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ON BONNESEN-TYPE INEQUALITIES FOR A SURFACE OF CONSTANT CURVATURE

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Abstract. New Bonnesen-type inequalities for simply connected domains on surfaces of constant curvature are proved by using integral formulas. These inequalities are generalizations of known inequalities of convex domains.

1. Introduction and preliminaries

The classical isoperimetric inequality says that for a compact set K bounded by a rectifiable simple closed curve in the Euclidean plane $\mathbb{R}^2$, denote by A_K and P_K the area and perimeter of K, respectively. Then

$$P_K^2 - 4\pi A_K \geq 0, \tag{1.1}$$

with equality if and only if K is a Euclidean disc.

A Bonnesen-type inequality is a sharp isoperimetric inequality that includes an error estimate in terms of inscribed and circumscribed regions. That is, there is a non-negative invariant B_K of geometric significance such that

$$P_K^2 - 4\pi A_K \geq B_K, \tag{1.2}$$

where B_K vanishes if and only if K is a Euclidean disc.

The typical example is the following Bonnesen isoperimetric inequality (see [3,4]):

$$P_K^2 - 4\pi A_K \geq \pi^2 (R_K - r_K)^2, \tag{1.3}$$

where R_K and r_K, respectively, denote the circumradius and inradius of K, with equality if and only if K is a Euclidean disc.

In the 1920's, Bonnesen first gave the inequality (1.3). Then many Bonnesen-type inequalities were found along with variations and generalizations (see [2–6, 8, 15, 20, 21, 24, 30–43]).

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Let $\mathbb{X}_\kappa$ be the surface of constant curvature κ, specifically:

$$\mathbb{X}_\kappa = \begin{cases} \mathbb{S}_\kappa, & \text{Euclidean 2-sphere of radius } 1/\sqrt{\kappa}, & \text{if } \kappa > 0; \\ \mathbb{R}^2, & \text{Euclidean plane}, & \text{if } \kappa = 0; \\ \mathbb{H}_\kappa, & \text{Hyperbolic plane of constant curvature } \kappa, & \text{if } \kappa < 0. \end{cases}$$

The isoperimetric inequality in $\mathbb{X}_\kappa$ has been established. For a compact set K bounded by a rectifiable simple closed curve with the area A_K and perimeter P_K in $\mathbb{X}_\kappa$, then (see [1, 7, 10, 13, 14, 17–23, 25–30, 32, 33, 35–37, 39]):

$$P_K^2 - (4\pi - \kappa A_K)A_K \geq 0, \tag{1.4}$$

with equality if and only if K is a geodesic disc.

The isoperimetric deficit of K is defined as

$$\Delta_\kappa(K) = P_K^2 - (4\pi - \kappa A_K)A_K. \tag{1.5}$$

The quantity $\Delta_\kappa(K)$ measures the deficit between K and a geodesic disc in $\mathbb{X}_\kappa$. Then the Bonnesen-type inequality of K takes the form

$$\Delta_\kappa(K) \geq B_K, \tag{1.6}$$

where the quantity B_K is a non-negative invariant of geometric significance and vanishes if and only if K is a geodesic disc (see [19, 20, 32, 42]).

Many Bonnesen-type inequalities in $\mathbb{X}_\kappa$ have been found (see [3, 4, 9, 14, 20, 21]). Santaló and Hadwiger obtain the isoperimetric inequality and Bonnesen-type inequalities by Blaschke and Poincaré's fundamental kinematic formulas in integral geometry (see [11, 12, 14, 22, 23]). Some new Bonnesen-type inequalities in $\mathbb{X}_\kappa$ are works of Klain and Zhou by kinematic methods (see [9, 14, 16, 32, 39, 42]).

A set K is said to be convex if for points $x, y \in K$, the shortest geodesic curve connecting x, y belongs to K. It should be noted that for a compact set K in $\mathbb{S}_\kappa$, we always assume that K lies in an open hemisphere of $\mathbb{S}_\kappa$; then $2\pi - \kappa A_K > 0$ follows.

