ON THE REVERSE DUAL LOOMIS-WHITNEY INEQUALITY

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Abstract. The dual Loomis-Whitney inequality provides the sharp lower bound for the volume of a convex body in terms of its $(n-1)$-dimensional coordinate sections. In this paper, some reverse forms of the dual Loomis-Whitney inequality are obtained. In particular, we show that the best universal DLW-constant for origin-symmetric planar convex bodies is 1.

1. Introduction

Throughout this paper, we shall use vol_k to denote k-dimensional volume (Lebesgue measure on the corresponding subspace) in Euclidean n-space $\mathbb{R}^n$, $1 \le k \le n$. We denote by $\mathrm{conv}\, A$ the convex hull of the set A and $\mathrm{lin}\, A$ the linear hull of the set A. The Euclidean norm $x \in \mathbb{R}^n$ is denoted by $\|x\|$ and the unit sphere of $\mathbb{R}^n$ is denoted by S^{n-1}.

The celebrated Loomis-Whitney inequality compares the volume of a Lebesgue measurable set with the geometric mean of the volumes of its $(n-1)$-dimensional coordinate projections. To be specific, let A be a Lebesgue measurable set in $\mathbb{R}^n$ and let $\{e_1, \dots, e_n\}$ be the standard orthogonal basis of $\mathbb{R}^n$. Then

$$\mathrm{vol}_n(A)^{n-1} \le \prod_{i=1}^{n} \mathrm{vol}_{n-1}(A|e_i^{\perp}), \tag{1.1}$$

with equality if and only if A is a coordinate box (a rectangular parallelepiped whose facets are parallel to the coordinate hyperplanes), where $A|e_i^{\perp}$ is the orthogonal projection of A onto the hyperplane $e_i^{\perp}$ perpendicular to e_i. This inequality,

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established by Loomis and Whitney [20] in 1949, is one of the fundamental inequalities in convex geometric analysis and has been widely used in many mathematical areas (see e.g., [8, 13–15, 24]). In recent years, the study of various extensions of the Loomis-Whitney inequality has received considerable attention (see, e.g., [1, 2, 4–6, 9–11, 16–19, 21]).

However, a direct way to reverse the Loomis-Whitney inequality (1.1) is not true, since we can take the volume of A arbitrarily small without changing its $(n-1)$-dimensional coordinate projections. A typical example can be found in the work of Campi, Gritzmann, and Gronchi [10]. Therefore, they [10] considered rotations of the standard orthogonal basis of $\mathbb{R}^n$ and defined the following LW-*constant* $\Lambda(K)$ of a convex body K (i.e., a compact convex set in $\mathbb{R}^n$ with nonempty interior) as

$$\Lambda(K) = \max_{F\in\mathcal{F}^n} \frac{\mathrm{vol}_n(K)^{n-1}}{\prod_{i=1}^n \mathrm{vol}_{n-1}(K|u_i^\perp)}, \tag{1.2}$$

where the frame $F = \{u_1, \ldots, u_n\}$ is an orthogonal basis of $\mathbb{R}^n$, and the set of all frames is denoted by $\mathcal{F}^n$. Thus, to reverse the Loomis-Whitney inequality means to find the greatest lower bound of the LW-constant. In [10], Campi, Gritzmann, and Gronchi showed that if K is a planar convex body, then

$$\Lambda(K) \geq \frac{1}{2}, \tag{1.3}$$

with equality if and only if K is a triangle. Some lower bounds of the LW-constant for special convex bodies in $\mathbb{R}^n$ were also provided in [10].

On the other hand, a dual version of the Loomis-Whitney inequality, in which the sharp lower bound of the volume of a convex body is given in terms of its $(n-1)$-dimensional coordinate sections, was obtained by Meyer [21]. He showed that, for a convex body K in $\mathbb{R}^n$,

$$\mathrm{vol}_n(K)^{n-1} \geq \frac{n!}{n^n} \prod_{i=1}^n \mathrm{vol}_{n-1}(K \cap e_i^\perp), \tag{1.4}$$

with equality if and only if K is a generalized cross-polytope (i.e., K is the convex hull of segments $[-b_i e_i, a_i e_i]$ with $a_i, b_i \geq 0$ and $a_i + b_i \neq 0$, $i = 1, \ldots, n$). Here $K \cap e_i^\perp$ is the intersection of K with the hyperplane $e_i^\perp$. Notice that there is duality between the extremal bodies in the Loomis-Whitney inequality (1.1) and Meyer's inequality (1.4); i.e., the polar body of a coordinate box that contains the origin in its interior is a generalized coordinate cross-polytope. More extensions of the dual Loomis-Whitney inequality can be found in, e.g., [9, 18, 19].

