# THE DUAL LOOMIS-WHITNEY INEQUALITY

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ABSTRACT. We establish a dual version of the Loomis-Whitney inequality for isotropic measures with complete equality conditions, where the sharp lower bound is given in terms of the volumes of hyperplane sections. For the special case of cross measures, we can drop the condition that the underlying body has centroid at the origin, yielding an independent proof of a result of Meyer's [43].

## 1. Introduction

Throughout this paper all Borel measures are understood to be nonnegative and finite, and a convex body $K$ in Euclidean $n$-space $\mathbb{R}^n$ is a compact convex set that contains the origin in its interior. We shall use $|\cdot|$ to denote $k$-dimensional volume (Lebesgue measure on the corresponding subspace) in $\mathbb{R}^k$ for $k = n, n-1$.

The classical Loomis-Whitney inequality [30] states that the volume of a convex body is dominated by the geometric mean of volumes of its coordinate shadows: Let $K$ be a convex body in $\mathbb{R}^n$, then

$$|K|^{n-1} \leq \prod_{i=1}^{n} |\mathrm{P}_{e_i^\perp} K|, \tag{1.1}$$

with equality if and only if $K$ is a coordinate box (a rectangular parallelepiped whose facets are parallel to the coordinate hyperplanes) in $\mathbb{R}^n$, where $\mathrm{P}_{e_i^\perp}$ denotes the orthogonal projection of $K$ onto the 1-codimensional subspace $e_i^\perp$ perpendicular to $e_i$ and $\{e_1, \ldots, e_n\}$ is the standard Euclidean basis of $\mathbb{R}^n$.

An elegant generalization of (1.1) in terms of shadows along a sequence of directions satisfying John's condition [27] was given by Ball [3], who showed that if there exist unit vectors $(u_i)_1^m$ and positive numbers $(c_i)_1^m$ satisfying John's condition

$$\sum_{i=1}^{m} c_i u_i \otimes u_i = I_n, \tag{1.2}$$

then

$$|K|^{n-1} \leq \prod_{i=1}^{m} |\mathrm{P}_{u_i^\perp} K|^{c_i}. \tag{1.3}$$

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2000 *Mathematics Subject Classification.* 52A40.

*Key words and phrases.* Loomis-Whitney inequality, hyperplane sections, generalized $\ell_1^n$-ball.

The first author was supported by NSFC-Henan Joint Fund (No. U1204102). The second author was supported in part by the National Natural Science Foundation of China (No. 11371239).

Here, $u_i \otimes u_i$ is the rank-one orthogonal projection onto the space spanned by $u_i$ and $I_n$ is the identity map on $\mathbb{R}^n$. Clearly, inequality (1.3) reduces to (1.1) if $m = n$, $u_i = e_i$ and $c_i = 1$ for all $i = 1, \dots, n$.

A Borel measure $\nu$ on the unit sphere $S^{n-1}$ is said to be *isotropic* if

$$\int_{S^{n-1}} u \otimes u d\nu(u) = I_n. \tag{1.4}$$

The measure $\nu$ is called *even* if it assumes the same value on antipodal sets. Notice that condition (1.4) reduces to (1.2) if the isotropic measure $\nu$ is of the form $(1/2)\sum_{i=1}^{m}(c_i\delta_{u_i} + c_i\delta_{-u_i})$ on $S^{n-1}$ ($\delta_x$ stands for the Dirac mass at $x$). An isotropic measure of the form $(1/2)\sum_{i=1}^{n}(\delta_{u_i} + \delta_{-u_i})$ with an orthonormal basis $(u_i)_1^n$ of $\mathbb{R}^n$ is called a *cross measure*.

Very recently, the authors [28] established the $L_p$ Loomis-Whitney inequality for isotropic measures with complete equality conditions. In particular, when $p = 1$, the authors showed that if $\nu$ is an even isotropic measure on $S^{n-1}$, then

$$|K|^{n-1} \le \exp\Big\{ \int_{S^{n-1}} \log |\mathrm{P}_{u^\perp} K| d\nu(u) \Big\}, \tag{1.5}$$

with equality if and only if $K$ is a box formed by the cross measure $\nu$, that is, there are positive numbers $(\alpha_i)_{i=1}^n$ and a vector $v_0 \in \mathbb{R}^n$ such that

$$K = \sum_{i=1}^{n} \alpha_i[-u_i, u_i] + v_0,$$

where the support set of $\nu$, $\operatorname{supp}\nu$, is $\{\pm u_1, \dots, \pm u_n\}$ and $(u_i)_1^n$ is an orthonormal basis of $\mathbb{R}^n$. Clearly, inequality (1.5) reduces to (1.3) if $\nu$ is of the form $(1/2)\sum_{i=1}^{m}(c_i\delta_{u_i} + c_i\delta_{-u_i})$ on $S^{n-1}$. When $\nu$ is replaced by the isotropic surface area measure of $K$, inequality (1.5) (without equality conditions) was proved by Giannopoulos and Papadimitrakis [20]. Further extensions and generalizations of the Loomis-Whitney inequality can be found in [5, 6, 8–10, 14].

The aim of this paper is to establish a dual version of the Loomis-Whitney inequality (1.5) with the sharp lower bound in terms of the volumes of hyperplane sections. Other dual versions of the Loomis-Whitney inequality for intrinsic volumes were given in [11].

**Theorem 1.1.** *Let $K$ be a convex body in $\mathbb{R}^n$ whose centroid is at the origin. If $\nu$ is an even isotropic measure on $S^{n-1}$, then*

$$|K|^{n-1} \ge \frac{n!}{n^n} \exp\Big\{ \int_{S^{n-1}} \log |K \cap u^\perp| d\nu(u) \Big\}, \tag{1.6}$$

*with equality if and only if $\nu$ is a cross measure on $S^{n-1}$ and $K$ is a generalized $\ell_1^n$-ball formed by $\nu$; i.e., there are positive numbers $(\alpha_i)_{i=1}^n$ such that*

$$K = \Big\{ x \in \mathbb{R}^n : \sum_{i=1}^{n} |x \cdot u_i| \alpha_i \le 1 \Big\},$$

*where $\operatorname{supp}\nu = \{\pm u_1, \dots, \pm u_n\}$ and $(u_i)_1^n$ is an orthonormal basis of $\mathbb{R}^n$.*

Note that the proof of our theorem highly relies on the work of Fradelizi [15] who gave a sharp estimate of the Banach-Mazur distance between the intersection body $IK$ and the polar $L_p$-centroid body $\Gamma_p^* K$. He showed that, for all $p \ge 1$, their Banach-Mazur distance is less than $\Gamma(p+2)^{\frac{1}{p}}$ and the bound is sharp for all convex bodies $K$ in $\mathbb{R}^n$ whose centroid is at the origin. In this paper we will use parts of his work stated in Lemma 4.2.