In [14], Klain proved the following Bonnesen-type inequality:

$$\Delta_\kappa(K) \geq \frac{\left((2\pi - \kappa A_K)^2 + \kappa P_K^2\right)^2}{4(2\pi - \kappa A_K)^2} \left(\mathrm{sn}_\kappa(R_K) - \mathrm{sn}_\kappa(r_K)\right)^2, \tag{1.7}$$

for compact convex set K satisfying the condition $(2\pi - \kappa A_K)^2 + \kappa P_K^2 \geq 0$ if $\kappa < 0$, where R_K and r_K are, respectively, the radius of the minimum circumscribed geodesic disc and the maximum inscribed geodesic disc of K.

Let K be a compact convex set in $\mathbb{S}_\kappa$. Klain showed the following inequality:

$$\Delta_\kappa(K) \geq \frac{1}{4}(2\pi - \kappa A_K)^2 \left(\mathrm{sn}_\kappa(R_K) - \mathrm{sn}_\kappa(r_K)\right)^2, \tag{1.8}$$

with equality if and only if K is a geodesic disc.

The function $\mathrm{sn}_\kappa(t)$ in (1.7) is defined as:

$$\mathrm{sn}_\kappa(t) = \begin{cases} \frac{1}{\sqrt{-\kappa}} \sinh(\sqrt{-\kappa}t), & \kappa < 0, \\ t, & \kappa = 0, \\ \frac{1}{\sqrt{\kappa}} \sin(\sqrt{\kappa}t), & \kappa > 0. \end{cases} \tag{1.9}$$

Similarly, one defines

$$\operatorname{cn}_\kappa(t)=\begin{cases}\cosh(\sqrt{-\kappa}t), & \kappa<0,\\ 1, & \kappa=0,\\ \cos(\sqrt{\kappa}t), & \kappa>0.\end{cases} \tag{1.10}$$

It is natural to define functions

$$\operatorname{tn}_\kappa(t)=\frac{\operatorname{sn}_\kappa(t)}{\operatorname{cn}_\kappa(t)},\quad \operatorname{ct}_\kappa(t)=\frac{\operatorname{cn}_\kappa(t)}{\operatorname{sn}_\kappa(t)}. \tag{1.11}$$

Hence

$$\kappa\cdot\operatorname{sn}_\kappa^2(t)+\operatorname{cn}_\kappa^2(t)=1. \tag{1.12}$$

Zhou and Chen obtained the following Bonnesen-type inequality (see [39]):

$$\Delta_\kappa(K)\quad\geq\quad\left(2\pi-\frac{\kappa}{2}\ A_K\right)^2\left(\operatorname{tn}_\kappa\frac{R_K}{2}-\operatorname{tn}_\kappa\frac{r_K}{2}\right)^2, \tag{1.13}$$

with equality if K is a geodesic disc.

Later, a strengthened version of (1.13) was given in [32] as follows:

$$\begin{aligned}\Delta_\kappa(K)\quad\geq\quad&\left(2\pi-\tfrac{\kappa}{2}\ A_K\right)^2\left(\operatorname{tn}_\kappa\tfrac{R_K}{2}-\operatorname{tn}_\kappa\tfrac{r_K}{2}\right)^2\\&+\left(2\pi-\tfrac{\kappa}{2}\ A_K\right)^2\left(\operatorname{tn}_\kappa\tfrac{R_K}{2}+\operatorname{tn}_\kappa\tfrac{r_K}{2}-\tfrac{2P_K}{4\pi-\kappa A_K}\right)^2,\end{aligned} \tag{1.14}$$

with equality if K is a geodesic disc.

If $\kappa=0$, inequality (1.14) immediately leads to a strengthened Bonnesen isoperimetric inequality:

$$P_K^2-4\pi A_K\quad\geq\quad\pi^2\left(R_K-r_K\right)^2+\left(P_K-\pi R_K-\pi r_K\right)^2,$$

with equality if K is a Euclidean disc.