We say a set is unconditional if it is symmetric with respect to the coordinate hyperplanes. Note that a reverse form of the dual Loomis-Whitney inequality (1.4) for unconditional convex bodies can be obtained by the Loomis-Whitney inequality (1.1) since $K|e_i^\perp = K \cap e_i^\perp$ for any unconditional convex body K. In general, a direct way to reverse the dual Loomis-Whitney inequality (1.4) is also not true, since we can take the volume of K arbitrarily large without changing its $(n-1)$-dimensional coordinate sections. In fact, let $a = (\tau, \tau, , \ldots, \tau) \in \mathbb{R}^n$ with $\tau > \frac{1}{n}$, and let $K = \operatorname{conv}\{\pm e_1, \ldots, \pm e_n, \pm a\}$. Then we have $\operatorname{vol}_{n-1}(K \cap e_i^\perp) = \frac{2^{n-1}}{(n-1)!}$, while the volume of K could be arbitrarily large since we can take the value of τ large enough. So we may wonder whether Campi, Gritzmann, and Gronchi's approach can be applied to this problem. To establish this, unlike $(n-1)$-dimensional coordinate projections, we may choose a suitable point as the center of hyperplane sections. In this paper, we let this point be the centroid of a convex body. In analogy to (1.2), we define the DLW-*constant* of a convex body K in $\mathbb{R}^n$ by

$$\widetilde{\Lambda}(K) = \min_{F \in \mathcal{F}^n} \frac{\operatorname{vol}_n(K)^{n-1}}{\prod_{i=1}^n \operatorname{vol}_{n-1}((K - c(K)) \cap u_i^\perp)}, \tag{1.5}$$

where the centroid $c(K) = \frac{1}{\operatorname{vol}_n(K)} \int_K x dx$. The compactness of S^{n-1} yields that the minimum is indeed attained. The frame that attains the minimum will be called a *best frame* for K.

Thus, to reverse the dual Loomis-Whitney inequality, we need to find the least upper bound of the DLW-constant; i.e., the infimum of all γ such that for each convex body K in $\mathbb{R}^n$, there exists an orthogonal basis $\{u_1, \ldots, u_n\}$ satisfying

$$\operatorname{vol}_n(K)^{n-1} \leq \gamma \prod_{i=1}^n \operatorname{vol}_{n-1}((K - c(K)) \cap u_i^\perp).$$

Any inequality of this type will be called a *reverse dual Loomis-Whitney inequality.*

Notice that the proof of inequality (1.3) is equivalent to finding a minimal area rectangle that contains the planar convex body K. Thus, in this paper, by searching a maximal area rhombus inscribed in K, we obtain the least upper bound of the DLW-constant for origin-symmetric planar convex bodies.

Theorem 1.1. *If K is an origin-symmetric planar convex body, then*

$$\widetilde{\Lambda}(K) \leq 1, \tag{1.6}$$

with equality if and only if K is a parallelogram with one of its diagonals perpendicular to its edges.

If we define the best *universal* DLW-constant $\widetilde{\Lambda}_e(n)$ for origin-symmetric convex bodies in $\mathbb{R}^n$ by

$$\widetilde{\Lambda}_e(n) = \sup_{K\in\mathcal{K}_e^n} \widetilde{\Lambda}(K),$$

where $\mathcal{K}_e^n$ denotes the class of origin-symmetric convex bodies in $\mathbb{R}^n$, then Theorem 1.1 immediately yields

$$\widetilde{\Lambda}_e(2) = 1.$$

Furthermore, a weaker upper bound of the DLW-constant for origin-symmetric convex bodies in $\mathbb{R}^n$ is given below.

Theorem 1.2. *If K is an origin-symmetric convex body in $\mathbb{R}^n$, then*

$$\widetilde{\Lambda}(K) \le ((n-1)!)^n. \tag{1.7}$$

Obviously, when $n = 2$, inequality (1.7) reduces to (1.6) but without the equality conditions.

Finally, we consider some special convex bodies in $\mathbb{R}^n$, for example, the unit cube $Q_n = [-\frac{1}{2}, \frac{1}{2}]^n$ in $\mathbb{R}^n$.

