VOLUME INEQUALITIES FOR SECTIONS AND PROJECTIONS OF WULFF SHAPES AND THEIR POLARS

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Abstract. Let $1 \leq k \leq n$. Sharp volume inequalities for k-dimensional sections of Wulff shapes and dual inequalities for projections are established. As their applications, several special Wulff shapes are investigated.

1. Introduction

Throughout, all Borel measures are understood to be nonnegative and finite. A convex body in $\mathbb{R}^n$ is a compact convex set containing the origin in its interior. The polar body of a convex body K is given by $K^* = \{x \in \mathbb{R}^n : x \cdot y \leq 1 \text{ for all } y \in K\}$, where $x \cdot y$ denotes the standard inner product of x and y in $\mathbb{R}^n$. We use $\|\cdot\|$ to denote the Euclidean norm on $\mathbb{R}^n$. When A is a compact convex set in $\mathbb{R}^n$, we write $|A|$ for the volume of A in the appropriate subspace. Let $\operatorname{supp}\nu$ denote the support of a measure ν and let P_H be the orthogonal projection onto a subspace H of $\mathbb{R}^n$.

Volume estimates for sections of convex bodies in $\mathbb{R}^n$ are not easy, even in specific cases. For the cube $Q_n = [-\frac{1}{2}, \frac{1}{2}]^n$, Hensley [13] first showed that if H is a hyperplane of $\mathbb{R}^n$ then $|H \cap Q_n|$ lies between 1 and 5, and conjectured that the upper bound is at most $\sqrt{2}$. This conjecture was solved by Ball [1,2], who also settled the more general case of k-dimensional sections.

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The example of the regular simplex is much more complicated. Webb [30] proved that the maximal central hyperplane section is the one containing $n-1$ vertices and the centroid. The question about the minimal central hyperplane section has not been completely solved yet. Brzezinski [10] proved a lower bound which differs from the conjectured minimal volume by a factor of approximately 1.27. For general k-dimensional sections, these questions were recently considered by Dirksen [11]. Other examples, such as ℓ_p^n-balls [7, 9, 25], complex cubes [27] and non-central sections of cubes [26], have also been investigated.

In this paper, we will study sections and projections of more general convex bodies than cubes and simplices. The main objects we consider are Wulff shapes [28], which were introduced by Wulff in 1901. Nowadays, it is an important notion in convex geometric analysis (see, e.g., [28]).

Definition. Suppose that ν is a Borel measure on S^{n-1} and that f is a positive, bounded, and measurable function on S^{n-1}. The *Wulff shape* $W_{\nu,f}$ determined by ν and f is defined by

$$W_{\nu,f} := \{x \in \mathbb{R}^n : x \cdot u \le f(u) \quad \text{for all} \quad u \in \operatorname{supp} \nu\}. \tag{1.1}$$

The measure ν is said to be *even* if it assumes the same value on antipodal sets. When ν and f are both even, then $W_{\nu,f}$ is origin-symmetric. It is easy to see that $W_{\nu,f}$ is always convex and may be unbounded. In order for a k-dimensional subspace H of $\mathbb{R}^n$, to guarantee that $|W_{\nu,f}|$ and $|H \cap W_{\nu,f}|$ are finite, we consider Wulff shapes determined by measures ν which are isotropic and f-centered with respect to H. A Borel measure ν on S^{n-1} is called *isotropic* if

$$\int_{S^{n-1}} u \otimes u d\nu(u) = I_n, \tag{1.2}$$

where $u \otimes u$ is the rank-one orthogonal projection onto the space spanned by u and I_n is the identity map on $\mathbb{R}^n$. The definition that the measure ν is f-centered with respect to H needs more description.

Let H be a k-dimensional subspace of $\mathbb{R}^n$ with $1 \le k \le n$ and let $\bar{\nu}$ be the Borel measure on $S^{n-1} \cap H$ defined by

$$\bar{\nu}(A) = \int_{S^{n-1} \setminus H^{\perp}} \mathbf{1}_A\Big(\frac{\mathrm{P}_H u}{\|\mathrm{P}_H u\|}\Big) \|\mathrm{P}_H u\|^2 d\nu(u) \tag{1.3}$$

for Borel sets $A \subset S^{n-1} \cap H$. It is easy to see that the support set of $\bar{\nu}$ is exactly the set of $\{\mathrm{P}_H u/\|\mathrm{P}_H u\| : u \in \operatorname{supp}\nu \backslash H^\perp\}$. The section $H \cap W_{\nu,f}$ can be expressed as

$$\begin{aligned} H \cap W_{\nu,f} &= \{y \in H : y \cdot u \leq f(u) \ \text{ for all } \ u \in \operatorname{supp}\nu\} \\ &= \Big\{ y \in H : y \cdot \frac{\mathrm{P}_H u}{\|\mathrm{P}_H u\|} \leq \frac{f(u)}{\|\mathrm{P}_H u\|} \ \text{ for all } \ u \in \operatorname{supp}\nu \backslash H^\perp \Big\} \\ &= \Big\{ y \in H : y \cdot w \leq \inf_{u \in \Xi_w} \frac{f(u)}{\|\mathrm{P}_H u\|} \ \text{ for all } \ w \in \operatorname{supp}\bar{\nu} \Big\}, \end{aligned} \tag{1.4}$$

where

$$\Xi_w = \Big\{ u \in \operatorname{supp}\nu \backslash H^\perp : \frac{\mathrm{P}_H u}{\|\mathrm{P}_H u\|} = w, \ w \in \operatorname{supp}\bar{\nu} \Big\}.$$

The measure ν is called *f-centered with respect to H* if

$$\int_{\Psi_H} f(u)\mathrm{P}_H u d\nu(u) = o, \tag{1.5}$$

where $\Psi_H \subseteq \operatorname{supp}\nu$ is the *support subset* (with respect to H) of ν, defined by

$$\Psi_H = \bigcup_{w \in \operatorname{supp}\bar{\nu}} \Psi_w,$$

where

$$\Psi_w = \Big\{ u_w \in \Xi_w : \frac{f(u_w)}{\|\mathrm{P}_H u_w\|} = \inf_{u \in \Xi_w} \frac{f(u)}{\|\mathrm{P}_H u\|} \Big\}.$$

Since f is measurable and P_H is continuous, Ψ_w is a measurable set in $\operatorname{supp}\nu \backslash H^\perp$. Obviously, if $\operatorname{supp}\bar{\nu}$ is countable or finite, then Ψ_H is a measurable set. So, when $1 \leq k < n$, we always assume that the measure ν, as well as $\bar{\nu}$, is discrete in our main results (Theorems 1.1 and 1.2) to guarantee that Ψ_H is measurable. If $k = n$, then $H = \mathbb{R}^n$, $\mathrm{P}_H = I_n$ and thus, $\Psi_H = \Psi_{\mathbb{R}^n} = \operatorname{supp}\nu$, a measurable set. In this case, (1.5) reduces to

$$\int_{S^{n-1}} f(u) u d\nu(u) = o, \tag{1.6}$$

and the measure ν is *f-centered* (see e.g., [29]). Note that the regular simplex (and the cube) is a Wulff shape determined by a (even) discrete measure ν which is f-centered and isotropic.

We also need the notion of displacement of $H \cap W_{\nu,f}$ defined by

$$\operatorname{disp}(H \cap W_{\nu,f}) = \frac{1}{|H \cap W_{\nu,f}|} \int_{H \cap W_{\nu,f}} x dx \cdot \int_{\Psi_H} \frac{\mathrm{P}_H u}{f(u)} \|\mathrm{P}_H u\|^2 d\nu(u), \tag{1.7}$$

and

$$\|f\|_{L_2(\Psi_H)} = \Big(\int_{\Psi_H} f(u)^2 d\nu(u)\Big)^{1/2}. \tag{1.8}$$

The aim of this paper is to establish volume inequalities for sections and projections of Wulff shapes and their polars. For the cases $1 \le k < n$ we make the additional assumption of discreteness of the underlying isotropic measure ν to guarantee that Ψ_H is measurable. When $k = n$, the set $\Psi_H = \operatorname{supp}\nu$ is measurable. Then the assumption of discreteness of ν is unnecessary and we recover the results by Schuster and Weberndorfer [29], and generalized and unified results of Ball [2,3], Barthe [6], and Lutwak, Yang, and Zhang [22, 23].