The geodesic disc of radius r with center x is defined as

$$B_\kappa(x,r)=\{y\in\mathbb{X}_\kappa:\ d(x,y)\leq r\},$$

where d is the geodesic distance function in $\mathbb{X}_\kappa$. The area, perimeter of $B_\kappa(x,r)$ in $\mathbb{X}_\kappa$ are, respectively (see [14]),

$$A(B_\kappa(x,r))=\frac{2\pi}{\kappa}(1-\operatorname{cn}_\kappa(r)),\quad P(B_\kappa(x,r))=2\pi\ \operatorname{sn}_\kappa(r). \tag{1.15}$$

The limiting cases of $\kappa\to 0$ yield to the Euclidean formulas $A(B(x,r))=\pi r^2$ and $P(B(x,r))=2\pi r$.

In this paper, we always consider a compact set K bounded by a rectifiable simple closed curve in $\mathbb{X}_\kappa$ without the convexity condition. Denote by A_K the area and P_K the perimeter of K. Let r_K and R_K be the radius of the maximum inscribed disc and the radius of the minimum circumscribed disc of K, respectively. Let $\mathcal{C}$ be the set of all compact sets bounded by a rectifiable simple closed curve with $P_K\leq\frac{2\pi}{\sqrt{\kappa}}$ if $\kappa>0$ in $\mathbb{X}_\kappa$. For simplicity, denote $B_\kappa(r)$ as a geodesic disc of radius r instead of $B_\kappa(x,r)$ in $\mathbb{X}_\kappa$. Denote by $\chi(K)$ the Euler-Poincaré characteristic of K. If K is a compact convex set, then $\chi(K)=1$, while $\chi(\emptyset)=0$.

By estimating the containment measure, we obtain a Bonnesen-type isoperimetric inequality with a quantity B_K larger than Klain's Bonnesen-type isoperimetric inequality (1.7), that is:

Theorem 1.1. *Suppose $K \in \mathcal{C}$. If $(2\pi - \kappa A_K)^2 + \kappa P_K^2 \geq 0$ for $\kappa < 0$, then*

(1.16)
$$\begin{aligned}\Delta_\kappa(K) \geq & \frac{\left((2\pi-\kappa A_K)^2+\kappa P_K^2\right)^2}{4(2\pi-\kappa A_K)^2}(\mathrm{sn}_\kappa(R_K) - \mathrm{sn}_\kappa(r_K))^2 \\ & + \frac{1}{4(2\pi-\kappa A_K)^2}\left[4\pi P_K - \left((2\pi - \kappa A_K)^2 + \kappa P_K^2\right)(\mathrm{sn}_\kappa(R_K) + \mathrm{sn}_\kappa(r_K))\right]^2,\end{aligned}$$

with equality if K is a geodesic disc.

Also, we obtain a new Bonnesen-type isoperimetric inequality for $K \in \mathcal{C}$, which is always true for any rectifiable simple closed curves in the hyperbolic plane.

Theorem 1.2. *Suppose $K \in \mathcal{C}$. Then*

$$\begin{aligned}\Delta_\kappa(K) \quad \geq \quad & \frac{A_K^2(4\pi-\kappa A_K)^2}{4(2\pi-\kappa A_K)^2}\left(\frac{1}{sn_\kappa(r_K)} - \frac{1}{sn_\kappa(R_K)}\right)^2 \\ & + \frac{A_K^2(4\pi-\kappa A_K)^2}{4(2\pi-\kappa A_K)^2}\left(\frac{1}{sn_\kappa(R_K)} + \frac{1}{sn_\kappa(r_K)} - \frac{4\pi P_K}{A_K(4\pi-\kappa A_K)}\right)^2,\end{aligned} \tag{1.17}$$

with equality if K is a geodesic disc.

2. The Bonnesen-type inequalities in $\mathbb{X}_\kappa$

Let K, L be compact sets of areas A_K, A_L bounded by rectifiable simple closed curves of perimeters P_K, P_L in $\mathbb{X}_\kappa$, respectively. Let G_κ be the group of isometries in $\mathbb{X}_\kappa$ and let dg be the Harr measure on G_κ. In the content of integral geometry, dg is called the kinematic density of G_κ. As is common in integral geometry we let K be fixed and gL moving via the isometry $g \in G_\kappa$. We have the fundamental kinematic formula of Blaschke (see [21]):

$$\int_{\{g:\ K\cap gL\neq\emptyset\}} \chi(K \cap gL)\ dg = 2\pi(A_K + A_L) + P_K P_L - \kappa A_K A_L. \tag{2.1}$$

