets with $p = 1$ are just convex. Moreover, a p_1-convex body is p_2-convex for all $0 < p_2 \leq p_1$.

Recently, a version of the Busemann theorem for p-convex bodies was obtained by Kim, Yaskin, Zvavitch [4], which can be formulated as follows:

Theorem 1.2. *Let K be an origin-symmetric p-convex body in $\mathbb{R}^n$ for $p \in (0, 1]$. Then the intersection body $I(K)$ of K is an origin symmetric γ-convex body with $\gamma = ((1/p - 1)(n - 1) + 1)^{-1}$.*

The study of complex convex bodies just appears recent years, and results appear only occasionally (see, e.g., [1, 6, 7, 8, 9, 10, 14, 17, 18]). To formulate the complex version, we need several definitions.

Origin-symmetric convex bodies in $\mathbb{C}^n$ are the unit balls of norms on $\mathbb{C}^n$. We denote by $\|\cdot\|_K$ the norm corresponding to the body K:

$$K = \{z \in \mathbb{C}^n : \|\cdot\|_K \leq 1\}.$$

We identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ using the standard mapping

$$\xi = (\xi_1, \cdots \xi_n) = (\xi_{11} + i\xi_{12}, \cdots, \xi_{n1} + i\xi_{n2}) \mapsto (\xi_{11}, \xi_{12}, \cdots, \xi_{n1}, \xi_{n2}).$$

Since norms on $\mathbb{C}^n$ satisfy the equality

$$\|\lambda z\| = |\lambda|\|z\|, \quad \forall z \in \mathbb{C}^n, \forall \lambda \in \mathbb{C},$$

origin symmetric complex convex bodies correspond to those origin symmetric convex bodies K in $\mathbb{R}^{2n}$ that are invariant with respect to any coordinate-wise two-dimensional rotation, namely for each $\theta \in [0, 2\pi]$ and each $(\xi_{11}, \xi_{12}, \cdots \xi_{n1}, \xi_{n2}) \in \mathbb{R}^{2n}$

$$\|\xi\|_K = \|R_\theta(\xi_{11}, \xi_{12}), \cdots, R_\theta(\xi_{n1}, \xi_{n2})\|_K, \tag{1.2}$$

where R_θ stands for the counterclockwise rotation of $\mathbb{R}^2$ by the angle θ with respect to the origin. We shall say that K is a complex convex body in $\mathbb{R}^{2n}$ if K is a convex body and satisfies equations (1.2).

For $\xi \in \mathbb{C}^n,\ |\xi| = 1$, denote by

$$H_\xi = \{z \in \mathbb{C}^n : (z, \xi) = \sum_{k=1}^{n} z_k \overline{\xi_k} = 0\}$$

the complex hyperplane through the origin, perpendicular to ξ. Under the standard mapping from $\mathbb{C}^n$ to $\mathbb{R}^{2n}$ the hyperplane H_ξ turns into a $(2n-2)$-dimensional subspace of $\mathbb{R}^{2n}$ orthogonal to the vectors

$$\xi = (\xi_{11}, \xi_{12}, \cdots, \xi_{n1}, \xi_{n2}) \quad \textit{and} \quad \xi^\perp = (-\xi_{12}, \xi_{11}, \cdots, -\xi_{n2}, \xi_{n1}).$$

The orthogonal two-dimensional subspace $H_\xi^\perp$ has orthonormal basis $\xi, \xi^\perp$. A star (convex) body K in $\mathbb{R}^{2n}$ is a complex star (convex) body if and only if, for every $\xi \in S^{2n-1}$, the section $K \cap H_\xi^\perp$ is a two-dimensional Euclidean circle with radius $\rho_K(\xi) = \|\xi\|_K^{-1}$.

Corresponding to the (real) intersection body, the complex counterpart was introduced recently by Kolodobsky, Paouris and Zymonopoulou [9].

Definition 1.3. Let K, L be origin-symmetric complex star bodies in $\mathbb{R}^{2n}$. We say that K is the complex intersection body of L and write $K = I_c(L)$ if for every $\xi \in \mathbb{R}^{2n}$

$$|K \cap H_\xi^\perp| = |L \cap H_\xi|. \tag{1.3}$$

Since $K \cap H_\xi^\perp$ is the two-dimensional Euclidean circle with radius $\|\xi\|_K^{-1}$, (1.3) can be written as

$$\pi \|\xi\|_{I_c(L)}^{-2} = |L \cap H_\xi|. \tag{1.4}$$

In [9], Kolodobsky, Paouris and Zymonopoulou also proved the complex version of the Busemann theorem.