Theorem 1.3. *If n is even, then*

$$\widetilde{\Lambda}(Q_n) = 2^{-\frac{n}{2}}.$$

If n is odd, then

$$2^{-\frac{n}{2}} < \widetilde{\Lambda}(Q_n) \le 2^{-\frac{n-1}{2}}.$$

The rest of this paper is organized as follows. In Section 2 we characterize all maximal area rhombuses inscribed in origin-symmetric polygons. By this, the reverse dual Loomis-Whitney inequality for origin-symmetric planar convex bodies is obtained. In Section 3 two types of upper bounds of the DLW-constant in $\mathbb{R}^n$ are given. Section 4 is devoted to estimating the DLW-constant for special convex bodies (i.e., unit cubes and regular simplexes).

2. Proof of Theorem 1.1

We list some basic notations about convex bodies. Good general references are Gardner [13] and Schneider [25]. A polytope is the convex hull of finitely many points. The 1-dimensional faces of a polytope are its edges, and the $(n-1)$-dimensional faces are its facets. A planar polytope is usually called a polygon. A cross-polytope in $\mathbb{R}^n$ is the convex hull of segments $[-\alpha_i u_i, \alpha_i u_i]$ with $\alpha_i > 0$, $i =$

$1, \dots, n$, and $\{u_1, \dots, u_n\}$ is a frame. A planar cross-polytope is also called a rhombus. We say that a set A is origin-symmetric if $x \in A$ implies that $-x \in A$. For a set $A \subset \mathbb{R}^n$, the relative interior of A is the interior relative to its affine hull.

Observe that the best frame for an origin-symmetric polygon is related to maximal area rhombuses inscribed in it. So we first establish the following characteristic theorem.

Theorem 2.1. *Let P be an origin-symmetric polygon. Then every maximal area rhombus inscribed in P has at least one pair of opposite vertices coinciding with that of P.*

In general, the rhombus of maximal area inscribed in a planar convex body may be not unique. Trivially, among all rhombuses inscribed in a disk every square has maximal area. Note that a dual version of Theorem 2.1 which characterizes all minimum area rectangles containing a polygon was proved by Fremann and Shapira [12]. To prove Theorem 2.1, we shall make use of the following lemma. However, it seems that Lemma 2.2 does not follow from Fremann and Shapira's result by polarity, so we give a direct and explicit construction.

Lemma 2.2. *Let P be an origin-symmetric polygon. If a rhombus C inscribed in P has all its four vertices in the relative interiors of edges of P, then there exists another rhombus C' inscribed in P such that the area of C' is larger than that of C.*

Proof. Since the vertices of C are all in the relative interiors of edges of P, we let α and β be two angles between the diagonals of C and the edges of P, respectively (illustrated in Figure 1 and 2).

The desired rhombus C' can be constructed by rotating C with an angle θ in the following two cases.

The first case is $0 < \alpha + \beta \leq \pi$. We will show that there exists a counterclockwise rotation θ such that $\mathrm{vol}_2(C) - \mathrm{vol}_2(C')$ is negative (see Figure 1). In fact, denote the half length of the diagonals of C and C' by a, b and a', b', respectively. Then,

$$\mathrm{vol}_2(C) - \mathrm{vol}_2(C') = 2ab - 2a'b'.$$

By the sine rule, we have

$$a' = \frac{a \sin \alpha}{\sin(\alpha - \theta)}, \quad b' = \frac{b \sin \beta}{\sin(\beta - \theta)},$$

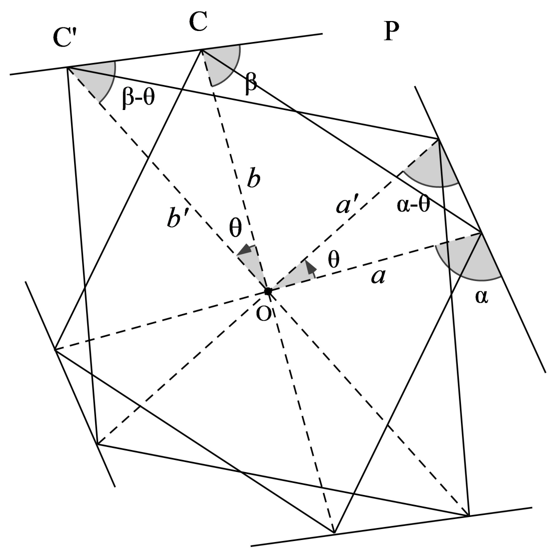

Figure 1: Formation of rhombus C' by a counterclockwise rotation.