As the limiting case, when K, L degenerate to curves ∂K, ∂L, respectively, then $A_K = A_L = 0$ and the perimeters are $2P_K$, $2P_L$. Then we have the kinematic formula of Poincaré (see [21]):

$$\int_{\{g:\ \partial K\cap\partial(gL)\neq\emptyset\}} \sharp(\partial K \cap \partial(gL))\ dg = 4P_K P_L. \tag{2.2}$$

Here $\sharp(\partial K\cap\partial(gL))$ is the number of points of the intersection $\partial K\cap\partial(gL)$. Since the compact sets are assumed to be simply connected and enclosed by simple curves, we have $\chi(K \cap gL) = n(g) \equiv$ the number of connected components of the intersection $K \cap gL$. Let $\mu = \{g \in G_\kappa :\ K \subset gL \text{ or } K \supset gL\}$; then the fundamental kinematic formula of Blaschke (2.1) can be rewritten as (see [30, 39]):

$$\int_\mu dg + \int_{\{g:\ \partial K\cap\partial(gL)\neq\emptyset\}} n(g)\ dg = 2\pi(A_K + A_L) + P_K P_L - \kappa A_K A_L. \tag{2.3}$$

When $\partial K \cap \partial(gL) \neq \emptyset$, each component of $K \cap gL$ is bounded by at least an arc of ∂K and an arc of $\partial(gL)$, and $n(g) \leq \sharp(\partial K \cap \partial(gL))/2$. Then the following containment measure inequality is an immediate consequence of Poincaré's formula (2.2) and Blaschke's formula (2.3) (see [9, 14, 21, 30]).

Proposition 2.1. *Let K, L be two compact sets in $\mathbb{X}_\kappa$, each set bounded by a rectifiable simple closed curve; then*

$$\int_\mu dg \geq 2\pi(A_K + A_L) - P_K P_L - \kappa A_K A_L. \tag{2.4}$$

If we let $K \equiv L$, then there is no $g \in G_\kappa$ such that $gK \supset K$ nor $gK \subset K$. Hence $\int_\mu dg = 0$ and the inequality (2.4) immediately results in the isoperimetric inequality (1.2).

Let L be a geodesic disc of radius r. Then there is no $g \in G_\kappa$ such that $gB_\kappa(r) \subset K$ nor $gB_\kappa(r) \supset K$ for $r_K \leq r \leq R_K$. Then by (2.4), a Bonnesen-type inequality follows:

Lemma 2.2. *Suppose $K \in \mathcal{C}$. If $r_K \leq r \leq R_K$, then*

$$\left[(2\pi - \kappa A_K)^2 + \kappa P_K^2\right] sn_\kappa^2(r) - 4\pi P_K\ sn_\kappa(r) - A_K(\kappa A_K - 4\pi) \leq 0. \tag{2.5}$$

Proof. Let L be a geodesic disc $B_\kappa(r)$ of radius r between the maximum inscribed disc of radius r_K and the minimum circumscribed disc of radius R_K of K. We have neither $gB_\kappa(r) \subset K$ nor $gB_\kappa(r) \supset K$ for any $g \in G_\kappa$. Then the measure $\int_\mu dg = 0$. Then by (1.15) and (2.4) we have

$$P_K\ \mathrm{sn}_\kappa(r) - \left(\frac{2\pi}{\kappa} - A_K\right)(1 - \mathrm{cn}_\kappa(r)) - A_K \geq 0. \tag{2.6}$$

Identity (1.12) shows $1 - \kappa \cdot \mathrm{sn}_\kappa^2(r) = \mathrm{cn}_\kappa^2(r) > 0$, and the inequality (2.6) can be rewritten as

$$P_K\ \mathrm{sn}_\kappa(r) - \frac{2\pi}{\kappa} \geq \left(A_K - \frac{2\pi}{\kappa}\right)\sqrt{1 - \kappa \cdot \mathrm{sn}_\kappa^2(r)}. \tag{2.7}$$