Theorem 1.4. *Let K be an origin-symmetric complex convex body in $\mathbb{C}^n$ and $I_c(K)$ the complex intersection body of K. Then $I_c(K)$ is also an origin symmetric convex body in $\mathbb{C}^n$.*

The main purpose of this paper is to give a version of the Busemann theorem for complex p-convex bodies, corresponding to the result for (real) p-convex bodies.

Theorem 1.5. *Let K be an origin-symmetric complex p-convex body in $\mathbb{C}^n$ and $I_c(K)$ the complex intersection body of K. Then $I_c(K)$ is also an origin symmetric $\gamma-$convex body in $\mathbb{C}^n$ with $\gamma = ((1/p-1)(n-1)+1)^{-1}$.*

Clearly, Theorem 1.5 reduces to Theorem 1.4 if $p = 1$. From Theorem 1.2 and Theorem 1.5, we see that given a p-convex body, the corresponding intersection body in real and complex spaces are both γ-convex with $\gamma = ((1/p-1)(n-1)+1)^{-1}$. In [4], Kim, Yaskin, Zvavitch obtained that this γ is asymptotically optimal in real space. However, the problems of the exact value of γ in real and complex spaces remain open.

The rest of this paper is organized as follows: In Section 2, a basic inequality is established, which is crucial for our arguments. Section 3 contains the proof of the Busemann theorem for (real) p-convex bodies due to Kim, Yaskin, Zvavitch [4]. In Section 4, analogously to the real case, we establish the Busemann theorem for complex p-convex bodies. Moreover, in Section 5 we introduce the concept of a generalized radial qth mean body, and prove that the generalized radial qth mean body of a p-convex body is γ-convex for certain γ, extending a result of Gardner and Zhang [13].

2. A Basic inequality

The following result given by Kolodobsky, Paouris and Zymonopoulou [9] is a variant origining from Busemann [3] and Ball [2].

Theorem 2.1. *Let $r_1, r_2 > 0$ and let $a > 0$. Define t_1, t_2, r_3 by:*

$$t_1 := \frac{r_2}{r_1 + r_2}, t_2 := \frac{r_1}{r_1 + r_2}, \tag{2.1}$$

$$\frac{a}{r_3} = \frac{1}{r_1} + \frac{1}{r_2}. \tag{2.2}$$

Suppose $f_1, f_2, f_3 : [0, \infty) \to [0, \infty)$ are three integrable functions, such that

$$f_3(r_3) \geq f_1^{t_1}(r_1) f_2^{t_2}(r_2), \quad \forall\ r_1, r_2 \geq 0. \tag{2.3}$$

Let $q > 0$ and denote

$$F_1 := \Big(\int_0^\infty r^{q-1} f_1(r) dr \Big)^{\frac{1}{q}},$$
$$F_2 := \Big(\int_0^\infty r^{q-1} f_2(r) dr \Big)^{\frac{1}{q}},$$
$$F_3 := \Big(\int_0^\infty r^{q-1} f_3(r) dr \Big)^{\frac{1}{q}}.$$

Then,

$$\frac{a}{F_3} \leq \frac{1}{F_1} + \frac{1}{F_2}. \tag{2.4}$$

Inspired by the results of Kim, Yaskin, Zvavitch [4], we extend the above inequality to the inequality for p-convex bodies, which can be stated as follows:

Theorem 2.2. *Let $r_1, r_2 > 0$ and let $a, p > 0$. Define t_1, t_2, r_3 by:*

$$t_1 := \frac{r_1^{-p}}{r_1^{-p} + r_2^{-p}}, t_2 := \frac{r_2^{-p}}{r_1^{-p} + r_2^{-p}}, \tag{2.5}$$

$$a^p r_3^{-p} = r_1^{-p} + r_2^{-p}. \tag{2.6}$$

Suppose $f_1, f_2, f_3 : [0, \infty) \to [0, \infty)$ are three integrable functions, and there exists $\alpha \geq 0$ such that

$$f_3(r_3) \geq (t_1^{t_1} t_2^{t_2})^{\alpha} f_1^{t_1}(r_1) f_2^{t_2}(r_2), \quad \forall\ r_1, r_2 \geq 0. \tag{2.7}$$