The asymmetric case can be stated as follows:

Theorem 1.1. *Let H be a k-dimensional subspace of $\mathbb{R}^n$ with $1 \le k \le n$ and let f be a positive bounded measurable function on S^{n-1}. Suppose that the measure ν on S^{n-1} is discrete, isotropic, and f-centered with respect to H when $1 \le k < n$, and that ν is isotropic and f-centered when $k = n$. Then*

$$|H \cap W_{\nu,f}| \le \frac{(k+1-\operatorname{disp}(H\cap W_{\nu,f}))^{k+1}}{k!(k+1)^{\frac{k+1}{2}}}\|f\|^k_{L_2(\Psi_H)},$$

and

$$|\mathrm{P}_H W^*_{\nu,f}| \ge \frac{(k+1)^{\frac{k+1}{2}}}{k!}\|f\|^{-k}_{L_2(\Psi_H)},$$

with equalities if and only if $H\cap W_{\nu,f}$ is a regular simplex inscribed in $\frac{\|f\|_{L_2(\Psi_H)}}{\sqrt{k}}(S^{n-1}\cap H)$ and $\frac{f(u)}{\|\mathrm{P}_H u\|}$ is constant for all $u \in \Psi_H$.

The symmetric case reads as follows:

Theorem 1.2. *Let H be a k-dimensional subspace of $\mathbb{R}^n$ with $1 \le k \le n$ and let f be an even positive bounded measurable function on S^{n-1}. Suppose that the measure ν on S^{n-1} is discrete and even isotropic when $1 \le k < n$, and that ν is even isotropic when $k = n$. Then*

$$|H \cap W_{\nu,f}| \le \Big(\frac{2}{\sqrt{k}}\Big)^k \|f\|^k_{L_2(\Psi_H)},$$

and

$$|\mathrm{P}_H W^*_{\nu,f}| \ge \frac{(2\sqrt{k})^k}{(k)!}\|f\|^{-k}_{L_2(\Psi_H)},$$

with equalities if and only if $H \cap W_{\nu,f}$ is an origin-symmetric cube with side length of $\frac{2}{\sqrt{k}}\|f\|_{L_2(\Psi_H)}$ and $\frac{f(u)}{\|\mathrm{P}_H u\|}$ is constant for all $u \in \Psi_H$.

Our proofs are based on a refinement of the approaches taken by Schuster and Weberndorfer [29], Lutwak, Yang, and Zhang [23], and Ball [3]. More precisely, the concept of isotropic embedding and the Ball-Barthe inequality play important roles in the proofs. For more applications of this approach, see e.g., [14, 16–19, 21, 22, 29].

This paper is organized as follows. In Section 2, background material is provided. Sections 3 and 4 contain the proofs of Theorem 1.1 and Theorem 1.2. As applications, several special Wulff shapes are investigated in Section 5.

2. Background material

We collect some background material. General references are the books of Gardner [12] and Schneider [28].

Let $e_1, \dots, e_n$ be the standard orthonormal basis of $\mathbb{R}^n$. The Minkowski functional $\|\cdot\|_K$ of a convex body K in $\mathbb{R}^n$ is defined by

$$\|x\|_K = \min\{t > 0 : x \in tK\}.$$

By using polar coordinates, we have

$$|K| = \frac{1}{\Gamma(1+\frac{n}{p})} \int_{\mathbb{R}^n} e^{-\|x\|_K^p} dx, \tag{2.1}$$

where integration is with respect to Lebesgue measure on $\mathbb{R}^n$.

It is well-known that the isotropy condition (1.2) is equivalent to

$$\|x\|^2 = \int_{S^{n-1}} |x \cdot u|^2 d\nu(u), \tag{2.2}$$

for all $x \in \mathbb{R}^n$. Special cases of even isotropic measures are the *cross measures*, which are concentrated uniformly on $\{\pm u_1, \dots, \pm u_n\}$, where $u_1, \dots, u_n$ is an orthonormal basis of $\mathbb{R}^n$.

Suppose that H is a k-dimensional subspace of $\mathbb{R}^n$ with $1 \le k \le n$. Let ν be a Borel measure on S^{n-1} and let $\bar{\nu}$ be the Borel measure on $S^{n-1} \cap H$ defined in (1.3). Then for any continuous function $g : S^{n-1} \cap H \to \mathbb{R}$,

$$\int_{S^{n-1}\cap H} g(w) d\bar{\nu}(w) = \int_{S^{n-1}\setminus H^{\perp}} g\Big(\frac{\mathrm{P}_H u}{\|\mathrm{P}_H u\|}\Big) \|\mathrm{P}_H u\|^2 d\nu(u). \tag{2.3}$$

If ν is an isotropic measure on S^{n-1}, then for an arbitrary $y \in H$, we have

$$\int_{S^{n-1}\cap H} |y\cdot w|^2 d\bar{\nu}(w) = \int_{S^{n-1}\setminus H^{\perp}} \left| y \cdot \frac{\mathrm{P}_H u}{\|\mathrm{P}_H u\|}\right|^2 \|\mathrm{P}_H u\|^2 d\nu(u)$$
$$= \int_{S^{n-1}} |y\cdot u|^2 d\nu(u) = \|y\|^2.$$

Hence, $\bar{\nu}$ is isotropic on $S^{n-1}\cap H$. Moreover, we have

$$\bar{\nu}(S^{n-1}\cap H) = k. \tag{2.4}$$

Let f be a positive, bounded, and measurable function on S^{n-1}. By (1.4), we have

$$\begin{aligned} H\cap W_{\nu,f} &= \left\{ y\in H : y\cdot w \le \inf_{u\in\Xi_w} \frac{f(u)}{\|\mathrm{P}_H u\|} \ \text{ for all } \ w\in \operatorname{supp}\bar{\nu}\right\} \\ &= \{y\in H : y\cdot w\le \bar{f}(w) \ \text{ for all } \ w\in\operatorname{supp}\bar{\nu}\} \\ &:= \bar{W}_{\bar{\nu},\bar{f}}, \end{aligned} \tag{2.5}$$

where the function $\bar{f} : S^{n-1}\cap H \to (0,\infty)$ is defined by, for $w\in \operatorname{supp}\bar{\nu}$,

$$\bar{f}(w) = \inf_{u\in\Xi_w}\frac{f(u)}{\|\mathrm{P}_H u\|} = \frac{f(u)}{\|\mathrm{P}_H u\|}, \quad u\in\Psi_w. \tag{2.6}$$

For a k-dimensional subspace H, define the $(k+1)$-dimensional subspace H' in $\mathbb{R}^{n+1} = \mathbb{R}^n\times\mathbb{R}$ by

$$H' = \operatorname{span}\{H, e_{n+1}\}. \tag{2.7}$$

Since we can identify $S^{n-1}\cap H$ with S^{k-1} in $\mathbb{R}^k$, and identify $S^n\cap H'$ with S^k in $\mathbb{R}^{k+1}$, a lemma due to Schuster and Weberndorfer [29, Lemma 4.1] can be restated as follows.

Lemma 2.1. *Let $\bar{\nu}$ be an isotropic measure on $S^{n-1}\cap H$ and let $\bar{f}$ be a positive bounded measurable function on $S^{n-1}\cap H$. For $w\in S^{n-1}\cap H$, define the functions $\varphi_{\pm} : S^{n-1}\cap H\to H'\setminus\{0\}$ by*

$$\varphi_{\pm}(w) = (\pm w, \bar{f}(w)). \tag{2.8}$$

Then the measure $\tilde{\nu}$ on $S^n\cap H'$, defined by

$$\int_{S^n\cap H'} \tilde{g}(\eta)d\tilde{\nu}(\eta) = \int_{S^{n-1}\cap H} \tilde{g}\Big(\frac{\varphi_{\pm}(w)}{\|\varphi_{\pm}(w)\|}\Big)\|\varphi_{\pm}(w)\|^2 d\bar{\nu}(w) \tag{2.9}$$

for every continuous $\tilde{g} : S^n\cap H'\to\mathbb{R}$, is isotropic if and only if $\bar{\nu}$ is $\bar{f}$-centered on H and $\|\bar{f}\|_{L_2(\bar{\nu})} = 1$.