For $\kappa \geq 0$, we have

$$P_K\ \mathrm{sn}_\kappa(r) - \frac{2\pi}{\kappa} \leq 0.$$

Squaring both sides of (2.7) we have

$$\left(P_K\ \mathrm{sn}_\kappa(r) - \frac{2\pi}{\kappa}\right)^2 \leq \left(A_K - \frac{2\pi}{\kappa}\right)^2 \left(1 - \kappa \cdot \mathrm{sn}_\kappa^2(r)\right),$$

that is,

$$\left((2\pi - \kappa A_K)^2 + \kappa P_K^2\right) \mathrm{sn}_\kappa^2(r) - 4\pi P_K\ \mathrm{sn}_\kappa(r) - A_K(\kappa A_K - 4\pi) \leq 0.$$

If $\kappa < 0$, then $A_K - \frac{2\pi}{\kappa} > 0$ and we have the following inequality by squaring both sides of (2.7):

$$\left(P_K\ \mathrm{sn}_\kappa(r) - \frac{2\pi}{\kappa}\right)^2 \geq \left(A_K - \frac{2\pi}{\kappa}\right)^2 \left(1 - \kappa \cdot \mathrm{sn}_\kappa^2(r)\right).$$

Hence, we have

$$\left((2\pi - \kappa A_K)^2 + \kappa P_K^2\right) \mathrm{sn}_\kappa^2(r) - 4\pi P_K\ \mathrm{sn}_\kappa(r) - A_K(\kappa A_K - 4\pi) \leq 0.$$

□

We are now in the position to establish Theorem 1.1.

Proof of Theorem 1.1. Since $(2\pi-\kappa A_K)^2+\kappa P_K^2\geq 0$ for arbitrary κ, by inequality (2.5) the quantity

$$\Big((2\pi-\kappa A_K)^2+\kappa P_K^2\Big)\Big\{\Big((2\pi-\kappa A_K)^2+\kappa P_K^2\Big)\operatorname{sn}_\kappa^2(r)-4\pi P_K\operatorname{sn}_\kappa(r) - A_K(\kappa A_K-4\pi)\Big\}$$

is non-positive for $r_K\leq r\leq R_K$, that is,

$$(2\pi-\kappa A_K)^2\Delta_\kappa(K)\geq\Big(2\pi P_K-\Big(2\pi-\kappa A_K)^2+\kappa P_K^2\Big)\operatorname{sn}_\kappa(r)\Big)^2.$$

Especially, for $r=r_K$, $r=R_K$, respectively,

$$(2.8)\quad (2\pi-\kappa A_K)^2\Delta_\kappa(K)\geq\Big(2\pi P_K-\Big((2\pi-\kappa A_K)^2+\kappa P_K^2\Big)\operatorname{sn}_\kappa(r_K)\Big)^2,$$

$$(2.9)\quad (2\pi-\kappa A_K)^2\Delta_\kappa(K)\geq\Big(\Big((2\pi-\kappa A_K)^2+\kappa P_K^2\Big)\operatorname{sn}_\kappa(R_K)-2\pi P_K\Big)^2.$$

Adding the two inequalities in (2.8) and (2.9) side by side, we have

$$\begin{aligned}2(2\pi-\kappa A_K)^2\Delta_\kappa(K)&\geq\Big(2\pi P_K-\Big((2\pi-\kappa A_K)^2+\kappa P_K^2\Big)\operatorname{sn}_\kappa(r_K)\Big)^2\\&+\Big(\Big((2\pi-\kappa A_K)^2+\kappa P_K^2\Big)\operatorname{sn}_\kappa(R_K)-2\pi P_K\Big)^2\\&=\frac{1}{2}\Big((2\pi-\kappa A_K)^2+\kappa P_K^2\Big)^2(\operatorname{sn}_\kappa(R_K)-\operatorname{sn}_\kappa(r_K))^2\\&+2\Big(2\pi P_K-\frac{1}{2}\Big((2\pi-\kappa A_K)^2+\kappa P_K^2\Big)(\operatorname{sn}_\kappa(R_K)+\operatorname{sn}_\kappa(r_K))\Big)^2.\end{aligned}$$