Let $q > 0$ and denote

$$F_1 := \Big(\int_0^\infty r^{q-1} f_1(r)dr\Big)^{\frac{1}{q}},$$
$$F_2 := \Big(\int_0^\infty r^{q-1} f_2(r)dr\Big)^{\frac{1}{q}},$$
$$F_3 := \Big(\int_0^\infty r^{q-1} f_3(r)dr\Big)^{\frac{1}{q}}.$$

Then,

$$\frac{a^\gamma}{F_3^\gamma} \le \frac{1}{F_1^\gamma} + \frac{1}{F_2^\gamma}, \tag{2.8}$$

where $\gamma = (\alpha/q + 1/p)^{-1}$.

Note that Theorem 2.2 reduces to Theorem 2.1 by setting $p = 1, \alpha = 0$.

Proof. For every $s \in [0,1]$ we define $r_i = r_i(s)$ for $i = 1, 2$ by

$$s = \frac{1}{F_1^q}\int_0^{r_1} r^{q-1} f_1(r)dr = \frac{1}{F_2^q}\int_0^{r_2} r^{q-1} f_2(r)dr,$$

then for $i = 1, 2$

$$\frac{dr_i}{ds} = \frac{F_i^q}{r_i^{q-1} f_i(r_i)}.$$

Denote $r_3 = r_3(s)$ by (2.6), then

$$\begin{aligned}\frac{dr_3}{ds} &= a^{-p} r_3^{p+1}\Big(r_1^{-p-1}\frac{dr_1}{ds} + r_2^{-p-1}\frac{dr_2}{ds}\Big) = r_3\Big[t_1(\frac{1}{r_1}\frac{dr_1}{ds}) + t_2(\frac{1}{r_2}\frac{dr_2}{ds})\Big]\\ &\ge r_3\Big(\frac{1}{r_1}\frac{dr_1}{ds}\Big)^{t_1}\Big(\frac{1}{r_2}\frac{dr_2}{ds}\Big)^{t_2} = a\Big(t_1^{\frac{1}{p}}\frac{dr_1}{ds}\Big)^{t_1}\Big(t_2^{\frac{1}{p}}\frac{dr_2}{ds}\Big)^{t_2}\\ &= a(t_1^{t_1}t_2^{t_2})^{\frac{1}{p}}\Big(\frac{F_1^q}{r_1^{q-1}f_1(r_1)}\Big)^{t_1}\Big(\frac{F_2^q}{r_2^{q-1}f_2(r_2)}\Big)^{t_2}.\end{aligned}$$

Together with (2.6) and (2.7), it follows that

$$\begin{aligned}\int_0^\infty r^{q-1} f_3(r)dr &\ge \int_0^1 r_3^{q-1} f_3(r_3)\frac{dr_3}{ds}ds\\ &\ge a^q\int_0^1 (t_1^{t_1}t_2^{t_2})^{\alpha+\frac{q}{p}} F_1^{qt_1}F_2^{qt_2}ds\\ &= a^q\int_0^1 \Big[(t_1F_1^{qm})^{t_1}(t_2F_2^{qm})^{t_2}\Big]^{\frac{1}{m}}ds\\ &\ge a^q\int_0^1\Big[\frac{t_1}{t_1F_1^{qm}} + \frac{t_2}{t_2F_2^{qm}}\Big]^{-\frac{1}{m}}ds\\ &= a^q\Big[\frac{1}{F_1^{qm}} + \frac{1}{F_2^{qm}}\Big]^{-\frac{1}{m}},\end{aligned}$$

where $m = (\alpha + q/p)^{-1}$. Therefore, (2.8) follows with $\gamma = qm = (\alpha/q + 1/p)^{-1}$. □

The following Lemma is crucial to connect Theorem 2.2 and the γ-convexity of a body, which was first observed by Kim, Yaskin, Zvavitch [4]. We present their result with minor modifications.

