THE L_p LOOMIS-WHITNEY INEQUALITY

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Abstract. In this paper, we establish the L_p Loomis-Whitney inequality for even isotropic measures in terms of the support function of L_p projection bodies with complete equality conditions. This generalizes Ball's Loomis-Whitney inequality to the L_p setting. In addition, the sharp upper bound of the minimal p-mean width of L_p zonoids is obtained.

1. Introduction

Throughout this paper all Borel measures are understood to be nonnegative and finite. A convex body is a compact convex set in $\mathbb{R}^n$ which is assumed to contain the origin in its interior. We use $|\cdot|$ to denote the volume of a convex body or its $(n-1)$-dimensional projection. Denote by $\mathcal{K}_o^n$ the space of convex bodies in $\mathbb{R}^n$ equipped with the Hausdorff metric. Each convex body K is uniquely determined by its support function $h_K(\cdot)$, defined by $h_K(x) = \max\{x \cdot y : y \in K\}$, for $x \in \mathbb{R}^n$, where $x \cdot y$ denotes the usual inner product of x and y in $\mathbb{R}^n$.

The classical Loomis-Whitney inequality [21] states that for a convex body K in $\mathbb{R}^n$,

$$|K|^{n-1} \le \prod_{i=1}^{n} |\mathrm{P}_{e_i^\perp} K|, \tag{1.1}$$

with equality if and only if K is a coordinate box (a rectangular parallelepiped whose facets are parallel to the coordinate hyperplanes), where $\mathrm{P}_{e_i^\perp} K$ denotes the orthogonal projection of K onto the 1-codimensional space $e_i^\perp$ perpendicular to e_i and $\{e_1, \dots, e_n\}$ is the standard orthonormal basis of $\mathbb{R}^n$. Note that the Loomis-Whitney inequality is of isoperimetric type. Indeed, denoting by $S(K)$ the surface

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area of K, then $S(K) \geq 2|\mathrm{P}_{e_i^\perp}K|$ for all $i = 1, \ldots, n$. Together with (1.1), we get

$$|K|^{n-1} \leq 2^{-n} S(K)^n,$$

an isoperimetric inequality without the best constant. The Loomis-Whitney inequality is one of the fundamental inequalities in convex geometry and has been studied intensively; see e.g., [3, 6–11, 19, 38].

In particular, Ball [3] showed that the Loomis-Whitney inequality still holds along a sequence of directions satisfying John's condition [17]. Specifically, for a convex body K in $\mathbb{R}^n$, if there are unit vectors $(u_i)_{i=1}^m$ and positive numbers $(c_i)_{i=1}^m$ satisfying John's condition

$$\sum_{i=1}^{m} c_i u_i \otimes u_i = I_n, \tag{1.2}$$

then

$$|K|^{n-1} \leq \prod_{i=1}^{m} |\mathrm{P}_{u_i^\perp}K|^{c_i}, \tag{1.3}$$

where $u_i \otimes u_i$ is the rank-one orthogonal projection onto the space spanned by the unit vector u_i and I_n is the identity map on $\mathbb{R}^n$. Obviously, the inequality (1.3) reduces to (1.1) when $m = n$ and taking $u_i = e_i$ with $c_i = 1$ for all $i = 1, \ldots, n$.

A Borel measure ν on the unit sphere S^{n-1} of $\mathbb{R}^n$ is said to be isotropic if

$$\int_{S^{n-1}} u \otimes u d\nu(u) = I_n. \tag{1.4}$$

Note that it is impossible for an isotropic measure to be concentrated on a proper subspace of $\mathbb{R}^n$. The measure ν is said to be even if it assumes the same value on antipodal sets. In particular, when choosing the isotropic measure $\nu = \frac{1}{2}\sum_{i=1}^m (c_i\delta_{u_i} + c_i\delta_{-u_i})$ on S^{n-1} (δ_x stands for the Dirac mass at x), the condition (1.4) reduces to (1.2).

The L_p Brunn-Minkowski theory had its origins in the early 1960s when Firey [12] introduced his concept of L_p combinations of convex bodies. In [22] and [23] these L_p Minkowski-Firey combinations were further investigated by Lutwak which lead to an embryonic L_p Brunn-Minkowski theory. This theory has expanded rapidly thereafte; for further details, as well as detailed bibliography on the topic we refer the reader to [32, Chapter 9] and the references therein. An important notion in the L_p Brunn-Minkowski theory is the L_p *projection body* $\Pi_p K$ introduced by Lutwak, Yang, and Zhang [24]. In this paper, the L_p projection body $\Pi_p K$ ($p \geq 1$) of $K \in \mathcal{K}_o^n$

is the origin-symmetric convex body defined by

$$h_{\Pi_p K}(v) = \Big(\frac{1}{|B^n_{p^*}|^{\frac{p}{n}}} \int_{S^{n-1}} |v \cdot u|^p dS_p(K, u)\Big)^{\frac{1}{p}}, \qquad v \in S^{n-1}, \tag{1.5}$$

where $dS_p(K, \cdot)$ is the L_p surface area measure of K and $B^n_{p^*}$ is the unit ball of the space $\ell^n_{p^*}$. Here p^* is the Hölder conjugate of p; i.e., $1/p + 1/p^* = 1$. The case $p = 1$ is the classical projection body ΠK. The normalization above is chosen so that for $p = 1$ we have

$$h_{\Pi K}(v) = |\mathrm{P}_{v^\perp} K| = \frac{1}{2} \int_{S^{n-1}} |v \cdot u| dS_K(u), \quad v \in S^{n-1}, \tag{1.6}$$

where $dS_K(\cdot)$ is the surface area measure of K.

The main purpose of this paper is to generalize Ball's Loomis-Whitney inequality (1.3) to the L_p setting (corresponding to the L_p Brunn-Minkowski theory); i.e., the L_p version of the Loomis-Whitney inequality in terms of the support function of L_p projection bodies with complete equality conditions is established. This inequality may be called as the L_p *Loomis-Whitney inequality.*

Theorem 1.1. *Suppose $p \geq 1$ and $K \in \mathcal{K}^n_o$. If ν is an even isotropic measure on S^{n-1}, then*

$$|K|^{\frac{n-p}{p}} \leq \exp\Big\{ \int_{S^{n-1}} \log h_{\Pi_p K}(u) d\nu(u)\Big\}. \tag{1.7}$$

For $1 < p \neq 2$, equality in (1.7) holds if and only if ν is a cross measure on S^{n-1} and K is the generalized $\ell^n_{p^*}$-ball formed by ν; i.e., there are positive numbers $(\alpha_i)^n_{i=1}$ such that

$$K = \Big\{ x \in \mathbb{R}^n : \Big(\sum_{i=1}^n |x \cdot u_i|^{p^*} \alpha_i \Big)^{\frac{1}{p^*}} \leq 1 \Big\},$$

where $\operatorname{supp} \nu = \{\pm u_1, \ldots, \pm u_n\}$ and $(u_i)^n_1$ is an orthonormal basis of $\mathbb{R}^n$. For $p = 1$, equality in (1.7) holds if and only if ν is a cross measure on S^{n-1} and K is a box formed by ν (up to translations); i.e., there is a vector $v_0 \in \mathbb{R}^n$ and positive numbers $(\alpha_i)^n_{i=1}$ such that

$$K = \sum_{i=1}^n \alpha_i [-u_i, u_i] + v_0,$$

where $\operatorname{supp} \nu = \{\pm u_1, \ldots, \pm u_n\}$ and $(u_i)^n_1$ is an orthonormal basis of $\mathbb{R}^n$.