Let K be a geodesic disc, that is, $R_K=r_K$; then both sides of (1.16) are 0. Indeed, since $R_K=r_K$, then $\Delta_\kappa(K)=0$ by (1.15). And (1.15) together with (1.12) shows that

$$\frac{4\pi P_K}{(2\pi-\kappa A_K)^2+\kappa P_K^2}-\operatorname{sn}_\kappa(R_K)-\operatorname{sn}_\kappa(r_K)=\frac{2\operatorname{sn}_\kappa(r_K)}{\kappa\operatorname{sn}_\kappa^2(r_K)+\operatorname{cn}_\kappa^2(r_K)}-2\operatorname{sn}_\kappa(r_K)=0.$$

Thus we complete the proof. □

A compact non-convex set bounded by a rectifiable simple closed curve in a hemisphere of $\mathbb{S}_\kappa$ may satisfy the condition $P_K\leq\frac{2\pi}{\sqrt{\kappa}}$ in $\mathbb{S}_\kappa$. For example, let $\kappa=\frac{1}{R^2}$; then $\frac{2\pi}{\sqrt{\kappa}}=2\pi R$ is the perimeter of a great circle. There are compact non-convex sets in a half hemisphere such that their perimeters $P_K\leq 2\pi R=\frac{2\pi}{\sqrt{\kappa}}$. All compact convex sets in $\mathbb{S}_\kappa$ satisfy $P_K\leq\frac{2\pi}{\sqrt{\kappa}}$. On the other hand, there are compact non-convex sets such that $(2\pi-\kappa A_K)^2+\kappa P_K^2\geq 0$ in $\mathbb{H}_\kappa$; a line segment gives an explicit counterexample to this condition (see [14]). The following Bonnesen-type inequality that strengthens the inequality (1.7) is an immediate consequence of the inequality (1.16) in Theorem 1.1.

Corollary 2.3. *Let K be a compact convex set bounded by a rectifiable simple closed curve in $\mathbb{X}_\kappa$. If $(2\pi-\kappa A_K)^2+\kappa P_K^2\geq 0$ for $\kappa<0$, then*

$$\begin{aligned}\Delta_\kappa(K)&\geq\frac{\big((2\pi-\kappa A_K)^2+\kappa P_K^2\big)^2}{4(2\pi-\kappa A_K)^2}(\operatorname{sn}_\kappa(R_K)-\operatorname{sn}_\kappa(r_K))^2\\&+\frac{\big((2\pi-\kappa A_K)^2+\kappa P_K^2\big)^2}{4(2\pi-\kappa A_K)^2}\Big(\frac{4\pi P_K}{(2\pi-\kappa A_K)^2+\kappa P_K^2}-\operatorname{sn}_\kappa(R_K)-\operatorname{sn}_\kappa(r_K)\Big)^2,\end{aligned}$$

with equality if K is a geodesic disc.

Proof of Theorem 1.2. For $r_K \le r \le R_K$, via (2.5) we have

$$\frac{A_K(4\pi-\kappa A_K)}{\mathrm{sn}_\kappa^2(r)} - \frac{4\pi P_K}{\mathrm{sn}_\kappa(r)} + 4\pi^2 + \kappa\Delta_\kappa(K) \le 0.$$

That is,

$$\Delta_\kappa(K) \ge \frac{A_K(4\pi-\kappa A_K)}{(2\pi-\kappa A_K)^2}\left(\frac{\sqrt{A_K(4\pi-\kappa A_K)}}{\mathrm{sn}_\kappa(r)} - \frac{2\pi P_K}{\sqrt{A_K(4\pi-\kappa A_K)}}\right)^2.$$

Especially, for $r = r_K$ and $r = R_K$, respectively, we have

$$\Delta_\kappa(K) \ \ge \ \frac{A_K(4\pi-\kappa A_K)}{(2\pi-\kappa A_K)^2}\left(\frac{\sqrt{A_K(4\pi-\kappa A_K)}}{\mathrm{sn}_\kappa(r_K)} - \frac{2\pi P_K}{\sqrt{A_K(4\pi-\kappa A_K)}}\right)^2,$$

$$\Delta_\kappa(K) \ \ge \ \frac{A_K(4\pi-\kappa A_K)}{(2\pi-\kappa A_K)^2}\left(\frac{\sqrt{A_K(4\pi-\kappa A_K)}}{\mathrm{sn}_\kappa(R_K)} - \frac{2\pi P_K}{\sqrt{A_K(4\pi-\kappa A_K)}}\right)^2.$$