Lemma 2.3. *Let K be a p-convex body and let L be a k-dimensional p-convex set in $\mathbb{R}^n$, and the function $f(x)$ be given by $f(x) = |K \cap (L + x)|$. Then for $x_1, x_2 \in \mathbb{R}^n, x_3 = a^{-1}(x_1 + x_2)$ and t_1, t_2, r_3 satisfying* (2.5), (2.6), *we have that*

$$f(r_3x_3) \geq (t_1^{t_1} t_2^{t_2})^{k(\frac{1}{p}-1)} f(r_1x_1)^{t_1} f(r_2x_2)^{t_2}. \tag{2.9}$$

Proof. Note that $r_3x_3 = t_1^{\frac{1}{p}} r_1x_1 + t_2^{\frac{1}{p}} r_2x_2$. Then

$$\begin{aligned} K \cap (L + r_3x_3) &= K \cap (L + t_1^{\frac{1}{p}} r_1x_1 + t_2^{\frac{1}{p}} r_2x_2) \\ &\supset t_1^{\frac{1}{p}} \Big(K \cap (L + r_1x_1)\Big) + t_2^{\frac{1}{p}} \Big(K \cap (L + r_2x_2)\Big) \\ &= t_1\Big(t_1^{\frac{1}{p}-1}(K \cap (L + r_1x_1))\Big) + t_2\Big(t_2^{\frac{1}{p}-1}(K \cap (L + r_2x_2))\Big). \end{aligned}$$

Using the Brunn-Minkowski inequality (see [11]), we have

$$\begin{aligned} f(r_3x_3) &\geq \Big|t_1^{\frac{1}{p}-1}(K \cap (L + r_1x_1))\Big|^{t_1} \Big|t_2^{\frac{1}{p}-1}(K \cap (L + r_2x_2))\Big|^{t_2} \\ &= (t_1^{t_1} t_2^{t_2})^{k(\frac{1}{p}-1)} \Big|K \cap (L + r_1x_1)\Big|^{t_1} \Big|K \cap (L + r_2x_2)\Big|^{t_2} \\ &= (t_1^{t_1} t_2^{t_2})^{k(\frac{1}{p}-1)} f(r_1x_1)^{t_1} f(r_2x_2)^{t_2}. \end{aligned}$$

□

3. The Busemann theorem for p-convex bodies

The following results were first obtained by Kim, Yaskin, Zvavitch [4] along the same approach. For reader's convenience, we present it here.

Lemma 3.1. *Let K be an origin-symmetric p-convex body in $\mathbb{R}^n$, $p \in (0, 1]$, and E an $(n-2)$-dimensional subspace of $\mathbb{R}^n$. Let $u \in E^\perp$ be a unit vector and $r(u) := |K \cap \operatorname{span}\{u, E\}|$. Then $r : E^\perp \cap S^{n-1} \to (0, \infty)$ is the boundary of a γ-convex body in $H^\perp$ with $\gamma = ((1/p - 1)(n - 1) + 1)^{-1}$.*

Proof. Note that E is $(n-2)$-dimensional p-convex set in $\mathbb{R}^n$ in Lemma 2.3. Let $u_1, u_2 \in S^{n-1} \cap E^\perp$, and $u_3 := \frac{u_1+u_2}{|u_1+u_2|} =: a^{-1}(u_1 + u_2)$. Denote $f_i(r) = 2|K \cap (E + ru_i)|$ for $i = 1, 2, 3$. By (1.1), it is sufficient to show that

$$\frac{a^\gamma}{r(u_3)^\gamma} \leq \frac{1}{r(u_1)^\gamma} + \frac{1}{r(u_2)^\gamma}. \tag{3.1}$$

Observe that

$$r(u_i) = 2\int_0^\infty |K \cap (E + ru_i)| dr = \int_0^\infty f_i(r) dr.$$

Now, Lemma 2.3 with $x_i = u_i$ for $i = 1, 2, 3$ and Theorem 2.2 with $q = 1$ yield (3.1). □

Proof of Theorem 1.2. Let E be any $(n-2)$-dimensional subspace. Let $u \in S^{n-1} \cap E^{\perp}$, and let $v \in S^{n-1} \cap E^{\perp}$ be orthogonal to u. Then

$$\rho_{I(K)}(v) = |K \cap \operatorname{span}\{u, E\}|.$$

From Lemma 3.1, we see that $I(K) \cap E^{\perp}$ is γ-convex with $\gamma = ((1/p-1)(n-1)+1)^{-1}$. This implies $I(K)$ is γ-convex. □

4. The Busemann theorem for complex p-convex bodies

In [9], Kolodobsky, Paouris and Zymonopoulou established the Busemann theorem for complex convex bodies. In the following, we will give the proof of the Busemann theorem for complex p-convex bodies (i.e. Theorem 1.5).