In particular, if $\nu$ is a cross measure on $S^{n-1}$, we can drop the condition in Theorem 1.1 that the underlying body has centroid at the origin, and obtain a result of Meyer's [43, p.151]:

**Theorem 1.2.** *Let $K$ be a convex body in $\mathbb{R}^n$. If $\nu$ is a cross measure on $S^{n-1}$ with $\operatorname{supp}\nu = \{\pm u_1, \cdots, \pm u_n\}$, then*

$$|K|^{n-1} \geq \frac{n!}{n^n} \prod_{i=1}^{n} |K \cap u_i^{\perp}|,$$

*with equality if and only if $K$ is a polytope with vertices $a_i u_i$ and $-b_i u_i$, where $a_i, b_i \geq 0$ and $a_i + b_i \neq 0$, $i = 1, \ldots, n$.*

The rest of this paper is organized as follows: In Section 2 the basic notations and preliminaries are provided. To prove Theorem 1.1, a crucial inequality is given in Section 3, which generalizes the $L_p$ volume ratio inequality due to Ball [2], Barthe [4], and Lutwak, Yang, and Zhang [38]. The dual Loomis-Whitney inequality for isotropic measures is proved in Section 4. In the final section, we focus on the dual Loomis-Whitney inequality for two important isotropic measures, namely the spherical Lebesgue measure and the cross measure.

## 2. Notations and Preliminaries

In this section we present the terminology and notation we shall use throughout. As a general reference, the reader may wish to consult the books of Gardner [16] and Schneider [47].

We use $\|\cdot\|$ to denote the standard Euclidean norm on $\mathbb{R}^n$. The volume of the corresponding unit ball is denoted by $\omega_n$. For $A \in \mathrm{GL}(n)$, denote by $A^{-t}$ the inverse of the transpose of $A$. For a convex body $K$ in $\mathbb{R}^n$, its centroid is $\frac{1}{|K|}\int_K x dx$. The polar body $K^*$ of $K$ is defined by

$$K^* = \{x \in \mathbb{R}^n : x \cdot y \leq 1 \text{ for all } y \in K\},$$

where $x \cdot y$ denotes the standard inner product of $x$ and $y$.

A compact set $K \subset \mathbb{R}^n$ is a star-shaped set (with respect to the origin) if the intersection of every straight line through the origin with $K$ is a line segment. Let $K \subset \mathbb{R}^n$ be a compact star shaped set (with respect to the origin), the radial function $\rho_K : \mathbb{R}^n \backslash \{o\} \to \mathbb{R}$ is defined by

$$\rho_K(x) = \max\{\lambda \geq 0 : \lambda x \in K\}.$$

If $\rho_K$ is positive and continuous, then we call $K$ a *star body* (with respect to the origin). Let $\mathcal{S}_o^n$ be the class of star bodies (with respect to the origin) in $\mathbb{R}^n$. Two star bodies $K$ and $L$ are said to be dilates (of each other) if $\rho_K(u)/\rho_L(u)$ is independent of $u \in S^{n-1}$.

The Minkowski functional $\|\cdot\|_K$ of $K \in \mathcal{S}_o^n$ is defined by

$$\|x\|_K = \min\{t > 0 : x \in tK\}. \tag{2.1}$$

It is easy to see that

$$\rho_K^{-1}(\cdot) = \|\cdot\|_K. \tag{2.2}$$

Next, we list some basic facts of the dual Brunn-Minkowski theory. The dual Brunn-Minkowski theory was initiated by Lutwak [32] and further details were provided in [33, 34]. The related developments can be found in [7, 17–19, 21–24, 31, 35–37, 41, 44, 50, 51].

For $p \in \mathbb{R}$, the *dual mixed volume* $\widetilde{V}_p(K, L)$ of $K, L \in \mathcal{S}_o^n$ was defined in [32] by

$$\widetilde{V}_p(K, L) = \frac{1}{n} \int_{S^{n-1}} \rho_K^{n-p}(u) \rho_L^p(u) du, \tag{2.3}$$

where the integration is with respect to spherical Lebesgue measure. In particular, $\widetilde{V}_p(K, K) = |K|$. A basic inequality for the dual mixed volumes $\widetilde{V}_p$ is the dual Minkowski inequality, which states that, for $K, L \in \mathcal{S}_o^n$,

$$\widetilde{V}_p(K, L)^n \le |K|^{n-p} |L|^p, \quad 0 < p < n, \tag{2.4}$$

and

$$\widetilde{V}_p(K, L)^n \ge |K|^{n-p} |L|^p, \quad p < 0 \ \text{ or } \ p > n. \tag{2.5}$$

Equality holds in each of the inequalities if and only if $K$ and $L$ are dilates.

It is easy to check that the isotropic assumption (1.4) is equivalent to

$$\|x\|^2 = \int_{S^{n-1}} |x \cdot u|^2 d\nu(u), \tag{2.6}$$

for all $x \in \mathbb{R}^n$. Note that an isotropic measure cannot be concentrated on a great subsphere of $S^{n-1}$. Taking the trace in (1.4) gives

$$\nu(S^{n-1}) = n. \tag{2.7}$$

The two most important examples of even isotropic measures on $S^{n-1}$ are the spherical Lebesgue measure and the cross measure. The *basic* cross measure on $S^{n-1}$ is an even isotropic discrete measure concentrated on $\pm e_1, \dots, \pm e_n$. A cross measure on $S^{n-1}$ is just a rotation of a basic cross measure; i.e., it is concentrated on $O\{\pm e_1, \dots, \pm e_n\}$, where $O \in \mathrm{O}(n)$. Note that each point in the support of a cross measure on $S^{n-1}$ is equally weighted.