Notice that when $p = 1$, by (1.6), the inequality (1.7) can be written as

$$|K|^{n-1} \leq \exp\Big\{ \int_{S^{n-1}} \log |\mathrm{P}_{u^\perp} K| d\nu(u)\Big\}, \tag{1.8}$$

with equality if and only if K is a box formed by the cross measure ν (up to translations). When taking $\nu=\frac{1}{2}\sum_{i=1}^{m}(c_i\delta_{u_i}+c_i\delta_{-u_i})$ on S^{n-1}, the inequality (1.8) is actually the inequality (1.3). When ν is replaced by the isotropic surface area measure of K, the inequality (1.8) (without equality conditions) was proved by Giannopoulos and Papadimitrakis [14].

By Theorem 1.1 we immediately get the solution of the dual L_p version of Vaaler's conjecture, extending Ball's result [3] ($p=1$).

Theorem 1.2. *Suppose* $1\le p\neq n$ *and* $K\in\mathcal{K}_o^n$. *Then there exists a nondegenerate affine transformation* T *of* $\mathbb{R}^n$ *such that the affine image* $\tilde{K}=TK$ *of* K *satisfies that for every* $v\in S^{n-1}$,

$$|\tilde{K}|^{\frac{n-p}{pn}}\le h_{\Pi_p\tilde{K}}(v). \tag{1.9}$$

The notion of L_p *zonoids* introduced by Schneider and Weil [33] is an important ingredient in the L_p Brunn-Minkowski theory. Suppose $p\ge 1$ and ν is an even Borel measure on S^{n-1} such that its support, $\operatorname{supp}\nu$, is not contained in a subsphere of S^{n-1}. The L_p zonoid $Z_p:=Z_p(\nu)$ is the origin-symmetric convex body defined by

$$h_{Z_p}(v)=\Big(\int_{S^{n-1}}|v\cdot u|^p d\nu(u)\Big)^{\frac{1}{p}},\qquad v\in S^{n-1}. \tag{1.10}$$

Furthermore, the Z_1 body is the classical zonoid, which is the limit of Minkowski sums of line segments.

For $K\in\mathcal{K}_o^n$, let $\omega_p(K,u)=h_K^p(u)+h_K^p(-u)$ denote the p-width of K in the direction of $u\in S^{n-1}$. Then the *p-mean width* of K defined in [37] is

$$\omega_p(K)=\int_{S^{n-1}}\omega_p(K,u)d\sigma(u)=2\int_{S^{n-1}}h_K^p(u)d\sigma(u), \tag{1.11}$$

where $d\sigma$ is the rotationally invariant probability measure on S^{n-1}. We say that K has *minimal p-mean width* if $\omega_p(K)\le\omega_p(AK)$ for every $A\in\mathrm{SL}(n)$.

Another purpose of this paper is to obtain the following sharp upper bound of the minimal p-mean width of Z_p.

Theorem 1.3. *Suppose* $p\ge 1$. *If the* L_p *zonoid* Z_p *has minimal p-mean width and* $|Z_p|=|B_{p^*}^n|$, *then*

$$\omega_p(Z_p)\le\omega_p(B_{p^*}^n), \tag{1.12}$$

with equality if ν *is a cross measure on* S^{n-1}. *Moreover, if* p *is not an even integer, the equality holds only if* ν *is a cross measure on* S^{n-1}.

The case $p = 1$ without the "only if " part was due to Giannopoulos, Milman, and Rudelson [15].

The rest of this paper is organized as follows: In Section 2 the background materials are provided. In Section 3, to prove Theorem 1.1, a crucial inequality (Lemma 3.2) is established. The proofs of Theorem 1.1 and Theorem 1.2 are given in Section 4. Section 5 is dedicated to proving Theorem 1.3.

2. Background materials

For quick later reference we recall some background materials from the L_p Brunn-Minkowski theory of convex bodies. Good general references are Gardner [13] and Schneider [32].

Let $K \in \mathcal{K}_o^n$. For $A \in \mathrm{GL}(n)$, write $AK = \{Ax : x \in K\}$ for the image of K under A. If $\lambda > 0$, then $\lambda K = \{\lambda x : x \in K\}$ is the dilation of K by a factor of λ. The polar body K^* of K is defined by

$$K^* = \{x \in \mathbb{R}^n : x \cdot y \leq 1 \text{ for all } y \in K\}.$$

It follows from the definition of the polar K^* of K that for $A \in \mathrm{GL}(n)$,

$$(AK)^* = A^{-t}K^*, \tag{2.1}$$

where A^{-t} is the inverse and transpose of A.

The Minkowski functional $\|\cdot\|_K$ of $K \in \mathcal{K}_o^n$ is defined by

$$\|x\|_K = \min\{t > 0 : x \in tK\}, \tag{2.2}$$

for $x \in \mathbb{R}^n$. It is easy to verify that

$$\|\cdot\|_K = h_{K^*}(\cdot). \tag{2.3}$$

For $p \geq 1$, $K, L \in \mathcal{K}_o^n$, and $\varepsilon > 0$, the L_p Minkowski-Firey combination $K +_p \varepsilon \cdot L$ is the convex body whose support function is given by

$$h(K +_p \varepsilon \cdot L, \cdot)^p = h(K, \cdot)^p + \varepsilon h(L, \cdot)^p.$$

The L_p mixed volume $V_p(K, L)$ of $K, L \in \mathcal{K}_o^n$, was defined in [22] by

$$V_p(K, L) = \frac{p}{n} \lim_{\varepsilon \to 0^+} \frac{|K +_p \varepsilon \cdot L| - |K|}{\varepsilon}. \tag{2.4}$$

In particular, $V_p(K, K) = |K|$. The L_p Minkowski inequality [22] states that for $K, L \in \mathcal{K}_o^n$,

$$V_p(K, L)^n \geq |K|^{n-p}|L|^p, \tag{2.5}$$

with equality if and only if K and L are dilates when $p > 1$ and if and only if K and L are homothetic (i.e. they coincide up to translations and dilatations) when $p = 1$.

It was shown in [22] that there is a positive Borel measure, $S_p(K, \cdot)$, on S^{n-1} so that

$$V_p(K, L) = \frac{1}{n} \int_{S^{n-1}} h_L^p(u) dS_p(K, u) \tag{2.6}$$

for $K, L \in \mathcal{K}_o^n$, where $dS_p(K, \cdot) = h_K^{1-p}(\cdot) dS_K(\cdot)$ is the L_p surface area measure of K and dS_K is the classical surface area measure of K. It is easy to verify that

$$dS_p(cK, \cdot) = c^{n-p} dS_p(K, \cdot), \quad c > 0. \tag{2.7}$$

When $L = B_2^n$, the L_p surface area $S_p(K)$ of K is given by

$$S_p(K) = nV_p(K, B_2^n) = \int_{S^{n-1}} dS_p(K, u).$$

The case $p = 1$ is the classical surface area $S(K)$ on S^{n-1} of K.

Let $\|\cdot\|$ denote the standard Euclidean norm in $\mathbb{R}^n$. It is evident that (1.4) is equivalent to

$$\|x\|^2 = \int_{S^{n-1}} |x \cdot u|^2 d\nu(u), \tag{2.8}$$

for all $x \in \mathbb{R}^n$. Taking the trace in (1.4) gives

$$\nu(S^{n-1}) = n. \tag{2.9}$$

The two most important examples of even isotropic measures on S^{n-1} are (suitably normalized) spherical Lebesgue measure and the cross measure, i.e., measures concentrated uniformly on $\{\pm u_1, \dots, \pm u_n\}$, where $u_1, \dots, u_n$ is an orthonormal basis of $\mathbb{R}^n$.