Note that the functions $\varphi_\pm$ lift the isotropic measure $\bar{\nu}$ on $S^{n-1}\cap H$ to the isotropic measure $\tilde{\nu}$ on $S^n \cap H'$ and are therefore usually called *isotropic embeddings* of $\bar{\nu}$. They were introduced by Lutwak, Yang, and Zhang [23].

Lemma 2.2. *If the measure ν on S^{n-1} is f-centered with respect to H and the support subset Ψ_H of ν is measurable such that $\|f\|_{L_2(\Psi_H)} = 1$, then the measure $\tilde{\nu}$ defined in (2.9) is isotropic on $S^n \cap H'$.*

Proof. By Lemma 2.1, it is sufficient to verify that $\bar{\nu}$ is $\bar{f}$-centered on H and $\|\bar{f}\|_{L_2(\bar{\nu})} = 1$. Indeed, it follows from (2.3), (2.6), and (1.5) that

$$\begin{aligned}\int_{S^{n-1}\cap H} \bar{f}(w) w d\bar{\nu}(w) &= \int_{S^{n-1}\setminus H^\perp} \bar{f}\Big(\frac{\mathrm{P}_H u}{\|\mathrm{P}_H u\|}\Big)\frac{\mathrm{P}_H u}{\|\mathrm{P}_H u\|}\|\mathrm{P}_H u\|^2 d\nu(u)\\ &= \int_{\Psi_H} \frac{f(u)}{\|\mathrm{P}_H u\|}\frac{\mathrm{P}_H u}{\|\mathrm{P}_H u\|}\|\mathrm{P}_H u\|^2 d\nu(u)\\ &= \int_{\Psi_H} f(u)\mathrm{P}_H u d\nu(u) = o. \end{aligned} \tag{2.10}$$

That is, $\bar{\nu}$ is $\bar{f}$-centered on H. Moreover, by (2.3) and (2.6), we have

$$\begin{aligned}\|\bar{f}\|_{L_2(\bar{\nu})} &= \Big(\int_{S^{n-1}\cap H} \bar{f}(w)^2 d\bar{\nu}(w)\Big)^{1/2}\\ &= \Big(\int_{S^{n-1}\setminus H^\perp} \bar{f}\Big(\frac{\mathrm{P}_H u}{\|\mathrm{P}_H u\|}\Big)^2 \|\mathrm{P}_H u\|^2 d\nu(u)\Big)^{1/2}\\ &= \Big(\int_{\Psi_H} \Big(\frac{f(u)}{\|\mathrm{P}_H u\|}\Big)^2 \|\mathrm{P}_H u\|^2 d\nu(u)\Big)^{1/2}\\ &= \|f\|_{L_2(\Psi_H)} = 1, \end{aligned} \tag{2.11}$$

which concludes the proof. □

Define the displacement of $\bar{W}_{\bar{\nu},\bar{f}}$ by

$$\mathrm{disp}(\bar{W}_{\bar{\nu},\bar{f}}) = \frac{1}{|\bar{W}_{\bar{\nu},\bar{f}}|}\int_{\bar{W}_{\bar{\nu},\bar{f}}} x dx \cdot \int_{S^{n-1}\cap H} \frac{w}{\bar{f}(w)} d\bar{\nu}(w). \tag{2.12}$$

Then, it follows from (2.5), (2.3), and (1.7) that

$$\mathrm{disp}(H \cap W_{\nu,f}) = \mathrm{disp}(\bar{W}_{\bar{\nu},\bar{f}}). \tag{2.13}$$

The following continuous version of the Ball-Barthe inequality was established by Lutwak, Yang, and Zhang [21], extending the discrete case due to Ball and Barthe [6].

Lemma 2.3. *If ν is an isotropic measure on S^{k-1} in $\mathbb{R}^k$ and t is a positive continuous function on* $\operatorname{supp}\nu$, *then*

$$\det \int_{S^{k-1}} t(u) u \otimes u d\nu(u) \geq \exp\Big\{ \int_{S^{k-1}} \log t(u) d\nu(u) \Big\}, \tag{2.14}$$

with equality if and only if $t(u_1)\cdots t(u_k)$ is constant for linearly independent unit vectors $u_1, \ldots, u_k \in \operatorname{supp}\nu$.

We shall need the following lemma due to Lutwak, Yang, and Zhang [22].

Lemma 2.4. *If ν is an isotropic measure on S^{k-1} in $\mathbb{R}^k$ and $h \in L_2(\nu)$, then*

$$\Big\| \int_{S^{k-1}} u h(u) d\nu(u) \Big\| \leq \Big(\int_{S^{k-1}} h(u)^2 d\nu(u) \Big)^{1/2}.$$

3. Asymmetric cases

Theorem 1.1 immediately follows from Theorems 3.1 and 3.2.

Theorem 3.1. *Let H be a k-dimensional subspace of $\mathbb{R}^n$ with $1 \leq k \leq n$. Suppose that f is a positive bounded measurable function on S^{n-1} and that the measure ν on S^{n-1} is isotropic and f-centered with respect to H. If the support subset Ψ_H of ν is measurable, then*

$$|H \cap W_{\nu,f}| \leq \frac{(k+1-\operatorname{disp}(H\cap W_{\nu,f}))^{k+1}}{k!(k+1)^{\frac{k+1}{2}}} \|f\|^k_{L_2(\Psi_H)}, \tag{3.1}$$

with equality if and only if $H \cap W_{\nu,f}$ is a regular simplex inscribed in $\frac{\|f\|_{L_2(\Psi_H)}}{\sqrt{k}}(S^{n-1}\cap H)$ and $\frac{f(u)}{\|\mathrm{P}_H u\|}$ is constant for all $u \in \Psi_H$.

Proof. By (1.1), (1.7) and (1.8), we may assume that $\|f\|_{L_2(\Psi_H)} = 1$.

Taking $\varphi_- = (-w, \bar{f}(w))$ in (2.8), Lemma 2.2 yields that $\tilde{\nu}$ defined in (2.9) is isotropic on $S^n \cap H'$.

By (2.5), we can consider $\bar{W}_{\bar{\nu},\bar{f}}$ instead of $H \cap W_{\nu,f}$. Define the cone $C \subset H' = H \times \operatorname{span}\{e_{n+1}\}$ by

$$C = \bigcup_{r>0} r\bar{W}_{\bar{\nu},\bar{f}} \times \{r\} \subset H'.$$

Obviously, $e_{n+1} \in C$. By (2.9), $\eta \in \operatorname{supp} \tilde{\nu}$ if and only if

$$\eta = \frac{(-w, \bar{f}(w))}{\sqrt{1 + \bar{f}^2(w)}} \tag{3.2}$$

for some $w \in \operatorname{supp} \bar{\nu}$. Moreover, it follows from the definition of $\bar{W}_{\bar{\nu},\bar{f}}$ that, for every $\eta \in \operatorname{supp} \tilde{\nu}$ and $z = (y, r) \in C$,

$$\eta \cdot z = \frac{-w \cdot y + r\bar{f}(w)}{\sqrt{1 + \bar{f}^2(w)}} \geq 0.$$

Now, for $\eta \in \operatorname{supp} \tilde{\nu}$, define the strictly increasing function $\phi_\eta : (0, \infty) \to \mathbb{R}$ by

$$\int_{-\infty}^{\phi_\eta(\tau)} e^{-\pi s^2} ds = \frac{1}{e_{n+1} \cdot \eta} \int_0^\tau \exp\Big(-\frac{s}{e_{n+1} \cdot \eta}\Big) ds.$$

Differentiating both sides with respect to τ, taking logarithms, and putting $\tau = \eta \cdot z$ for $\eta \in \operatorname{supp} \tilde{\nu}$ and $z \in \operatorname{int} C$, we get

$$\log \phi_\eta'(\eta \cdot z) - \pi \phi_\eta^2(\eta \cdot z) = -\log(e_{n+1} \cdot \eta) - \frac{\eta \cdot z}{e_{n+1} \cdot \eta}. \tag{3.3}$$