For each $p \in (0, \infty)$, the volume of $K \in \mathcal{S}_o^n$ is given by

$$|K| = \frac{1}{\Gamma(1 + \frac{n}{p})} \int_{\mathbb{R}^n} e^{-\|x\|_K^p} dx, \tag{2.8}$$

where the integral is with respect to Lebesgue measure. Let $B_p^n$ denote the unit ball of the $\ell_p^n$ space, that is,

$$B_p^n = \Big\{ x \in \mathbb{R}^n : \Big( \sum_{i=1}^n |x \cdot e_i|^p \Big)^{\frac{1}{p}} \le 1 \Big\}, \quad 0 < p < \infty,$$

and

$$B_\infty^n = \{ x \in \mathbb{R}^n : |x \cdot e_i| \le 1, \ i = 1, \dots, n \}, \quad p = \infty.$$

Then by (2.8) we have

$$|B_p^n| = \frac{(2\Gamma(1 + \frac{1}{p}))^n}{\Gamma(1 + \frac{n}{p})} \quad \text{and} \quad |B_\infty^n| = 2^n. \tag{2.9}$$

The notion of the *generalized* $\ell_p^n$*-ball*, $B_{p,\alpha}^n = B_{p,\alpha}^n(\nu)$, formed by a cross measure $\nu$ on $S^{n-1}$ is defined by

$$B_{p,\alpha}^n = \Big\{x \in \mathbb{R}^n : \Big(\sum_{i=1}^n |x \cdot u_i|^p \alpha(u_i)\Big)^{\frac{1}{p}} \le 1\Big\}, \quad 0 < p < \infty, \tag{2.10}$$

where $(\alpha(u_i))_{i=1}^n > 0$ and $\operatorname{supp}\nu = \{\pm u_1, \dots, \pm u_n\} = O\{\pm e_1, \dots, \pm e_n\}$ for some $O \in \mathrm{O}(n)$. Thus,

$$\begin{aligned} B_{p,\alpha}^n &= \Big\{x \in \mathbb{R}^n : \Big(\sum_{i=1}^n |x \cdot Oe_i|^p \alpha(u_i)\Big)^{\frac{1}{p}} \le 1\Big\} \\ &= \Big\{x \in \mathbb{R}^n : \Big(\sum_{i=1}^n |AO^t x \cdot e_i|^p\Big)^{\frac{1}{p}} \le 1\Big\} \\ &= \Big\{O^{-t}A^{-1}x \in \mathbb{R}^n : \Big(\sum_{i=1}^n |x \cdot e_i|^p\Big)^{\frac{1}{p}} \le 1\Big\} \\ &= OA^{-1}B_p^n, \end{aligned} \tag{2.11}$$

where $A = \operatorname{diag}\{\alpha(u_1)^{1/p}, \dots, \alpha(u_n)^{1/p}\}$ is a diagonal matrix. Then we have

$$|B_{p,\alpha}^n| = |OA^{-1}B_p^n| = |B_p^n|\Big(\prod_{i=1}^n \alpha(u_i)\Big)^{-\frac{1}{p}}. \tag{2.12}$$

The following continuous version (along with the equality conditions) of Ball-Barthe inequality was given by Lutwak, Yang, and Zhang [38], extending the discrete case due to Ball and Barthe [4, Proposition 9].

**Lemma 2.1.** *If $\nu$ is an isotropic measure on $S^{n-1}$, then for each continuous $f : S^{n-1} \to (0,\infty)$,*

$$\det \int_{S^{n-1}} f(u) u \otimes u \, d\nu(u) \ge \exp\Big\{\int_{S^{n-1}} \log f(u) d\nu(u)\Big\}, \tag{2.13}$$

*with equality if and only if $f(u_1)\cdots f(u_n)$ is constant for linearly independent unit vectors $u_1, \dots, u_n \in \operatorname{supp}\nu$.*

## 3. A volume inequality

Suppose that $\nu$ is an even isotropic measure on $S^{n-1}$ and $\alpha : S^{n-1} \to (0,+\infty)$ is an even continuous positive function. For $0 < p \le \infty$, define the origin-symmetric star body $Z_{p,\alpha}^*$ in $\mathbb{R}^n$ whose Minkowski functional is given, for each $x \in \mathbb{R}^n$, by

$$\|x\|_{Z_{p,\alpha}^*} = \Big(\int_{S^{n-1}} |x \cdot u|^p \alpha(u) d\nu(u)\Big)^{\frac{1}{p}}, \quad 0 < p < \infty, \tag{3.1}$$

and

$$\|x\|_{Z_{\infty,\alpha}^*} = \sup_{u \in \operatorname{supp}\nu} |x \cdot u|, \quad p = \infty. \tag{3.2}$$

When $1 \le p \le \infty$, the convex body $Z_{p,\alpha}^*$, without the isotropic assumption on $\nu$, is the polar of the $L_p$ *zonoid* $Z_{p,\alpha}$ introduced by Schneider and Weil [48].

In particular, if $\nu$ is a cross measure on $S^{n-1}$ with $\operatorname{supp}\nu = \{\pm u_1, \dots, \pm u_n\}$, then by (3.1) and (2.10), we have

$$\|x\|_{Z_{p,\alpha}^*} = \Big(\sum_{i=1}^n |x \cdot u_i|^p \alpha(u_i)\Big)^{\frac{1}{p}} = \|x\|_{B_{p,\alpha}}, \tag{3.3}$$

for each $x \in \mathbb{R}^n$. Thus, it follows from (2.12) that

$$|Z^*_{p,\alpha}| = |B^n_{p,\alpha}| = |B^n_p|\Big(\prod_{i=1}^n \alpha(u_i)\Big)^{-\frac{1}{p}}. \tag{3.4}$$

Moreover, it is easy to see from (3.2) that

$$|Z^*_{\infty,\alpha}| = |B^n_\infty|.$$

The following lemma was proved by Lutwak, Yang, and Zhang [39].

**Lemma 3.1.** *If $\nu$ is an isotropic measure on $S^{n-1}$ and $h \in L_2(\nu)$, then*

$$\Big\| \int_{S^{n-1}} uh(u)d\nu(u)\Big\| \le \Big(\int_{S^{n-1}} |h(u)|^p d\nu(u)\Big)^{\frac{1}{p}}.$$

The following lemma, generalizing the $L_p$ volume ratio inequality due to Ball [2] and Barthe [4], is crucial in the proof of Theorem 1.1. When $\alpha = 1$ and $1 \le p \le \infty$, Lemma 3.2 was proved by Lutwak, Yang, and Zhang [38]. Note that our proof is based on a refinement of the approach by Lutwak, Yang, and Zhang [38], which, in turn, uses some ideas of Ball [2] and Barthe [4], for example, the Ball-Barthe inequality (2.13) and the technique of mass transportation. For more information, see e.g., [26, 29, 39, 40, 49].