Using the polar coordinate formula for volume, it is easy to see that for each $p \in (0, \infty)$, the volume of a convex body $K \in \mathbb{R}^n$ is given by

$$|K| = \frac{1}{\Gamma(1 + \frac{n}{p})} \int_{\mathbb{R}^n} e^{-\|x\|_K^p} dx, \tag{2.10}$$

where integration is with respect to Lebesgue measure on $\mathbb{R}^n$. Let B_p^n denote the unit ball of ℓ_p^n-space, understood as

$$B_p^n = \Big\{ x \in \mathbb{R}^n : \Big(\sum_{i=1}^n |x \cdot e_i|^p \Big)^{\frac{1}{p}} \le 1 \Big\}, \quad 1 \le p < \infty,$$

and

$$B_\infty^n = \{ x \in \mathbb{R}^n : |x \cdot e_i| \le 1, \text{ for all } i = 1, \dots, n \}, \quad p = \infty.$$

From (2.10), we get

$$|B_p^n| = \frac{(2\Gamma(1+\frac{1}{p}))^n}{\Gamma(1+\frac{n}{p})} \quad \text{and} \quad |B_\infty^n| = 2^n. \tag{2.11}$$

As mentioned in Theorem 1.1, the notion of the *generalized ℓ_p^n-ball* $B_{p,\alpha}^n := B_{p,\alpha}^n(\nu)$ formed by ν is defined as

$$B_{p,\alpha}^n = \Big\{x \in \mathbb{R}^n : \Big(\sum_{i=1}^n |x \cdot u_i|^p \alpha(u_i)\Big)^{\frac{1}{p}} \le 1\Big\}, \quad 1 \le p < \infty, \tag{2.12}$$

and

$$B_{\infty,\alpha}^n = \Big\{x \in \mathbb{R}^n : |x \cdot u_i|\alpha(u_i) \le 1 \text{ for all } i = 1, \dots, n\Big\}, \quad p = \infty, \tag{2.13}$$

where $(\alpha(u_i))_{i=1}^n > 0$ and ν is a cross measure on S^{n-1} such that

$$\operatorname{supp} \nu = \{\pm u_1, \dots, \pm u_n\} = O\{\pm e_1, \dots, \pm e_n\},$$

for some $O \in \mathrm{O}(n)$. We shall mention that $B_{\infty,\alpha}^n$ can be called as the box formed by ν. Thus,

$$\begin{aligned}
B_{p,\alpha}^n &= \Big\{x \in \mathbb{R}^n : \Big(\sum_{i=1}^n |x \cdot Oe_i|^p \alpha(u_i)\Big)^{\frac{1}{p}} \le 1\Big\} \\
&= \Big\{x \in \mathbb{R}^n : \Big(\sum_{i=1}^n |AO^t x \cdot e_i|^p\Big)^{\frac{1}{p}} \le 1\Big\} \\
&= \Big\{O^{-t}A^{-1}x \in \mathbb{R}^n : \Big(\sum_{i=1}^n |x \cdot e_i|^p\Big)^{\frac{1}{p}} \le 1\Big\} \\
&= OA^{-1}B_p^n, \qquad (2.14)
\end{aligned}$$

where $A = \operatorname{diag}\{\alpha(u_1)^{1/p}, \dots, \alpha(u_n)^{1/p}\}$ is a diagonal matrix. Then we immediately get

$$|B_{p,\alpha}^n| = |OA^{-1}B_p^n| = |B_p^n|\Big(\prod_{i=1}^n \alpha(u_i)\Big)^{-\frac{1}{p}}. \tag{2.15}$$

It follows from (2.3) and (2.12) that for $p \ge 1$

$$h_{(B_{p,\alpha}^n)^*}(x) = \Big(\sum_{i=1}^n |x \cdot u_i|^p \alpha(u_i)\Big)^{\frac{1}{p}}. \tag{2.16}$$

Moreover, by (2.14) and (2.1), for $p > 1$, we have

$$(B_{p,\alpha}^n)^* = (OA^{-1}B_p^n)^* = OA^t B_{p^*}^n$$

$$= \Big\{x \in \mathbb{R}^n : \Big(\sum_{i=1}^n |A^{-t}O^{-1}x \cdot e_i|^{p^*}\Big)^{\frac{1}{p^*}} \le 1\Big\}$$
$$= \Big\{x \in \mathbb{R}^n : \Big(\sum_{i=1}^n |x \cdot u_i|^{p^*} \alpha(u_i)^{-p^*/p}\Big)^{\frac{1}{p^*}} \le 1\Big\}$$
$$= B^n_{p^*,\alpha^{-p^*/p}}. \tag{2.17}$$

For $p = 1$, by the same way, we have

$$(B^n_{1,\alpha})^* = B^n_{\infty,1/\alpha}. \tag{2.18}$$

Then from (2.15) we have

$$|(B^n_{p,\alpha})^*| = |B^n_{p^*,\alpha^{-p^*/p}}| = |B^n_{p^*}|\Big(\prod_{i=1}^n \alpha(u_i)\Big)^{\frac{1}{p}}. \tag{2.19}$$

For each $p \ge 1$ and each even Borel measure ν on S^{n-1}, let $\mathrm{C}_p\nu$ denote the spherical L_p cosine transform of ν, which is a continuous function on S^{n-1} defined by

$$(\mathrm{C}_p\nu)(u) = \Big(\int_{S^{n-1}} |u \cdot v|^p d\nu(v)\Big)^{\frac{1}{p}},$$

for each $u \in S^{n-1}$. A basic fact is that for p not an even integer the L_p cosine transform (see e.g., Alexandrov [1], Lonke [20] and Neyman [30]) is injective; i.e., if $p \ge 1$ is not an even integer and the measures ν and $\bar{\nu}$ are even Borel measure on S^{n-1} such that $\mathrm{C}_p(\nu) = \mathrm{C}_p(\bar{\nu})$, then $\nu = \bar{\nu}$.

The following continuous version of the Ball-Barthe inequality was given by Lutwak, Yang, and Zhang [26], extending the discrete case due to Ball and Barthe [5, Proposition 9].

Lemma 2.1. *If $f : S^{n-1} \to (0,\infty)$ is continuous and ν is an isotropic measure on S^{n-1}, then*

$$\det \int_{S^{n-1}} f(u) u \otimes u d\nu(u) \ge \exp\Big\{\int_{S^{n-1}} \log f(u) d\nu(u)\Big\}, \tag{2.20}$$

with equality if and only if $f(u_1)\cdots f(u_n)$ is constant for linearly independent unit vectors $u_1, \dots, u_n \in \operatorname{supp}\nu$.

3. A volume inequality

Suppose ν is an even isotropic measure on S^{n-1} and $\alpha : S^{n-1} \to (0, +\infty)$ is an even positive continuous function. In this paper, for $p \geq 1$, we define the *"general" L_p zonoid* $Z_{p,\alpha} := Z_{p,\alpha}(\nu)$ to be the origin-symmetric convex body whose support function is given by

$$h_{Z_{p,\alpha}}(v) = \Big(\int_{S^{n-1}} |v \cdot u|^p \alpha(u) d\nu(u) \Big)^{\frac{1}{p}}, \quad v \in S^{n-1}. \tag{3.1}$$

Without the isotropic assumption of ν, the definition (3.1) coincides with the definition of the L_p zonoid (1.10) introduced by Schneider and Weil [33].