Recall that we identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$. Assume that $n \geq 3$. Let $u_1, u_2 \in \mathbb{C}^n$, $|u_1| = |u_2| = 1$, with $H_{u_i}^{\perp} = \operatorname{span}\{u_i, u_i^{\perp}\}$, $\theta_i \in S_{H_{u_i}^{\perp}}$, $i = 1, 2$. We define $u_3 := \frac{u_1+u_2}{|u_1+u_2|} \in S_{H_{u_3}^{\perp}}$, with $H_{u_3}^{\perp}$ and $\theta_3 := \frac{\theta_1+\theta_2}{|\theta_1+\theta_2|}$ such that $|\theta_1+\theta_2| = |u_1+u_2|$. We can assume that $H_{u_1}^{\perp} \bigcap H_{u_2}^{\perp} = \{0\}$.

Let $r_1, r_2 > 0$. We define r_3, t_i such that for $i = 1, 2$,

$$t_i := \frac{r_i^{-p}}{r_1^{-p} + r_2^{-p}}, r_3^p = \frac{|u_1+u_2|^p}{r_1^{-p} + r_2^{-p}}.$$

If $S := (H_{u_1}^{\perp} + H_{u_2}^{\perp})^{\perp}$, we denote $E_i := \operatorname{span}\{H_{u_i}^{\perp}, S\}$, $i = 1, 2, 3$. The function $g_i : H_{u_i}^{\perp} \to \mathbb{R}$, $f_i : [0, \infty) \to [0, \infty)$, $i = 1, 2, 3$ is defined by

$$g_i(x) := \int_{S+x} \mathbf{1}_K(y) dy = |K \cap (S+x)|,$$

$$f_i(r) := g_i(r\theta_i).$$

To prove Theorem 1.5, the following Lemma will be needed.

Lemma 4.1. ([9]) *For $i = 1, 2, 3$, we have that*

$$|K \cap E_i| = 2\pi \int_0^{\infty} r f_i(r) dr.$$

Lemma 4.2. *With the above notation, if K is a p-convex body, then*

$$f_3(r_3) \geq (t_1^{t_1} t_2^{t_2})^{\alpha} f_1^{t_1}(r_1) f_2^{t_2}(r_1),$$

where $\alpha = (1/p-1)(2n-4)$.

Proof. Note that S is $(2n-4)$-dimensional p-convex set in $\mathbb{R}^{2n}$. Taking $a = |\theta_1+\theta_2| = |u_1+u_2|$ and $x_i = \theta_i$ for $i = 1, 2, 3$ in Lemma 2.3, we complete the proof. □

Theorem 4.3. *With the above notation, if $a := |\theta_1+\theta_2|$, then*

$$\frac{a^{\gamma}}{|K \cap E_3|^{\gamma/2}} \leq \frac{1}{|K \cap E_1|^{\gamma/2}} + \frac{1}{|K \cap E_2|^{\gamma/2}},$$

where $\gamma = ((1/p-1)(n-1)+1)^{-1}$.

Proof. From Lemma 4.1, it follows that

$$F_i^2 = \int_0^\infty r f_i(r)dr = \frac{|K \cap E_i|}{2\pi}.$$

Then, applying Lemma 4.2 and Theorem 2.2 with $q = 2$, the result follows. □

Corollary 4.4. *Let K be an origin-symmetric complex p-convex body in $\mathbb{C}^n$. Let H be an $(n-2)$-dimensional subspace of $\mathbb{C}^n$. Let $u \in H^\perp$ be a complex unit vector and let $H_u := \mathrm{span}\{H, u\}$ and $r(u) := |K \cap H_u|^{\frac{1}{2}}$. Then $r : H^\perp \cap S^{2n-1} \to (0, \infty)$ is the boundary of a complex γ-convex body in $H^\perp$ with $\gamma = ((1/p - 1)(n-1) + 1)^{-1}$.*

Proof. Let u_1, u_2 are two non-parallel unit vectors in $H^\perp$, and $u_3 := \frac{u_1+u_2}{|u_1+u_2|} =: a^{-1}(u_1 + u_2)$. As in the proofs of Lemma 3.1, it is sufficient to show that

$$\frac{a^\gamma}{r(u_3)^\gamma} \le \frac{1}{r(u_1)^\gamma} + \frac{1}{r(u_2)^\gamma}.$$

In the notation of this section we have that $H_{u_i} = E_i$, $r(u_i) = |K \cap E_i|^{\frac{1}{2}}$. The result follows from Theorem 4.3. □

We are now ready to prove Theorem 1.5.