Adding the two inequalities side by side, we have

$$\begin{aligned}\Delta_\kappa(K) \ &\ge \ \frac{A_K(4\pi-\kappa A_K)}{2\,(2\pi-\kappa A_K)^2}\left\{\left(\frac{\sqrt{A_K(4\pi-\kappa A_K)}}{\mathrm{sn}_\kappa(r_K)} - \frac{2\pi P_K}{\sqrt{A_K(4\pi-\kappa A_K)}}\right)^2\right.\\ &\qquad \left. + \left(\frac{\sqrt{A_K(4\pi-\kappa A_K)}}{\mathrm{sn}_\kappa(R_K)} - \frac{2\pi P_K}{\sqrt{A_K(4\pi-\kappa A_K)}}\right)^2\right\}\\ &= \ \frac{A_K^2(4\pi-\kappa A_K)^2}{4\,(2\pi-\kappa A_K)^2}\left(\frac{1}{\mathrm{sn}_\kappa(r_K)} - \frac{1}{\mathrm{sn}_\kappa(R_K)}\right)^2\\ &\qquad + \frac{A_K^2(4\pi-\kappa A_K)^2}{4\,(2\pi-\kappa A_K)^2}\left(\frac{1}{\mathrm{sn}_\kappa(R_K)} + \frac{1}{\mathrm{sn}_\kappa(r_K)} - \frac{4\pi P_K}{A_K(4\pi-\kappa A_K)}\right)^2.\end{aligned}$$

Let K be a geodesic disc, that is, $R_K = r_K$; then both sides of (1.17) are 0. Indeed, since $R_K = r_K$, then $\Delta_\kappa(K) = 0$ by (1.15). And (1.15) together with (1.12) shows that

$$\frac{1}{\mathrm{sn}_\kappa(R_K)} + \frac{1}{\mathrm{sn}_\kappa(r_K)} - \frac{4\pi P_K}{A_K(4\pi-\kappa A_K)} = \frac{2}{\mathrm{sn}_\kappa(r_K)} - \frac{2\kappa\ \mathrm{sn}_\kappa(r_K)}{1-\mathrm{cn}_\kappa^2(r_K)} = 0.$$

We complete the proof of Theorem 1.2. □

The following Bonnesen-type inequality is an immediate consequence of the inequality (1.17) in Theorem 1.2 with equality condition.

Corollary 2.4. *Suppose $K \in \mathcal{C}$. Then*

$$\Delta_\kappa(K) \ \ge \ \frac{A_K^2(4\pi-\kappa A_K)^2}{4(2\pi-\kappa A_K)^2}\left(\frac{1}{sn_\kappa(R_K)} + \frac{1}{sn_\kappa(r_K)} - \frac{4\pi P_K}{A_K(4\pi-\kappa A_K)}\right)^2, \tag{2.10}$$

with equality if and only if K is a geodesic disc.

Proof. The inequality (2.10) follows from (1.17) and

$$\frac{A_K^2(4\pi-\kappa A_K)^2}{4\,(2\pi-\kappa A_K)^2}\left(\frac{1}{\mathrm{sn}_\kappa(r_K)} - \frac{1}{\mathrm{sn}_\kappa(R_K)}\right)^2 \ge 0. \tag{2.11}$$

Equality holds in (2.10) if and only if equalities hold in (1.17) and (2.11) at the same time. That is, $R_K = r_K$ and K must be a geodesic disc. □

3. The limiting cases of the Euclidean plane $\mathbb{R}^2$

In this section, we consider the limiting cases of these Bonnesen-type inequalities obtained.