which gives

$$\mathrm{vol}_2(C) - \mathrm{vol}_2(C') = 2ab\Big(1 - \frac{\sin\alpha\sin\beta}{\sin(\alpha-\theta)\sin(\beta-\theta)}\Big). \tag{2.1}$$

Let

$$f(\theta) = \sin\alpha\sin\beta - \sin(\alpha-\theta)\sin(\beta-\theta).$$

Clearly,

$$f(0) = 0,$$

and

$$f'(\theta) = \cos(\alpha-\theta)\sin(\beta-\theta) + \sin(\alpha-\theta)\cos(\beta-\theta) = \sin(\alpha+\beta-2\theta).$$

Since $0 < \alpha + \beta \leq \pi$, then there exists a sufficient small $\varepsilon > 0$ such that $\sin(\alpha + \beta - 2\varepsilon) > 0$, which implies that $f(\theta)$ is a strictly increasing function on $[0, \varepsilon]$. Thus, there exists $\theta \in (0, \varepsilon)$ such that

$$f(\theta) > f(0) = 0,$$

which yields $\mathrm{vol}_2(C) - \mathrm{vol}_2(C') < 0$.

The second case is $\pi < \alpha + \beta < 2\pi$. We will show that there exists a clockwise rotation θ such that $\mathrm{vol}_2(C) - \mathrm{vol}_2(C')$ is negative (see Figure 2). By a similar

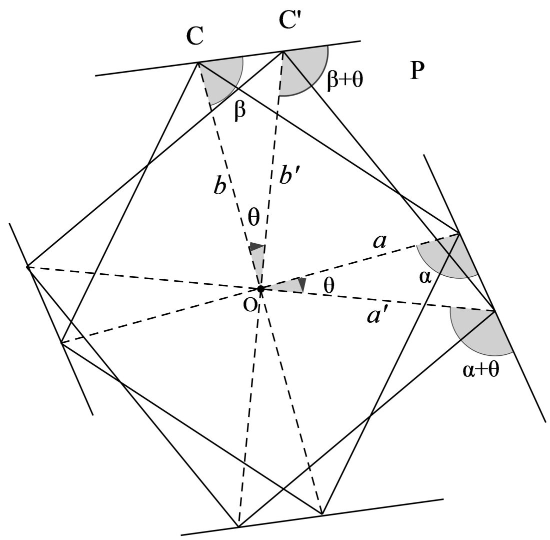

Figure 2: Formation of rhombus C' by a clockwise rotation.

computation of (2.1), we have

$$\mathrm{vol}_2(C) - \mathrm{vol}_2(C') = 2ab - 2a'b' = 2ab\Big(1 - \frac{\sin\alpha\sin\beta}{\sin(\alpha+\theta)\sin(\beta+\theta)}\Big).$$

Let

$$g(\theta) = \sin\alpha\sin\beta - \sin(\alpha+\theta)\sin(\beta+\theta).$$

Clearly

$$g(0) = 0,$$

and

$$g'(\theta) = -\sin(\alpha+\beta+2\theta).$$

Since $\pi < \alpha+\beta < 2\pi$, then there exists a sufficient small $\varepsilon > 0$ such that $\sin(\alpha+\beta+2\varepsilon) < 0$, which implies that $g(\theta)$ is a strictly increasing function on $[0,\varepsilon]$. Thus, there exists $\theta \in (0,\varepsilon)$ such that

$$g(\theta) > g(0) = 0,$$

which gives $\mathrm{vol}_2(C) - \mathrm{vol}_2(C') < 0$.

Therefore, we can construct another rhombus C' inscribed in P by rotating C such that the area of C' is larger than that of C. □

Proof of Theorem 2.1. Suppose the theorem is false; i.e., there is a maximal area rhombus inscribed in P has all its four vertices in the relative interiors of edges of P. But, it follows from Lemma 2.2 that there exists another rhombus inscribed in P with a larger area. That is a contradiction. □

We are now in a position to prove Theorem 1.1 by using Theorem 2.1.