Define the transformation $T : \operatorname{int} C \to H'$ by

$$T(z) = \int_{S^n \cap H'} \phi_\eta(\eta \cdot z) \eta d\tilde{\nu}(\eta).$$

Hence, for every $z \in \operatorname{int} C$,

$$dT(z) = \int_{S^n \cap H'} \phi_\eta'(\eta \cdot z) \eta \otimes \eta d\tilde{\nu}(\eta). \tag{3.4}$$

Since $\phi_\eta' > 0$, the matrix $dT(z)$ is positive definite for $z \in \operatorname{int} C$. Therefore, $T : \operatorname{int} C \to H'$ is injective. Moreover, applying Lemma 2.4 with $h(\eta) = \phi_\eta(\eta \cdot z)$ yields

$$\|T(z)\|^2 \leq \int_{S^n \cap H'} \phi_\eta(\eta \cdot z)^2 d\tilde{\nu}(\eta). \tag{3.5}$$

From (3.3), the Ball-Barthe inequality (2.14) with $t(\eta) = \phi_\eta'(\eta \cdot z)$, (3.4), (3.5), and the change of variables $x = T(z)$, we obtain

$$\begin{aligned}
&\int_{\operatorname{int} C} \exp\Big(\int_{S^n \cap H'} \Big(-\log(e_{n+1} \cdot \eta) - \frac{\eta \cdot z}{e_{n+1} \cdot \eta}\Big) d\tilde{\nu}(\eta)\Big) dz \\
&= \int_{\operatorname{int} C} \exp\Big(\int_{S^n \cap H'} -\pi \phi_\eta^2(\eta \cdot z) d\tilde{\nu}(\eta)\Big) \exp\Big(\int_{S^n \cap H'} \log \phi_\eta'(\eta \cdot z) d\tilde{\nu}(\eta)\Big) dz \\
&\leq \int_{\operatorname{int} C} \exp(-\pi \|T(z)\|^2) \det dT(z) dz \leq \int_{H'} e^{-\pi \|x\|^2} dx = 1.
\end{aligned} \tag{3.6}$$

On the other hand, the isotropy of $\tilde{\nu}$ on $S^n \cap H'$ yields $\tilde{\nu}(S^n \cap H') = k+1$. So applying Jensen's inequality, we arrive at

$$\begin{aligned}
\exp\Big(\int_{S^n\cap H'} \log(e_{n+1}\cdot\eta)d\tilde{\nu}(\eta)\Big) &= \exp\Big(\int_{S^n\cap H'} \frac{1}{k+1}\log(e_{n+1}\cdot\eta)^2 d\tilde{\nu}(\eta)\Big)^{\frac{k+1}{2}} \\
&\le \exp\Big[\log\Big(\int_{S^n\cap H'} \frac{1}{k+1}(e_{n+1}\cdot\eta)^2 d\tilde{\nu}(\eta)\Big)\Big]^{\frac{k+1}{2}} \\
&= \Big(\frac{1}{k+1}\Big)^{\frac{k+1}{2}}. \qquad (3.7)
\end{aligned}$$

By (2.9), (3.2), (2.10), and (2.11), we obtain that, for $z = (y, r) \in C$,

$$\begin{aligned}
\int_{S^n\cap H'} \frac{(y,r)\cdot\eta}{e_{n+1}\cdot\eta} d\tilde{\nu}(\eta) &= \int_{S^{n-1}\cap H} \frac{(y,r)\cdot(-w,\bar{f}(w))/\sqrt{1+\bar{f}^2(w)}}{e_{n+1}\cdot(-w,\bar{f}(w))/\sqrt{1+\bar{f}^2(w)}}(1+\bar{f}^2(w))d\bar{\nu}(w) \\
&= \int_{S^{n-1}\cap H} \Big(-\frac{y\cdot w}{\bar{f}(w)} + r - (y\cdot w)\bar{f}(w) + r\bar{f}^2(w)\Big)d\bar{\nu}(w) \\
&= -\int_{S^{n-1}\cap H} \frac{y\cdot w}{\bar{f}(w)} d\bar{\nu}(w) + (k+1)r. \qquad (3.8)
\end{aligned}$$

Applying Jensen's inequality and (2.12), we get

$$\begin{aligned}
&\frac{1}{|r\bar{W}_{\bar{\nu},\bar{f}}|}\int_{r\bar{W}_{\bar{\nu},\bar{f}}} \exp\Big(\int_{S^{n-1}\cap H} \frac{y\cdot w}{\bar{f}(w)} d\bar{\nu}(w)\Big)dy \\
&\qquad \ge \exp\Big(\frac{1}{|r\bar{W}_{\bar{\nu},\bar{f}}|}\int_{r\bar{W}_{\bar{\nu},\bar{f}}} \int_{S^{n-1}\cap H} \frac{y\cdot w}{\bar{f}(w)} d\bar{\nu}(w)dy\Big) \\
&\qquad = \exp(r\mathrm{disp}(\bar{W}_{\bar{\nu},\bar{f}})). \qquad (3.9)
\end{aligned}$$

Thus, by (3.7), (3.8), and (3.9), we have

$$\begin{aligned}
&\int_{\text{int } C} \exp\Big(\int_{S^n\cap H'}\Big(-\log(e_{n+1}\cdot\eta)-\frac{\eta\cdot z}{e_{n+1}\cdot\eta}\Big)d\tilde\nu(\eta)\Big)dz\\
&\geq (k+1)^{\frac{k+1}{2}}\int_{\text{int } C}\exp\Big(\int_{S^{n-1}\cap H}\frac{y\cdot w}{\bar f(w)}d\bar\nu(w)-(k+1)r\Big)dz\\
&=(k+1)^{\frac{k+1}{2}}\int_0^\infty e^{-(k+1)r}\int_{r\bar W_{\bar\nu,\bar f}}\exp\Big(\int_{S^{n-1}\cap H}\frac{y\cdot w}{\bar f(w)}d\bar\nu(w)\Big)dydr\\
&\geq (k+1)^{\frac{k+1}{2}}\int_0^\infty e^{-(k+1)r}\exp\Big(r\text{disp}(\bar W_{\bar\nu,\bar f})\Big)|r\bar W_{\bar\nu,\bar f}|dr.\\
&=(k+1)^{\frac{k+1}{2}}|\bar W_{\bar\nu,\bar f}|\int_0^\infty e^{-(k+1-\text{disp}(\bar W_{\bar\nu,\bar f}))r}r^k dr.\\
&=(k+1)^{\frac{k+1}{2}}|\bar W_{\bar\nu,\bar f}|k!(k+1-\text{disp}(\bar W_{\bar\nu,\bar f}))^{-(k+1)}.
\end{aligned}$$

Note that by (2.12), $\text{disp}(\bar W_{\bar\nu,\bar f})\leq k$. This, together with (3.6), (2.5), and (2.13), yields that

$$|H\cap W_{\nu,f}|\leq\frac{(k+1-\text{disp}(H\cap W_{v,f}))^{k+1}}{k!(k+1)^{\frac{k+1}{2}}},$$

which is the desired inequality.

Assume that there is equality in inequality (3.1). By the equality conditions of Jensen's inequality, equality in (3.7) holds if and only if $e_{n+1}\cdot\eta$ is constant for any $\eta\in\text{supp}\,\tilde\nu$. It follows from (3.2) that $\bar f$ is constant on $\text{supp}\,\bar\nu$, which, by (2.6), means that the function $\frac{f(u)}{\|\mathrm{P}_H u\|}$ is constant for all $u\in\Psi_H$. Moreover, by (2.11), we must have $\bar f(w)=\frac{1}{\sqrt{k}}\|f\|_{L_2(\Psi_H)}$ on $\text{supp}\,\bar\nu$.