In particular, if ν is a cross measure such that $\operatorname{supp} \nu = \{\pm u_1, \dots, \pm u_n\}$. By (3.1), (2.16), (2.17) and (2.18), we have

$$h_{Z_{p,\alpha}}(x) = \Big(\sum_{i=1}^{n} |x \cdot u_i|^p \alpha(u_i) \Big)^{\frac{1}{p}} = h_{(B^n_{p,\alpha})^*}(x) = h_{B^n_{p^*, \alpha^{-p^*/p}}}(x), \tag{3.2}$$

for each $x \in \mathbb{R}^n$. From (2.19), we obtain

$$|Z_{p,\alpha}| = |B^n_{p^*, \alpha^{-p^*/p}}| = |B^n_{p^*}| \Big(\prod_{i=1}^{n} \alpha(u_i) \Big)^{\frac{1}{p}}. \tag{3.3}$$

The following lemma was proved by Lutwak, Yang, and Zhang [26, Lemma 3.1].

Lemma 3.1. *Suppose $p \geq 1$ and α is an even continuous positive function on S^{n-1}. Let ν be an even Borel measure on S^{n-1}. If $t \in L_{p^*}(\nu)$, then*

$$\Big\| \int_{S^{n-1}} u t(u) \alpha(u) d\nu(u) \Big\|_{Z_{p,\alpha}} \leq \Big(\int_{S^{n-1}} |t(u)|^{p^*} \alpha(u) d\nu(u) \Big)^{\frac{1}{p^*}}. \tag{3.4}$$

The proof of Theorem 1.1 relies on the following sharp volume estimates of the "general" L_p zonoids $Z_{p,\alpha}$. When $\alpha(\cdot) \equiv 1$, Lemma 3.2 is the well-known L_p volume ratio inequality due to Ball [2], Barthe [4,5], and Lutwak, Yang, and Zhang [26]. The proof of Lemma 3.2 is based on a refinement of the approach by Lutwak, Yang, and Zhang [26], which uses the Ball-Barthe inequality (2.20) and the technique of mass transportation. For more applications about this approach, see e.g., [16,18,28,29,34].

Lemma 3.2. *Suppose $p \geq 1$ and α is an even continuous positive function on S^{n-1}. If ν is an even isotropic measure on S^{n-1}, then*

$$\frac{|Z_{p,\alpha}|}{|B^n_{p^*}|} \geq \Big(\exp \int_{S^{n-1}} \log \alpha(u) d\nu(u) \Big)^{\frac{1}{p}}. \tag{3.5}$$

For $p \neq 2$, there is equality if and only if ν is a cross measure on S^{n-1}.

Proof. **Case $p>1$:** Define the strictly increasing function $\psi:\mathbb{R}\to\mathbb{R}$ by

$$\frac{1}{\Gamma(\frac{3}{2})}\int_{-\infty}^{t} e^{-s^2}ds=\frac{1}{\Gamma(1+\frac{1}{p^*})}\int_{-\infty}^{\psi(t)} e^{-|s|^{p^*}}ds.$$

Differentiating both sides with respect to t gives

$$e^{-t^2}=\frac{\Gamma(\frac{3}{2})}{\Gamma(1+\frac{1}{p^*})}e^{-|\psi(t)|^{p^*}}\psi'(t).$$

Taking the log of both sides, we get

$$-t^2=\log\Gamma\Big(\frac{3}{2}\Big)-\log\Gamma(1+\frac{1}{p^*})-|\psi(t)|^{p^*}+\log\psi'(t). \tag{3.6}$$

Define $T:\mathbb{R}^n\to\mathbb{R}^n$ by

$$T(x)=\int_{S^{n-1}} u\psi(x\cdot u)\alpha(u)^{\frac{1}{p}}d\nu(u),$$

for each $x\in\mathbb{R}^n$. The differential of T is given by

$$dT(x)=\int_{S^{n-1}} u\otimes u\psi'(x\cdot u)\alpha(u)^{\frac{1}{p}}d\nu(u). \tag{3.7}$$

Since $\psi'>0$ and $\alpha>0$, the matrix $dT(x)$ is positive definite for each $x\in\mathbb{R}^n$. Hence, the transformation T: $\mathbb{R}^n\to\mathbb{R}^n$ is injective. Moreover, by Lemma 3.1 with $t(u)=\psi(x\cdot u)\alpha(u)^{-\frac{1}{p^*}}$, we obtain

$$\begin{aligned}\|T(x)\|_{Z_{p,\alpha}}^{p^*}&\le\int_{S^{n-1}}|\psi(x\cdot u)\alpha(u)^{-\frac{1}{p^*}}|^{p^*}\alpha(u)d\nu(u)\\&=\int_{S^{n-1}}|\psi(x\cdot u)|^{p^*}d\nu(u).\end{aligned} \tag{3.8}$$

From (2.10), (2.8), (3.6) with $t=x\cdot u$, (2.9), the Ball-Barthe inequality (2.20) with $f(u)=\psi'(x\cdot u)\alpha(u)^{\frac{1}{p}}$, (3.7), (3.8), making the change of variable $y=T(x)$, and again (2.10), we have

$$\begin{aligned}\Gamma\Big(\frac{1}{2}\Big)^n&=\int_{\mathbb{R}^n}e^{-\|x\|^2}dx\\&=\int_{\mathbb{R}^n}\exp\Big\{-\int_{S^{n-1}}|x\cdot u|^2d\nu(u)\Big\}dx\\&=\int_{\mathbb{R}^n}\exp\Big\{\int_{S^{n-1}}\Big[\log\Gamma\Big(\frac{3}{2}\Big)-\log\Gamma\Big(1+\frac{1}{p^*}\Big)-|\psi(x\cdot u)|^{p^*}+\log\psi'(x\cdot u)\Big]d\nu(u)\Big\}dx\\&=\Big(\frac{\Gamma(\frac{3}{2})}{\Gamma(1+\frac{1}{p^*})}\Big)^n\int_{\mathbb{R}^n}\exp\Big\{\int_{S^{n-1}}-|\psi(x\cdot u)|^{p^*}d\nu(u)\Big\}\\&\quad\times\exp\Big\{\int_{S^{n-1}}-\frac{1}{p}\log\alpha(u)d\nu(u)\Big\}\exp\Big\{\int_{S^{n-1}}\log\big(\psi'(x\cdot u)\alpha(u)^{\frac{1}{p}}\big)d\nu(u)\Big\}dx\end{aligned}$$

$$
\begin{aligned}
&\le \Big(\frac{\Gamma\big(\frac{3}{2}\big)}{\Gamma(1+\frac{1}{p^*})}\Big)^n \exp\Big\{\int_{S^{n-1}} -\frac{1}{p}\log\alpha(u)d\nu(u)\Big\} \\
&\qquad\qquad \times \int_{\mathbb{R}^n} \exp\Big\{\int_{S^{n-1}} -|\psi(x\cdot u)|^{p^*} d\nu(u)\Big\}|dT(x)|dx \\
&\le \Big(\frac{\Gamma\big(\frac{3}{2}\big)}{\Gamma(1+\frac{1}{p^*})}\Big)^n \exp\Big\{\int_{S^{n-1}} -\frac{1}{p}\log\alpha(u)d\nu(u)\Big\}\int_{\mathbb{R}^n} e^{-\|Tx\|^{p^*}_{Z_{p,\alpha}}}|dT(x)|dx \\
&\le \Big(\frac{\Gamma\big(\frac{3}{2}\big)}{\Gamma(1+\frac{1}{p^*})}\Big)^n \Big(\exp\int_{S^{n-1}}\log\alpha(u)d\nu(u)\Big)^{-\frac{1}{p}}\int_{\mathbb{R}^n} e^{-\|y\|^{p^*}_{Z_{p,\alpha}}}dy \\
&= \Big(\frac{\Gamma\big(\frac{3}{2}\big)}{\Gamma(1+\frac{1}{p^*})}\Big)^n \Big(\exp\int_{S^{n-1}}\log\alpha(u)d\nu(u)\Big)^{-\frac{1}{p}}|Z_{p,\alpha}|\Gamma\Big(1+\frac{n}{p^*}\Big).
\end{aligned}
$$