Proof of Theorem 1.1. Since K is origin-symmetric, there exist two points $A_1, A_2 \in K$ such that $\|A_1 - A_2\|$ is the diameter of K and the the origin O is the midpoint of the segment A_1A_2. Let $v_1 = (A_1 - A_2)/\|A_1 - A_2\|$ and let $v_2 \in S^1$ be perpendicular to v_1. Draw a line along v_2 intersecting K at points B_1, B_2. Note that the lines $A_1 + \mathrm{lin}\{v_2\}$ and $A_2 + \mathrm{lin}\{v_2\}$ support K. Through B_1, B_2 there are also two parallel supporting lines to K. Thus, we can construct a parallelogram Q with vertices E_1, F_1, E_2, F_2 such that $K \subseteq Q$, as illustrated in Figure 3.

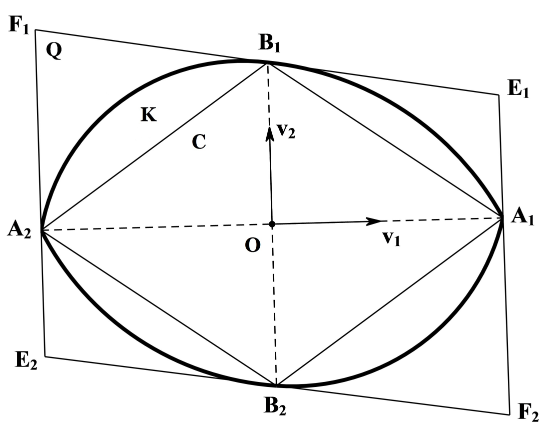

Figure 3:

Denote the length of OA_1 and OB_1 by a, b, respectively. Then we have

$$\mathrm{vol}_1(K \cap v_1^{\perp})\mathrm{vol}_1(K \cap v_2^{\perp}) = \mathrm{vol}_2(Q) = 4ab.$$

Thus, it follows from (1.5) that

$$\widetilde{\Lambda}(K) \leq \frac{\mathrm{vol}_2(K)}{\mathrm{vol}_1(K \cap v_1^{\perp})\mathrm{vol}_1(K \cap v_2^{\perp})} \leq \frac{\mathrm{vol}_2(Q)}{4ab} = 1.$$

Equality of the second inequality yields that $K = Q$. Let C be the rhombus with vertices A_1, B_1, A_2, B_2. Then, equality of the first inequality yields that C is a maximal area rhombus inscribed in K. By Theorem 2.1, there is at least one pair

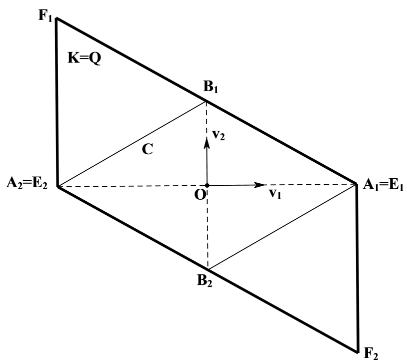

Figure 4: The equality case 1.

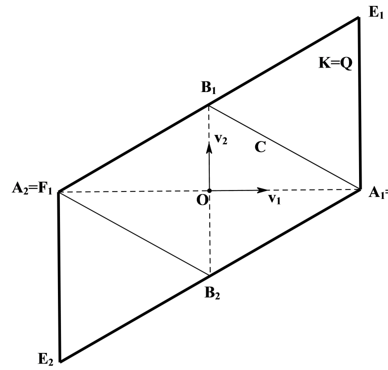

Figure 5: The equality case 2.

of opposite vertices of C coinciding with that of Q. Thus, one diagonal of C is perpendicular to a pair of opposite edges of Q, as illustrated in Figures 4 and 5.

□

As a byproduct of Theorem 2.1, we are able to characterize all maximal volume cross-polytopes inscribed in an origin-symmetric polytope in $\mathbb{R}^n$.

Theorem 2.3. *Let P be an origin-symmetric polytope in $\mathbb{R}^n$. If C is a maximal volume cross-polytope inscribed in P, then C has at least $n-1$ diagonals passing through the edges (possible the vertices) of P.*