**Lemma 3.2.** *Suppose $0 < p \le \infty$ and $\alpha$ is an even continuous positive function on $S^{n-1}$. If $\nu$ is an even isotropic measure on $S^{n-1}$, then*

$$\frac{|Z^*_{p,\alpha}|}{|B^n_p|} \le \Big(\exp\int_{S^{n-1}} \log\alpha(u)d\nu(u)\Big)^{-\frac{1}{p}}, \tag{3.5}$$

*For $p \ne 2$, there is equality if and only if $\nu$ is a cross measure on $S^{n-1}$.*

*Proof.* **Case $0 < p < \infty$:** For each $u \in S^{n-1}$, define the strictly increasing function $\phi_u : \mathbb{R} \to \mathbb{R}$ by

$$\frac{\alpha(u)^{\frac{1}{p}}}{\Gamma(1+\frac{1}{p})}\int_{-\infty}^{t} e^{-\left|\alpha(u)^{\frac{1}{p}}s\right|^p} ds = \frac{1}{\Gamma(\frac{3}{2})}\int_{-\infty}^{\alpha(u)\phi_u(t)} e^{-|s|^2} ds.$$

Differentiating both sides with respect to $t$ gives

$$\frac{\Gamma(\frac{3}{2})\alpha(u)^{\frac{1}{p}}}{\Gamma(1+\frac{1}{p})} e^{-\alpha(u)|t|^p} = e^{-|\alpha(u)\phi_u(t)|^2}\phi'_u(t)\alpha(u).$$

Taking the log of both sides, we get

$$\log\Gamma\Big(\frac{3}{2}\Big) + \frac{1}{p}\log\alpha(u) - \log\Gamma\Big(1+\frac{1}{p}\Big) - \alpha(u)|t|^p = -|\alpha(u)\phi_u(t)|^2 + \log\big(\phi'_u(t)\alpha(u)\big). \tag{3.6}$$

Define $T : \mathbb{R}^n \to \mathbb{R}^n$ by

$$Tx = \int_{S^{n-1}} u\phi_u(x\cdot u)\alpha(u)d\nu(u), \tag{3.7}$$

for each $x \in \mathbb{R}^n$. Differential of $T$ is given by

$$dT(x) = \int_{S^{n-1}} u\otimes u\phi'_u(x\cdot u)\alpha(u)d\nu(u). \tag{3.8}$$

Since $\phi'_u > 0$ and $\alpha > 0$, the matrix $dT(x)$ is positive definite for each $x \in \mathbb{R}^n$. Hence, the transformation $T$: $\mathbb{R}^n \to \mathbb{R}^n$ is injective.

From (2.8), (3.1), (3.6), (2.7), (3.8), the Ball-Barthe inequality (2.13) with $f(u) = \phi'_u(x\cdot u)\alpha(u)$, Lemma 3.1 with $h(u) = \phi_u(x\cdot u)\alpha(u)$, making the change of variable $y = Tx$, and again (2.8), we have

$$\begin{aligned}
&\Gamma\big(1+\frac{n}{p}\big)|Z^*_{p,\alpha}| = \int_{\mathbb{R}^n} e^{-\|x\|^p_{Z^*_{p,\alpha}}}dx\\
&= \int_{\mathbb{R}^n} \exp\Big\{-\int_{S^{n-1}} |x\cdot u|^p\alpha(u)d\nu(u)\Big\}dx\\
&= \int_{\mathbb{R}^n} \exp\Big\{-\int_{S^{n-1}} \Big[|\alpha(u)\phi_u(x\cdot u)|^2 - \log\big(\phi'_u(x\cdot u)\alpha(u)\big) + \log\Gamma\Big(\frac{3}{2}\Big)\\
&\qquad\qquad + \frac{1}{p}\log\alpha(u) - \log\Gamma\Big(1+\frac{1}{p}\Big)\Big]d\nu(u)\Big\}dx\\
&= \Big(\frac{\Gamma(1+\frac{1}{p})}{\Gamma(\frac{3}{2})}\Big)^n \exp\Big\{-\frac{1}{p}\int_{S^{n-1}}\log\alpha(u)d\nu(u)\Big\}\\
&\quad\times\int_{\mathbb{R}^n}\exp\Big\{-\int_{S^{n-1}}|\alpha(u)\phi_u(x\cdot u)|^2d\nu(u)\Big\}\exp\Big\{\int_{S^{n-1}}\log\big(\phi'_u(x\cdot u)\alpha(u)\big)d\nu(u)\Big\}dx\\
&\le \Big(\frac{\Gamma(1+\frac{1}{p})}{\Gamma(\frac{3}{2})}\Big)^n \exp\Big\{-\frac{1}{p}\int_{S^{n-1}}\log\alpha(u)d\nu(u)\Big\}\int_{\mathbb{R}^n}\exp\Big\{-\int_{S^{n-1}}|\alpha(u)\phi_u(x\cdot u)|^2d\nu(u)\Big\}|dT(x)|dx\\
&\le \Big(\frac{\Gamma(1+\frac{1}{p})}{\Gamma(\frac{3}{2})}\Big)^n \exp\Big\{-\frac{1}{p}\int_{S^{n-1}}\log\alpha(u)d\nu(u)\Big\}\int_{\mathbb{R}^n}e^{-\|Tx\|^2}|dT(x)|dx\\
&\le \Big(\frac{\Gamma(1+\frac{1}{p})}{\Gamma(\frac{3}{2})}\Big)^n \Big(\exp\int_{S^{n-1}}\log\alpha(u)d\nu(u)\Big)^{-\frac{1}{p}}\int_{\mathbb{R}^n}e^{-\|y\|^2}dy\\
&= \Big(2\Gamma\big(1+\frac{1}{p}\big)\Big)^n\Big(\exp\int_{S^{n-1}}\log\alpha(u)d\nu(u)\Big)^{-\frac{1}{p}}.
\end{aligned}$$

Thus, from (2.9) we have

$$\frac{|Z^*_{p,\alpha}|}{|B^n_p|} = \frac{\Gamma(1+\frac{n}{p})|Z^*_{p,\alpha}|}{(2\Gamma(1+\frac{1}{p}))^n} \le \Big(\exp\int_{S^{n-1}}\log\alpha(u)d\nu(u)\Big)^{-\frac{1}{p}}. \tag{3.9}$$

Assume that equality holds in (3.9). Since $\nu$ is isotropic on $S^{n-1}$, the measure $\nu$ is not concentrated on any subsphere of $S^{n-1}$. Then there exist linearly independent $u_1,\dots,u_n \in \operatorname{supp}\nu$. Since $\nu$ is even, we have

$$\{\pm u_1,\dots,\pm u_n\} \subseteq \operatorname{supp}\nu.$$

Assume that there exists a vector $v \in \operatorname{supp}\nu$ such that

$$v \notin \{\pm u_1,\dots,\pm u_n\}.$$

Let $v = \lambda_1 u_1 + \cdots + \lambda_n u_n$ such that at least one coefficient, say $\lambda_1$, is not zero. Then the equality conditions of the Ball-Barthe inequality imply that

$$\phi'_{u_1}(x\cdot u_1)\alpha(u_1)\phi'_{u_2}(x\cdot u_2)\alpha(u_2)\cdots\phi'_{u_n}(x\cdot u_n)\alpha(u_n) = \phi'_v(x\cdot v)\alpha(v)\phi'_{u_2}(x\cdot u_2)\alpha(u_2)\cdots\phi'_{u_n}(x\cdot u_n)\alpha(u_n), \tag{3.10}$$

for all $x \in \mathbb{R}^n$. But $\phi'_u > 0$ and $\alpha > 0$, and hence

$$\phi'_{u_1}(x\cdot u_1)\alpha(u) = \phi'_v(x\cdot v)\alpha(v)$$

for all $x \in \mathbb{R}^n$.