Thus, from (2.11) we have

$$
\frac{|Z_{p,\alpha}|}{|B^n_{p^*}|} = \frac{\Gamma(1+\frac{n}{p^*})|Z_{p,\alpha}|}{(2\Gamma(1+\frac{1}{p^*}))^n} \ge \Big(\exp\int_{S^{n-1}}\log\alpha(u)d\nu(u)\Big)^{\frac{1}{p}}. \tag{3.9}
$$

Assume that equality holds in (3.9). Since ν is isotropic on S^{n-1}, the measure ν is not concentrated on any subsphere of S^{n-1}. Then there exist linearly independent $u_1,\dots,u_n \in \operatorname{supp}\nu$. Since μ is even, we have

$$
\{\pm u_1,\dots,\pm u_n\} \subseteq \operatorname{supp}\nu.
$$

Assume that there exists a vector $v \in \operatorname{supp}\nu$ such that

$$
v \notin \{\pm u_1,\dots,\pm u_n\}.
$$

Let $v = \lambda_1 u_1 + \cdots + \lambda_n u_n$ such that at least one coefficient, say λ_1, is not zero. Then the equality condition of the Ball-Barthe inequality implies that

$$
\begin{aligned}
&\psi'(x\cdot u_1)\alpha(u_1)^{\frac{1}{p}}\psi'(x\cdot u_2)\alpha(u_2)^{\frac{1}{p}}\cdots\psi'(x\cdot u_n)\alpha(u_n)^{\frac{1}{p}} \\
&= \psi'(x\cdot v)\alpha(v)^{\frac{1}{p}}\psi'(x\cdot u_2)\alpha(u_2)^{\frac{1}{p}}\cdots\psi'(x\cdot u_n)\alpha(u_n)^{\frac{1}{p}},
\end{aligned} \tag{3.10}
$$

for all $x \in \mathbb{R}^n$. However, $\psi' > 0$ and $\alpha > 0$ yield that

$$
\psi'(x\cdot u_1)\alpha(u_1)^{\frac{1}{p}} = \psi'(x\cdot v)\alpha(v)^{\frac{1}{p}}
$$

for all $x \in \mathbb{R}^n$.

If $p \neq 2$, then the function ψ' is not constant. Differentiating both sides with respect to x gives that

$$
\psi''(x\cdot u_1)\alpha(u_1)^{\frac{1}{p}}u_1 = \psi''(x\cdot v)\alpha(v)^{\frac{1}{p}}v,
$$

for all $x \in \mathbb{R}^n$. Since $\alpha > 0$ and there exists $x \in \mathbb{R}^n$ such that $\psi''(x \cdot u_1) \neq 0$, this is the desired contradiction. So we must have $v = \pm u_1$, and hence

$$\{\pm u_1, \ldots, \pm u_n\} = \operatorname{supp} \nu.$$

Therefore, we have for $x \in \mathbb{R}^n$,

$$|x|^2 = \sum_{i=1}^{n} \nu(\{\pm u_i\}) |x \cdot u_i|^2. \tag{3.11}$$

Substituting $x = u_j$, we see that $\nu(\{\pm u_j\}) \leq 1$. From the fact that $\sum_{i=1}^{n} \nu(\{\pm u_i\}) = n$, we get $\nu(\{\pm u_j\}) = 1$. By (3.11), we can see that $u_j \cdot u_i = 0$ for $j \neq i$, and hence ν is a cross measure on S^{n-1}.

Conversely, if ν is a cross measure on S^{n-1} such that $\operatorname{supp} \nu = \{\pm u_1, \ldots, \pm u_n\}$, the equality of (3.5) immediately follows from (3.3).

Case $p = 1$: Define the strictly increasing function $\phi : \mathbb{R} \to (-1, 1)$ by

$$\frac{1}{\Gamma(\frac{3}{2})} \int_{-\infty}^{t} e^{-s^2} ds = \int_{-\infty}^{\phi(t)} \mathbf{1}_{[-1,1]}(s) ds.$$

Note that $|\phi(t)| < 1$. Differentiating both sides with respect to t and taking the log give

$$-t^2 = \log \Gamma\left(\frac{3}{2}\right) + \log \phi'(t). \tag{3.12}$$

Define $T : \mathbb{R}^n \to \mathbb{R}^n$ by

$$T(x) = \int_{S^{n-1}} u \phi(x \cdot u) \alpha(u) d\nu(u),$$

for each $x \in \mathbb{R}^n$. In fact, $T : \mathbb{R}^n \to Z_{1,\alpha}$; i.e.,

$$T(\mathbb{R}^n) \subseteq Z_{1,\alpha}. \tag{3.13}$$

To see this, by Lemma 3.1 with $t(u) = \phi(x \cdot u)$ and the fact that $|\phi| < 1$, we obtain

$$\|T(x)\|_{Z_{1,\alpha}} \leq \max_{u \in S^{n-1}} |\phi(x \cdot u)| < 1.$$

The definition of the Minkowski functional (2.2) shows that $Tx \in Z_{1,\alpha}$ for all $x \in \mathbb{R}^n$. The differential of T gives that

$$dT(x) = \int_{S^{n-1}} u \otimes u \phi'(y \cdot u) \alpha(u) d\nu(u). \tag{3.14}$$

Since $\phi' > 0$ and $\alpha > 0$, the matrix $dT(y)$ is positive definite for each $x \in H$. Hence, the transformation T: $\mathbb{R}^n \to \mathbb{R}^n$ is injective.

From (2.10), (2.8), (3.12) with $t = x \cdot u$, (2.9), (3.14), the Ball-Barthe inequality (2.20) with $f(u) = \phi'(x \cdot u)\alpha(u)$, (3.14), making the change of variable $y = T(x)$ and (3.13), we have

$$\begin{aligned}
&\Gamma\Big(\frac{1}{2}\Big)^n = \int_{\mathbb{R}^n} e^{-\|x\|^2} dx \\
&= \int_{\mathbb{R}^n} \exp\Big\{-\int_{S^{n-1}} |x \cdot u|^2 d\nu(u)\Big\} dy \\
&= \int_{\mathbb{R}^n} \exp\Big\{\int_{S^{n-1}} \Big(\log\Gamma\Big(\frac{3}{2}\Big) + \log\phi'(x \cdot u)\Big) d\nu(u)\Big\} dx \\
&= \Gamma\Big(\frac{3}{2}\Big)^n \exp\Big\{-\int_{S^{n-1}} \log\alpha(u) d\nu(u)\Big\} \int_{\mathbb{R}^n} \exp\Big\{\int_{S^{n-1}} \log\big(\phi'(x \cdot u)\alpha(u)\big) d\nu(u)\Big\} dx \\
&\le \Gamma\Big(\frac{3}{2}\Big)^n \Big(\exp\int_{S^{n-1}} \log\alpha(u) d\nu(u)\Big)^{-1} \int_{\mathbb{R}^n} |dT(x)| dx \\
&\le \Gamma\Big(\frac{3}{2}\Big)^n \Big(\exp\int_{S^{n-1}} \log\alpha(u) d\nu(u)\Big)^{-1} \int_{Z_{1,\alpha}} dy \\
&= \Gamma\Big(\frac{3}{2}\Big)^n \Big(\exp\int_{S^{n-1}} \log\alpha(u) d\nu(u)\Big)^{-1} |Z_{1,\alpha}|.
\end{aligned}$$