Since $\tilde\nu$ is isotropic on $S^n\cap H'$, there exist linearly independent $\eta_1,\ldots,\eta_{k+1}\in\text{supp}\,\tilde\nu$ such that

$$\{\eta_1,\ldots,\eta_{k+1}\}\subseteq\text{supp}\,\tilde\nu.$$

Assume that there exists a different vector $\eta_0\in\text{supp}\,\tilde\nu$. Write $\eta_0=\lambda_1\eta_1+\cdots+\lambda_{k+1}\eta_{k+1}$ such that at least one coefficient, say λ_1, is not zero. Then the equality conditions of the Ball-Barthe inequality imply that

$$\phi'_{\eta_1}(z\cdot\eta_1)\phi'_{\eta_2}(z\cdot\eta_2)\cdots\phi'_{\eta_{k+1}}(z\cdot\eta_{k+1})=\phi'_{\eta_0}(z\cdot\eta_0)\phi'_{\eta_2}(z\cdot\eta_2)\cdots\phi'_{\eta_{k+1}}(z\cdot\eta_{k+1}),$$

for all $z\in H'$. But $\phi'_\eta>0$, and hence

$$\phi'_{\eta_1}(z\cdot\eta_1)=\phi'_{\eta_0}(z\cdot\eta_0)$$

for all $z \in H'$. Differentiating both sides with respect to z shows that

$$\phi''_{\eta_1}(z\cdot\eta_1)\eta_1 = \phi''_{\eta_0}(z\cdot\eta_0)\eta_0,$$

for all $z \in H'$. Since there exists $z \in H'$ such that $\phi''_{\eta_1}(z\cdot\eta_1) \neq 0$, it follows that $\eta_0 = \pm\eta_1$. But $\tilde{\nu}$ is supported inside a hemisphere of $S^n \cap H'$, so $\eta_0 = \eta_1$. Consequently,

$$\{\eta_1, \dots, \eta_{k+1}\} = \operatorname{supp}\tilde{\nu}.$$

Therefore, we have for $y \in H'$

$$|y|^2 = \sum_{i=1}^{k+1} \tilde{\nu}(\{\eta_i\})|y\cdot\eta_i|^2.$$

Substituting $y = \eta_j \in S^n \cap H'$, we see that necessarily $\tilde{\nu}(\{\eta_j\}) \le 1$. From the fact that $\sum_{i=1}^{k+1}\tilde{\nu}(\{\eta_i\}) = k+1$ we get $\tilde{\nu}(\{\eta_j\}) = 1$. Thus, $\eta_j\cdot\eta_i = 0$ for $j \neq i$. That is, $\eta_1, \dots, \eta_{k+1}$ is an orthonormal basis of H'.

From (2.9) and (3.2), it follows that $\operatorname{supp}\bar{\nu} = \{w_1, \dots, w_{k+1}\}$, where

$$\eta_i = \frac{(-w_i, \bar{f}(w_i))}{\sqrt{1+\bar{f}^2(w_i)}}, \quad 1 \le i \le k+1.$$

Moreover, we have

$$0 = \eta_i\cdot\eta_j = \frac{(-w_i, \bar{f}(w_i))}{\sqrt{1+\bar{f}^2(w_i)}} \cdot \frac{(-w_j, \bar{f}(w_j))}{\sqrt{1+\bar{f}^2(w_j)}}, \quad 1 \le i \neq j \le k+1.$$

That is, $w_i\cdot w_j = -\bar{f}(w_i)\bar{f}(w_j)$ for all $i \neq j$. Since $\bar{f}(w) = \frac{\|f\|_{L_2(\Psi_H)}}{\sqrt{k}}$, we obtain that $w_i\cdot w_j = -\frac{\|f\|^2_{L_2(\Psi_H)}}{k}$, $i \neq j$. By the normalization $\|f\|_{L_2(\Psi_H)} = 1$, we have $w_i\cdot w_j = -\frac{1}{k}$ for all $i \neq j$. Hence, $\operatorname{conv}(\operatorname{supp}\bar{\nu})$ must be a regular simplex inscribed in the subsphere $S^{n-1}\cap H$. Now (2.5) implies that $H \cap W_{\nu,f}$ is a regular simplex inscribed in $\frac{\|f\|_{L_2(\Psi_H)}}{\sqrt{k}}(S^{n-1}\cap H)$. □

Theorem 3.2. *Under the conditions of Theorem 3.1, we have*

$$|\mathrm{P}_H W^*_{\nu,f}| \ge \frac{(k+1)^{\frac{k+1}{2}}}{k!}\|f\|^{-k}_{L_2(\Psi_H)},$$

with equality if and only if $H\cap W_{\nu,f}$ is a regular simplex inscribed in $\frac{\|f\|_{L_2(\Psi_H)}}{\sqrt{k}}(S^{n-1}\cap H)$ and $\frac{f(u)}{\|\mathrm{P}_H u\|}$ is constant for all $u \in \Psi_H$.

Proof. By (1.1) and (1.8), we may assume that $\|f\|_{L_2(\Psi_H)} = 1$.

Taking $\varphi_+ = (w, \bar{f}(w))$ in (2.8), Lemma 2.2 yields that $\tilde{\nu}$ defined in (2.9) is isotropic on $S^n \cap H'$. For $\eta \in \operatorname{supp}\tilde{\nu}$, define the strictly increasing function $\phi_\eta : \mathbb{R} \to (0, \infty)$ by

$$e_{n+1} \cdot \eta \int_0^{\phi_\eta(\tau)} e^{-(e_{n+1}\cdot\eta)s} ds = \int_{-\infty}^{\tau} e^{-\pi s^2} ds.$$

Differentiating both sides with respect to τ, taking logarithms, and putting $\tau = \eta \cdot z$ for $\eta \in \operatorname{supp}\tilde{\nu}$ and $z \in H'$, we get

$$\log(e_{n+1} \cdot \eta) - (e_{n+1} \cdot \eta)\phi_\eta(\eta \cdot z) + \log \phi'_\eta(\eta \cdot z) = -\pi(\eta \cdot z)^2. \tag{3.10}$$

Define the transformation $T : H' \to H'$ by

$$T(z) = \int_{S^n \cap H'} \phi_\eta(\eta \cdot z)\eta d\tilde{\nu}(\eta), \tag{3.11}$$

for $z \in H'$. The differential of T is given by

$$dT(z) = \int_{S^n \cap H'} \phi'_\eta(\eta \cdot z)\eta \otimes \eta d\tilde{\nu}(\eta). \tag{3.12}$$

Since $\phi'_\eta > 0$, the matrix $dT(z)$ is positive definite for every $z \in H'$. Therefore, $T : H' \to H'$ is injective.

Define the cone $C \subset H' = H \times \operatorname{span}\{e_{n+1}\}$ by

$$C = \bigcup_{r>0} r\bar{W}^*_{\bar{\nu},\bar{f}} \times \{r\}.$$

Note that $T(z) \subset C$ for all $z \in H'$. To see this, it is sufficient to show that if $T(z) = (y, r) \in H'$ and $x \in \bar{W}_{\bar{\nu},\bar{f}}$, then $y \cdot x \leq r$. By (2.9) with φ_+ given in(2.8), (3.11), and the fact that $w \cdot x \leq \bar{f}(w)$ for every $w \in \operatorname{supp}\bar{\nu}$, we have

$$\begin{aligned} y \cdot x &= \int_{S^{n-1}\cap H} \phi_\eta\Big(\frac{(w, \bar{f}(w))}{\sqrt{1+\bar{f}^2(w)}} \cdot z\Big)\Big(\frac{w}{\sqrt{1+\bar{f}^2(w)}} \cdot x\Big)(1+\bar{f}^2(w))d\bar{\nu}(w) \\ &\leq \int_{S^{n-1}\cap H} \phi_\eta\Big(\frac{(w, \bar{f}(w))}{\sqrt{1+\bar{f}^2(w)}} \cdot z\Big)\frac{\bar{f}(w)}{\sqrt{1+\bar{f}^2(w)}}(1+\bar{f}^2(w))d\bar{\nu}(w) \\ &= \int_{S^n\cap H'} \phi_\eta(\eta \cdot z)(\eta \cdot e_{n+1})d\tilde{\nu}(\eta) \\ &= T(z) \cdot e_{n+1} = r. \end{aligned}$$