Proof. Let C be a cross-polytope of maximal volume inscribed in P, and let $\pm v_1, \dots, \pm v_n$ be the diagonal unit vectors of C. Arguing by contradiction, we assume that there are two diagonals of C which do not pass through the edges of P; i.e., they pass through the relative interiors of two pairs of opposite k-dimensional faces $(2 \le k \le n-1)$ of P. Without loss of generality, let $\pm v_1, \pm v_2$ be these diagonal vectors and let $\xi = \mathrm{lin}\{v_1, v_2\}$. By Theorem 2.1, we see that $C \cap \xi$ is not a maximal area rhombus inscribed in the polygon $P \cap \xi$. Thus, there exists another rhombus C' inscribed in $P \cap \xi$ such that $\mathrm{vol}_2(C') > \mathrm{vol}_2(C \cap \xi)$. Note that $\widetilde{C} = \mathrm{conv}\{C', C \cap \xi^{\perp}\}$ is still a cross-polytope in $\mathbb{R}^n$ and $\mathrm{vol}_n(\widetilde{C}) > \mathrm{vol}_n(C)$, which leads a contradiction to the assumption of C. □

The dual version of Theorem 2.3 which characterizes all minimum volume rectangular boxes containing a polytope was proved by Fremann and Shapira [12] for $n = 2$, by O'Rourke [22] for $n = 3$, and by Campi, Gritzmann, and Gronchi [10] for arbitrary dimensions.

3. Proof of Theorem 1.2

The following lemma is due to Campi, Gritzmann, and Gronchi [10, Lemma 5.5].

Lemma 3.1. *If K is an origin-symmetric convex body in $\mathbb{R}^n$, then there exists a cross-polytope C contained in K with*

$$\mathrm{vol}_n(K) \leq n!\mathrm{vol}_n(C).$$

By Busemann's theorem (see, e.g., [13, Theorem 8.1.10]), the intersection body IK of an origin-symmetric convex body K is the origin symmetric convex body whose radial function at $u \in S^{n-1}$ is given by

$$\rho_{IK}(u) = \max\{\lambda : \lambda u \in IK\} = \mathrm{vol}_{n-1}(K \cap u^{\perp}).$$

Suppose K is an origin-symmetric convex body in $\mathbb{R}^n$ and C is the maximal volume cross-polytope inscribed in IK whose diagonal unit vectors are $\pm u_1, \dots, \pm u_n$. Then

$$\mathrm{vol}_n(C) = \frac{2^n}{n!}\prod_{i=1}^{n}\rho_{IK}(u_i) = \frac{2^n}{n!}\prod_{i=1}^{n}\mathrm{vol}_{n-1}(K \cap u_i^{\perp}).$$

Thus, it follows from (1.5) that

$$\widetilde{\Lambda}(K) = \frac{2^n}{n!}\min_{F\in\mathcal{F}^n}\frac{\mathrm{vol}_n(K)^{n-1}}{\mathrm{vol}_n(C(IK;F))}, \tag{3.1}$$

where $C(IK;F)$ is the maximal volume cross-polytope inscribed in IK with the diagonal unit vectors in F. Using this relation, we can give upper bounds of $\widetilde{\Lambda}(K)$ for an origin-symmetric convex body K in $\mathbb{R}^n$ in terms of its intersection body IK.

Theorem 3.2. *If K is an origin-symmetric convex body in $\mathbb{R}^n$, then*

$$\widetilde{\Lambda}(K) \leq \frac{2^n\mathrm{vol}_n(K)^{n-1}}{\mathrm{vol}_n(IK)}.$$

Proof. Using Lemma 3.1, we have

$$\mathrm{vol}_n(IK) \leq n!\mathrm{vol}_n(C(IK;F)),$$

and thus,

$$\frac{\mathrm{vol}_n(K)^{n-1}}{\mathrm{vol}_n(C(IK;F))} \leq \frac{n!\mathrm{vol}_n(K)^{n-1}}{\mathrm{vol}_n(IK)}.$$

Hence, by (3.1), we have

$$\widetilde{\Lambda}(K) = \frac{2^n}{n!}\min_{F\in\mathcal{F}^n}\frac{\mathrm{vol}_n(K)^{n-1}}{\mathrm{vol}_n(C(IK;F))} \leq \frac{2^n\mathrm{vol}_n(K)^{n-1}}{\mathrm{vol}_n(IK)}.$$

□

Here the quantity

$$\widetilde{\Theta}(K) = \frac{\mathrm{vol}_n(K)^{n-1}}{\mathrm{vol}_n(IK)}$$

is an important functional in convex geometric analysis, which is dual to the Petty functional [23]. The sharp upper bound of $\widetilde{\Theta}$ is still unknown, but a sharp lower bound comes from the classical Busemann intersection inequality [25, p. 581], which states that for a convex body K in $\mathbb{R}^n$,

$$\widetilde{\Theta}(K) \geq \frac{\omega_n^{n-1}}{\omega_{n-1}^n},$$

with equality for $n = 2$ if and only if K is origin-symmetric, and for $n \geq 3$ if and only if K is an origin-symmetric ellipsoid. Here ω_n is the volume of the Euclidean unit ball in $\mathbb{R}^n$.