If $p \neq 2$, then the function $\phi'_u$ is not constant. Differentiating both sides with respect to $x$ shows that

$$\phi''_{u_1}(x \cdot u_1)\alpha(u_1)u_1 = \phi''_v(x \cdot v)\alpha(v)v,$$

for all $x \in \mathbb{R}^n$. Since $\alpha > 0$ and there exists $x \in \mathbb{R}^n$ such that $\phi''_{u_1}(x \cdot u_1) \neq 0$, this is the desired contradiction. So we have $v = \pm u_1$. Hence

$$\{\pm u_1, \ldots, \pm u_n\} = \operatorname{supp} \nu.$$

Therefore, for $x \in \mathbb{R}^n$,

$$\|x\|^2 = \sum_{i=1}^{n} \nu(\{\pm u_i\})(x \cdot u_i)^2.$$

Substituting $x = u_j$, we see that necessarily $\nu(\{\pm u_j\}) \leq 1$. From the fact that $\sum_{i=1}^{n} \nu(\{\pm u_i\}) = n$ we get $\nu(\{\pm u_j\}) = 1$. Thus, $u_j \cdot u_i = 0$ for $j \neq i$, and we obtain that $\nu$ is a cross measure on $S^{n-1}$.

Conversely, if $\nu$ is a cross measure on $S^{n-1}$ with $\operatorname{supp} \nu = \{\pm u_1, \ldots, \pm u_n\}$, then equality in (3.5) immediately follows from (3.4).

**Case** $p = \infty$**:** For each $u \in S^{n-1}$, define the strictly increasing function $\phi_u : (-1, 1) \to \mathbb{R}$ by

$$\int_{-1}^{t} \mathbf{1}_{[-1,1]}(s) ds = \frac{1}{\Gamma(\frac{3}{2})} \int_{-\infty}^{\phi_u(t)\alpha(u)} e^{-|s|^2} ds$$

Differentiating both sides with respect to $t$ and taking the log yield

$$\log \Gamma\Big(\frac{3}{2}\Big) + \log \mathbf{1}_{[-1,1]}(t) = -|\phi_u(t)\alpha(u)|^2 + \log\big(\phi'_u(t)\alpha(u)\big). \tag{3.11}$$

Now (3.2) gives

$$\operatorname{int} Z^*_{\infty,\alpha} = \{x \in \mathbb{R}^n : \sup_{u \in \operatorname{supp} \nu} |x \cdot u| < 1\}. \tag{3.12}$$

Thus, for each $x \in \operatorname{int} Z^*_{\infty,\alpha}$,

$$\exp\Big\{\int_{S^{n-1}} \log \mathbf{1}_{[-1,1]}(x \cdot u) d\nu(u)\Big\} = 1. \tag{3.13}$$

Define $T : \operatorname{int} Z^*_{\infty,\alpha} \to \mathbb{R}^n$ by

$$Tx = \int_{S^{n-1}} u\phi_u(x \cdot u)\alpha(u) d\nu(u), \tag{3.14}$$

for each $x \in \mathbb{R}^n$. Note that (3.12) shows that $x \cdot u$ is in the domain of $\phi_u$ for all $x \in \operatorname{int} Z^*_{\infty,\alpha}$ and all $u \in \operatorname{supp} \nu$. Differential of $T$ gives that

$$dT(x) = \int_{S^{n-1}} u \otimes u\phi'_u(x \cdot u)\alpha(u) d\nu(u). \tag{3.15}$$

Since $\phi'_u > 0$ and $\alpha > 0$, the matrix $dT(x)$ is positive definite for each $x \in \operatorname{int} Z^*_{\infty,\alpha}$. Hence, the transformation $T$: $\operatorname{int} Z^*_{\infty,\alpha} \to \mathbb{R}^n$ is injective.

From (3.13), (3.11), (2.7), (3.15), the Ball-Barthe inequality (2.13) with $f(u) = \phi'_u(x \cdot u)\alpha(u)$, Lemma 3.1 with $h(u) = \phi_u(x \cdot u)\alpha(u)$, making the change of variable $y = Tx$, and finally (2.8), we have

$$|Z^*_{\infty,\alpha}| = \int_{\operatorname{int} Z^*_{\infty,\alpha}} dx$$

$$
\begin{aligned}
&= \int_{\mathrm{int} Z^*_{\infty,\alpha}} \exp\Big\{ \int_{S^{n-1}} \log \mathbf{1}_{[-1,1]}(x\cdot u) d\nu(u) \Big\} dx \\
&= \int_{\mathrm{int} Z^*_{\infty,\alpha}} \exp\Big\{ \int_{S^{n-1}} \Big( -\log\Gamma\Big(\frac{3}{2}\Big) - |\phi_u(x\cdot u)\alpha(u)|^2 + \log\big(\phi'_u(x\cdot u)\alpha(u)\big)\Big) d\nu(u) \Big\} dx \\
&= \Gamma\Big(\frac{3}{2}\Big)^{-n} \int_{\mathrm{int} Z^*_{\infty,\alpha}} \exp\Big\{ -\int_{S^{n-1}} |\phi_u(x\cdot u)\alpha(u)|^2 d\nu(u)\Big\} \exp\Big\{ \int_{S^{n-1}} \log\big(\phi'_u(x\cdot u)\alpha(u)\big) d\nu(u)\Big\} dx \\
&\le \Gamma\Big(\frac{3}{2}\Big)^{-n} \int_{\mathrm{int} Z^*_{\infty,\alpha}} \exp\Big\{ -\int_{S^{n-1}} |\phi_u(x\cdot u)\alpha(u)|^2 d\nu(u)\Big\} |dT(x)| dx \\
&\le \Gamma\Big(\frac{3}{2}\Big)^{-n} \int_{\mathrm{int} Z^*_{\infty,\alpha}} e^{-\|Tx\|^2} |dT(x)| dx \\
&\le \Gamma\Big(\frac{3}{2}\Big)^{-n} \int_{\mathbb{R}^n} e^{-\|y\|^2} dy \\
&= 2^n = |B^n_\infty|.
\end{aligned}
$$