Therefore, we obtain

$$|Z_{1,\alpha}| \ge 2^n \Big(\exp\int_{S^{n-1}} \log\alpha(u) d\nu(u)\Big).$$

The equality conditions are basically the same as the case of $p > 1$. □

4. The L_p Loomis-Whitney inequality

Recall that the L_p projection body $\Pi_p K$ of $K \in \mathcal{K}_o^n$, for $p \ge 1$, is the origin-symmetric convex body defined by

$$h_{\Pi_p K}(u) = \Big(\frac{1}{|B_{p^*}^n|^{\frac{p}{n}}} \int_{S^{n-1}} |u \cdot v|^p dS_p(K, v)\Big)^{\frac{1}{p}}, \qquad u \in S^{n-1}. \tag{4.1}$$

The following intertwining properties of Π_p and Π_p^* with linear transformations were established by Lutwak, Yang, and Zhang [24] for $p > 1$, and by Petty [31] for $p = 1$.

Lemma 4.1. *Suppose $p \ge 1$ and $K \in \mathcal{K}_o^n$. Then for $A \in \mathrm{GL}(n)$,*

$$\Pi_p AK = |\det A|^{1/p} A^{-t} \Pi_p K \quad \textit{and} \quad \Pi_p^* AK = |\det A|^{-1/p} A \Pi_p^* K. \tag{4.2}$$

In particular,

$$\Pi_p(cK) = c^{\frac{n-p}{p}} \Pi_p K, \quad c > 0. \tag{4.3}$$

Inspired by the method of Ball [3], we establish Theorem 1.1; i.e., the L_p Loomis-Whitney inequality.

Theorem 4.2. *Suppose $p \geq 1$ and $K \in \mathcal{K}_o^n$. If ν is an even isotropic measure on S^{n-1}, then*

$$|K|^{\frac{n-p}{p}} \leq \exp\Big\{ \int_{S^{n-1}} \log h_{\Pi_p K}(u) d\nu(u) \Big\}. \tag{4.4}$$

For $1 < p \neq 2$, equality in (4.4) holds if and only if ν is a cross measure on S^{n-1} and K is the generalized $\ell_{p^*}^n$-ball formed by ν. For $p = 1$, equality in (4.4) holds if and only if ν is a cross measure on S^{n-1} and K is a box formed by ν (up to translations).

Proof. Let

$$\alpha(u) = h_{\Pi_p K}^{-p}(u) \tag{4.5}$$

for $u \in S^{n-1}$. From (2.5), (2.6), the definitions of $Z_{p,\alpha}$ (3.1) and $\Pi_p K$ (4.1), Fubini's theorem and (2.9), we have

$$\begin{aligned}
|K|^{n-p} &\leq |Z_{p,\alpha}|^{-p} V_p(K, Z_{p,\alpha})^n = |Z_{p,\alpha}|^{-p} \Big(\frac{1}{n} \int_{S^{n-1}} h_{Z_{p,\alpha}}^p(v) dS_p(K, v) \Big)^n \\
&= |Z_{p,\alpha}|^{-p} \Big(\frac{1}{n} \int_{S^{n-1}} \int_{S^{n-1}} |u \cdot v|^p \alpha(u) d\nu(u) dS_p(K, v) \Big)^n \\
&= |Z_{p,\alpha}|^{-p} \Big(\frac{1}{n} \int_{S^{n-1}} \int_{S^{n-1}} |u \cdot v|^p dS_p(K, v) \alpha(u) d\nu(u) \Big)^n \\
&= |Z_{p,\alpha}|^{-p} \Big(\frac{1}{n} \int_{S^{n-1}} |B_{p^*}^n|^{\frac{p}{n}} h_{\Pi_p K}^p(u) \alpha(u) d\nu(u) \Big)^n \\
&= |Z_{p,\alpha}|^{-p} |B_{p^*}^n|^p.
\end{aligned}$$

Combining with Lemma 3.2 and (4.5), we have

$$\begin{aligned}
|K|^{n-p} &\leq |Z_{p,\alpha}|^{-p} |B_{p^*}^n|^p \leq \Big[|B_{p^*}^n| \Big(\exp \int_{S^{n-1}} \log \alpha(u) d\nu(u) \Big)^{\frac{1}{p}} \Big]^{-p} |B_{p^*}^n|^p \\
&= \Big(\exp \int_{S^{n-1}} \log \alpha(u) d\nu(u) \Big)^{-1} \\
&= \Big(\exp \int_{S^{n-1}} \log h_{\Pi_p K}(u) d\nu(u) \Big)^p,
\end{aligned} \tag{4.6}$$

which is the desired inequality.

For the equality conditions of (4.6), by the L_p Minkowski inequality (2.5), the equality of the first inequality in (4.6) holds if and only if K and $Z_{p,\alpha}$ are dilates when $p > 1$ (K and $Z_{p,\alpha}$ are homothetic when $p = 1$). Lemma 3.2 implies that equality of the second inequality in (4.6) holds if and only if ν is a cross measure on S^{n-1} when $p \neq 2$, and thus by (3.2), $Z_{p,\alpha}$ is the generalized $\ell^n_{p^*}$-ball $B^n_{p^*,\alpha^{-p^*/p}}$ formed by ν. Hence K is a dilation of the generalized $\ell^n_{p^*}$-ball formed by the cross measure ν, which is still the generalized $\ell^n_{p^*}$-ball formed by ν when $2 \neq p > 1$ (K coincides with the box formed by ν up to translations when $p = 1$).

Conversely, when $1 < p \neq 2$, we will show that equality in (4.6) holds if K is the generalized $\ell^n_{p^*}$-ball formed by ν; i.e., there are positive numbers $(\alpha_i)^n_{i=1}$ such that

$$K = \Big\{ x \in \mathbb{R}^n : \Big(\sum_{i=1}^n |x \cdot u_i|^{p^*} \alpha_i \Big)^{\frac{1}{p^*}} \leq 1 \Big\}, \tag{4.7}$$

where $\operatorname{supp} \nu = \{\pm u_1, \ldots, \pm u_n\}$ and $(u_i)^n_1$ is an orthonormal basis of $\mathbb{R}^n$. From (4.6), it is sufficient to verify that K and $Z_{p,\alpha}$ are dilates. From (2.14), we have

$$K = B^n_{p^*,\alpha_i} = OA^{-1} B^n_{p^*},$$

where O is an orthogonal matrix such that $Oe_i = u_i$ for $i = 1, \ldots, n$ and $A = \operatorname{diag}\{\alpha_1^{1/p^*}, \ldots, \alpha_n^{1/p^*}\}$ is a diagonal matrix. From (4.5) and (4.2), we get