In [10], Campi, Gritzmann, and Gronchi defined the functional $\Phi(K)$ of a convex body K as

$$\Phi(K) = \max_{F \in \mathcal{F}^n} \frac{\mathrm{vol}_n(K)}{\mathrm{vol}_n(B(K;F))},$$

where $B(K;F)$ is the minimal volume rectangular box containing K with edges parallel to the vectors in F. They [10, Lemma 7.2] showed that

$$\Phi(K) \geq \frac{1}{n!}. \tag{3.2}$$

Similarly, we define the functional $\widetilde{\Phi}(K)$ of a convex body K as

$$\widetilde{\Phi}(K) = \min_{F \in \mathcal{F}^n} \frac{\mathrm{vol}_n(K)}{\mathrm{vol}_n(C(K;F))}, \tag{3.3}$$

where $C(K;F)$ is the maximal volume cross-polytope inscribed in K with the diagonal unit vectors in F. Thus, by Lemma 3.1, we have the following dual inequality of (3.2): for an origin-symmetric convex body in $\mathbb{R}^n$,

$$\widetilde{\Phi}(K) \leq n!. \tag{3.4}$$

It was also proved in [10, Lemma 7.1] that for a convex body K in $\mathbb{R}^n$,

$$\Lambda(K) \geq \Phi(K)^{n-1}.$$

Similarly, we obtain the following dual inequality. Obviously, by (1.5) and (3.3), $\widetilde{\Lambda}(K) = \widetilde{\Phi}(K)/2$ holds for the origin-symmetric planar convex body K.

Theorem 3.3. *If K is an origin-symmetric convex body in $\mathbb{R}^n$, then*

$$\widetilde{\Lambda}(K) \leq \frac{n!}{n^n}\widetilde{\Phi}(K)^{n-1}.$$

Proof. Let $F=\{u_1,\dots,u_n\}$ be a frame such that $\widetilde{\Phi}(K)=\mathrm{vol}_n(K)/\mathrm{vol}_n(C(K;F))$. Then it follows from (1.5) and the equality conditions of Meyer's inequality (1.4) that

$$\begin{aligned}\widetilde{\Lambda}(K) &\le \frac{\mathrm{vol}_n(K)^{n-1}}{\prod_{i=1}^n \mathrm{vol}_{n-1}(K\cap u_i^\perp)} \le \frac{\mathrm{vol}_n(K)^{n-1}}{\prod_{i=1}^n \mathrm{vol}_{n-1}(C(K;F)\cap u_i^\perp)}\\ &= \frac{n!\mathrm{vol}_n(K)^{n-1}}{n^n\mathrm{vol}_n(C(K;F))^{n-1}} = \frac{n!}{n^n}\widetilde{\Phi}(K)^{n-1},\end{aligned}$$

which yields the desired inequality. □

Now, Theorem 1.2 immediately follows from Theorem 3.3 and inequality (3.4).

4. Upper bounds of $\widetilde{\Lambda}(K)$ for special convex bodies

Obviously, it follows from Meyer's inequality (1.4) that the DLW-constant of a cross-polytope is $n!/n^n$. Now let us estimate the bounds of $\widetilde{\Lambda}(Q_n)$ for the unit cube $Q_n=[-\frac{1}{2},\frac{1}{2}]^n$ in $\mathbb{R}^n$.

Theorem 4.1. *If n is even, then*

$$\widetilde{\Lambda}(Q_n)=2^{-\frac{n}{2}}.$$

If n is odd, then

$$2^{-\frac{n}{2}}<\widetilde{\Lambda}(Q_n)\le 2^{-\frac{n-1}{2}}.$$

Proof. In [3], Ball proved that for every $u\in S^{n-1}$,

$$\mathrm{vol}_{n-1}(Q_n\cap u^\perp)\le\sqrt{2}, \tag{4.1}$$

with equality if and only if the hyperplane $u^\perp$ contains an $(n-2)$-dimensional face of Q_n.