The equality conditions are basically the same as the case $0 < p < \infty$. This completes the proof. □

## 4. The dual Loomis-Whitney inequality

Suppose $p > 0$ and $K \in \mathcal{S}^n_o$. The *polar $L_p$-centroid body*, $\Gamma^*_p K$, of $K$ is the body whose Minkowski functional is given by, for $x \in \mathbb{R}^n$,

$$\|x\|_{\Gamma^*_p K} = \Big( \frac{1}{|K|} \int_K |x\cdot y|^p dy \Big)^{\frac{1}{p}}.$$

A normalized definition was introduced by Lutwak and Zhang [42] for $p \ge 1$. When $p = 1$, the body $\Gamma K$ is the classical centroid body, which was defined and investigated by Petty [46]. For more information about the $L_p$-centroid body, see, e.g., [12, 13, 22, 37, 45].

By the polar coordinate formula, for $u \in S^{n-1}$, we have

$$
\begin{aligned}
\|u\|_{\Gamma^*_p K} &= \Big( \frac{1}{|K|} \int_K |u\cdot y|^p dy \Big)^{\frac{1}{p}} \\
&= \Big( \frac{1}{(n+p)|K|} \int_{S^{n-1}} |u\cdot v|^p \rho_K^{n+p}(v) dv \Big)^{\frac{1}{p}}.
\end{aligned} \tag{4.1}
$$

**Lemma 4.1.** *Let $p > 0$ and let $K \in \mathcal{S}^n_o$. If $\nu$ is an even isotropic measure on $S^{n-1}$, then*

$$|K| \le |B^n_p| (n+p)^{\frac{n}{p}} \exp\Big\{ \int_{S^{n-1}} \log \|u\|_{\Gamma^*_p K} d\nu(u) \Big\}. \tag{4.2}$$

*For $p \ne 2$, there is equality if and only if $\nu$ is a cross measure on $S^{n-1}$ and $K$ is a generalized $\ell^n_p$-ball formed by $\nu$.*

*Proof.* Let

$$\alpha(u)^{-1} = \int_{S^{n-1}} |u\cdot v|^p \rho_K^{n+p}(v) dv \tag{4.3}$$

for $u \in S^{n-1}$. From (2.5), (2.3), (2.2), the definition of $Z^*_{p,\alpha}$ (3.1), Fubini's theorem and (2.7), we have

$$|K|^{n+p} \le |Z^*_{p,\alpha}|^p \widetilde{V}_{-p}(K, Z^*_p)^n = |Z^*_{p,\alpha}|^p \Big( \frac{1}{n} \int_{S^{n-1}} \rho_K^{n+p}(v) \rho_{Z^*_{p,\alpha}}^{-p}(v) dv \Big)^n$$

$$= |Z^*_{p,\alpha}|^p \Big( \frac{1}{n} \int_{S^{n-1}} \rho_K^{n+p}(v) \|v\|^p_{Z^*_{p,\alpha}} dv \Big)^n$$
$$= |Z^*_{p,\alpha}|^p \Big( \frac{1}{n} \int_{S^{n-1}} \Big( \int_{S^{n-1}} |u \cdot v|^p \alpha(u) d\nu(u) \Big) \rho_K^{n+p}(v) dv \Big)^n$$
$$= |Z^*_{p,\alpha}|^p \Big( \frac{1}{n} \int_{S^{n-1}} \Big( \int_{S^{n-1}} |u \cdot v|^p \rho_K^{n+p}(v) dv \Big) \alpha(u) d\nu(u) \Big)^n$$
$$= |Z^*_{p,\alpha}|^p.$$

Combining this with Lemma 3.2 and (4.1) yields

$$\begin{aligned} |K|^{\frac{n+p}{p}} \le |Z^*_{p,\alpha}| &\le |B^n_p| \Big( \exp \int_{S^{n-1}} \log \alpha(u) d\nu(u) \Big)^{-\frac{1}{p}} \\ &= |B^n_p| \exp \Big\{ \int_{S^{n-1}} \log \alpha^{-\frac{1}{p}}(u) d\nu(u) \Big\} \\ &= |B^n_p| \exp \Big\{ \int_{S^{n-1}} \log \Big[ \int_{S^{n-1}} |u \cdot v|^p \rho_K^{n+p}(v) dv \Big]^{\frac{1}{p}} d\nu(u) \Big\} \\ &= |B^n_p| \exp \Big\{ \int_{S^{n-1}} \log \big[ ((n+p)|K|)^{\frac{1}{p}} \|u\|_{\Gamma^*_p K} \big] d\nu(u) \Big\} \\ &= |B^n_p| (n+p)^{\frac{n}{p}} |K|^{\frac{n}{p}} \exp \Big\{ \int_{S^{n-1}} \log \|u\|_{\Gamma^*_p K} d\nu(u) \Big\}, \end{aligned} \tag{4.4}$$

which is the desired inequality.

Now, we deal with the characterization of equalities in (4.4). By the dual Minkowski inequality (2.5), equality of the first inequality in (4.4) holds if and only if $K$ and $Z^*_{p,\alpha}$ are dilates. Lemma 3.2 shows that equality of the second inequality in (4.4) holds if and only if $\nu$ is a cross measure on $S^{n-1}$ when $p \neq 2$, and thus $Z^*_{p,\alpha}$ is a generalized $\ell^n_p$-ball $B^n_{p,\alpha}$ formed by $\nu$ when $p \neq 2$. Therefore, we obtain that equality in (4.4) holds if and only if $K$ is a dilation of the generalized $\ell^n_p$-ball formed by the cross measure $\nu$, which is still a generalized $\ell^n_p$-ball formed by the cross measure $\nu$, when $p \neq 2$.