For $\kappa > 0$, let $\kappa = \frac{1}{R^2}$. Then the inequality (1.16) becomes

$$\begin{aligned} P_K^2 &- 4\pi A_K + \frac{A_K^2}{R^2} \\ &\geq \frac{\left(\left(2\pi - \frac{A_K}{R^2}\right)^2 + \frac{P_K^2}{R^2}\right)^2}{4\left(\frac{A_K}{R^2} - 2\pi\right)^2} R^2 \left(\sin\frac{R_K}{R} - \sin\frac{r_K}{R}\right)^2 \\ &+ \frac{1}{4\left(\frac{A_K}{R^2} - 2\pi\right)^2}\left[4\pi P_K - \left(\left(2\pi - \frac{A_K}{R^2}\right)^2 + \frac{P_K^2}{R^2}\right)\left(R\sin\frac{R_K}{R} + R\sin\frac{r_K}{R}\right)\right]. \end{aligned}$$

As $R \to \infty$, by L'Hôpital's rule we have

$$\begin{aligned} P_K^2 &- 4\pi A_K \\ &\geq \lim_{R\to\infty}\left\{\pi^2 R^2\left(\sin\frac{R_K}{R} - \sin\frac{r_K}{R}\right)^2 + \pi^2\left(\frac{P_K}{\pi} - R\sin\frac{R_K}{R} - R\sin\frac{r_K}{R}\right)^2\right\} \\ &= \pi^2\left(R_K - r_K\right)^2 + \left(P_K - \pi R_K - \pi r_K\right)^2, \end{aligned}$$

a strengthened Bonnesen isoperimetric inequality (see [32, 33]).

Let $\kappa = -\frac{1}{R^2} < 0$; then (1.16) can be rewritten as

$$\begin{aligned} P_K^2 - 4\pi A_K - \frac{A_K^2}{R^2} & \\ \geq \frac{\left(\left(2\pi - \frac{A_K}{R^2}\right)^2 + \frac{P_K^2}{R^2}\right)^2}{4\left(\frac{A_K}{R^2} - 2\pi\right)^2} & R^2\left(\sinh\frac{R_K}{R} - \sinh\frac{r_K}{R}\right)^2 \\ + \frac{1}{4\left(\frac{A_K}{R^2} - 2\pi\right)^2} & \left[4\pi P_K - \left(\left(2\pi - \frac{A_K}{R^2}\right)^2 + \frac{P_K^2}{R^2}\right)\left(R\sinh\frac{R_K}{R} + R\sinh\frac{r_K}{R}\right)\right]. \end{aligned}$$

As $R \to \infty$, it also leads to

$$P_K^2 - 4\pi A_K \geq \pi^2\left(R_K - r_K\right)^2 + \left(P_K - \pi R_K - \pi r_K\right)^2.$$

For $\kappa = \frac{1}{R^2}$, inequality (1.17) can be rewritten as:

$$\begin{aligned} P_K^2 - 4\pi A_K + \frac{A_K^2}{R^2} \geq \frac{A_K^2\left(4\pi - \frac{A_K}{R^2}\right)^2}{4\left(2\pi - \frac{A_K}{R^2}\right)^2}\frac{1}{R^2}\left(\frac{1}{\sin\frac{R_K}{R}} - \frac{1}{\sin\frac{r_K}{R}}\right)^2 & \\ + \frac{A_K^2\left(4\pi - \frac{A_K}{R^2}\right)^2}{4\left(2\pi - \frac{A_K}{R^2}\right)^2}\left(\frac{1}{R\sin\frac{R_K}{R}} + \frac{1}{R\sin\frac{r_K}{R}} - \frac{4\pi P_K}{A_K(4\pi - \frac{A_K}{R^2})}\right)^2. & \end{aligned}$$

As $R \to \infty$, we have the following Bonnesen-type inequality in $\mathbb{R}^2$ (see [32]) that strengthens a Bonnesen-type inequality

$$P_K^2 - 4\pi A_K \geq A_K^2\left(\frac{1}{r_K} - \frac{1}{R_K}\right)^2$$

in [38].

Corollary 3.1. *Let K be a compact set bounded by a rectifiable simple closed curve in $\mathbb{R}^2$. Then we have*

$$P_K^2 - 4\pi A_K \geq A_K^2 \left(\frac{1}{r_K} - \frac{1}{R_K}\right)^2 + A_K^2 \left(\frac{1}{R_K} + \frac{1}{r_K} - \frac{P_K}{A_K}\right)^2,$$

with equality if K is a Euclidean disc.

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