From (2.1), the fact that $T(z) \subset C$ for all $z \in H'$, the Ball-Barthe inequality (2.14) with $t(\eta) = \phi'_\eta(\eta \cdot z)$, (3.12), (3.11), (3.10), the isotropy of $\tilde{\nu}$ on H', Jensen's inequality, and the isotropy of $\tilde{\nu}$ again, we have

$$\begin{aligned}
k!|\bar{W}^*_{\tilde{\nu},\bar{f}}| &= \int_H e^{-\|x\|_{\bar{W}^*_{\tilde{\nu},\bar{f}}}} dx = \int_0^\infty \int_{r\bar{W}^*_{\tilde{\nu},\bar{f}}} e^{-r} dx dr = \int_C e^{-e_{n+1}\cdot x} dx \\
&\geq \int_{H'} e^{-e_{n+1}\cdot T(z)} \det dT(z) dz \\
&\geq \int_{H'} \exp\Big(- \int_{S^n \cap H'} \phi_\eta(\eta \cdot z)(e_{n+1}\cdot\eta) d\tilde{\nu}(\eta)\Big) \exp\Big(\int_{S^n\cap H'} \log \phi'_\eta(\eta\cdot z) d\tilde{\nu}(\eta)\Big) dz \\
&= \exp\Big(- \int_{S^n\cap H'} \log(e_{n+1}\cdot\eta) d\tilde{\nu}(\eta)\Big) \int_{H'} \exp\Big(-\pi \int_{S^n\cap H'} (\eta\cdot z)^2 d\tilde{\nu}(\eta)\Big) dz \\
&= \exp\Big(- \int_{S^n\cap H'} \log(e_{n+1}\cdot\eta) d\tilde{\nu}(\eta)\Big) \int_{H'} e^{-\pi\|z\|^2} dz \\
&= \exp\Big(\int_{S^n\cap H'} \frac{1}{k+1} \log(e_{n+1}\cdot\eta)^2 d\tilde{\nu}(\eta)\Big)^{-\frac{k+1}{2}} \\
&\geq (k+1)^{\frac{k+1}{2}}.
\end{aligned}$$

Note that (see e.g., [12, (0.38)])

$$(H \cap W_{\nu,f})^* = \mathrm{P}_H W^*_{\nu,f}, \tag{3.13}$$

where the polar operation on the left is taken in H. Thus, we have

$$|\mathrm{P}_H W^*_{\nu,f}| = |(H\cap W_{\nu,f})^*| = |\bar{W}^*_{\tilde{\nu},\bar{f}}| \geq \frac{(k+1)^{\frac{k+1}{2}}}{k!}.$$

The equality conditions are proved in basically the same way as in the proof of Theorem 3.1.

□

4. Symmetric cases

Theorem 1.2 immediately follows from the following two theorems.

Theorem 4.1. *Let H be a k-dimensional subspace of $\mathbb{R}^n$ with $1 \leq k \leq n$. Suppose that f is an even positive bounded measurable function on S^{n-1} and that the measure ν on S^{n-1} is even isotropic. If the support subset Ψ_H of ν is measurable, then*

$$|H \cap W_{\nu,f}| \leq \Big(\frac{2}{\sqrt{k}}\Big)^k \|f\|^k_{L_2(\Psi_H)},$$

with equality if and only if $H \cap W_{\nu,f}$ *is an origin-symmetric cube with side length of* $\frac{2}{\sqrt{k}}\|f\|_{L_2(\Psi_H)}$ *and* $\frac{f(u)}{\|\mathrm{P}_H u\|}$ *is constant for all* $u \in \Psi_H$*.*

Proof. Clearly, $H \cap W_{\nu,f} = \bar{W}_{\bar{\nu},\bar{f}}$ is origin-symmetric and $\bar{\nu}$ and $\bar{f}$ are both even.

For $w \in \operatorname{supp}\bar{\nu}$, define the strictly increasing function $\phi_w : (-\bar{f}(w), \bar{f}(w)) \to \mathbb{R}$ by

$$\frac{1}{\bar{f}(w)} \int_{-\bar{f}(w)}^{\tau} \mathbb{1}_{[-\bar{f}(w),\bar{f}(w)]}(s)ds = \frac{1}{\Gamma(\frac{3}{2})} \int_{-\infty}^{\phi_w(\tau)} e^{-s^2} ds.$$

Differentiating both sides with respect to τ, taking logarithms on both sides and putting $\tau = y \cdot w$ for $y \in \bar{W}_{\bar{\nu},\bar{f}}$ and $w \in \operatorname{supp}\bar{\nu}$, we get

$$\log \Gamma\left(\frac{3}{2}\right) - \log \bar{f}(w) + \log \mathbb{1}_{[-\bar{f}(w),\bar{f}(w)]}(y \cdot w) = -\phi_w(y \cdot w)^2 + \log \phi_w'(y \cdot w). \quad (4.1)$$

Define the transformation $T : \operatorname{int} \bar{W}_{\bar{\nu},\bar{f}} \to H$ by

$$T(y) = \int_{S^{n-1} \cap H} w \phi_w(y \cdot w) d\bar{\nu}(w). \quad (4.2)$$

Then, the differential of T is given by

$$dT(y) = \int_{S^{n-1} \cap H} w \otimes w \phi_w'(y \cdot w) d\bar{\nu}(w). \quad (4.3)$$

Since $\phi_w' > 0$, the matrix $dT(y)$ is positive definite for each $y \in \operatorname{int} \bar{W}_{\bar{\nu},\bar{f}}$. Hence, the transformation $T : \operatorname{int} \bar{W}_{\bar{\nu},\bar{f}} \to H$ is injective.

Applying Lemma 2.4 with $h(w) = \phi_w(y \cdot w)$ yields

$$\|T(y)\|^2 \le \int_{S^{n-1} \cap H} \phi_w(y \cdot w)^2 d\bar{\nu}(w). \quad (4.4)$$

The definition of $\bar{W}_{\bar{\nu},\bar{f}}$ implies that, for all $y \in \bar{W}_{\bar{\nu},\bar{f}}$,

$$\exp\Big(\int_{S^{n-1} \cap H} \log \mathbb{1}_{[-\bar{f}(w),\bar{f}(w)]}(y \cdot w) d\bar{\nu}(w)\Big) = 1. \quad (4.5)$$

From (4.5), (4.1), (2.4), (4.4), the Ball-Barthe inequality (2.14) with $t(w) = \phi_w'(w \cdot y)$, (4.3), and the change of variable $z = Ty$, we have

$$\begin{aligned} |\bar{W}_{\bar{\nu},\bar{f}}| &= \int_{\operatorname{int} \bar{W}_{\bar{\nu},\bar{f}}} dy \\ &= \int_{\operatorname{int} \bar{W}_{\bar{\nu},\bar{f}}} \exp\Big(\int_{S^{n-1} \cap H} \log \mathbb{1}_{[-\bar{f}(w),\bar{f}(w)]}(y \cdot w) d\bar{\nu}(w)\Big) dy \end{aligned}$$

$$
\begin{aligned}
&= \Gamma\Big(\frac{3}{2}\Big)^{k-n} \exp\Big(\int_{S^{n-1}\cap H} \log \bar{f}(w) d\bar{\nu}(w)\Big)\\
&\times \int_{\mathrm{int}\,\bar{W}_{\bar{\nu},\bar{f}}} \exp\Big(\int_{S^{n-1}\cap H} -\phi_w(y\cdot w)^2 d\bar{\nu}(w)\Big) \exp\Big(\int_{S^{n-1}\cap H} \log \phi_w'(y\cdot w) d\bar{\nu}(w)\Big) dy\\
&\le \Gamma\Big(\frac{3}{2}\Big)^{k-n} \exp\Big(\int_{S^{n-1}\cap H} \log \bar{f}(w) d\bar{\nu}(w)\Big) \int_{\mathrm{int}\,\bar{W}_{\bar{\nu},\bar{f}}} e^{-\|Ty\|^2} \det dT(y) dy\\
&\le \Gamma\Big(\frac{3}{2}\Big)^{k-n} \exp\Big(\int_{S^{n-1}\cap H} \log \bar{f}(w) d\bar{\nu}(w)\Big) \int_H e^{-\|z\|^2} dz\\
&= 2^k \exp\Big(\int_{S^{n-1}\cap H} \log \bar{f}(w) d\bar{\nu}(w)\Big).
\end{aligned}
$$