If n is even, then, for $i\in\{1,\dots,n/2\}$, the vectors

$$v_{2i-1}=\frac{1}{\sqrt{2}}(0,\dots,0,1,1,0,\dots,0)^T,\quad v_{2i}=\frac{1}{\sqrt{2}}(0,\dots,0,1,-1,0,\dots,0)^T$$

form a frame F, where the first nonzero entry is in the $(2i-1)$th position. Then by (4.1), we have

$$\mathrm{vol}_{n-1}(Q_n\cap v_{2i-1}^\perp)=\mathrm{vol}_{n-1}(Q_n\cap v_{2i}^\perp)=\sqrt{2}.$$

and $Q_n\cap v_{2i-1}^\perp$, $Q_n\cap v_{2i}^\perp$ are the largest $(n-1)$-dimensional sections of Q_n. Thus, it follows from (1.5) that

$$\widetilde{\Lambda}(Q_n)=\min_{F\in\mathcal{F}^n}\frac{\mathrm{vol}_n(Q_n)^{n-1}}{\prod_{i=1}^n \mathrm{vol}_{n-1}(Q_n\cap u_i^\perp)}$$

$$= \frac{\mathrm{vol}_n(Q_n)^{n-1}}{\prod_{i=1}^{n/2} \mathrm{vol}_{n-1}(Q_n \cap v_{2i-1}^{\perp})\mathrm{vol}_{n-1}(Q_n \cap v_{2i}^{\perp})} = 2^{-\frac{n}{2}}.$$

If n is odd, then, for $i \in \{1, \dots, (n-1)/2\}$, the vectors

$$v_{2i-1} = \frac{1}{\sqrt{2}}(0, \dots, 0, 1, 1, 0, \dots, 0)^T, \quad v_{2i} = \frac{1}{\sqrt{2}}(0, \dots, 0, 1, -1, 0, \dots, 0)^T$$

and the vector $v_n = e_n = (0, \dots, 0, 1)^T$ form a frame F. Thus, it follows from (1.5) and (4.1) that

$$\begin{aligned} 2^{-\frac{n}{2}} < \widetilde{\Lambda}(Q_n) &= \min_{F\in\mathcal{F}^n} \frac{\mathrm{vol}_n(Q_n)^{n-1}}{\prod_{i=1}^{n} \mathrm{vol}_{n-1}(Q_n \cap u_i^{\perp})} \\ &\leq \frac{\mathrm{vol}_n(Q_n)^{n-1}}{\mathrm{vol}_{n-1}(Q_n \cap v_n^{\perp}) \prod_{i=1}^{(n-1)/2} \mathrm{vol}_{n-1}(Q_n \cap v_{2i-1}^{\perp})\mathrm{vol}_{n-1}(Q_n \cap v_{2i}^{\perp})} = 2^{-\frac{n-1}{2}}. \end{aligned}$$

□

Searching the best frame for general convex bodies is a difficult problem even in the planar case. The following rough bounds for regular simplexes are given below.

Theorem 4.2. *Let T_n be a regular simplex in $\mathbb{R}^n$ with edges of length $\sqrt{2}$ whose centroid is at origin. Then*

$$\frac{(\sqrt{2})^n n!}{n^n\sqrt{n+1}} < \widetilde{\Lambda}(T_n) < \frac{(2\sqrt{3})^n n!}{n^n\sqrt{n+1}}.$$

Proof. In [26], Webb established the following inequality

$$\mathrm{vol}_{n-1}(T_n \cap u^{\perp}) \leq \frac{\sqrt{n+1}}{\sqrt{2}(n-1)!}, \quad u \in S^{n-1}, \tag{4.2}$$

with equality if and only if the section contains $n-1$ vertices of T_n. On the other hand, Brzezinski [7] showed that, for $u \in S^{n-1}$,

$$\mathrm{vol}_{n-1}(T_n \cap u^{\perp}) \geq \frac{\sqrt{n+1}}{(n-1)!}\frac{1}{2\sqrt{3}}.$$

Observe that

$$\mathrm{vol}_n(T_n) = \frac{\sqrt{n+1}}{n!}.$$

Thus, it follows from (1.5) that

$$\frac{(\sqrt{2})^n n!}{n^n\sqrt{n+1}} < \widetilde{\Lambda}(T_n) = \min_{F\in\mathcal{F}^n} \frac{\mathrm{vol}_n(T_n)^{n-1}}{\prod_{i=1}^{n} \mathrm{vol}_{n-1}(T_n \cap u_i^{\perp})} < \frac{(2\sqrt{3})^n n!}{n^n\sqrt{n+1}}.$$

□

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