$$\begin{aligned}
\alpha(u_k) &= h^{-p}_{\Pi_p K}(u_k) = h^{-p}_{\Pi_p (OA^{-1} B^n_{p^*})}(u_k) \\
&= h^{-p}_{|\det A|^{-1/p} (OA^{-1})^{-t} \Pi_p (B^n_{p^*})}(u_k) \\
&= |\det A| h^{-p}_{\Pi_p (B^n_{p^*})}(AO^{-1} u_k) \\
&= |\det A| h^{-p}_{\Pi_p (B^n_{p^*})}(A e_k) \\
&= |\det A| h^{-p}_{\Pi_p (B^n_{p^*})}(\alpha_k^{1/p^*} e_k) \\
&= |\det A| h^{-p}_{\Pi_p (B^n_{p^*})}(e_k) \alpha_k^{-p/p^*}
\end{aligned}$$

for every $k = 1, \ldots, n$. Notice that $h^{-p}_{\Pi_p (B^n_{p^*})}(e_k)$ is a constant for all $k = 1, \ldots, n$. Thus, there exists a constant $c > 0$ such that $\alpha(u_k) = c\alpha_k^{-p/p^*}$ for every $k = 1, \ldots, n$. Now, it follows from (3.2) and (4.7) that

$$\begin{aligned}
Z_{p,\alpha} &= B^n_{p^*,\alpha^{-p^*/p}} \\
&= \Big\{ x \in \mathbb{R}^n : \Big(\sum_{i=1}^n |x \cdot u_i|^{p^*} \alpha(u_i)^{-p^*/p} \Big)^{\frac{1}{p^*}} \leq 1 \Big\} \\
&= \Big\{ x \in \mathbb{R}^n : \Big(\sum_{i=1}^n |x \cdot u_i|^{p^*} c^{-p^*/p} \alpha_i \Big)^{\frac{1}{p^*}} \leq 1 \Big\} = c^{\frac{1}{p}} K.
\end{aligned}$$

That is, K and $Z_{p,\alpha}$ are dilates when $1 < p \neq 2$. When $p = 1$, the proof is the same, together with the observation that $\Pi(K + v_0) = \Pi K$ for every $v_0 \in \mathbb{R}^n$.

□

In [35], Vaaler conjectured that for every origin-symmetric convex body K, there is an affine image $\tilde{K}$ of K so that for every 1-codimensional space $v^\perp$ perpendicular to v,

$$|\tilde{K}|^{\frac{n-1}{n}} \leq |\tilde{K} \cap v^\perp|.$$

It is natural to ask whether there is a dual version of Vaaler's conjecture. Ball [3] answered this question and obtained that there is an affine image $\tilde{K}$ of K such that for every $v \in S^{n-1}$,

$$|\tilde{K}|^{\frac{n-1}{n}} \leq |\mathrm{P}_{v^\perp} \tilde{K}|.$$

Now, we extend Ball's result to the L_p setting (Theorem 1.2).

Theorem 4.3. *Suppose* $1 \leq p \neq n$ *and* $K \in \mathcal{K}_o^n$. *Then there exists a nondegenerate affine image* $\tilde{K}$ *of* K *such that for every* $v \in S^{n-1}$,

$$|\tilde{K}|^{\frac{n-p}{pn}} \leq h_{\Pi_p \tilde{K}}(v). \tag{4.8}$$

Proof. By Lemma 4.1, for $1 \leq p \neq n$, there is an affine image $\tilde{K}$ of K so that the ellipsoid of maximal volume contained in $\Pi_p \tilde{K}$ is the Euclidean ball B_2^n. Thus, for every $v \in S^{n-1}$,

$$h_{\Pi_p \tilde{K}}(v) \geq h_{B_2^n}(v) = 1.$$

Since $\Pi_p \tilde{K}$ is origin-symmetric, it follows from John's theorem [17] that the contact points form an even discrete isotropic measure ν such that for all $u \in \operatorname{supp} \nu$ we have

$$h_{\Pi_p \tilde{K}}(u) = 1.$$

Therefore, from Theorem 4.2, we have for every $v \in S^{n-1}$,

$$|\tilde{K}|^{\frac{n-p}{p}} \leq \exp\left\{ \int_{\operatorname{supp} \nu} \log h_{\Pi_p \tilde{K}}(u) d\nu(u) \right\} = 1 \leq (h_{\Pi_p \tilde{K}}(v))^n. \tag{4.9}$$

□

5. Minimal p-mean width

Recall that a convex body K has minimal p-mean width if $\omega_p(K) \leq \omega_p(A_1 K)$ for every $A_1 \in \mathrm{SL}(n)$, where $\omega_p(K)$ is the p-mean width of K defined in (1.11). We say that the body K has minimal L_p surface area if $S_p(K) \leq S_p(A_2 K)$ for every $A_2 \in \mathrm{SL}(n)$. The following two lemmas will be needed.

Lemma 5.1. [27,36] *Suppose $p \geq 1$ and $K \in \mathcal{K}_o^n$. Then K has minimal L_p surface area if and only if the measure $nS_p(K,\cdot)/S_p(K)$ is isotropic on S^{n-1}.*

Lemma 5.2. [37] *Suppose $p \geq 1$ and $K \in \mathcal{K}_o^n$. Then $\Pi_p K$ has minimal p-mean width if and only if K has minimal L_p surface area.*

We also need the following remarkable result due to Lutwak, Yang, and Zhang [25].

The solution of normalized even L_p Minkowski problem. Let $p \geq 1$. If ν is an even Borel measure on S^{n-1} whose support is not contained in a subsphere of S^{n-1}, then there exists an unique origin-symmetric convex body $K \in \mathcal{K}_o^n$ such that $S_p(K,\cdot)/|K| = \nu$.

Applying Lemma 3.2 to the L_p projection body with minimal L_p surface area, we have

Lemma 5.3. *Suppose $p \geq 1$ and K is an origin-symmetric convex body. If K has minimal L_p surface area, then*

$$S_p(K) \leq n|\Pi_p K|^{\frac{p}{n}}. \tag{5.1}$$

For $p \neq 2$, there is equality if and only if K is a centred cube.

Proof. By the homogeneity of the desired inequality, we may assume $S_p(K) = n$. It is sufficient to prove that $|\Pi_p K| \geq 1$. Let $d\nu(\cdot) = dS_p(K,\cdot)$. It follows from Lemma 5.1 that ν is an even isotropic measure on S^{n-1}. Comparing the definition of $Z_{p,\alpha}$ (3.1) with $\alpha = |B_{p^*}^n|^{-\frac{p}{n}}$ to that of $\Pi_p K$ (4.1), we get $Z_{p,\alpha} = \Pi_p K$. By Lemma 3.2 and (2.9), we obtain

$$|\Pi_p K| \geq |B_{p^*}^n| \Big(\frac{1}{|B_{p^*}^n|^{\frac{p}{n}}} \Big)^{\frac{n}{p}} = 1. \tag{5.2}$$

The equality condition of Lemma 3.2 gives that for $p \neq 2$, there is equality in (5.2) if and only if $ndS_p(K,\cdot)/S(K)$ is a cross measure on S^{n-1}. On the other hand, it is easy to verify that $nS_p(OC_0,\cdot)/S_p(OC_0)$ is a cross measure on S^{n-1} for some $O \in \mathrm{O}(n)$, where $C_0 = [-1,1]^n$ is the unit cube in $\mathbb{R}^n$. Moreover, from (2.7), we have

$$\frac{nS_p(OC_0,\cdot)}{S_p(OC_0)} = \frac{S_p(\lambda OC_0,\cdot)}{|\lambda OC_0|} \quad \text{with } \lambda = \Big(\frac{n|C_0|}{S_p(C_0)} \Big)^{-\frac{1}{p}}.$$

Thus, the equality condition follows from the uniqueness of the solution of the normalized even L_p Minkowski problem and the fact that

$$\Big(\frac{n|tK|}{S_p(tK)} \Big)^{-\frac{1}{p}} tK = \Big(\frac{n|K|}{S_p(K)} \Big)^{-\frac{1}{p}} K$$

for all $t > 0$.