Applying Jensen's inequality and (2.11), we get

$$
\begin{aligned}
\exp\Big(\int_{S^{n-1}\cap H} \log \bar{f}(w) d\bar{\nu}(w)\Big) &= \Big[\exp\Big(\frac{1}{k}\int_{S^{n-1}\cap H} \log \bar{f}(w) d\bar{\nu}(w)\Big)\Big]^k\\
&\le \Big(\frac{1}{k}\int_{S^{n-1}\cap H} \bar{f}^2(w) d\bar{\nu}(w)\Big)^{k/2}\\
&= \frac{1}{k^{k/2}} \|\bar{f}\|^k_{L_2(\bar{\nu})} = \frac{1}{k^{k/2}} \|f\|^k_{L_2(\Psi_H)}. \qquad (4.6)
\end{aligned}
$$

Therefore, we obtain the desired inequality

$$
|H\cap W_{\nu,f}| = |\bar{W}_{\bar{\nu},\bar{f}}| \le \Big(\frac{2}{\sqrt{k}}\Big)^k \|f\|^k_{L_2(\Psi_H)}.
$$

Assume that there is equality in inequality (3.1). As in the proof of Theorem 3.1, it is easy to verify that $\bar{\nu}$ is a cross measure on $S^{n-1}\cap H$. By the equality conditions of Jensen's inequality, equality in (4.6) holds if only if $\bar{f}(w)$ is constant for every $w \in \operatorname{supp}\bar{\nu}$. Hence, by (2.6), the function $\frac{f(u)}{\|\mathrm{P}_H u\|}$ is constant for all $u \in \Psi_H$. Moreover, by (2.11), we must have $\bar{f}(w) = \frac{1}{\sqrt{k}}\|f\|_{L_2(\Psi_H)}$ on $\operatorname{supp}\bar{\nu}$. It follows from (2.5) that $H\cap W_{\nu,f}$ is an origin-symmetric cube with side length of $\frac{2}{\sqrt{k}}\|f\|_{L_2(\Psi_H)}$. □

The following lemma was proved by Lutwak, Yang, and Zhang [21, Lemma 3.1].

Lemma 4.2. *Let $\bar{\nu}$ be an even Borel measure on $S^{n-1}\cap H$ and let $\bar{W}_{\bar{\nu},\bar{f}}$ be a Wulff shape on H. If $h \in L_1(\nu)$, then*

$$
\Big\| \int_{S^{n-1}\cap H} w h(w) d\nu(w)\Big\|_{\bar{W}^*_{\bar{\nu},\bar{f}}} \le \int_{S^{n-1}\cap H} \bar{f}(w)|h(w)| d\nu(w), \qquad (4.7)
$$

where the polar operation on $\bar{W}_{\bar{\nu},\bar{f}}$ is taken in H.

Theorem 4.3. *Under the conditions of Theorem 4.1, we have*

$$|\mathrm{P}_H W^*_{\nu,f}| \geq \frac{(2\sqrt{k})^k}{k!}\|f\|^{-k}_{L_2(\Psi_H)},$$

with equality if and only if $H \cap W_{\nu,f}$ is an origin-symmetric cube with side length of $\frac{2}{\sqrt{k}}\|f\|_{L_2(\Psi_H)}$ and $\frac{f(u)}{\|\mathrm{P}_H u\|}$ is constant for all $u \in \Psi_H$.

Proof. For $w \in \operatorname{supp}\bar{\nu}$, define the strictly increasing function $\phi_w : \mathbb{R} \to \mathbb{R}$ by

$$\frac{1}{\Gamma(\frac{3}{2})}\int_{-\infty}^{\tau} e^{-s^2}ds = \bar{f}(w)\int_{-\infty}^{\phi_w(\tau)} e^{-\bar{f}(w)|s|}ds.$$

Differentiating both sides with respect to τ, taking logarithms and putting $\tau = y \cdot w$ for $y \in H \cap W_{\nu,f}$ and $w \in \operatorname{supp}\bar{\nu}$, we get

$$-(y\cdot w)^2 = \log\Gamma\Big(\frac{3}{2}\Big) + \log\bar{f}(w) - \bar{f}(w)|\phi_w(y\cdot w)| + \log\phi_w'(y\cdot w). \tag{4.8}$$

Define the transformation $T : H \to H$ by

$$Ty = \int_{S^{n-1}\cap H} w\phi_w(y\cdot w)d\bar{\nu}(w),$$

for each $y \in H$. The differential of T is given by

$$dT(y) = \int_{S^{n-1}\cap H} w\otimes w\phi_w'(y\cdot w)d\bar{\nu}(w). \tag{4.9}$$

Since $\phi_w' > 0$, the matrix $dT(y)$ is positive definite for each $y \in H$. Hence, the transformation $T : H \to H$ is injective.

Taking $h(w) = \phi_w(y\cdot w)$ in (4.7) gives

$$\|Ty\|_{\bar{W}^*_{\bar{\nu},\bar{f}}} \leq \int_{S^{n-1}\cap H} \bar{f}(w)|\phi_w(y\cdot w)|d\bar{\nu}(w). \tag{4.10}$$

From the isotropy of $\bar{\nu}$, (4.8), (2.4), the Ball-Barthe inequality (2.14) with $t(w) = \phi_w'(w\cdot y)$, (4.9), (4.10), the change of variables $z = Ty$ and (2.1), we have

$$\begin{aligned}
\pi^{\frac{k}{2}} &= \int_H e^{-\|y\|^2}dy \\
&= \int_H \exp\Big\{-\int_{S^{n-1}\cap H}(y\cdot w)^2 d\bar{\nu}(w)\Big\}dy \\
&= \Gamma\Big(\frac{3}{2}\Big)^k \exp\Big(\int_{S^{n-1}\cap H}\log\bar{f}(w)d\bar{\nu}(w)\Big) \\
&\times \int_H \exp\Big\{-\int_{S^{n-1}\cap H}\bar{f}(w)|\phi_w(y\cdot w)|d\bar{\nu}(w)\Big\}\exp\Big\{\int_{S^{n-1}\cap H}\log\phi_w'(y\cdot w)d\bar{\nu}(w)\Big\}dy
\end{aligned}$$

$$\leq \Gamma\Big(\frac{3}{2}\Big)^k \exp\Big(\int_{S^{n-1}\cap H} \log \bar{f}(w) d\bar{\nu}(w)\Big) \int_H e^{-\|Ty\|_{\bar{W}^*_{\bar{\nu},\bar{f}}}} \det dT(y) dy$$
$$\leq \Gamma\Big(\frac{3}{2}\Big)^k \exp\Big(\int_{S^{n-1}\cap H} \log \bar{f}(w) d\bar{\nu}(w)\Big) \int_H e^{-\|z\|_{\bar{W}^*_{\bar{\nu},\bar{f}}}} dz$$
$$= \Gamma\Big(\frac{3}{2}\Big)^k \exp\Big(\int_{S^{n-1}\cap H} \log \bar{f}(w) d\bar{\nu}(w)\Big) (k)! |\bar{W}^*_{\bar{\nu},\bar{f}}|.$$

Together with (3.13) and (4.6), this yields

$$|\mathrm{P}_H W^*_{\nu,f}| = |(H\cap W_{\nu,f})^*| = |\bar{W}^*_{\bar{\nu},\bar{f}}|$$
$$\geq \frac{2^k}{k!} \exp\Big(\frac{1}{k}\int_{S^{n-1}\cap H} \log \bar{f}(w) d\bar{\nu}(w)\Big)^{-k} \geq \frac{(2\sqrt{k})^k}{k!} \|f\|^{-k}_{L^2(\Psi_H)},$$

which gives the desired inequality.

The equality conditions are proved in basically the same way as in the proof of Theorem 4.1.