Conversely, we will show equality in (4.4) holds if $K$ is a generalized $\ell^n_p$-ball formed by the cross measure $\nu$; i.e., there are positive numbers $(\alpha_i)^n_{i=1}$ such that

$$K = \Big\{ x \in \mathbb{R}^n : \Big( \sum_{i=1}^n |x \cdot u_i|^p \alpha_i \Big)^{\frac{1}{p}} \le 1 \Big\},$$

where $\operatorname{supp} \nu = \{\pm u_1, \dots, \pm u_n\}$ and $(u_i)^n_{i=1}$ is an orthonormal basis of $\mathbb{R}^n$. From (4.4), it is sufficient to verify that $K$ and $Z^*_{p,\alpha}$ are dilates when $p \neq 2$. From (2.11), we have

$$K = OA^{-1} B^n_p,$$

where $O \in \mathrm{O}(n)$ such that $Oe_i = u_i$ and $A = \operatorname{diag}\{\alpha_1^{1/p}, \dots, \alpha_n^{1/p}\}$ is a diagonal matrix. By (4.3) and (4.1) we get

$$\begin{aligned} \alpha(u_i)^{-1} &= \int_{S^{n-1}} |u_i \cdot v|^p \rho_K^{n+p}(v) dv = (n+p) \int_K |u_i \cdot y|^p dy = (n+p) \int_{OA^{-1} B^n_p} |u_i \cdot y|^p dy \\ &= (n+p) |\det A|^{-1} \int_{B^n_p} |A^{-t} O^t u_i \cdot z|^p dz \end{aligned}$$

$$= (n+p)|\det A|^{-1}\alpha_i^{-1}\int_{B_p^n}|e_i\cdot z|^p dz$$

for every $i = 1,\dots,n$. Notice that $\int_{B_p^n}|e_i\cdot z|^p dz$ is a constant for all $i = 1,\dots,n$. Thus, there exists a constant $c > 0$ such that $\alpha(u_i) = c\alpha_i$ for every $i = 1,\dots,n$. Recall that

$$\begin{aligned} Z^*_{p,\alpha} &= \Big\{x\in\mathbb{R}^n : \Big(\int_{S^{n-1}}|x\cdot u|^p\alpha(u)d\nu(u)\Big)^{\frac{1}{p}} \le 1\Big\} \\ &= \Big\{x\in\mathbb{R}^n : \Big(\sum_{i=1}^n|x\cdot u_i|^p\alpha(u_i)\Big)^{\frac{1}{p}} \le 1\Big\} \\ &= \Big\{x\in\mathbb{R}^n : \Big(\sum_{i=1}^n|x\cdot u_i|^p c\alpha_i\Big)^{\frac{1}{p}} \le 1\Big\} = c^{-\frac{1}{p}}K. \end{aligned}$$

That is, $K$ and $Z^*_{p,\alpha}$ are dilates when $p \neq 2$.

□

The following sharp estimates, vital to the proof of our theorem, were established by Fradelizi [15, Theorem 3, 4]. The symmetric case ($p > 0$) of (4.5) was given by Milman and Pajor [45, Corollary 2.7] and the case $p = 2$ is due to Hensley [25]. See also [1] for related inequalities.

**Lemma 4.2.** *Let $p \ge 1$ and let $K$ be a convex body in $\mathbb{R}^n$ whose centroid is at the origin. For $u \in S^{n-1}$, we have*

$$\|u\|_{\Gamma_p^*K} \le \frac{n|K|}{2|K\cap u^\perp|}\binom{n+p}{n}^{-\frac{1}{p}}, \tag{4.5}$$

*with equality if and only if $K$ is a double cone in the direction $u$; and*

$$\|u\|_{\Gamma_p^*K} \le c_{n,p}\frac{|K|}{\max_{x\in\mathbb{R}^n}|(K+x)\cap u^\perp|}, \tag{4.6}$$

*with equality if and only if $K$ is a cone in the direction $u$, where*

$$c_{n,p} = \Big(\Big(\frac{n}{n+1}\Big)^{n+p}\int_{-1}^{n}|t|^p\Big(1-\frac{t}{n}\Big)^{n-1}dt\Big)^{\frac{1}{p}}.$$

Combining (4.2), (4.5) and (2.9), we immediately obtain the following dual Loomis-Whitney inequality.

**Lemma 4.3.** *Let $p \ge 1$ and let $K$ be a convex body in $\mathbb{R}^n$ whose centroid is at the origin. If $\nu$ is an even isotropic measure on $S^{n-1}$, then*

$$|K|^{n-1} \ge \frac{\Gamma(1+\frac{n}{p})}{n^n(n+p)^{\frac{n}{p}}(\Gamma(1+\frac{1}{p}))^n}\binom{n+p}{n}^{\frac{n}{p}}\exp\Big\{\int_{S^{n-1}}\log|K\cap u^\perp|d\nu(u)\Big\}. \tag{4.7}$$

Lemma 4.3 gives a whole family of inequalities when $p$ varies. We show that the strongest is the one for $p = 1$. Thanks to the equality conditions in the results above, Lemma 4.3 with $p = 1$ gives a sharp inequality, with equality condition characterized.

**Theorem 4.4.** *Let $K$ be a convex body in $\mathbb{R}^n$ whose centroid is at the origin. If $\nu$ is an even isotropic measure on $S^{n-1}$, then*

$$|K|^{n-1} \ge \frac{n!}{n^n}\exp\Big\{\int_{S^{n-1}}\log|K\cap u^\perp|d\nu(u)\Big\}, \tag{4.8}$$

*with equality if and only if $\nu$ is a cross measure on $S^{n-1}$ and $K$ is a generalized $\ell_1^n$-ball formed by $\nu$.*

*Proof.* We only need to examine the equality conditions of (4.7). Note that (4.7) follows from (4.2) and (4.5). So, by the equality condition of (4.2), it follows that $\nu$ is a cross measure on $S^{n-1}$ and $K$ is a generalized $\ell_p^n$-ball formed by $\nu$ when $p \neq 2$. Note that the centroid of the generalized $\ell_p^n$-ball lies on the origin. The equality conditions of (4.5) yield that $K$ is a double cone in the direction of the support of $\nu$. Obviously, only the generalized $\ell_1^n$-ball formed by the cross measure $\nu$ is satisfied. Hence $K$ is a generalized $\ell_1^n$-ball formed by the cross measure $\nu$, which is the desired equality conditions of (4.8).

□

The same argument, together with (4.2) and (4.6), yields the following inequality.

**Theorem 4.5.** *Let $K$ be a convex body in $\mathbb{R}^n$ whose centroid is at the origin. If $\nu$ is an even isotropic measure on $S^{n-1}$, then*

$$|K|^{n-1} \geq \frac{\Gamma(1+\frac{n}{p})}{2^n(n+p)^{\frac{n}{p}}(\Gamma(1+\frac{1}{p}))^n c_{n,p}^n} \exp\Big\{\int_{S^{n-1}} \log\big(\max_{x\in\mathbb{R}^n} |(K+x)\cap u^{\perp}|\big) d\nu(u)\Big\}.$$

## 5. Special cases

As shown in Section 2, the two most important examples of even isotropic measures on $S^{n-1}$ are the (suitably normalized) spherical Lebesgue measure and the cross measure.