□

Recall that the L_p zonoid Z_p is defined by, for $v \in S^{n-1}$,

$$h_{Z_p}(v) = \Big(\int_{S^{n-1}} |v \cdot u|^p d\nu(u) \Big)^{\frac{1}{p}}, \tag{5.3}$$

where ν is an even Borel measure on S^{n-1} such that $\operatorname{supp} \nu$ is not contained in a subsphere of S^{n-1}. We finally establish Theorem 1.3.

Theorem 5.4. *Suppose $p \geq 1$. If the L_p zonoid Z_p has minimal p-mean width and $|Z_p| = |B^n_{p^*}|$, then*

$$\omega_p(Z_p) \leq \omega_p(B^n_{p^*}), \tag{5.4}$$

with equality if ν is a cross measure on S^{n-1}. Moreover, if p is not an even integer, the equality holds only if ν is a cross measure on S^{n-1}.

Proof. From (1.11), the definition of the L_p projection body (4.1) and Fubini's theorem, we have

$$\begin{aligned} \omega_p(\Pi_p K) &= 2 \int_{S^{n-1}} h^p_{\Pi_p K}(u) d\sigma(u) \\ &= \frac{2}{|B^n_{p^*}|^{\frac{p}{n}}} \int_{S^{n-1}} \int_{S^{n-1}} |u \cdot v|^p dS_p(K, v) d\sigma(u) \\ &= \frac{2}{|B^n_{p^*}|^{\frac{p}{n}}} \int_{S^{n-1}} \int_{S^{n-1}} |u \cdot v|^p d\sigma(u) dS_p(K, v) \\ &= \frac{2c_p}{n|B^n_{p^*}|^{\frac{p}{n}}} S_p(K), \end{aligned} \tag{5.5}$$

where

$$c_p = \frac{\Gamma(1+\frac{n}{2})\Gamma(\frac{1+p}{2})}{\Gamma(1+\frac{1}{2})\Gamma(\frac{n+p}{2})}.$$

Assume that $p \geq 1$ and $p \neq 2$. By the solution of the normalized even L_p Minkowski problem, for the measure $d\nu(\cdot)$ in Z_p, there exists a unique origin-symmetric convex body L such that $dS_p(L, \cdot)/|L| = d\nu(\cdot)$. So from (5.3) and (4.1), we have

$$h_{Z_p}(x) = \Big(\int_{S^{n-1}} |x{\cdot}u|^p d\nu(u) \Big)^{\frac{1}{p}} = \Big(\frac{1}{|L|} \int_{S^{n-1}} |x{\cdot}u|^p dS_p(L, u) \Big)^{\frac{1}{p}} = \Big(\frac{|B^n_{p^*}|^{\frac{p}{n}}}{|L|} \Big)^{\frac{1}{p}} h_{\Pi_p L}(x)$$

for $x \in \mathbb{R}^n$. That is,

$$Z_p = \Big(\frac{|B_{p^*}^n|^{\frac{p}{n}}}{|L|}\Big)^{\frac{1}{p}} \Pi_p L. \tag{5.6}$$

It follows from Lemma 5.2 that Z_p has minimal p-mean width if and only if L has minimal L_p surface area. From the assumption $|Z_p| = |B_{p^*}^n|$ and (5.6), we have $|L| = |\Pi_p L|^{p/n}$. Then, (5.6) becomes

$$Z_p = \Big(\frac{|B_{p^*}^n|}{|\Pi_p L|}\Big)^{1/n} \Pi_p L. \tag{5.7}$$

Using the inequality (5.1) for L, (5.5) and (5.7), we have

$$|\Pi_p L| \geq \Big(\frac{S_p(L)}{n}\Big)^{\frac{n}{p}} = \Big(\frac{|B_{p^*}^n|^{\frac{p}{n}} \omega_p(\Pi_p L)}{2c_p}\Big)^{\frac{n}{p}} = \Big(\frac{|\Pi_p L|^{\frac{p}{n}} \omega_p(Z_p)}{2c_p}\Big)^{\frac{n}{p}}. \tag{5.8}$$

That is,

$$\omega_p(Z_p) \leq 2c_p. \tag{5.9}$$

Let $C_0 = [-1,1]^n$. It is easy to verify that

$$\Big(\frac{|B_{p^*}^n|^{\frac{p}{n}}}{|C_0|}\Big)^{1/p} \Pi_p C_0 = \Big(\frac{|B_{p^*}^n|}{|\Pi_p C_0|}\Big)^{1/n} \Pi_p C_0 = B_{p^*}^n. \tag{5.10}$$

Thus, by the equality condition of (5.1), together with (5.10) and (5.9), we immediately get

$$\omega_p(Z_p) \leq \omega_p(B_{p^*}^n). \tag{5.11}$$

Suppose ν is a cross measure in (5.3), then there exists an orthogonal transform O such that $d\nu(\cdot) = dS_p(OC_0, \cdot)/|C_0|$. From (4.1), (4.2) and (5.10), we have

$$\begin{aligned} h_{Z_p}(x) &= \Big(\int_{S^{n-1}} |x\cdot u|^p d\nu(u)\Big)^{\frac{1}{p}} = \Big(\frac{1}{|C_0|}\int_{S^{n-1}} |x\cdot u|^p dS_p(OC_0,u)\Big)^{\frac{1}{p}} \\ &= \Big(\frac{|B_{p^*}^n|^{\frac{p}{n}}}{|C_0|}\Big)^{\frac{1}{p}} h_{\Pi_p(OC_0)}(x) = h_{OB_{p^*}^n}(x), \end{aligned}$$

for $x \in \mathbb{R}^n$. In other words, $Z_p = OB_{p^*}^n$. By (1.11), the equality of (5.11) follows.

Conversely, suppose the equality of (5.11) holds. By Lemma 5.3, the equality of (5.8) implies that L is a centred cube in $\mathbb{R}^n$. Hence we can write $L = aOC_0$ for some $a > 0$ and $O \in \mathrm{O}(n)$. From (5.6), (4.3), and (5.10), we have

$$|B_{p^*}^n| = |Z_p| = \Big|\Big(\frac{|B_{p^*}^n|^{\frac{p}{n}}}{|aOC_0|}\Big)^{1/p} \Pi_p(aOC_0)\Big| = a^{-n}\Big|\Big(\frac{|B_{p^*}^n|^{\frac{p}{n}}}{|C_0|}\Big)^{1/p} \Pi_p C_0\Big| = a^{-n}|B_{p^*}^n|.$$

Thus we have $a = 1$ and

$$Z_p = \Big(\frac{|B_{p^*}^n|^{\frac{p}{n}}}{|C_0|}\Big)^{1/p} \Pi_p(OC_0)$$

for some $O \in O(n)$. That is,

$$h_{Z_p}(x) = \Big(\frac{1}{|C_0|}\int_{S^{n-1}} |x\cdot u|^p dS_p(OC_0,u)\Big)^{\frac{1}{p}}.$$

Observe that $dS_p(OC_0,\cdot)/|C_0|$ is a cross measure on S^{n-1}. Note that the spherical L_p cosine transform is injective if $1 \leq p < \infty$ is not an even integer (see Section 2). Hence the measure ν in Z_p is exactly a cross measure on S^{n-1}. □

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