□

5. Applications

Let H be a k-dimensional subspace of $\mathbb{R}^n$ with $1 \leq k \leq n$ and let ν be an isotropic measure on S^{n-1}. Denote by $W_{\nu,\|\mathrm{P}_H\|}$ the special Wulff shape with $f(u) = \|\mathrm{P}_H u\|$ for all $u \in \operatorname{supp}\nu \setminus H^\perp$. It follows from (2.5) that

$$H \cap W_{\nu,\|\mathrm{P}_H\|} = \bar{W}_{\bar{\nu},1}.$$

Note that $\Psi_H = \operatorname{supp}\nu \setminus H^\perp$ is a measurable set. Moreover, from (1.8), (2.3), and (2.4), we have

$$\|f\|_{L_2(\Psi_H)} = \Big(\int_{\operatorname{supp}\nu\setminus H^\perp} \|\mathrm{P}_H u\|^2 d\nu(u)\Big)^{1/2} = \bar{\nu}(S^{n-1}\cap H) = k^{1/2}.$$

By (1.5), we may say that the measure ν is $\|\mathrm{P}_H\|$-centered with respect to H if

$$\int_{\operatorname{supp}\nu\setminus H^\perp} \|\mathrm{P}_H u\| \mathrm{P}_H u \, d\nu(u) = o. \tag{5.1}$$

In this case, it follows from (1.7) that $\operatorname{disp}(H \cap W_{\nu,\|\mathrm{P}_H\|}) = o$.

Applying Theorems 3.1 and 3.2 to $W_{\nu,\|\mathrm{P}_H\|}$, we obtain the following result.

Theorem 5.1. *Let H be a k-dimensional subspace of $\mathbb{R}^n$ with $1 \le k \le n$. If ν is isotropic on S^{n-1} and $\|\mathrm{P}_H\|$-centered with respect to H, then*

$$|H \cap W_{\nu,\|\mathrm{P}_H\|}| \le \frac{k^{\frac{k}{2}}(k+1)^{\frac{k+1}{2}}}{k!},$$

and

$$|\mathrm{P}_H W^*_{\nu,\|\mathrm{P}_H\|}| \ge \frac{(k+1)^{\frac{k+1}{2}}}{k^{\frac{k}{2}}k!},$$

with equality if and only if $H \cap W_{\nu,\|\mathrm{P}_H\|}$ is a regular simplex inscribed in $S^{n-1} \cap H$.

Similarly, applying Theorems 4.1 and 4.3 to $W_{\nu,\|\mathrm{P}_H\|}$, we obtain the following theorem.

Theorem 5.2. *Let H be a k-dimensional subspace of $\mathbb{R}^n$ with $1 \le k \le n$. If ν is even isotropic on S^{n-1}, then*

$$|H \cap W_{\nu,\|\mathrm{P}_H\|}| \le 2^k, \tag{5.2}$$

and

$$|\mathrm{P}_H W^*_{\nu,\|\mathrm{P}_H\|}| \ge \frac{2^k}{k!},$$

with equality if and only if $H \cap W_{\nu,\|\mathrm{P}_H\|}$ is an origin-symmetric cube with side length of 2.

Next, we consider another two special cases of Wulff shapes, namely, regular simplices and cubes.

Unfortunately, the inequalities in Theorem 1.1 are not sharp for regular simplices. Webb [30] proved that the maximal central hyperplane section of regular simplices in $\mathbb{R}^n$ contains $n-1$ vertices and the centroid, which is no longer a regular simplex in H. However, Theorem 1.1 still provides an upper bound for the volume sections of regular simplices by any subspace H.

Now, we consider the cube $\hat{Q}_n = [-\sqrt{k/n}, \sqrt{k/n}]^n$. It was shown in [2] that if k divides n, then the maximal k-dimensional section is attained by the subspace $H_{max} = \mathrm{span}\{w_1, \dots, w_k\}$, where $w_j = v_j/|v_j|$ and $v_j = e_{(j-1)l+1} + \cdots + e_{jl}$, $l = n/k$, $j = 1, \dots, k$. Let ν be a cross measure concentrated uniformly on $\{\pm e_1, \dots, \pm e_n\}$. We will show that $W_{\nu,\|\mathrm{P}_{H_{max}}\|} = \hat{Q}_n$. It suffices to verify that $\|\mathrm{P}_{H_{max}} e_i\| = \sqrt{k/n}$ for $i = 1, \ldots, n$. Notice that $\omega_1, \ldots, \omega_k$ form an orthonormal basis of H_{max}. If $i \in (j-1)l+1, \ldots, jl$, then we have

$$\mathrm{P}_{H_{max}} e_i = (e_i \cdot w_j) w_j = \sqrt{\frac{k}{n}} w_j, \tag{5.3}$$

which is the desired result. Furthermore, we have

$$\begin{aligned} H_{max} \cap \hat{Q}_n &= \{y \in H_{max} : |y \cdot \mathrm{P}_{H_{max}} e_i| \leq \sqrt{\frac{k}{n}} \text{ for all } i = 1, \ldots, n\} \\ &= \left\{y \in H_{max} : \left|y \cdot \sqrt{\frac{k}{n}} w_j\right| \leq \sqrt{\frac{k}{n}} \text{ for all } j = 1, \ldots, k\right\} \\ &= \left\{y \in H_{max} : |y \cdot w_j| \leq 1, \text{ for all } j = 1, \ldots, k\right\}, \end{aligned} \tag{5.4}$$

that is, $H_{max} \cap \hat{Q}_n$ is isometric to B^k_∞. By the equality conditions of Theorem 5.2, equality in (5.2) holds in this case.

We now recover the volume inequality for sections of $\hat{Q}_n$ due to Ball [2] and Barthe [7].

Corollary 5.3. *Let H be a k-dimensional subspace of $\mathbb{R}^n$ with $1 \leq k \leq n$. Then*

$$|H \cap \hat{Q}_n| \leq 2^k. \tag{5.5}$$

There is equality if and only if k divides n.

Proof. Obviously,

$$\|f\|_{L_2(\Psi_H)} \leq \|f\|_{L_2(\nu)} = \sqrt{k}. \tag{5.6}$$

Thus, Theorem 4.1 implies that

$$|H \cap Q_n| \leq \left(\frac{2}{\sqrt{k}}\right)^k \|f\|^k_{L_2(\Psi_H)} \leq 2^k. \tag{5.7}$$

If k divides n, inequality (5.5) holds with equality for H_{max}. Conversely, equality in (5.6) shows that $\Psi_H = \operatorname{supp} \nu = \{\pm e_1, \ldots, \pm e_n\}$. The equality conditions of Theorem 4.1 yield that $\bar{f}(w) = \frac{1}{\sqrt{k}} \|f\|_{L_2(\Psi_H)} = 1$ on $\operatorname{supp} \bar{\nu}$, and thus $\frac{f(e_i)}{\|\mathrm{P}_H e_i\|} = \frac{\sqrt{k/n}}{\|\mathrm{P}_H e_i\|} = 1$ for all $i = 1, \ldots, n$. Hence we get $\|\mathrm{P}_H e_i\| = \sqrt{k/n}$ for all $i = 1, \ldots, n$, which implies $\mathrm{P}_H e_i \in \{\pm\sqrt{k/n}\,\omega_{j(i)}\}$ for a $j(i) \in \{1, \ldots, k\}$. Therefore,

$$e_i \cdot \omega_{j(i)} = \pm \|\mathrm{P}_H e_i\| = \pm\sqrt{\frac{k}{n}}, \tag{5.8}$$

and $e_i \cdot \omega_{j'} = 0$ for $j' \neq j(i)$. In other words, we can divide $e_1, \cdots, e_n$ into k parts such that each $\omega_{j(i)}$ is a linear combination of some elements of $\{e_1, \ldots, e_n\}$. Write $\omega_{j(i)} = \lambda_{j_1} e_{j_1} + \cdots + \lambda_{j_m} e_{j_m}$ with $\lambda_{j_1}^2 + \cdots + \lambda_{j_m}^2 = 1$. By (5.8), we know that $\lambda_{j_1} = \cdots = \lambda_{j_m} = \pm\sqrt{k/n}$. Together with $\lambda_{j_1}^2 + \cdots + \lambda_{j_m}^2 = 1$, we must have $m = n/k$, which is an integer.

□

The same argument as above, together with Theorem 4.3, yields the following corollary due to Barthe [7].

Corollary 5.4. *Let H be a k-dimensional subspace of $\mathbb{R}^n$ with $1 \leq k \leq n$. Then*

$$|\mathrm{P}_H \hat{Q}_n^*| \geq \frac{2^k}{k!}.$$

There is equality if and only if k divides n.

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