**Case 1:** The dual Loomis-Whitney inequality for the spherical Lebesgue measure.

Clearly, when $\nu$ is a multiple of Lebesgue measure, inequality (4.8) is not sharp, and we thus establish a sharp dual Loomis-Whitney inequality by direct arguments.

The Busemann intersection inequality (see e.g., [47, p.580]) states that if $K$ is a convex body in $\mathbb{R}^n$, then

$$\frac{1}{n}\int_{S^{n-1}} |K\cap u^{\perp}|^n du \leq \frac{\omega_{n-1}^n}{\omega_n^{n-2}}|K|^{n-1}. \tag{5.1}$$

Equality holds for $n=2$ if and only if $K$ is origin-symmetric, and for $n\geq 3$ if and only if $K$ is a centered ellipsoid. Inequality (5.1) immediately yields the following dual Loomis-Whitney inequality for the spherical Lebesgue measure.

**Theorem 5.1.** *Let $K$ be a convex body in $\mathbb{R}^n$. Then*

$$|K|^{n-1} \geq \frac{\omega_n^{n-1}}{\omega_{n-1}^n} \exp\Big\{\frac{1}{\omega_n}\int_{S^{n-1}} \log|K\cap u^{\perp}| du\Big\}, \tag{5.2}$$

*with equality if and only if $K$ is a centered Euclidean ball.*

*Proof.* Inequality (5.2) follows from Jensen's inequality and (5.1):

$$\begin{aligned}\exp\Big\{\frac{1}{\omega_n}\int_{S^{n-1}} \log|K\cap u^{\perp}| du\Big\} &= \exp\Big\{\frac{1}{n\omega_n}\int_{S^{n-1}} \log|K\cap u^{\perp}|^n du\Big\}\\ &\leq \frac{1}{n\omega_n}\int_{S^{n-1}} |K\cap u^{\perp}|^n du\\ &\leq \frac{\omega_{n-1}^n}{\omega_n^{n-1}}|K|^{n-1}.\end{aligned}$$

The equality conditions of Jensen's inequality imply that $|K \cap u^\perp|$ is constant for all $u \in S^{n-1}$. Note that

$$|K \cap u^\perp| = \frac{1}{n-1} \int_{S^{n-1}\cap u^\perp} \rho_K^{n-1}(v)dv.$$

Clearly, if equality in (5.1) holds, then $\rho_K$ is an even continuous function on $S^{n-1}$. Then, by the injective of the spherical Radon transform (see e.g., [16, Theorem C.2.4]), we obtain that $\rho_K(u)$ is constant for all $u \in S^{n-1}$, which implies that $K$ is a centered Euclidean ball. □

**Case 2:** The dual Loomis-Whitney inequality for the cross measure.

Theorem 4.4 gives a sharp estimate when $\nu$ is a cross measure, but only if the centroid of $K$ is at the origin. We show that this latter condition can be dropped by using the technique of Steiner symmetrization, yielding an independent proof of a result of Meyer's.

**Theorem 5.2.** *Let $K$ be a convex body in $\mathbb{R}^n$. If $\nu$ is a cross measure on $S^{n-1}$ with $\operatorname{supp}\nu = \{\pm u_1, \cdots, \pm u_n\}$, then*

$$|K|^{n-1} \geq \frac{n!}{n^n} \prod_{i=1}^{n} |K \cap u_i^\perp|, \tag{5.3}$$

*with equality if and only if $K$ is a polytope with vertices $a_i u_i$ and $-b_i u_i$, where $a_i, b_i \geq 0$ and $a_i + b_i \neq 0$, $i = 1, \ldots, n$.*

*Proof.* We denote by $S_u K$ the Steiner symmetrization of $K$ with respect to $u^\perp$. Let $K' = S_{u_n} S_{u_{n-1}} \cdots S_{u_1} K$, which is after $n$ Steiner symmetrizations about $u_i^\perp$, $i = 1, \ldots, n$. Clearly, the centroid of $K'$ is the origin. Indeed, the Steiner symmetrizations ensure that the centroid of $S_{u_1} K$ lies on the hyperplane $u_1^\perp$, and the centroid of $S_{u_2} S_{u_1} K$ lies on $u_1^\perp \cap u_2^\perp$. Therefore, after $n$ times, the centroid of $K'$ lies on $u_1^\perp \cap u_2^\perp \cap \cdots \cap u_n^\perp = \{o\}$. Moreover, by the Steiner symmetrization, we have $|K'| = |K|$, $|S_{u_j} K \cap u_i^\perp| = |K \cap u_i^\perp|$, and $|S_{u_i} K \cap u_i^\perp| = |\mathrm{P}_{u_i^\perp} K| \geq |K \cap u_i^\perp|$ for $j \neq i$. Hence, it follows from Theorem 4.4 that

$$\begin{aligned} |K|^{n-1} = |K'|^{n-1} &\geq \frac{n!}{n^n} \exp\Big\{ \int_{S^{n-1}} \log |K' \cap u^\perp| d\nu(u) \Big\} \\ &= \frac{n!}{n^n} \prod_{i=1}^{n} |K' \cap u_i^\perp| = \frac{n!}{n^n} \prod_{i=1}^{n} |S_{u_n} S_{u_{n-1}} \cdots S_{u_1} K \cap u_i^\perp| \\ &\geq \frac{n!}{n^n} \prod_{i=1}^{n} |K \cap u_i^\perp|. \end{aligned} \tag{5.4}$$

By Theorem 4.4, equality of the first inequality in (5.4) holds if and only if $K'$ is a generalized $\ell_1^n$-ball formed by $\nu$. Equality of the second inequality in (5.4) forces that $|K \cap u_i^\perp| = |\mathrm{P}_{u_i^\perp} K|$, $i = 1, \ldots, n$. Since $K'$ is a generalized $\ell_1^n$-ball after $n$ Steiner symmtrizations of $K$, it follows that $S_{u_i} K = \operatorname{conv}(\alpha_i u_i, -\alpha_i u_i, K \cap u_i^\perp)$, $\alpha_i > 0$, for each $i = 1, \ldots, n$. Then there exist $a_i, b_i \geq 0$ such that $a_i + b_i = 2\alpha_i$ and $K = \operatorname{conv}(a_i u_i, -b_i u_i, K \cap u_i^\perp)$. Thus the necessary condition of equality in (5.3) immediately follows by applying $n$ times Steiner symmtrizations. The sufficient condition follows by an easy calculation. □

**Acknowledgements**

The authors are indebted to the referee for many valuable suggestions and comments.

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