Proof of Theorem 1.5. In the case where $n = 2$ the body $I_c(K)$ is simply a rotation of K, so the result is obvious. Let $n \ge 3$. Then equation (1.4) and Corollary 4.4 imply that $I_c(K) \cap H^\perp$ is γ-convex for every $(n-2)$-dimensional subspace H of $\mathbb{C}^n$. This implies that $I_c(K)$ is γ-convex. Finally, it is not difficult to see that $I_c(K)$ satisfies (1.2). This implies that $I_c(K)$ is a complex γ-convex body. □

5. The generalized radial qth mean bodies $R_q(K)$

Let K be a body in $\mathbb{R}^n$. The covariogram g_K of K is the function

$$g_K(x) = |K \cap (K + x)|, \qquad x \in \mathbb{R}^n.$$

The concept of a radial qth mean body $R_q(K)$ was introduced by Gardner and Zhang [13]. We introduce a generalization of $R_q(K)$ for a body K, which is identical with the case that K is a convex body (see [13, Lemma 3.1]).

Definition 5.1. Let K be a body in $\mathbb{R}^n$ and let $q > 0$. Then the generalized radial qth mean body $R_q(K)$ is defined as the body whose radial function is given by

$$\rho_{R_q(K)}(x) = \Big(\frac{q}{|K|}\int_0^\infty r^{q-1} g_K(rx)dr\Big)^{1/q}, \qquad x \in \mathbb{R}^n \backslash \{o\}.$$

Note that the above definition of $R_q(K)$ is well defined, since $\rho_{R_q(K)}(x)$ is homogeneous of degree -1.

Lemma 5.2. *Let f be a nonnegative integrable function in $\mathbb{R}^n$ satisfying* (2.9). *Then the function ρ defined by*

$$\rho(x) = \Big(\int_0^\infty r^{q-1} f(rx)dr\Big)^{\frac{1}{q}},$$

for $x \in \mathbb{R}^n \backslash \{o\}$, is the radial function of a γ-convex body in $\mathbb{R}^n$ with $\gamma = \Big(\frac{k}{q}(\frac{1}{p} - 1) + \frac{1}{p}\Big)^{-1}$.

Proof. Obviously, $\rho(x)$ is homogeneous of degree -1. For $x_1, x_2 \in \mathbb{R}^n$, let $x_3 = a^{-1}(x_1 + x_2)$ and $f_i(r) = f(rx_i)$ for $i = 1, 2, 3$. By Theorem 2.2 with $\alpha = k(\frac{1}{p} - 1)$, we obtain

$$\frac{a^\gamma}{\rho^\gamma(x_3)} \le \frac{1}{\rho^\gamma(x_1)} + \frac{1}{\rho^\gamma(x_2)},$$

where $\gamma = \Big(\frac{k}{q}(\frac{1}{p} - 1) + \frac{1}{p}\Big)^{-1}$. Consequently,

$$\frac{1}{\rho^\gamma(x_1 + x_2)} \le \frac{1}{\rho^\gamma(x_1)} + \frac{1}{\rho^\gamma(x_2)}.$$

The result follows from (1.1). □

Theorem 5.3. *If $q > 0$, then the generalized radial qth mean body $R_q(K)$ of a p-convex body K, $p \in (0, 1]$, is an origin-symmetric γ-convex body with $\gamma = \Big(\frac{n}{q}(\frac{1}{p} - 1) + \frac{1}{p}\Big)^{-1}$.*

Proof. From Lemma 2.3, we see the covariogram g_K of K satisfies (2.9) with $k = n$. From Lemma 5.2 with $f(rx) = qg_K(rx)/|K|$, we complete the proof. □

Corollary 5.4. ([13]) *If $q \geq 0$, then the radial qth mean body $R_q(K)$ of a convex body K is an origin-symmetric convex body.*

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